IBM Research

Sieving for shortest lattice vectors using near neighbor techniques

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(April 26, 2017)

Outline

Lattices
- Basics
- Cryptography

Enumeration algorithms
- Fincke–Pohst enumeration
- Kannan enumeration
- Pruned enumeration

Sieving algorithms
- Basic sieving
- Leveled sieving
- Near neighbor searching

Practical comparison

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Lattices

What is a lattice?

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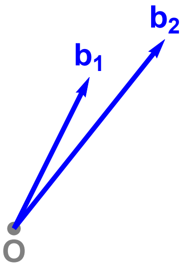

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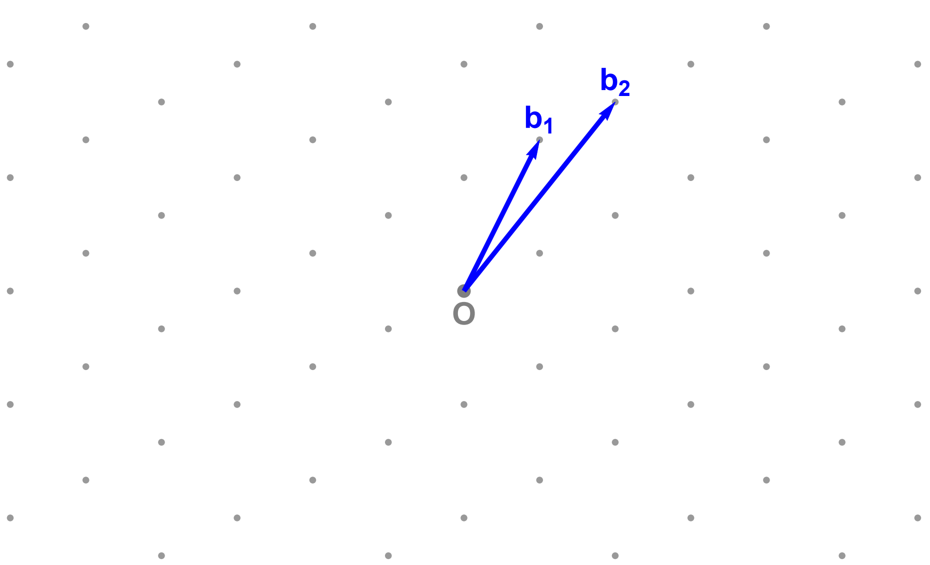

Lattices

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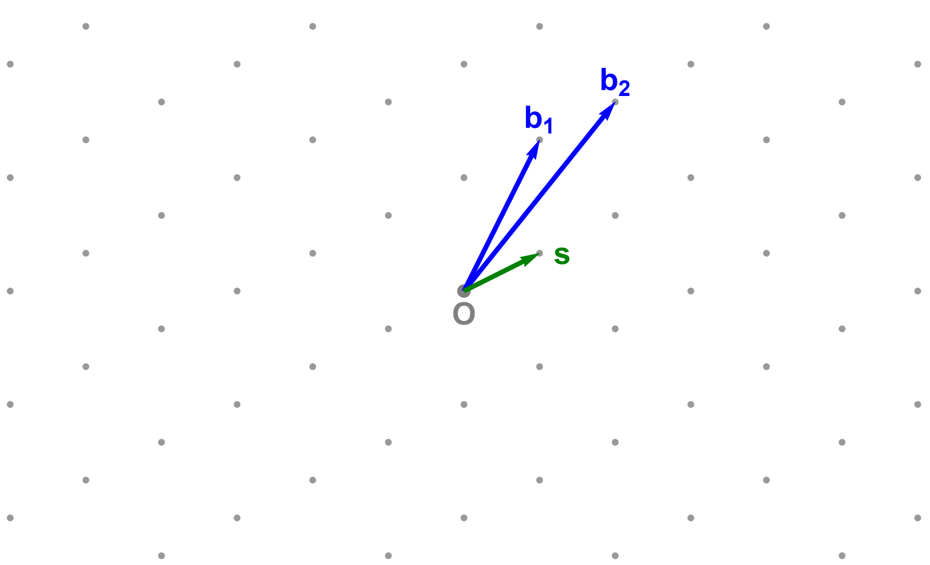

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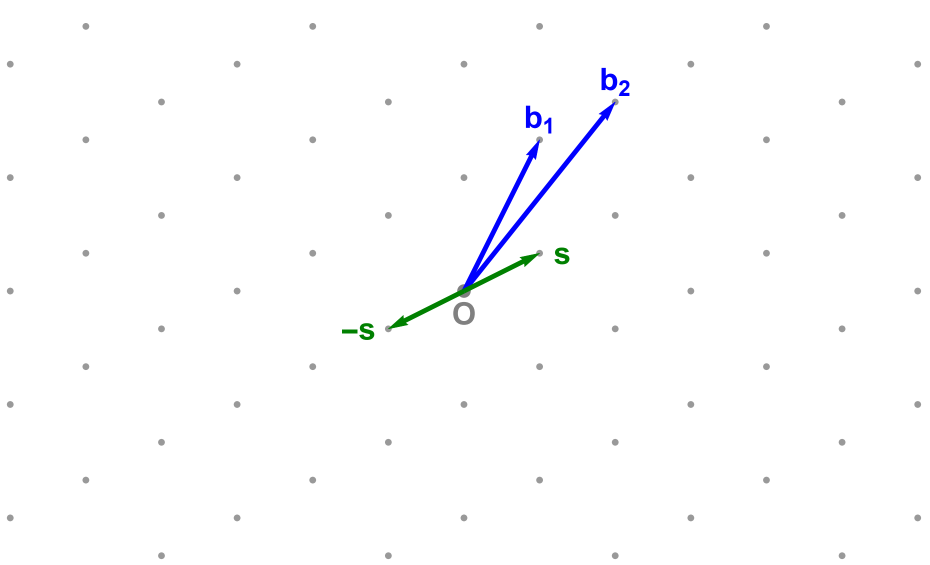

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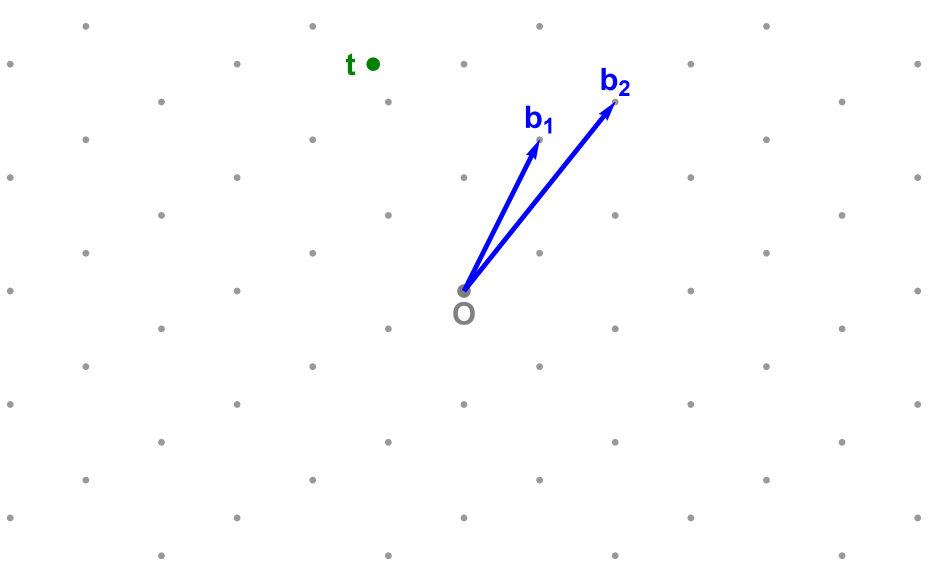

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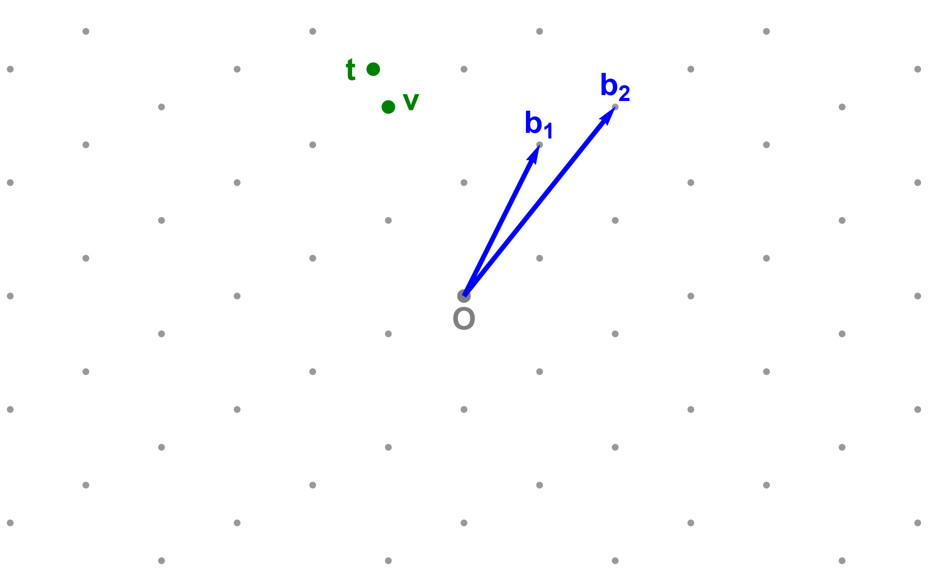
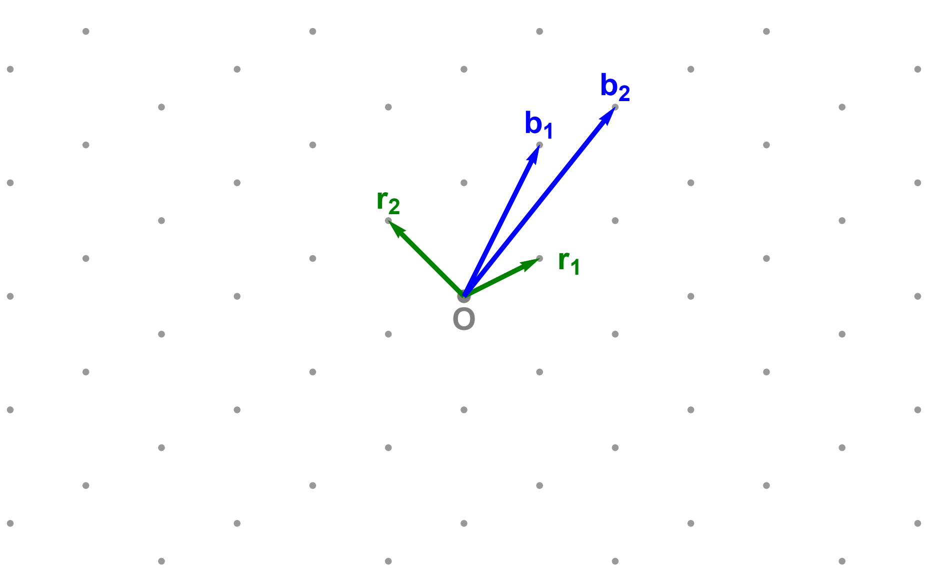


Lattices

Lattice basis reduction



Cryptography

GGH cryptosystem [GGH97]

Private key: $R = \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \end{pmatrix}$

Public key: $B = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}$

Encrypt $\boldsymbol{m}$:

$$\boldsymbol{v} = \boldsymbol{m}B$$
$$\boldsymbol{c} = \boldsymbol{v} + \boldsymbol{e}$$

Decrypt $\boldsymbol{c}$:

$$\boldsymbol{v}' = \lfloor \boldsymbol{c}R^{-1} \rceil R$$
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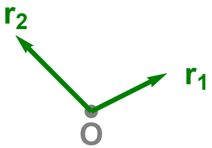

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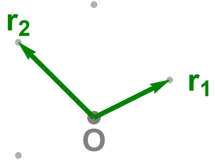

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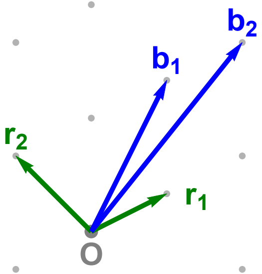

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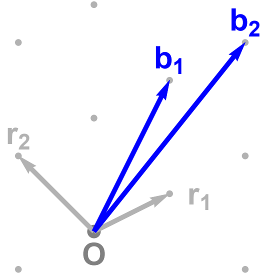

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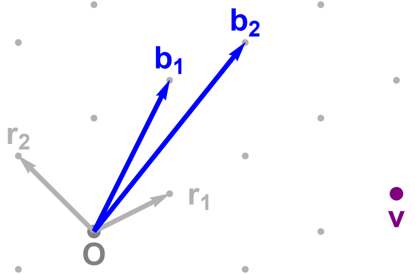

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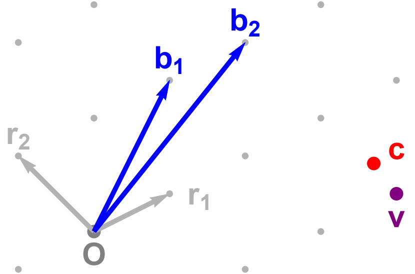

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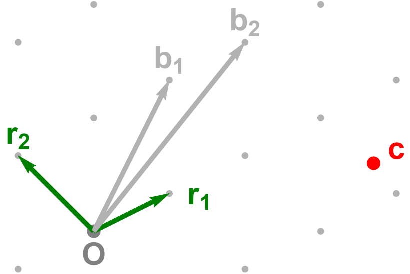

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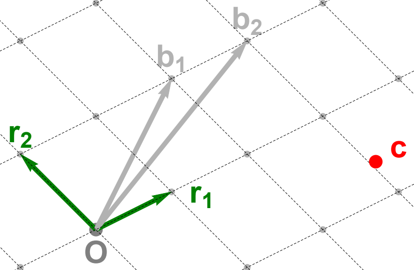

