Technische Universiteit
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Collusion-resistant fingerprinting and group testing

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Van der Meulen seminar, Enschede, The Netherlands
(November 27, 2014)

Outline

Search problems

Introduction

Problem: Given a universe $\mathcal{U}$ of n elements and a model $\vec{\theta}$, find a hidden subset $\mathcal{C} \subset \mathcal{U}$ of size $c \ll n$ using *subset queries*.

Subset query: Given a subset $\mathcal{S} \subseteq \mathcal{U}$, an oracle returns a bit $y_{\mathcal{S}}$:

$$\mathbb{P}(Y_{\mathcal{S}} = 1) = \theta_z. \qquad (z = |\mathcal{S} \cap \mathcal{C}| \in \{0, \dots, c\})$$

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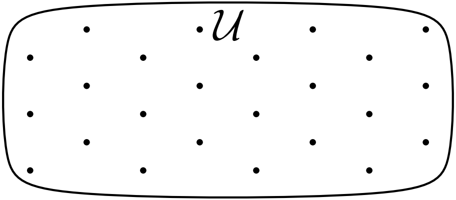

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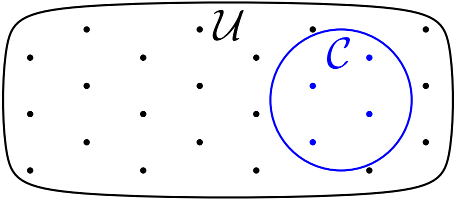

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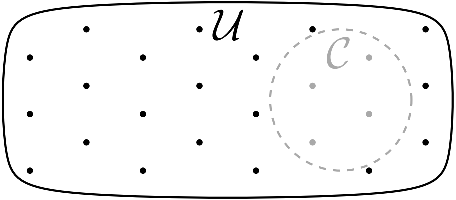

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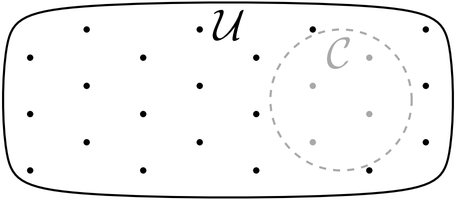

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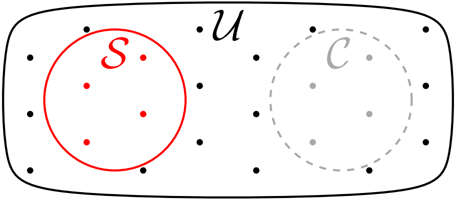

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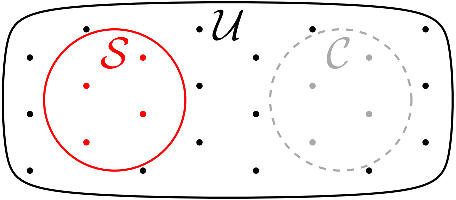

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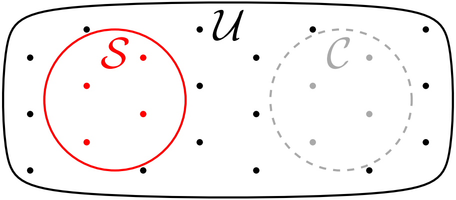

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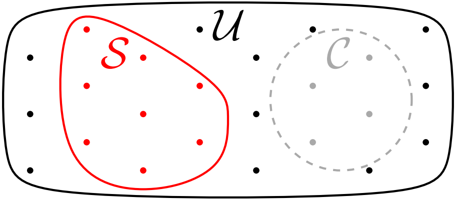

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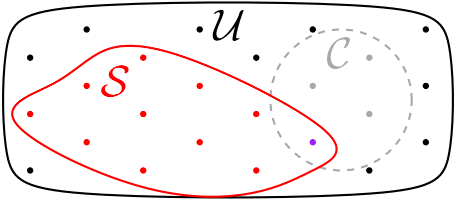