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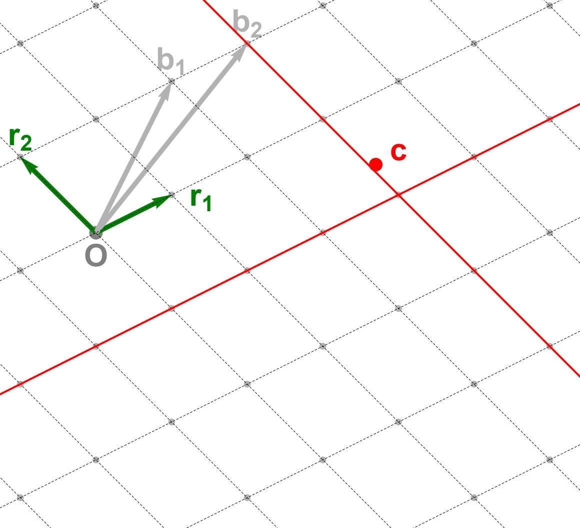

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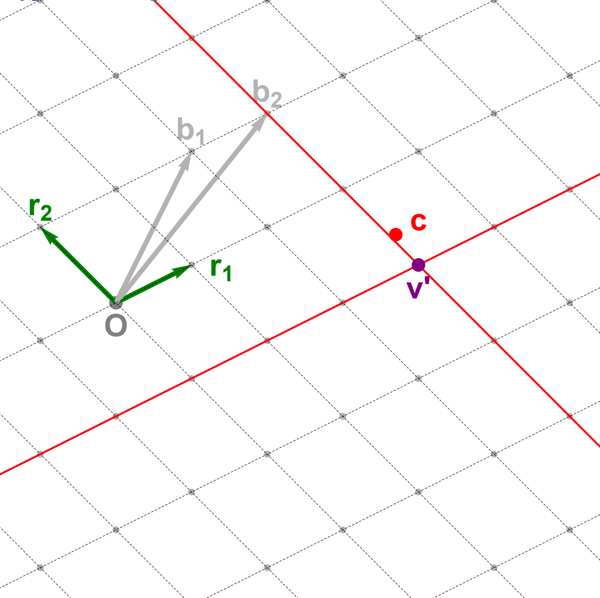

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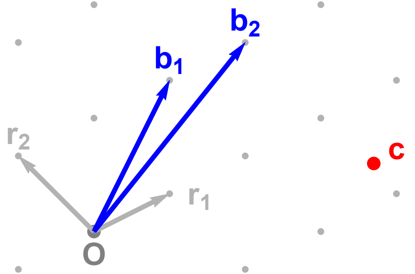

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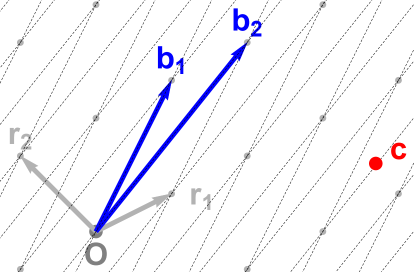

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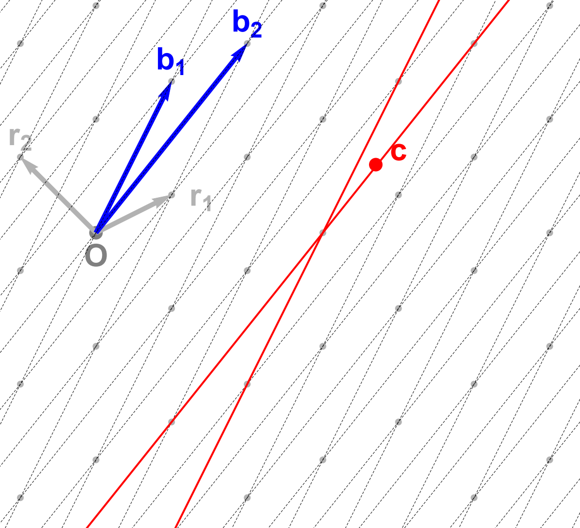

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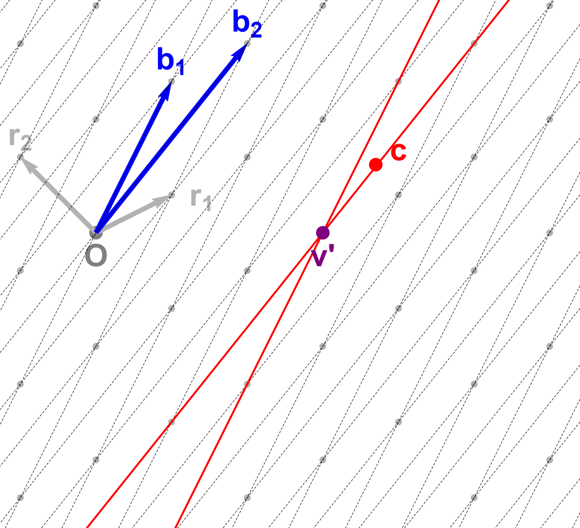

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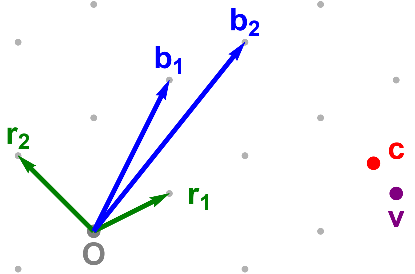

Cryptography

GGH signatures

Private key: $R = \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \end{pmatrix}$

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Sign $\boldsymbol{m}$:

$\boldsymbol{c} = H(\boldsymbol{m})$

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Verify $(\boldsymbol{m}, \boldsymbol{s})$:

$\boldsymbol{s}$ lies on the lattice

$\|\boldsymbol{s} - H(\boldsymbol{m})\|$ is small

Cryptography

Private and public keys

Private key: $R = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$

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Sign m:

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Verify (m, s):

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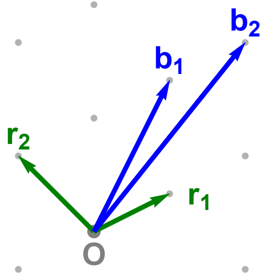

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Cryptography

Signing messages

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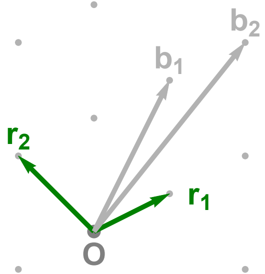

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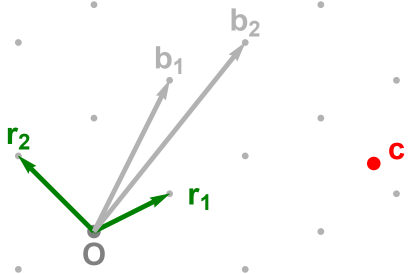

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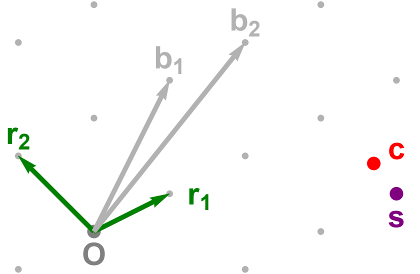

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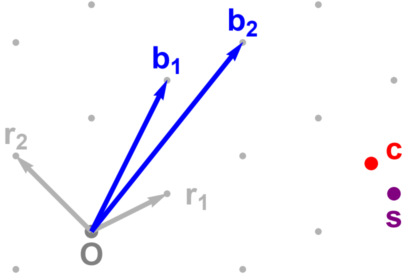

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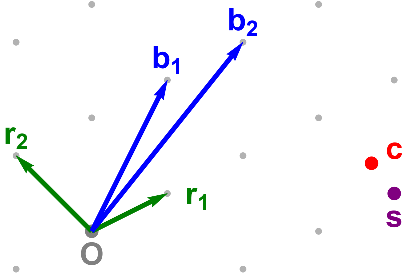
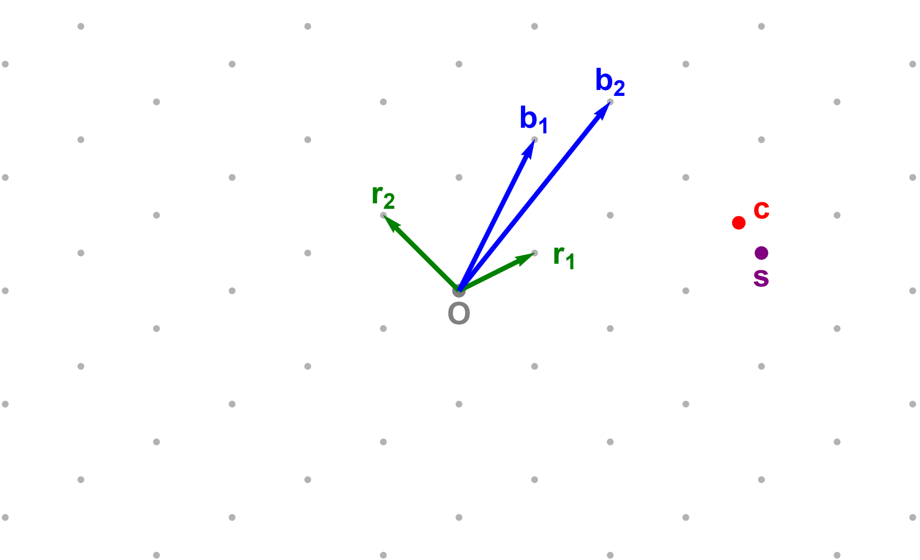


Cryptography

Breaking the scheme [NR06]



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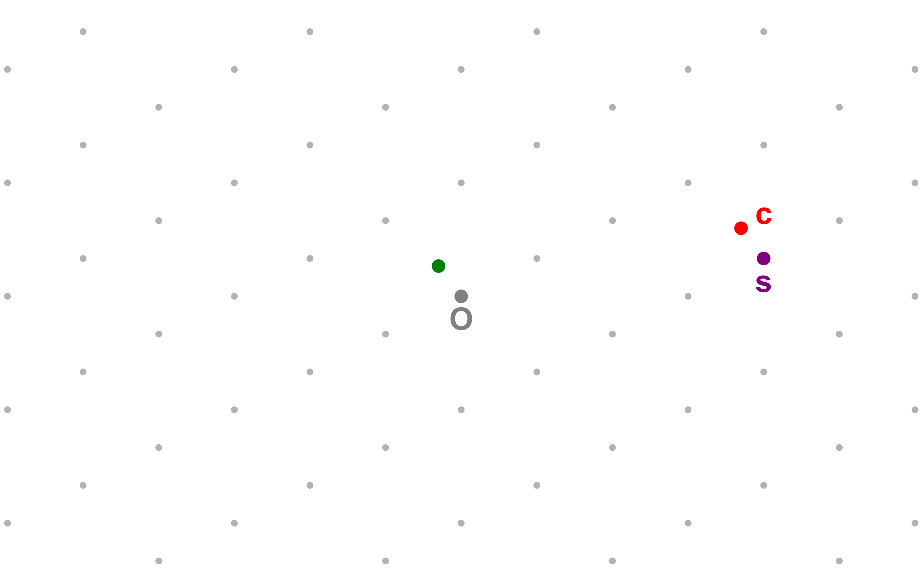

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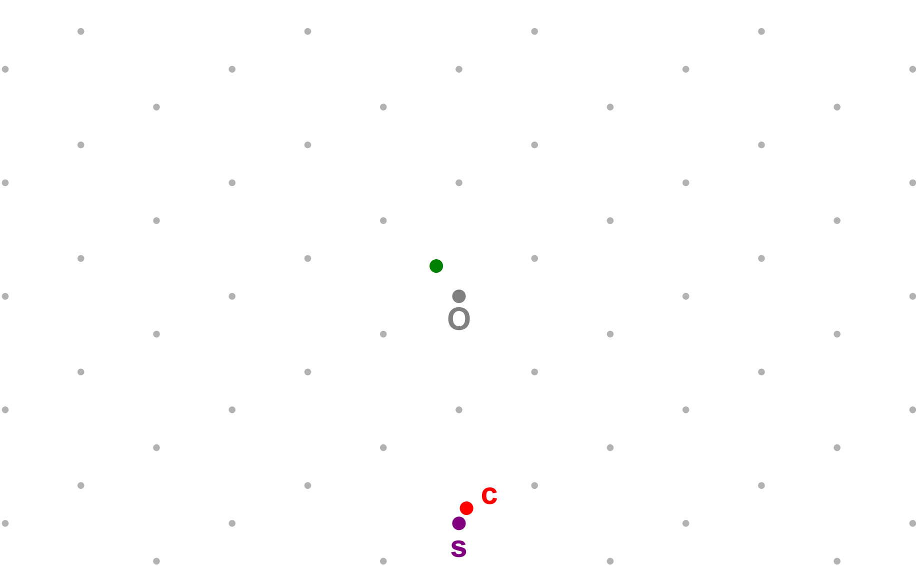

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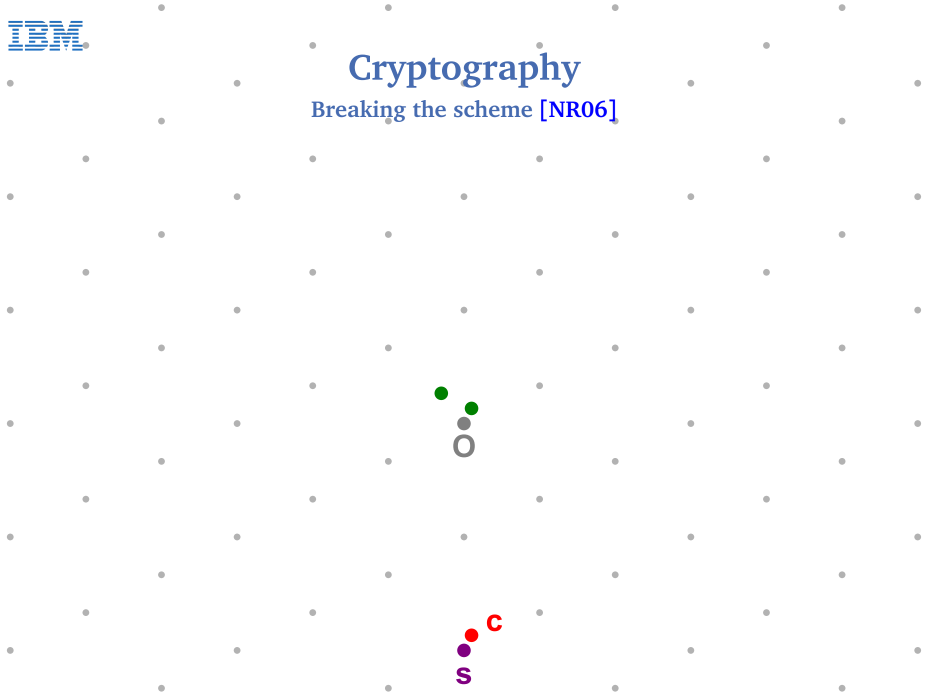

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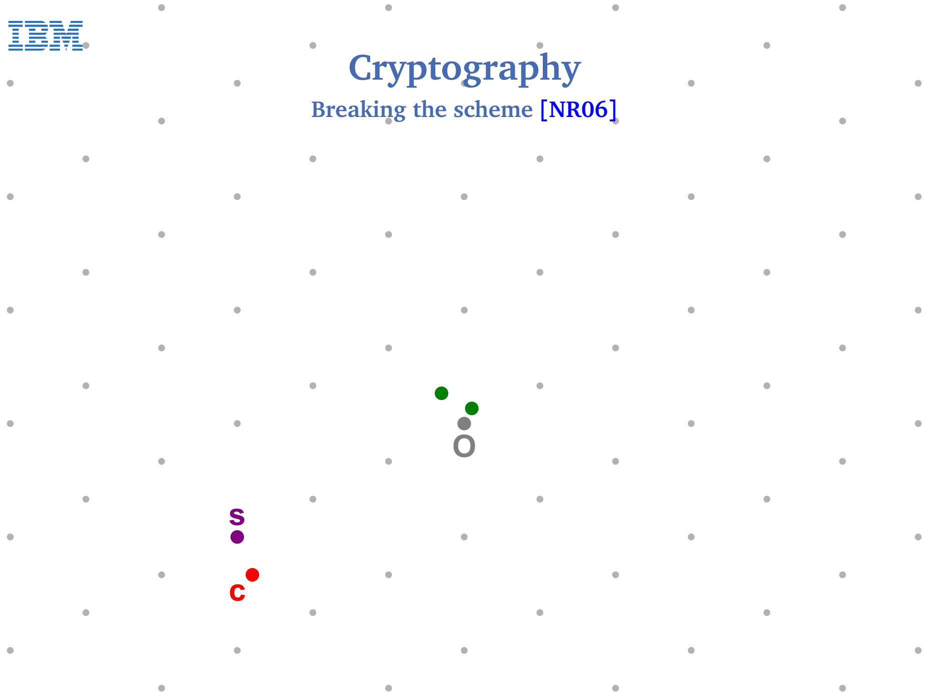

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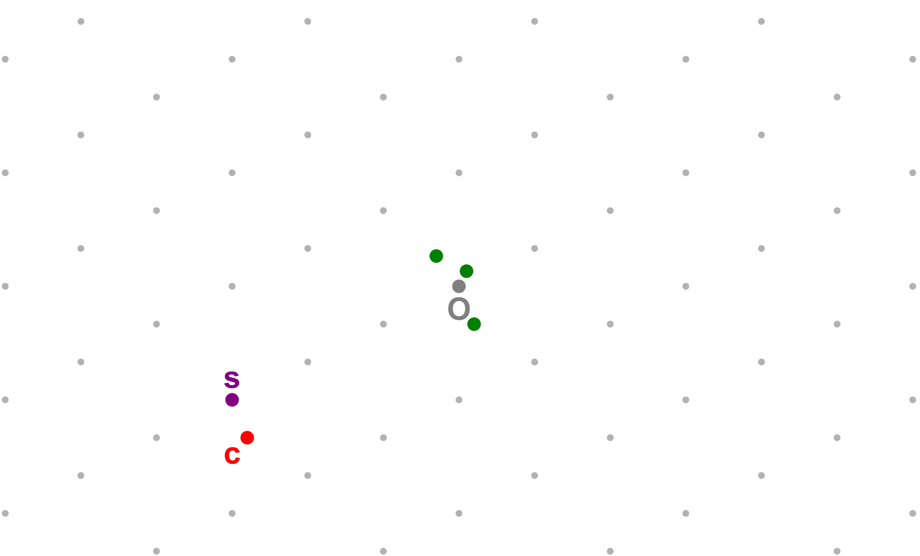

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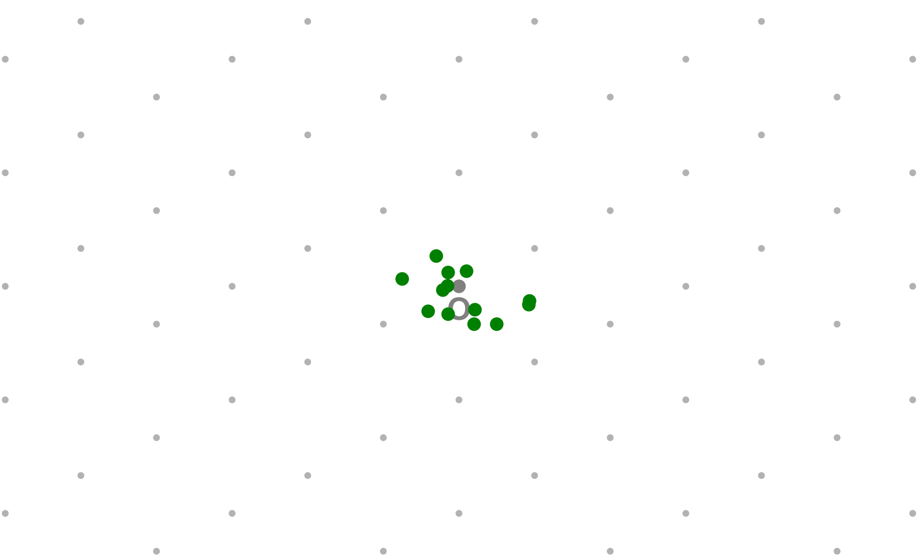

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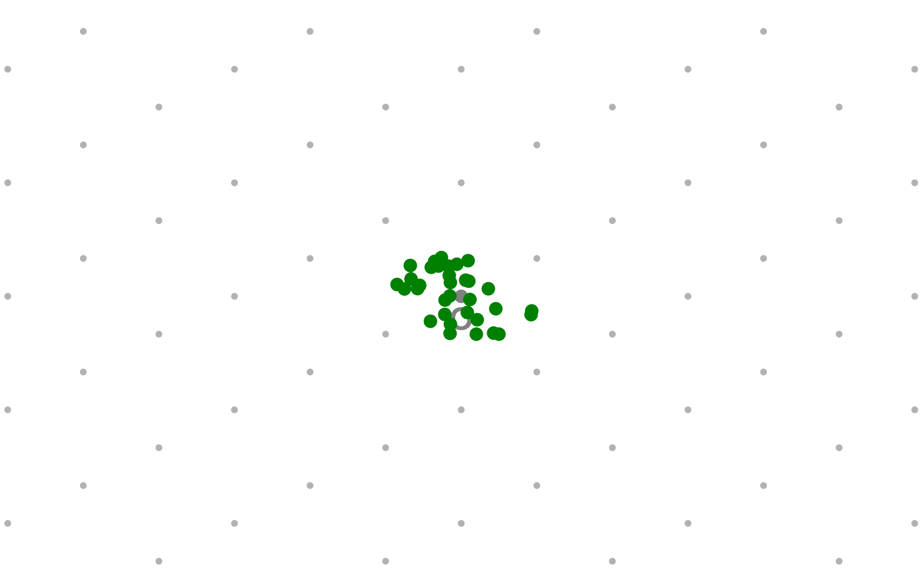

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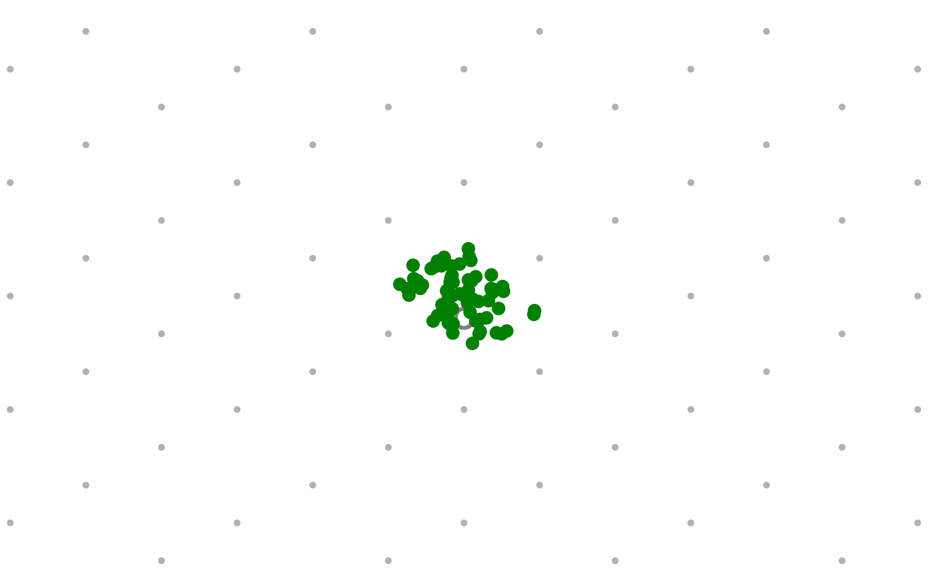

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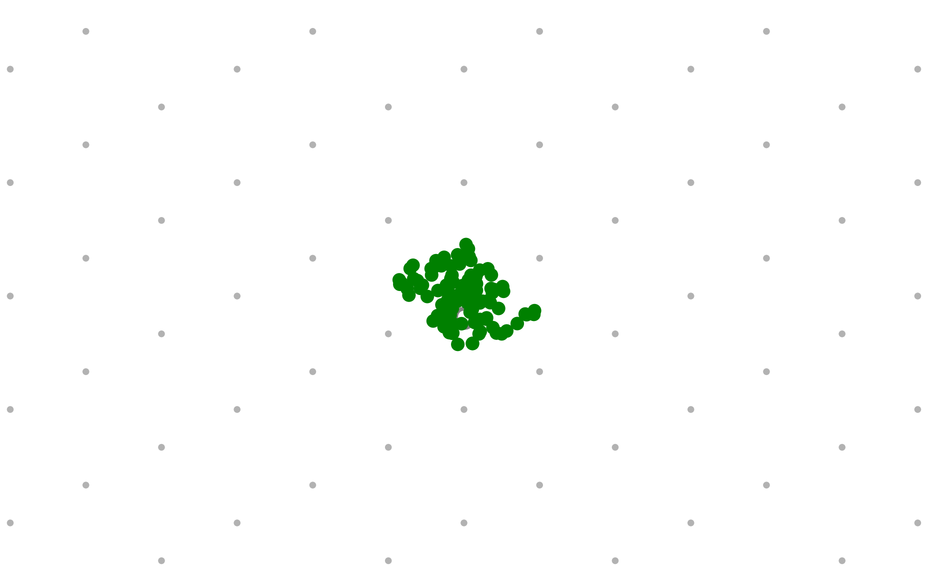

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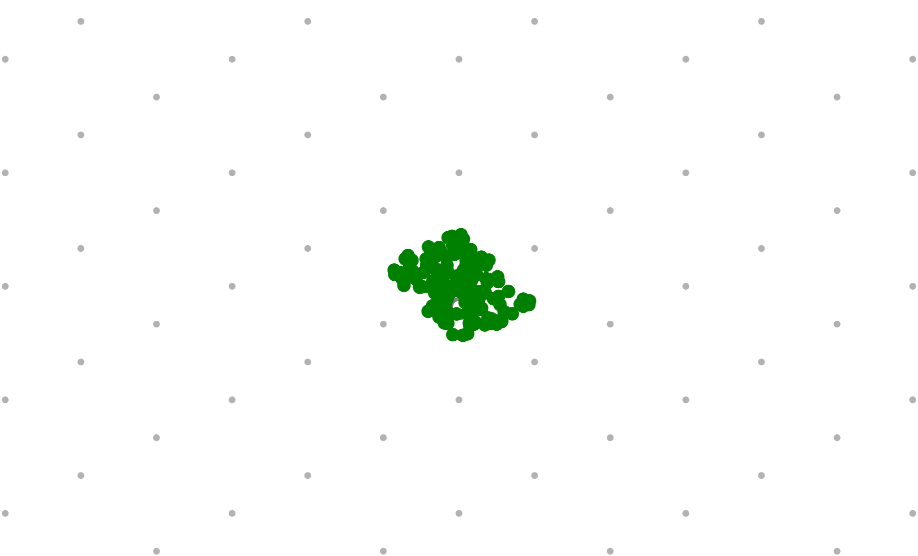

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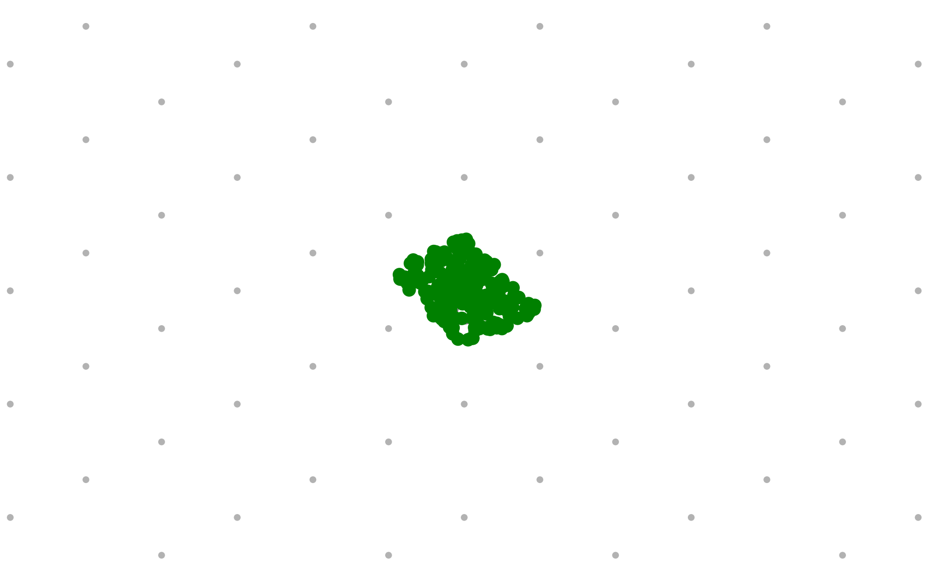