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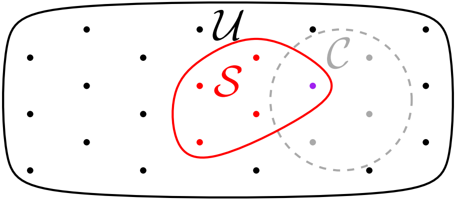

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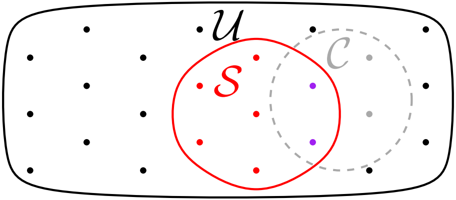

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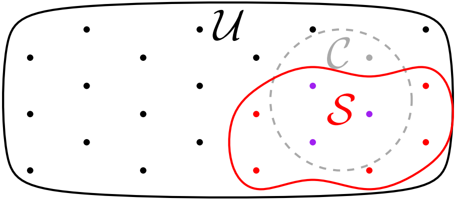

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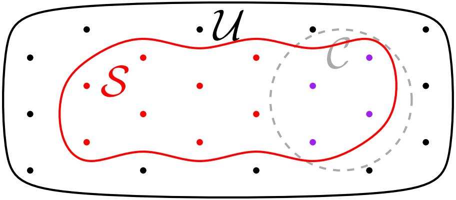

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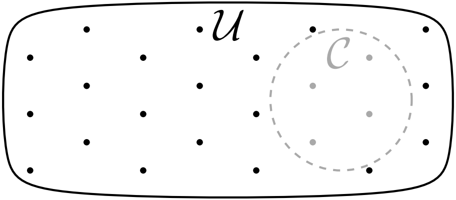

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Search problems

Fingerprinting and group testing models

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \theta_{c-1} \\ \theta_c \end{bmatrix} =$$

Search problems

Fingerprinting and group testing models

Fingerprinting

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \theta_{c-1} \\ \theta_c \end{bmatrix} = \begin{bmatrix} 0 \\ * \\ * \\ * \\ \vdots \\ * \\ 1 \end{bmatrix}$$

Search problems

Fingerprinting and group testing models

$$
\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \theta_{c-1} \\ \theta_c \end{bmatrix}
=
\underset{\textbf{Fingerprinting}}{\begin{bmatrix} 0 \\ * \\ * \\ * \\ \vdots \\ * \\ 1 \end{bmatrix}}
\quad
\underset{\text{...coinflip atk.}}{\begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1/2 \\ \vdots \\ 1/2 \\ 1 \end{bmatrix}}
\quad
\underset{\text{...majority atk.}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix}}
\quad
\underset{\text{...linear atk.}}{\begin{bmatrix} 0 \\ 1/c \\ 2/c \\ 3/c \\ \vdots \\ (c-1)/c \\ 1 \end{bmatrix}}
$$

Search problems

Fingerprinting and group testing models

Fingerprinting | ...coinflip atk. | ...majority atk. | ...linear atk.

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \theta_{c-1} \\ \theta_c \end{bmatrix} = \begin{bmatrix} 0 \\ * \\ * \\ * \\ \vdots \\ * \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1/2 \\ \vdots \\ 1/2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1/c \\ 2/c \\ 3/c \\ \vdots \\ (c-1)/c \\ 1 \end{bmatrix}$$

- **Fingerprinting**: Model corresponds to adversary; unknown

Search problems

Fingerprinting and group testing models

Fingerprinting | ...coinflip atk. | ...majority atk. | ...linear atk. | **Group testing**

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \theta_{c-1} \\ \theta_c \end{bmatrix} = \begin{bmatrix} 0 \\ * \\ * \\ * \\ \vdots \\ * \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1/2 \\ \vdots \\ 1/2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1/c \\ 2/c \\ 3/c \\ \vdots \\ (c-1)/c \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$