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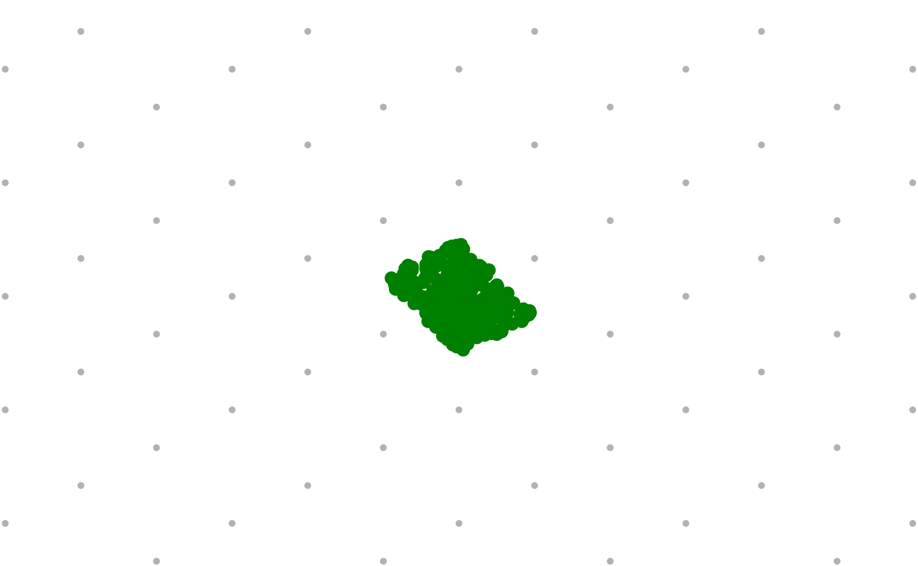

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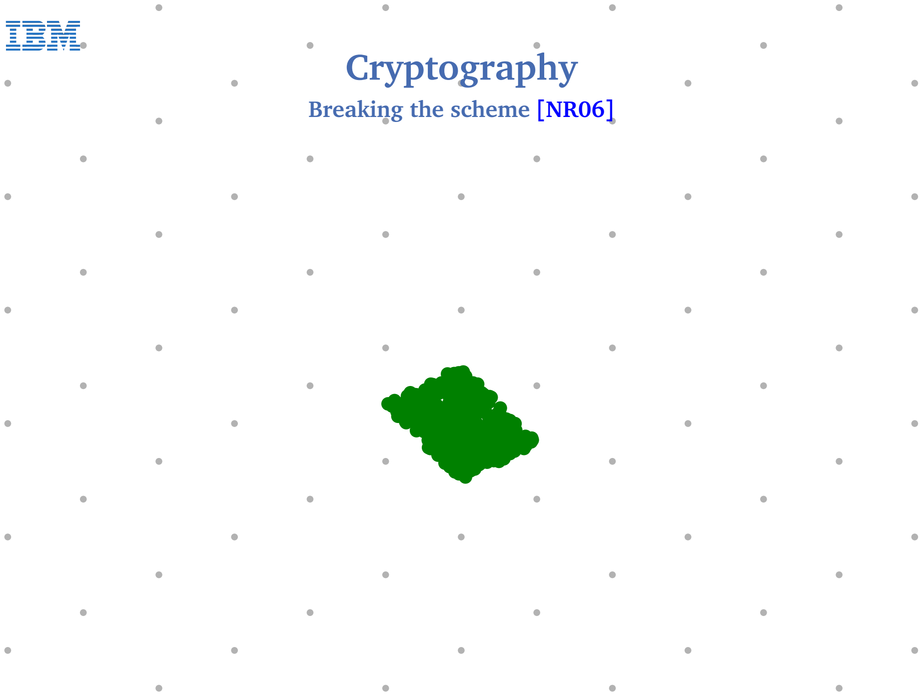

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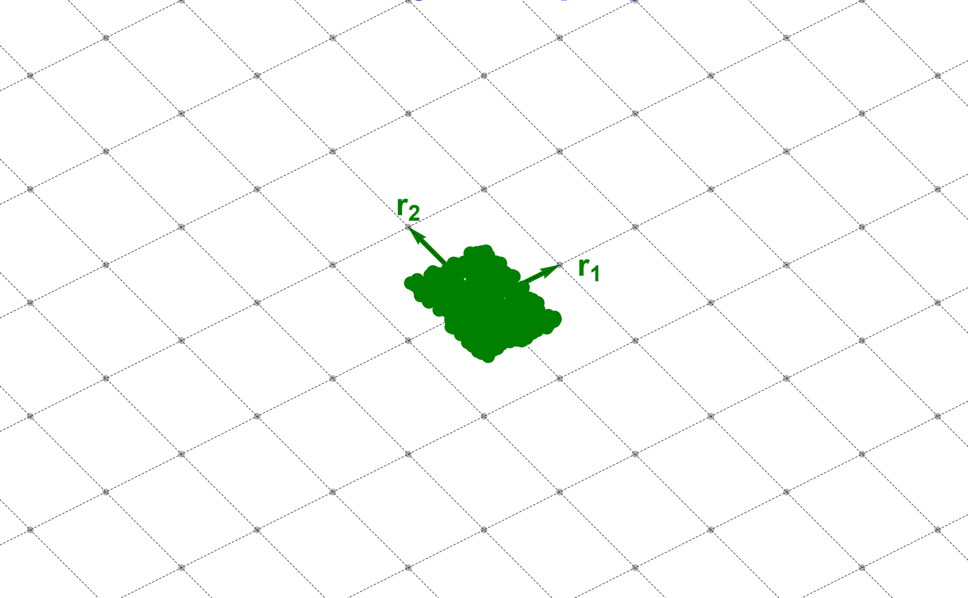

Cryptography

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- Finding short bases implies breaking these schemes

Cryptography

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- Estimate hardness based on state-of-the-art basis reduction
  - LLL [LLL83] - fast, but poor quality in high dimensions
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- Question: What is the computational cost of exact SVP?

Lattices

Exact SVP algorithms

	Algorithm	$\log_2(\text{Time})$	$\log_2(\text{Space})$
Provable SVP	Enumeration [Poh81, Kan83, ..., MW15, AN17]	$O(n \log n)$	$O(\log n)$
	AKS-sieve [AKS01, NV08, MV10, HPS11]	$3.398n$	$1.985n$
	ListSieve [MV10, MDB14]	$3.199n$	$1.327n$
	Birthday sieves [PS09, HPS11]	$2.465n$	$1.233n$
	Voronoi cell algorithm [AEVZ02, MV10b]	$2.000n$	$1.000n$
	Discrete Gaussians [ADRS15, ADS15, Ste16]	$1.000n$	$1.000n$
Heuristic SVP	Nguyen–Vidick sieve [NV08]	$0.415n$	$0.208n$
	GaussSieve [MV10, ..., IKMT14, BNvdP14]	$0.415n$	$0.208n$
	Leveled sieving [WLTB11, ZPH13]	$0.3778n$	$0.283n$
	Overlattice sieve [BGJ14]	$0.3774n$	$0.293n$
	Hyperplane LSH [Laa15, MLB15, Mar15]	$0.337n$	$0.208n^*$
	May and Ozerov's NNS method [BGJ15]	$0.311n$	$0.208n^*$
	Spherical/cross-polytope LSH [LdW15, BL16]	$0.298n$	$0.208n^*$
	Spherical filtering [BDGL16, MLB17]	$0.293n$	$0.208n^*$
	Triple sieve [BLS16, HK17, Laa17]	$0.359n$	$0.188n$

Outline

Lattices
- Basics
- Cryptography

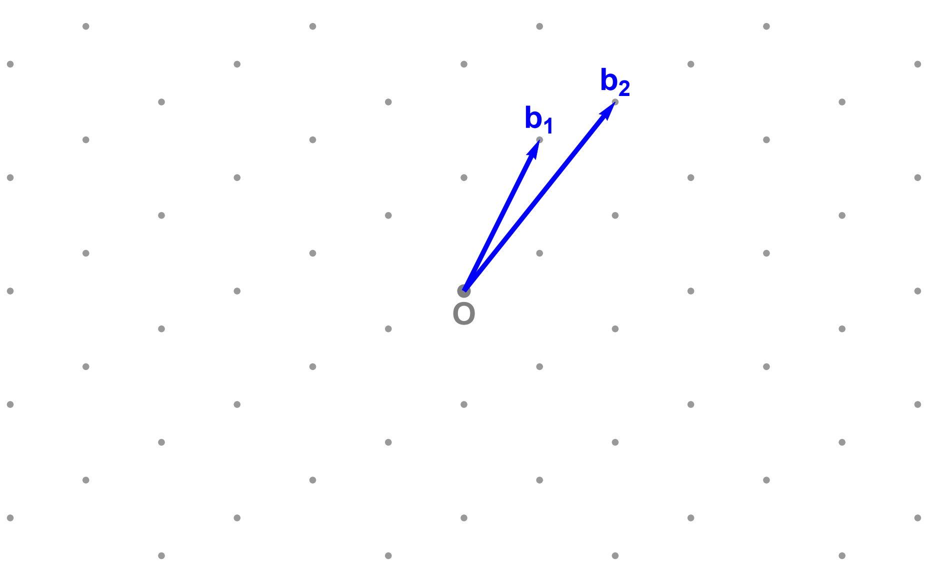
Enumeration algorithms
- Fincke–Pohst enumeration
- Kannan enumeration
- Pruned enumeration

Sieving algorithms
- Basic sieving
- Leveled sieving
- Near neighbor searching

Practical comparison

Fincke-Pohst enumeration

Determine possible coefficients of b_2



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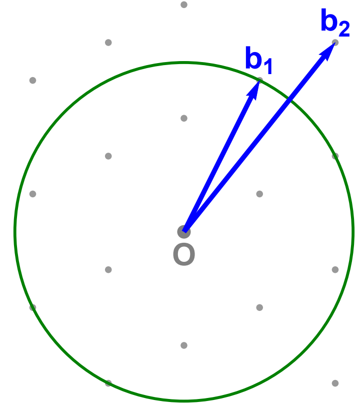

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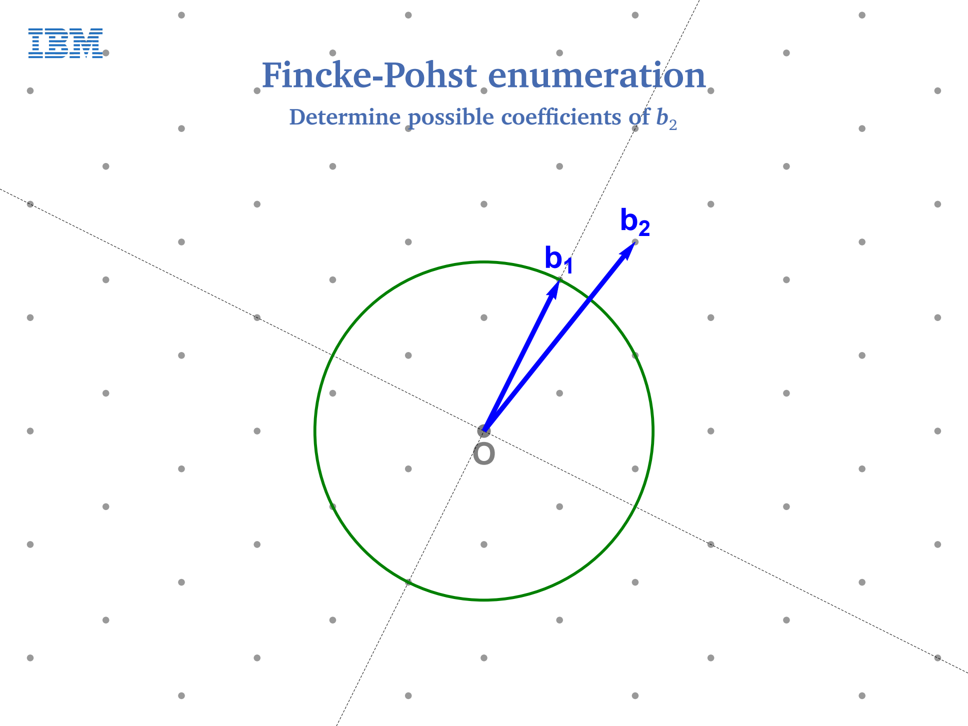

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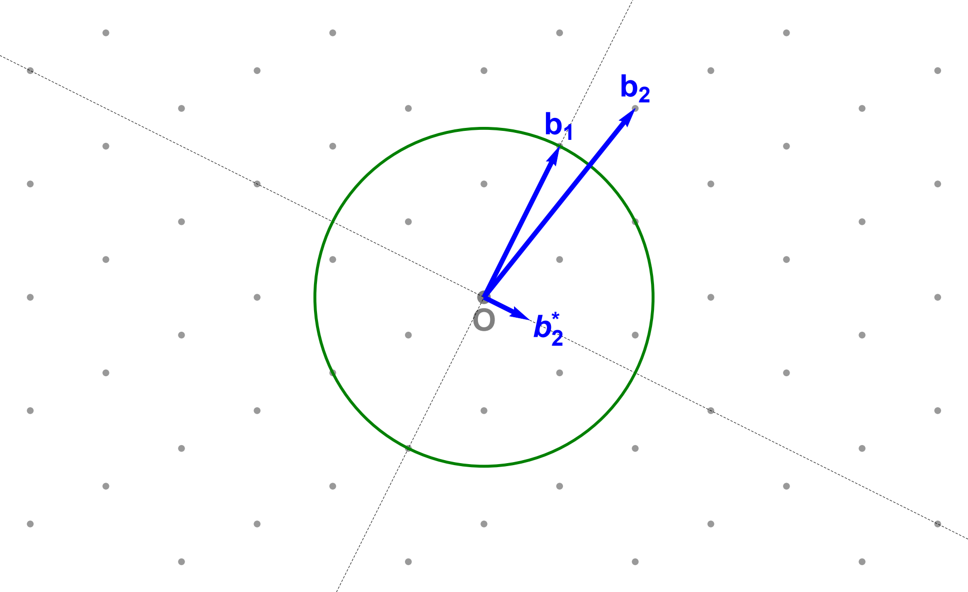

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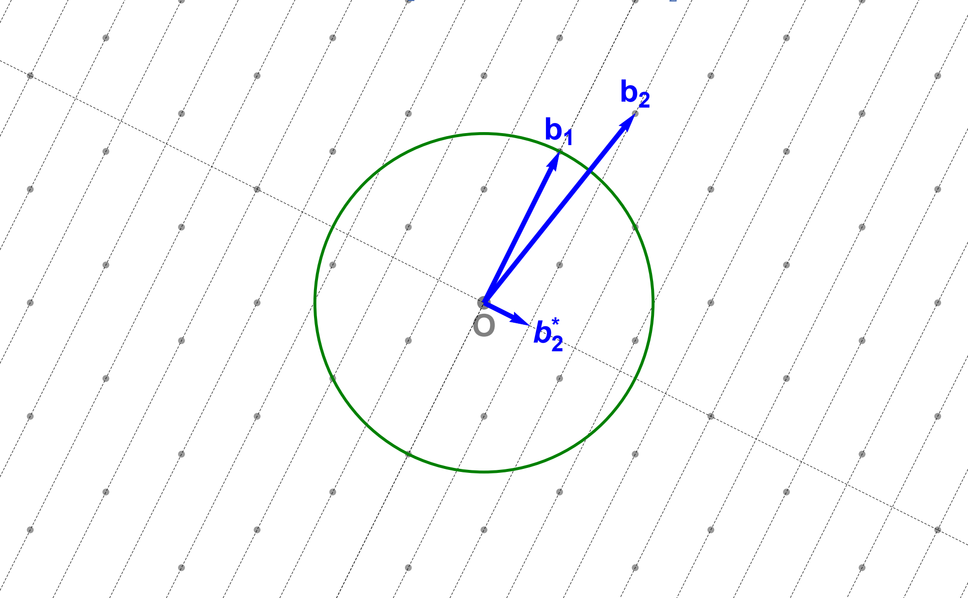
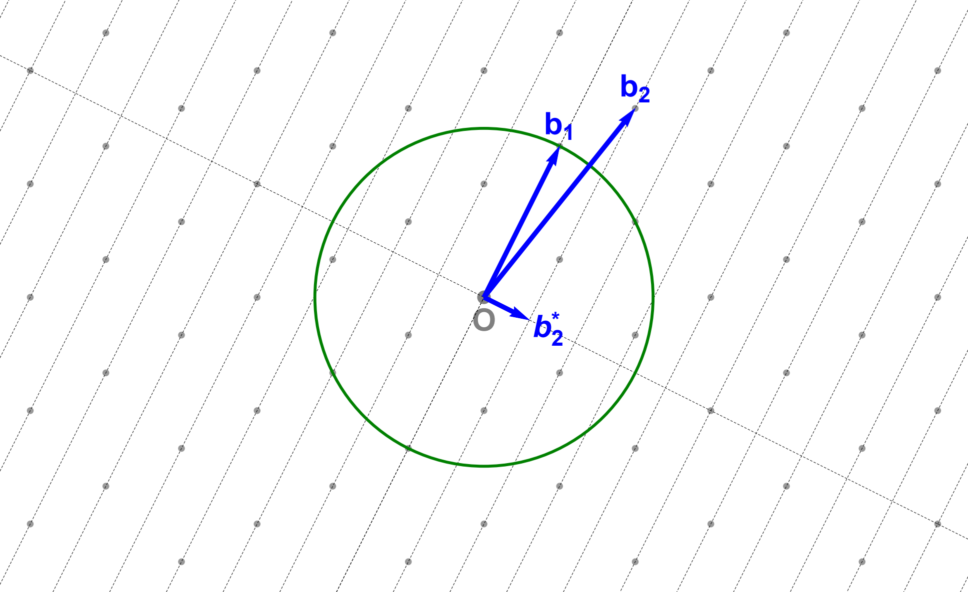


Fincke-Pohst enumeration

Find short vectors for each coefficient of b_2



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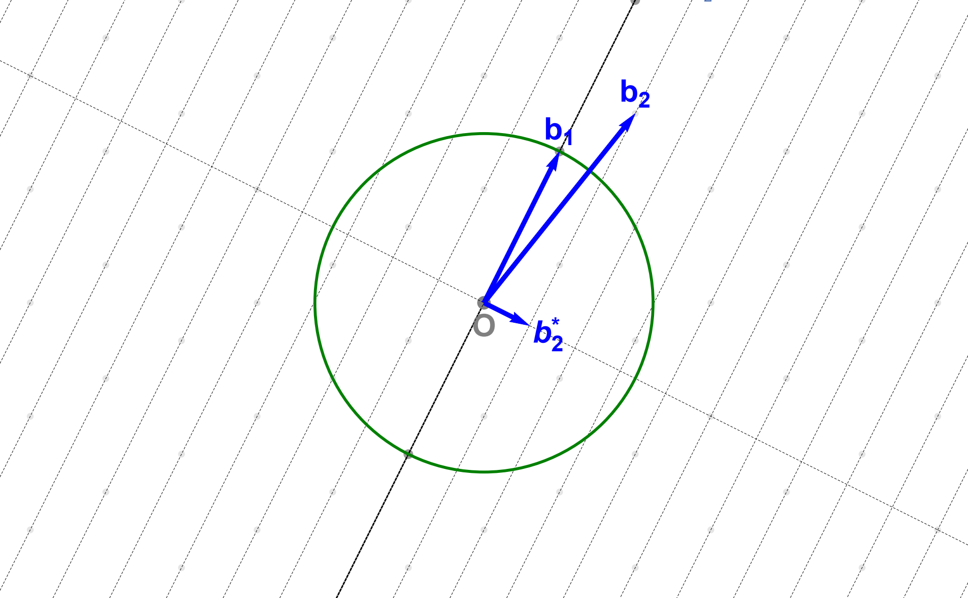

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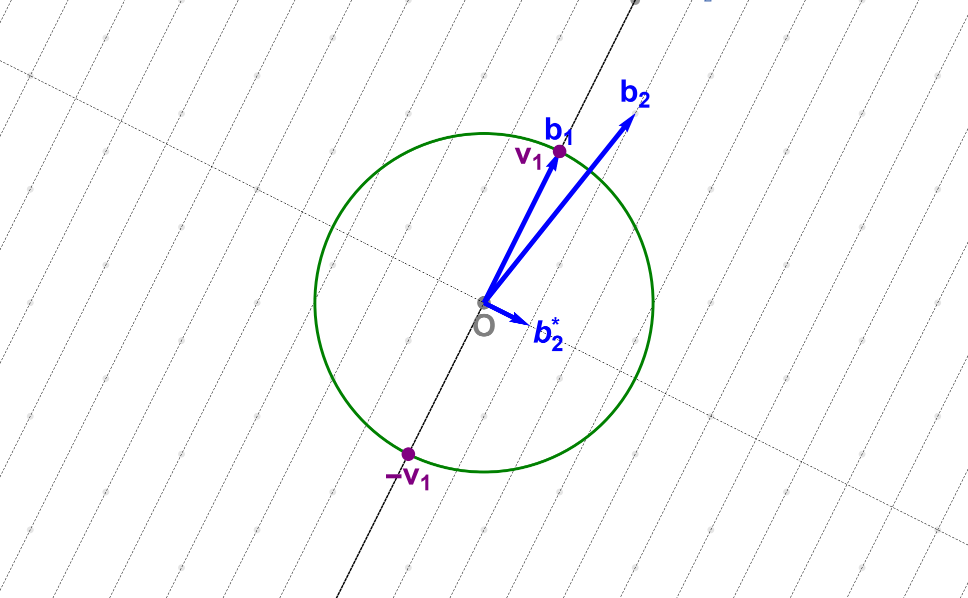

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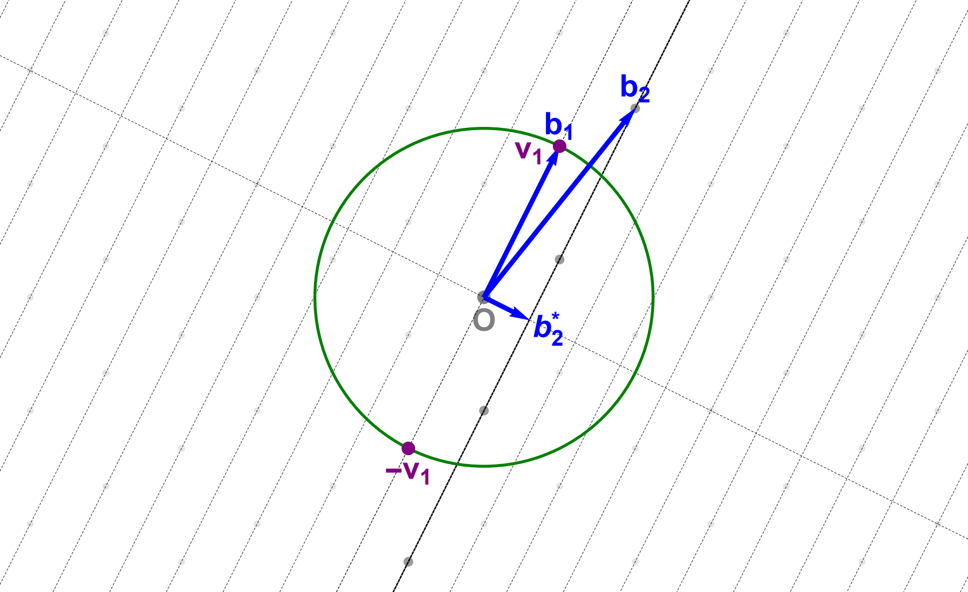

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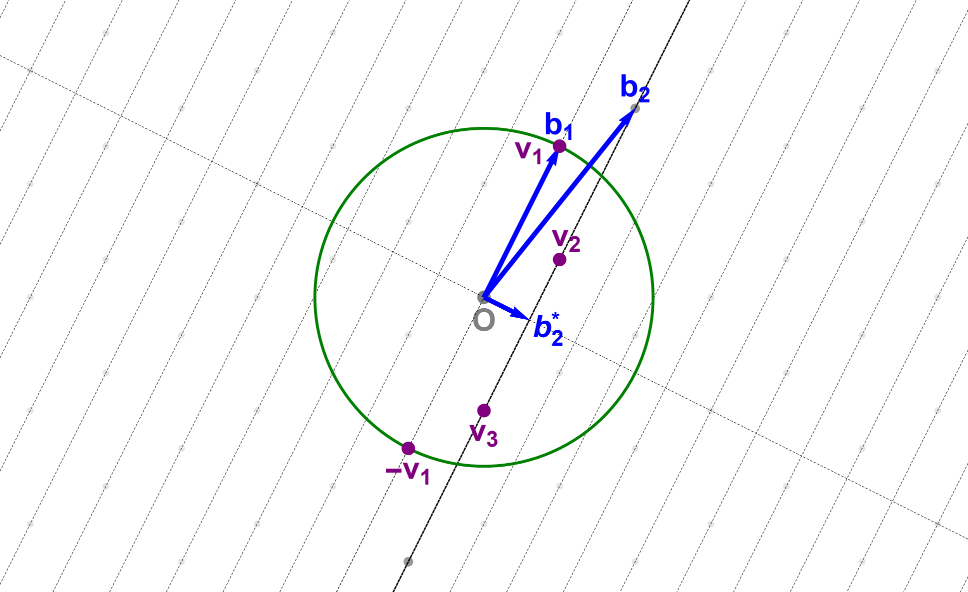

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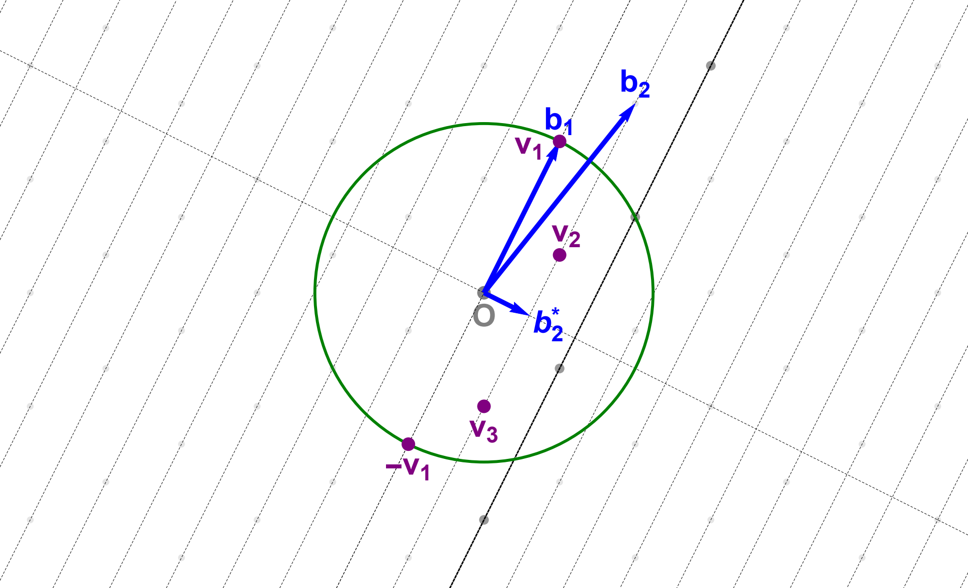