- **Fingerprinting**: Model corresponds to adversary; unknown

Search problems

Fingerprinting and group testing models

		Fingerprinting	...coinflip atk.	...majority atk.	...linear atk.	**Group testing**	...with noise	...with thresholds
θ_0		0	0	0	0	0	r	0
θ_1		$*$	$1/2$	0	$1/c$	1	1	$*$
θ_2		$*$	$1/2$	0	$2/c$	1	1	$*$
θ_3	$=$	$*$	$1/2$	0	$3/c$	1	1	1
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
θ_{c-1}		$*$	$1/2$	1	$(c-1)/c$	1	1	1
θ_c		1	1	1	1	1	1	1

- **Fingerprinting**: Model corresponds to adversary; unknown

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Fingerprinting and group testing models

θ	**Fingerprinting**	...coinflip atk.	...majority atk.	...linear atk.	**Group testing**	...with noise	...with thresholds
θ_0	0	0	0	0	0	r	0
θ_1	$*$	$1/2$	0	$1/c$	1	1	$*$
θ_2	$*$	$1/2$	0	$2/c$	1	1	$*$
θ_3	$*$	$1/2$	0	$3/c$	1	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
θ_{c-1}	$*$	$1/2$	1	$(c-1)/c$	1	1	1
θ_c	1	1	1	1	1	1	1

- **Fingerprinting**: Model corresponds to adversary; unknown
- **Group testing**: Model generally known but possibly noisy

Overview

Lower bounds on fingerprinting

How many queries ℓ are necessary for non-adaptive fingerprinting?

- 1998: $\ell = \Omega(c \log n)$[1]
- 2003: $\ell = \Omega(c^2 \log \frac{n}{c})$[2]
- 2003: $\ell = \Omega(c^2 \log n)$[3]
- 2009: $\ell \overset{?}{\sim} 2c^2 \ln n$[4]
- 2012: $\ell \sim 2c^2 \ln n$[5]
  - asymptotic optimal attack is the linear attack ($\theta_z = {}^z/_c$)

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[2] C. Peikert *et al.*, "Lower bounds for collusion-secure fingerprinting," *ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 2003, 472–479.

[3] G. Tardos, "Optimal probabilistic fingerprint codes," *ACM Symposium on Theory of Computing (STOC)*, 2003, 116–125.

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[5] Y.-W. Huang P. Moulin, "On the saddle-point solution and the large-coalition asymptotics of fingerprinting games," *IEEE Transactions on Information Forensics and Security*, 7, 1, 160–175, 2012.

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Overview

Upper bounds on fingerprinting

How many queries ℓ are sufficient for non-adaptive fingerprinting?

- 1995: $\ell = O(c^4 \log n)$[1]
- 2003: $\ell = 100c^2 \ln n$[2] ("the Tardos scheme")
- 2006: $\ell \sim 4\pi^2 c^2 \ln n$[6]
- 2008: $\ell \sim \pi^2 c^2 \ln n$[7]
- 2011: $\ell \sim \frac{1}{2}\pi^2 c^2 \ln n$[8]
- 2013: $\ell \sim 2c^2 \ln n$[9]
  - decoder designed against the linear attack is 'optimal'

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[6] B. Skoric *et al.*, "Tardos fingerprinting is better than we thought," *IEEE Transactions on Information Theory*, 54, 8, 3663–3676, 2008.

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[9] J.-J. Oosterwijk *et al.*, "A capacity-achieving simple decoder for bias-based traitor tracing schemes," *Cryptology ePrint Archive*, 2013.

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Overview

Lower bounds on group testing

How many queries ℓ are necessary for non-adaptive group testing?

- 1985: $\ell \sim c \log_2 n$[10]
- 1989: $\ell = \Omega(c^2 \log n)$[11] (deterministic tracing)
- 2009: $\ell = \Omega(\frac{c \log n}{(1-r)^2})$[12] (noise)
- ...

[10] A. Sebő, "On two random search problems," *Journal of Statistical Planning and Inference*, 11, 1, 23–31, 1985.

[11] A. G. Dyachkov *et al.*, "Superimposed distance codes," *Problems of Control and Information Theory*, 18, 4, 237–250, 1989.