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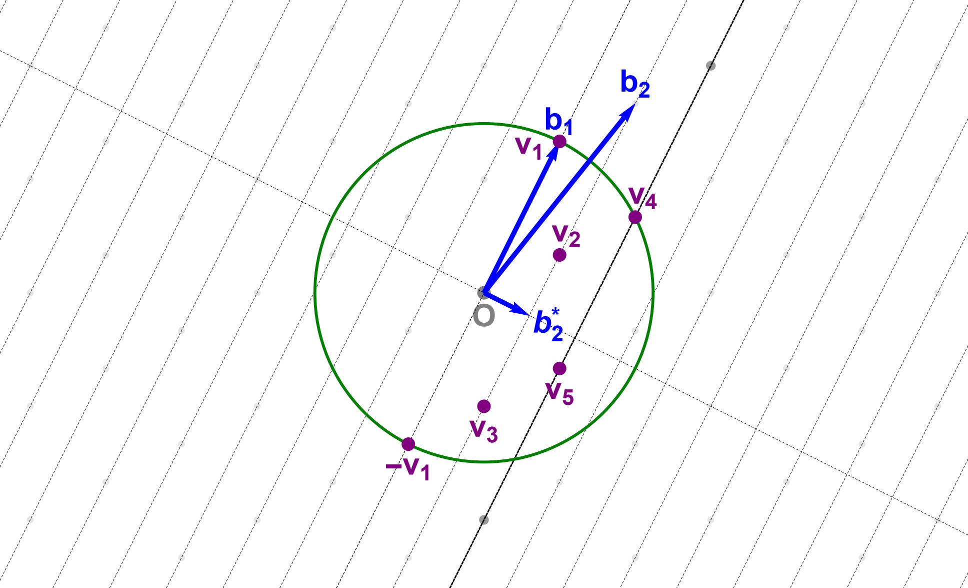

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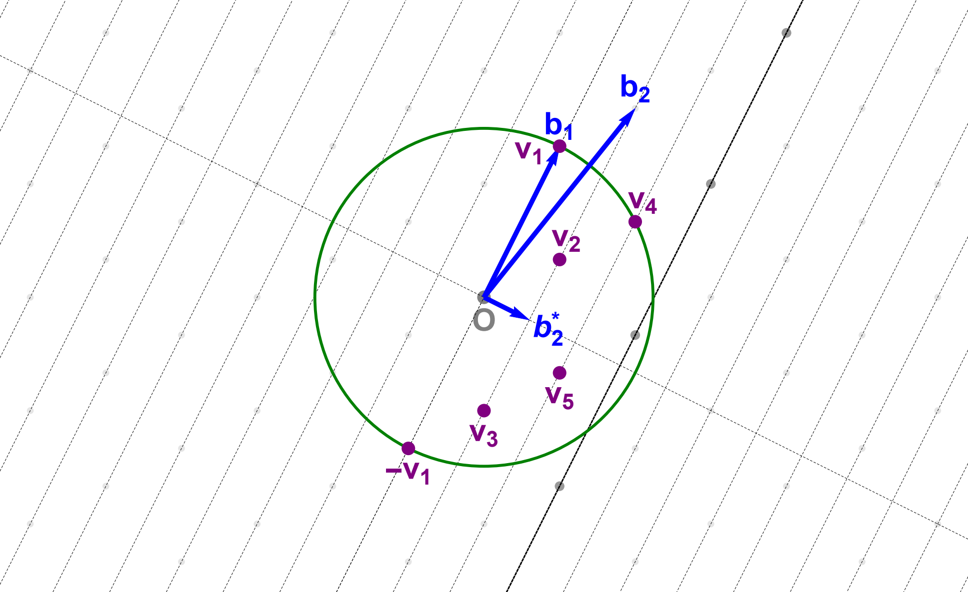

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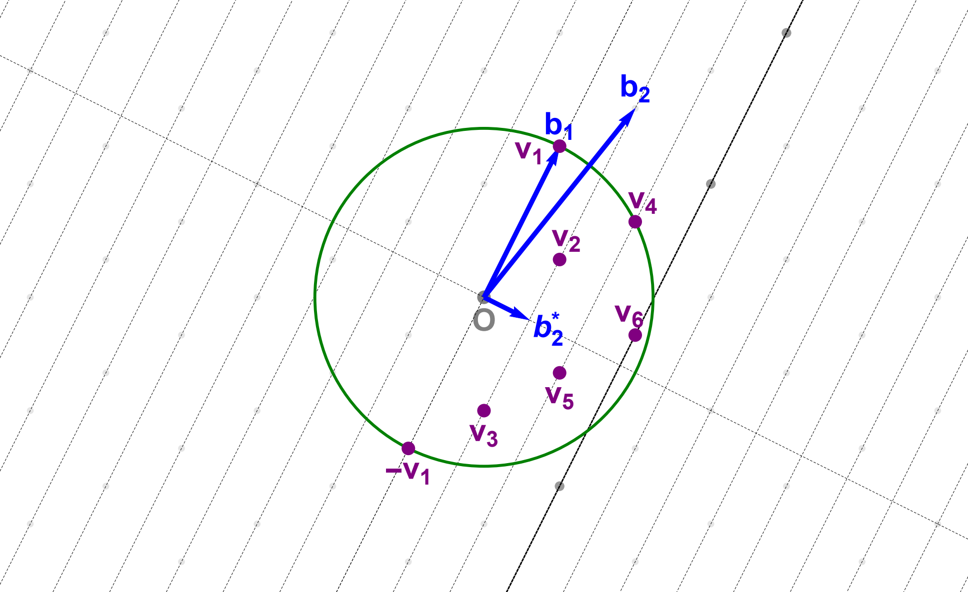

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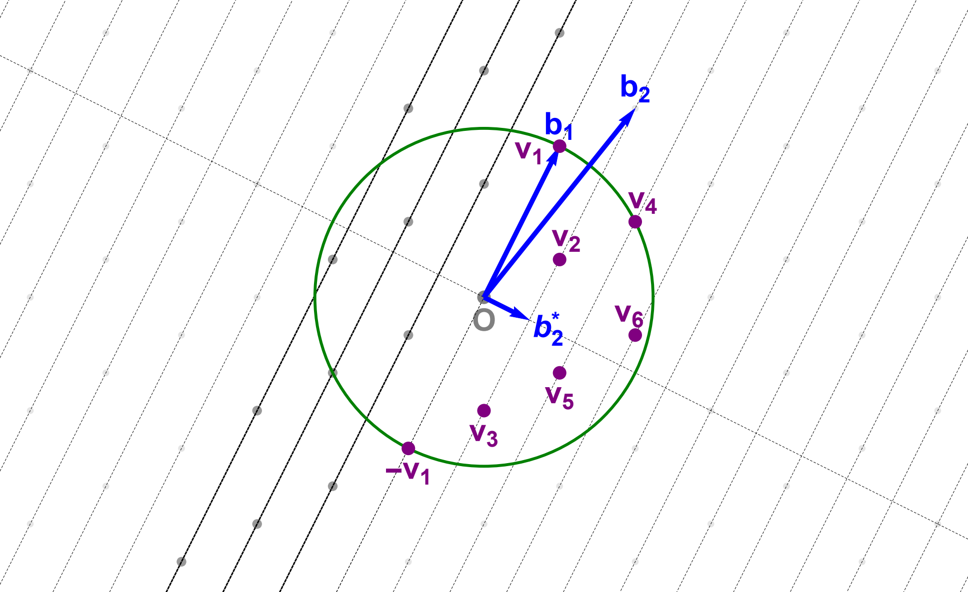

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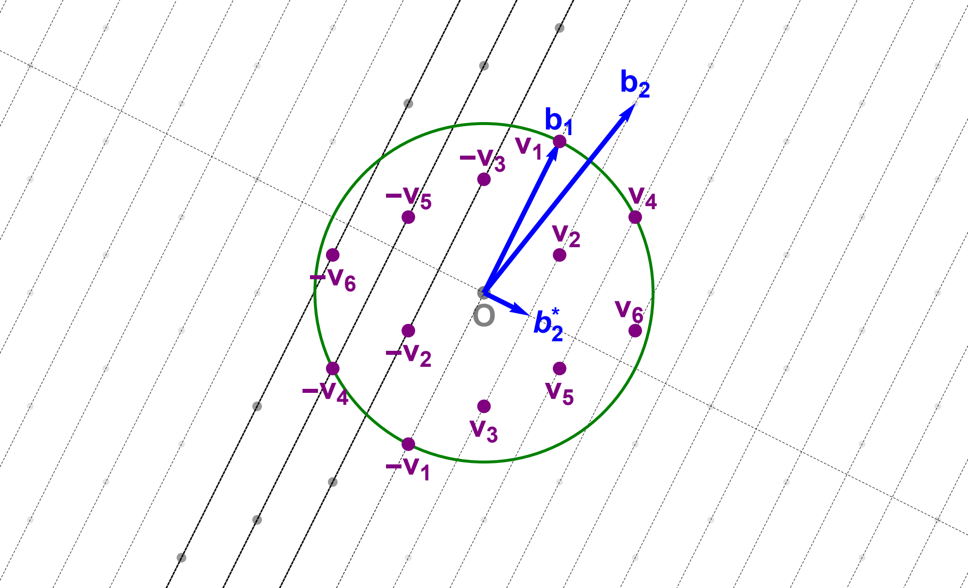

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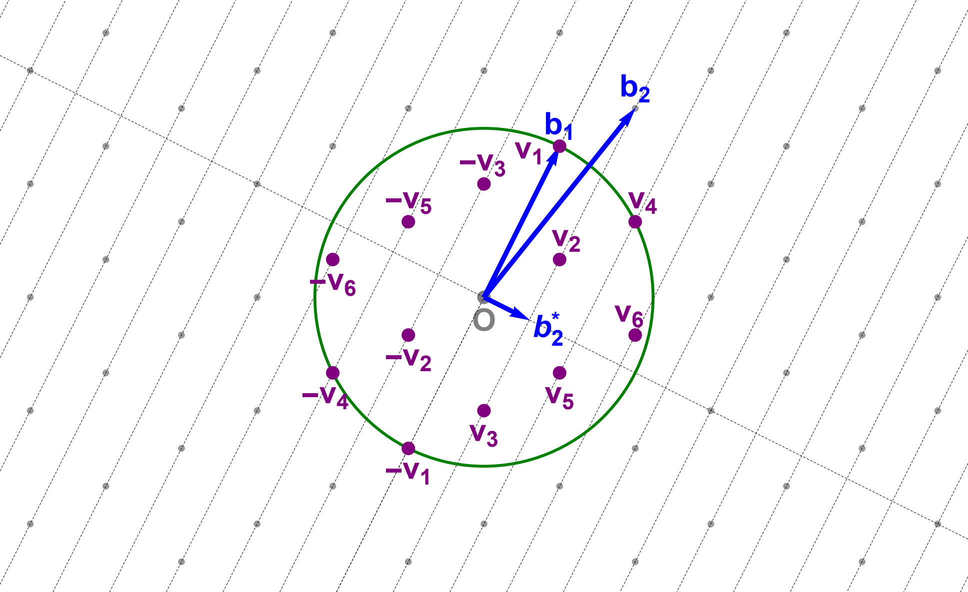
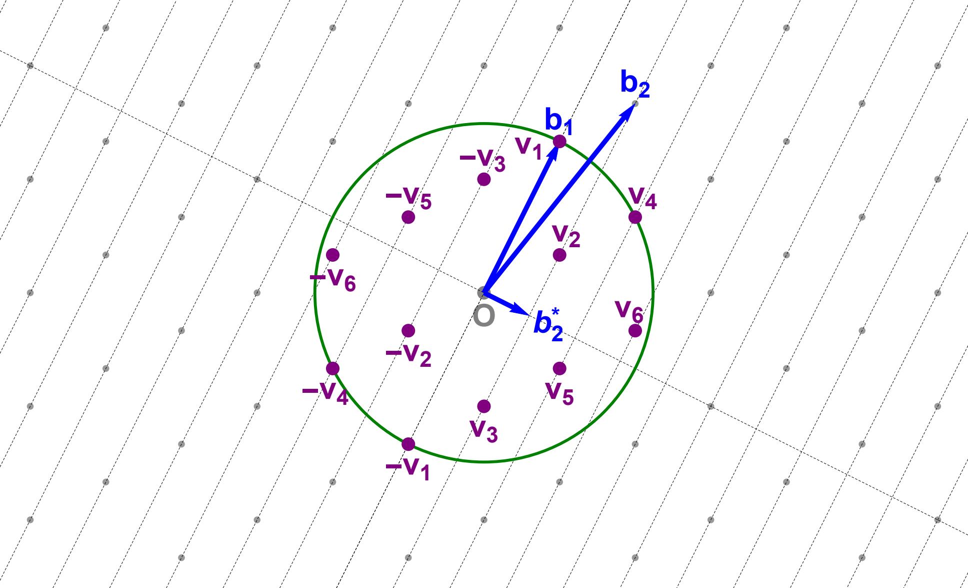


Fincke-Pohst enumeration

Find a shortest vector among all found vectors



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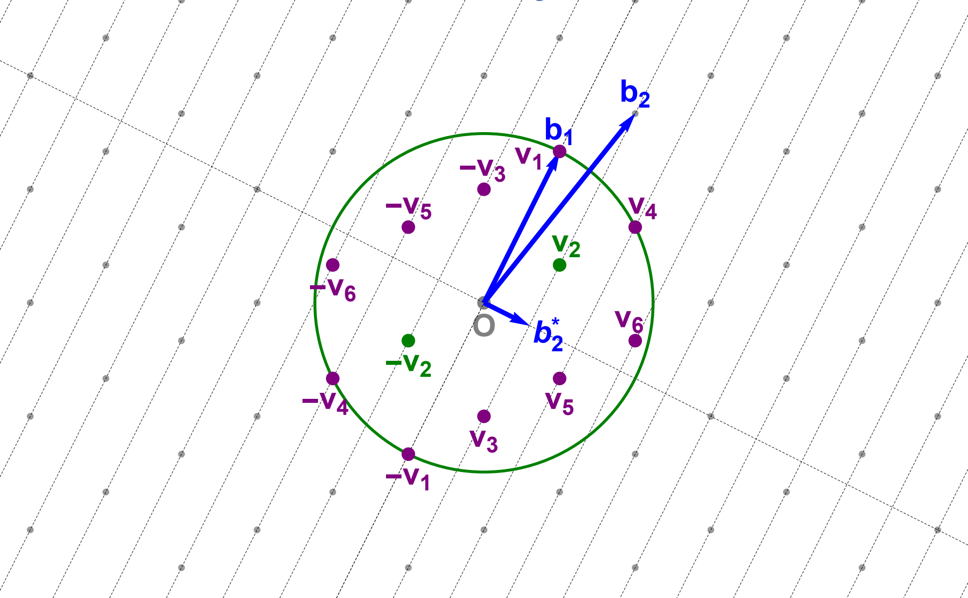

Fincke-Pohst enumeration

Overview

Fincke-Pohst enumeration

Overview

Theorem (Fincke–Pohst, Math. of Comp. '85)

Fincke-Pohst enumeration runs in time $2^{O(n^2)}$ *and space* $\mathrm{poly}(n)$.

Fincke-Pohst enumeration

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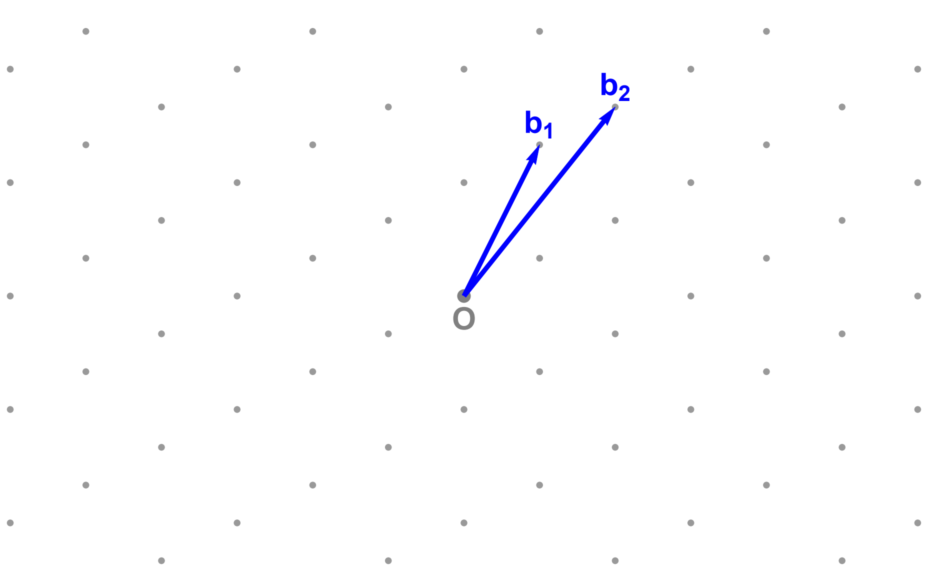
Theorem (Fincke–Pohst, Math. of Comp. '85)

Fincke-Pohst enumeration runs in time $2^{O(n^2)}$ *and space* $\text{poly}(n)$.

Essentially reduces SVP_n (CVP_n) to $2^{O(n)}$ instances of CVP_{n-1}

Kannan enumeration

Better bases



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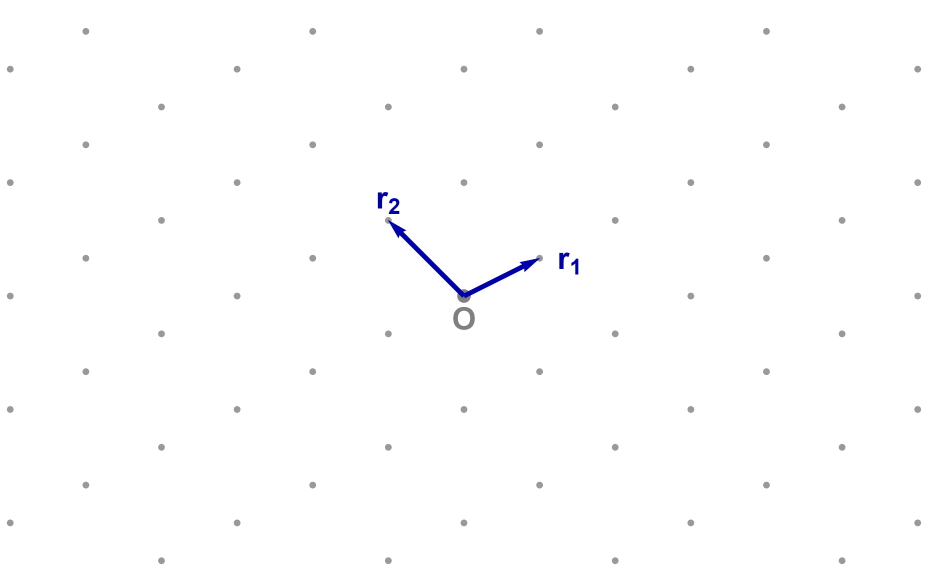

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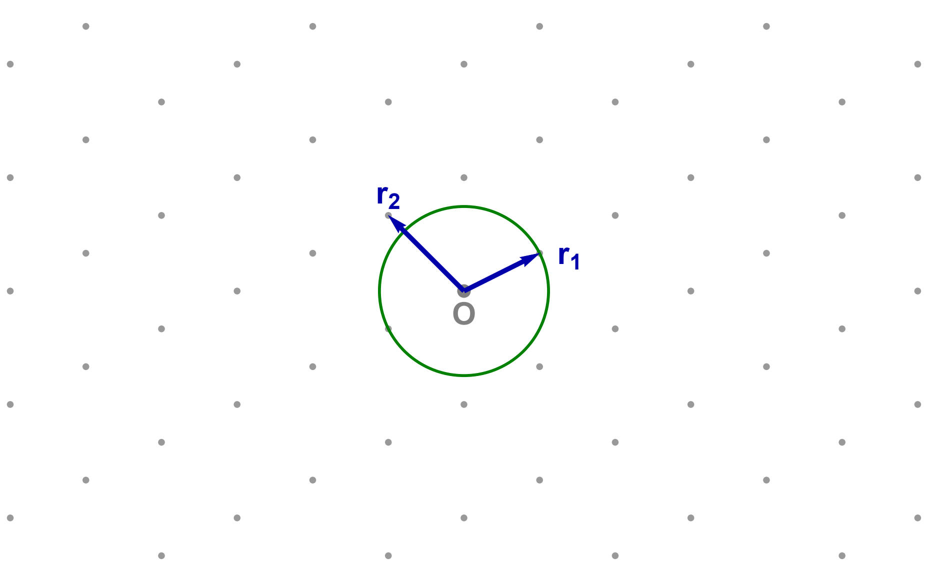

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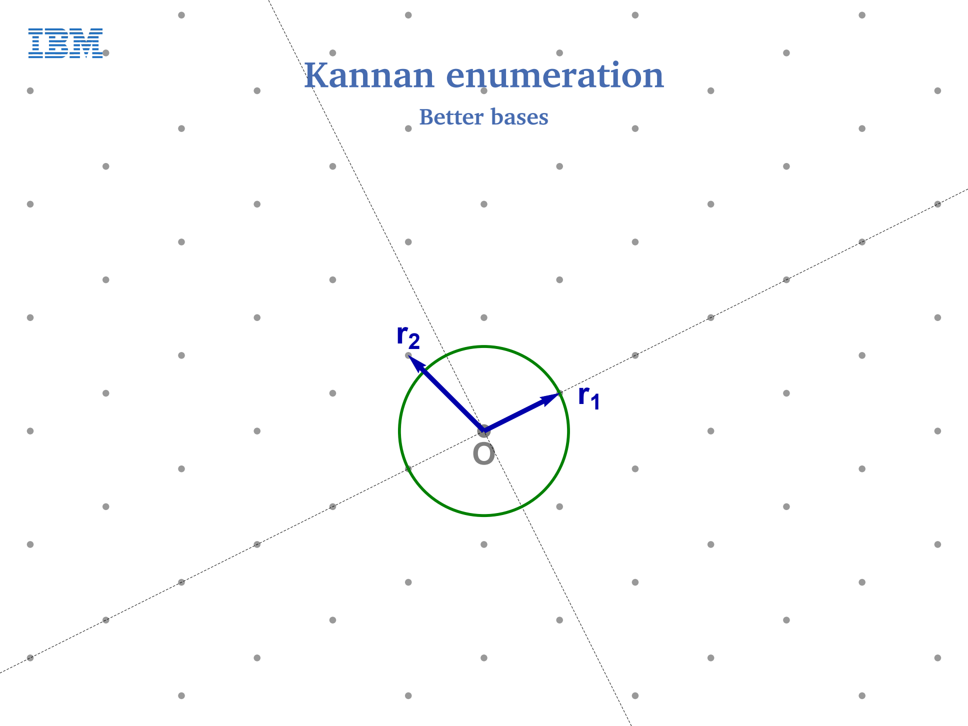

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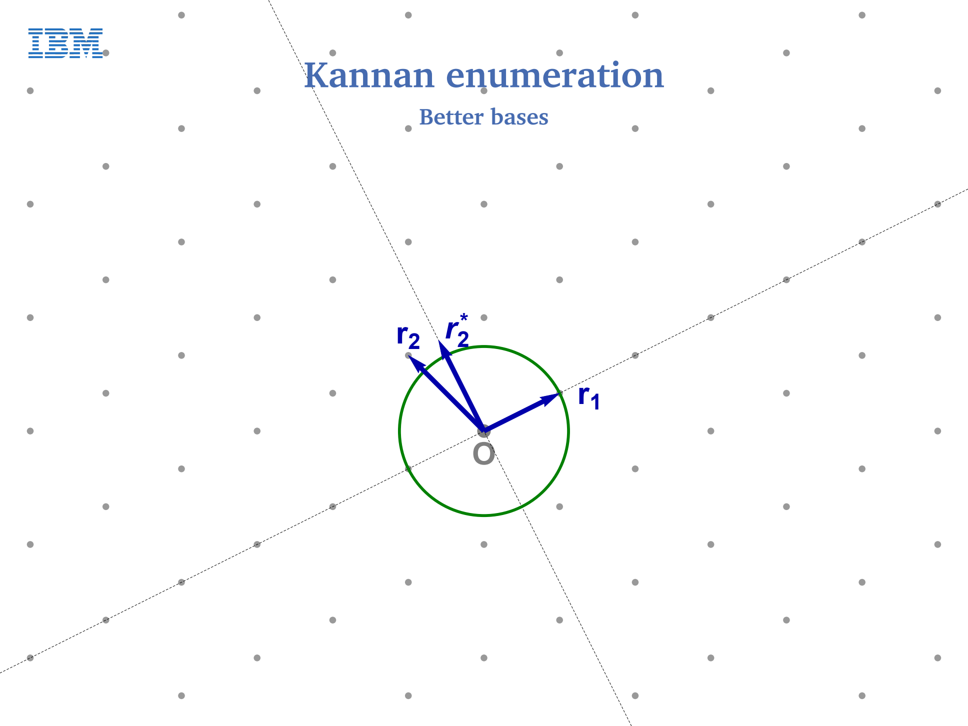

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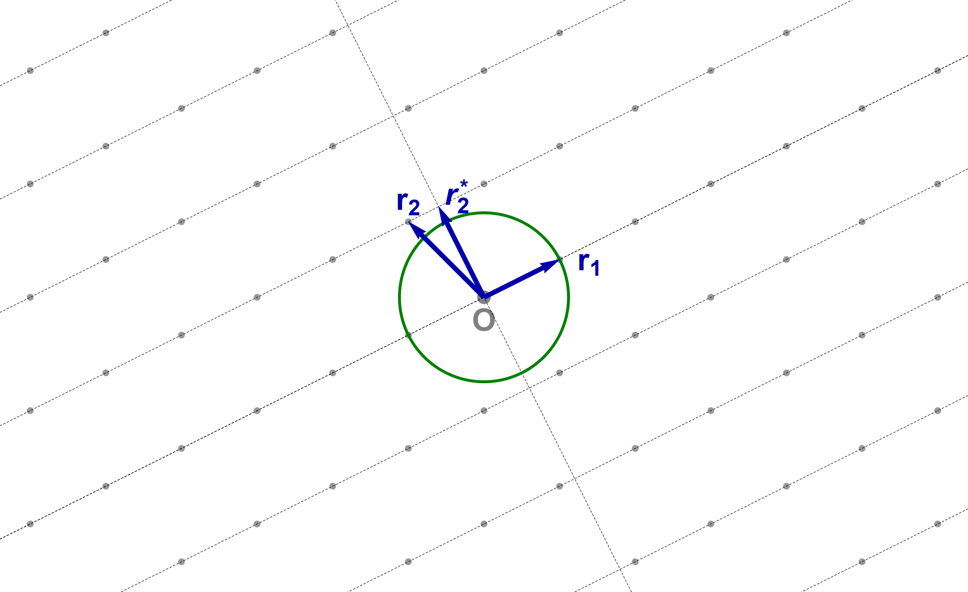

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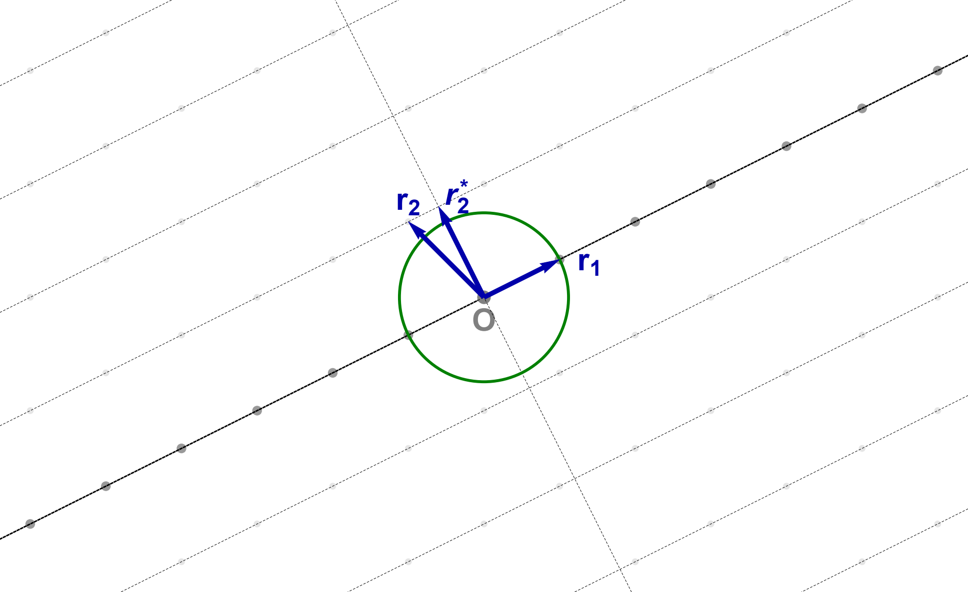