[12] G. K. Atia V. Saligrama, "Boolean compressed sensing and noisy group testing," *IEEE Transactions on Information Theory*, 58, 3, 1880–1901, 2012.

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Overview

Upper bounds on group testing

How many queries ℓ are sufficient for non-adaptive group testing?

- 2005: $\ell \sim 2c^2 \log_2 n$[13] (linear gap)
- 2009: $\ell = O(\frac{c \log n}{(1-r)^3})$[14] (noise)
- 2011: $\ell \sim ec \ln n$[15]
- 2013: $\ell \sim O(\sqrt{g} c \log n)$[16] (coinflip gap)
- 2013: $\ell \sim \pi c \ln n$[17] (majority)
- ...

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[16] C. L. Chan *et al.*, "Stochastic threshold group testing," *IEEE Information Theory Workshop (ITW)*, 2013, 1–5.

[17] T. Laarhoven, "Efficient probabilistic group testing based on traitor tracing," *Annual Allerton Conference on Communication, Control and Computing (Allerton)*, 2013, 1358–1365.

Contributions

Explicit asymptotics of the capacities of various models[18]

- Information-theoretic approach: Mutual information game
- Both simple (efficient) and joint (optimal) decoding
- Can be applied to arbitrary models $\vec{\theta}$

[18] T. Laarhoven, "Asymptotics of fingerprinting and group testing: tight bounds from channel capacities," *submitted to IEEE Transactions on Information Theory*, 1–14, 2014.

[19] T. Laarhoven, "Asymptotics of fingerprinting and group testing: capacity-achieving log-likelihood decoders," *submitted to IEEE Transactions on Information Theory*, 1–13, 2014.

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Capacity-achieving decoders for arbitrary models[19]

- Statistical approach: Neyman-Pearson hypothesis testing
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- Asymptotically optimal regardless of the model

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- **Focus of this talk**

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[18] T. Laarhoven, "Asymptotics of fingerprinting and group testing: tight bounds from channel capacities," *submitted to IEEE Transactions on Information Theory*, 1–14, 2014.

[19] T. Laarhoven, "Asymptotics of fingerprinting and group testing: capacity-achieving log-likelihood decoders," *submitted to IEEE Transactions on Information Theory*, 1–13, 2014.

Capacities

Randomized construction

Choosing the subsets $\mathcal{S} \subseteq \mathcal{U}$ to query

- Choose a parameter $p \in (0, 1)$
- For every query and element j: $\mathbb{P}(j \in \mathcal{S}) = p$
  - Different queries and elements are independent

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Finding the hidden subset $\mathcal{C} \subseteq \mathcal{U}$

- Simple decoding: Decide whether $j \in \mathcal{C}$ based on...
  - X: The information whether $j \in \mathcal{S}$ or not
  - Y: The output bits $y_{\mathcal{S}}$
  - P: The randomly drawn parameters p
- Joint decoding: Decide whether $j \in \mathcal{C}$ based on...
  - X': The information whether $j' \in \mathcal{S}$ or not, for all $j' \in \mathcal{U}$
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$$C^{\text{simple}}(\vec{\theta}) = \max_{p \in (0,1)} I(X; Y | P = p).$$

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Capacities

Joint decoding

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Capacities

Joint decoding

- Joint decoding: Decide whether $j \in \mathcal{C}$ based on...
  - Z: The size of $\mathcal{S} \cap \mathcal{C}$
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TU/e

Capacities

Results

Model $\vec{\theta}$	$C^{\text{simple}}(\vec{\theta})$	$C^{\text{joint}}(\vec{\theta})$
$(0, *, *, *, \ldots, *, *, *, 1)$	$1/(2c^2 \ln 2)$ [5]	$1/(2c^2 \ln 2)$ [5]
$(0, 1/c, 2/c, \ldots, (c-1)/c, 1)$	$1/(2c^2 \ln 2)$ [5]	$1/(2c^2 \ln 2)$ [5]
$(0, 1/2, 1/2, \ldots, 1/2, 1/2, 1)$	$\ln 2/(4c)$	$\log_2(\frac{5}{4})/c$
$(0, 0, 0, 0, \ldots, 1, 1, 1, 1)$	$1/(\pi c \ln 2)$	$1/c$
$(0, 1, 1, 1, \ldots, 1, 1, 1, 1)$	$\ln 2/c$	$1/c$ [10]
$(r, 1, 1, 1, \ldots, 1, 1, 1, 1)$	$[\ln 2 - r + O(r^2)]/c$	$[1 - \frac{1}{2}h(r) + O(r^2)]/c$
$\ldots$	$\ldots$	$\ldots$

Capacities

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$\ldots$	$\ldots$	$\ldots$

$$C^{\text{joint}}(\vec{\theta}) = \frac{1}{c} C(Z\text{-channel with } p = \tfrac{1}{2}) = \log_2(\tfrac{5}{4})/c.$$

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$\ldots$	$\ldots$	$\ldots$

Results

Lower bounds on fingerprinting

How many queries ℓ are necessary for non-adaptive fingerprinting?

- 1998: $\ell = \Omega(c \log n)$[1]
- 2003: $\ell = \Omega(c^2 \log \frac{n}{c})$[2]
- 2003: $\ell = \Omega(c^2 \log n)$[3]
- 2009: $\ell \overset{?}{\sim} 2c^2 \ln n$[4]
- 2012: $\ell \sim 2c^2 \ln n$[5]

Results

Upper bounds on fingerprinting

How many queries ℓ are sufficient for non-adaptive fingerprinting?

- 1995: $\ell = O(c^4 \log n)$[1]
- 2003: $\ell = 100c^2 \ln n$[2] ("the Tardos scheme")
- 2006: $\ell \sim 4\pi^2 c^2 \ln n$[6]
- 2008: $\ell \sim \pi^2 c^2 \ln n$[7]
- 2011: $\ell \sim \frac{1}{2}\pi^2 c^2 \ln n$[8]
- 2013: $\ell \sim 2c^2 \ln n$[9][19]

Results

Lower bounds on group testing

How many queries ℓ are necessary for non-adaptive group testing?

- 1985: $\ell \sim c \log_2 n$[10]
- 1989: $\ell = \Omega(c^2 \log n)$[11] (deterministic tracing)
- 2009: $\ell = \Omega(\frac{c \log n}{(1-r)^2})$[12] (noise)
- 2014: $\ell \sim \frac{c \log_2 n}{\ln 2}$[18] (simple decoding)
- 2014: $\ell \sim \frac{c \log_2 n}{\ln 2 - r + O(r^2)}$[18] (simple decoding, noise)
- 2014: $\ell \sim \frac{c \log_2 n}{1 - \frac{1}{2} h(r) + O(r^2)}$[18] (joint decoding, noise)
- ...

Results

Upper bounds on group testing

How many queries ℓ are sufficient for non-adaptive group testing?

- 2005: $\ell \sim 2c^2 \log_2 n$[13] (linear gap)
- 2009: $\ell = O(\frac{c \log n}{(1-r)^3})$[14] (noise)
- 2011: $\ell \sim ec \ln n$[15] (simple decoding)
- 2013: $\ell \sim O(\sqrt{g} c \log n)$[16] (coinflip gap)
- 2013: $\ell \sim \pi c \ln n$[17] (majority)
- 2014: $\ell \sim \frac{c \log_2 n}{\ln 2}$[19] (simple decoding)
- 2014: $\ell \sim \frac{4c \log_2 n}{\ln 2}$[19] (simple decoding, coinflip)
- 2014: $\ell \sim c \log_{5/4} n$[19] (joint decoding, coinflip)
- ...

Conclusion

Explicit asymptotics of the capacities of various models [18]

- Information-theoretic approach: Mutual information game
- Both simple (efficient) and joint (optimal) decoding
- Can be applied to arbitrary models $\vec{\theta}$

Capacity-achieving decoders for arbitrary models [19]

- Statistical approach: Neyman-Pearson hypothesis testing
- Both simple and joint decoding
- Asymptotically optimal regardless of the model

Questions?