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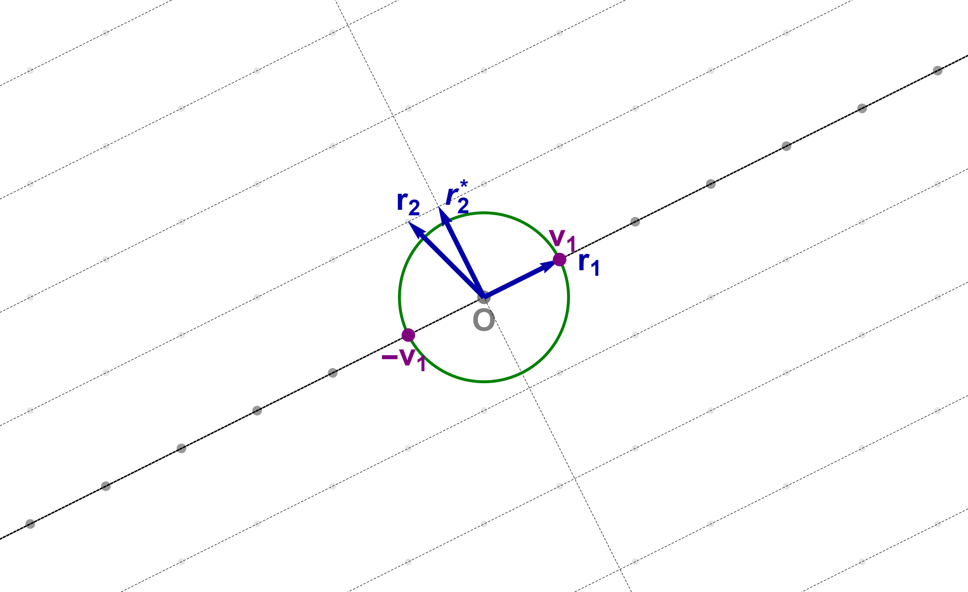

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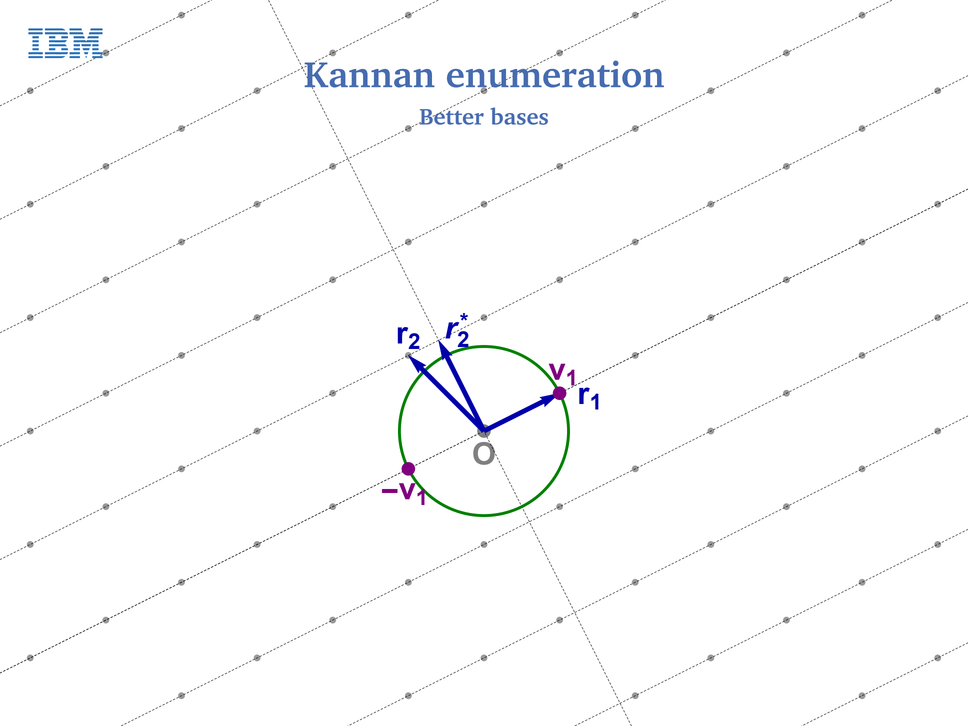

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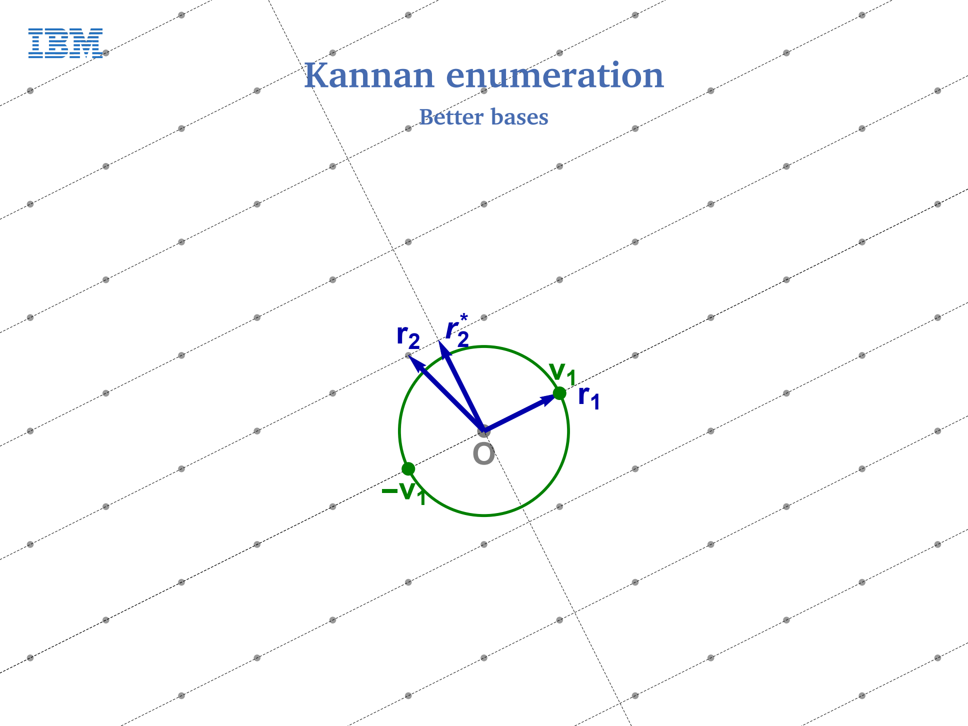

Kannan enumeration

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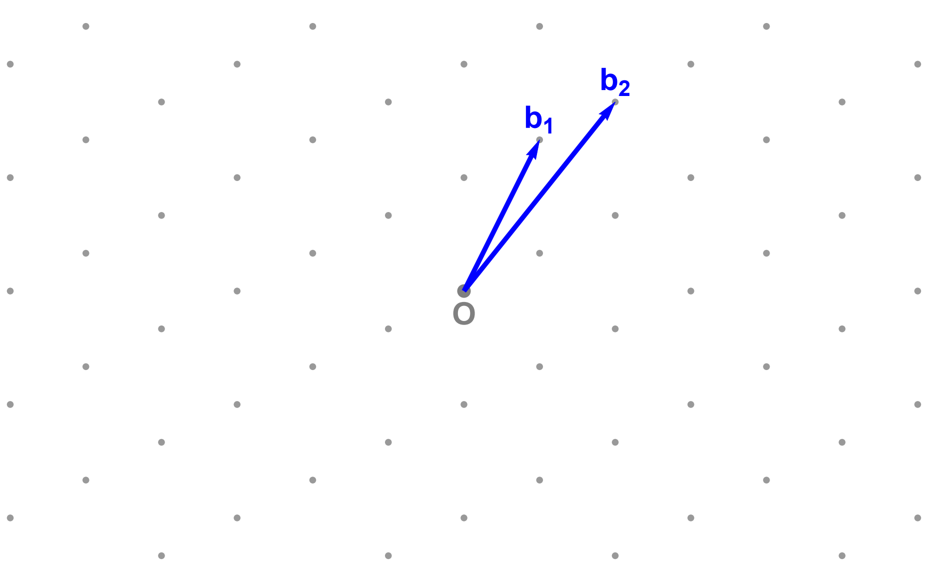
> *"Our algorithm reduces an n-dimensional problem to polynomially many (instead of* $2^{O(n)}$*)* $(n-1)$*-dimensional problems. [...] The algorithm we propose, first finds a more orthogonal basis for a lattice in time* $2^{O(n \log n)}$*."*
>
> — Kannan, *STOC'83*

Pruned enumeration

Reducing the search space



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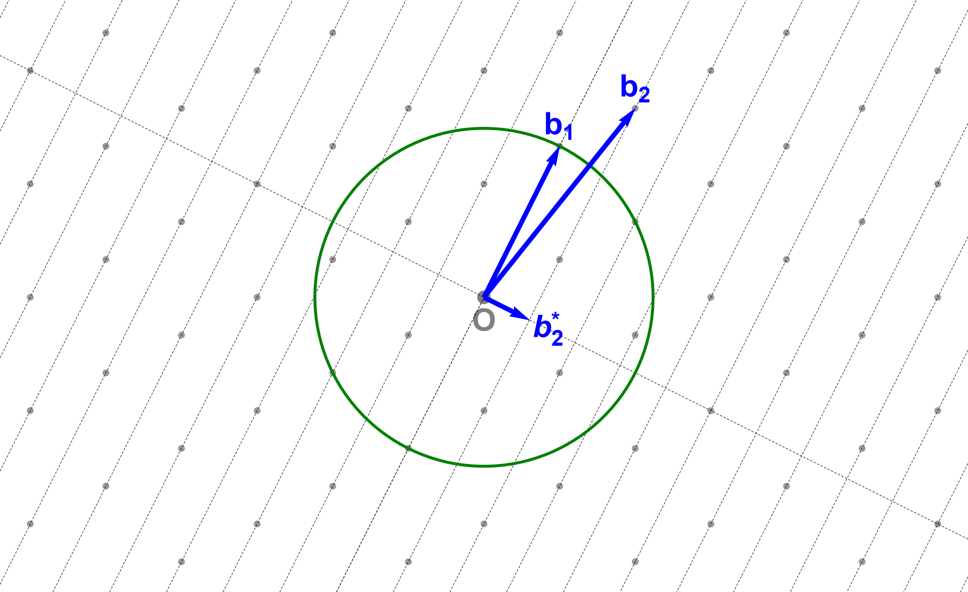

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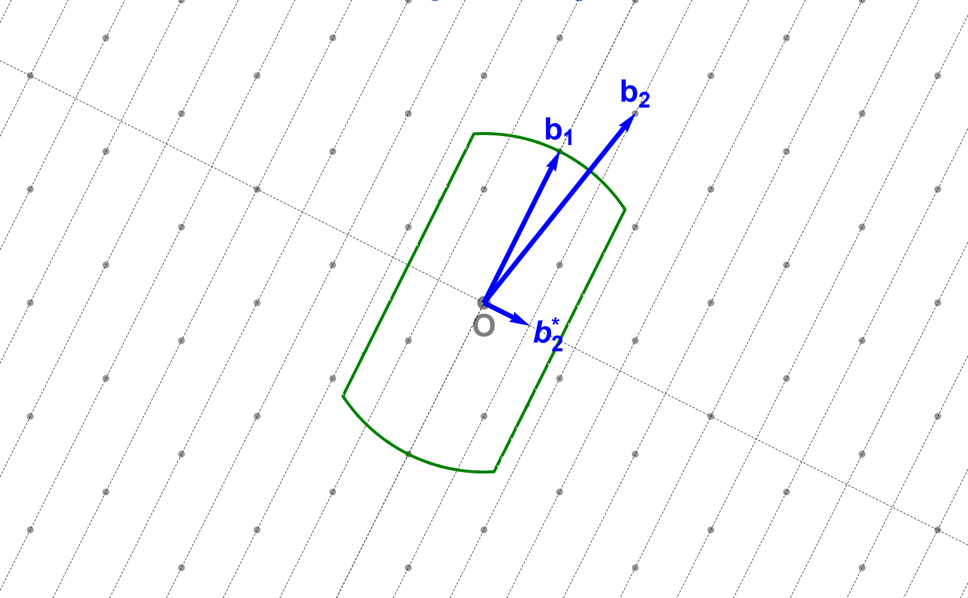

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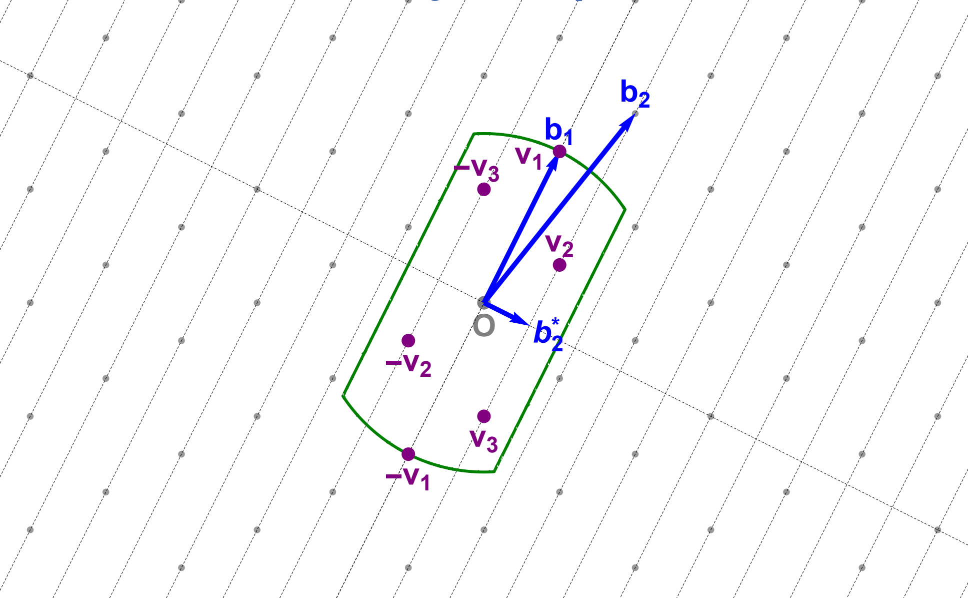

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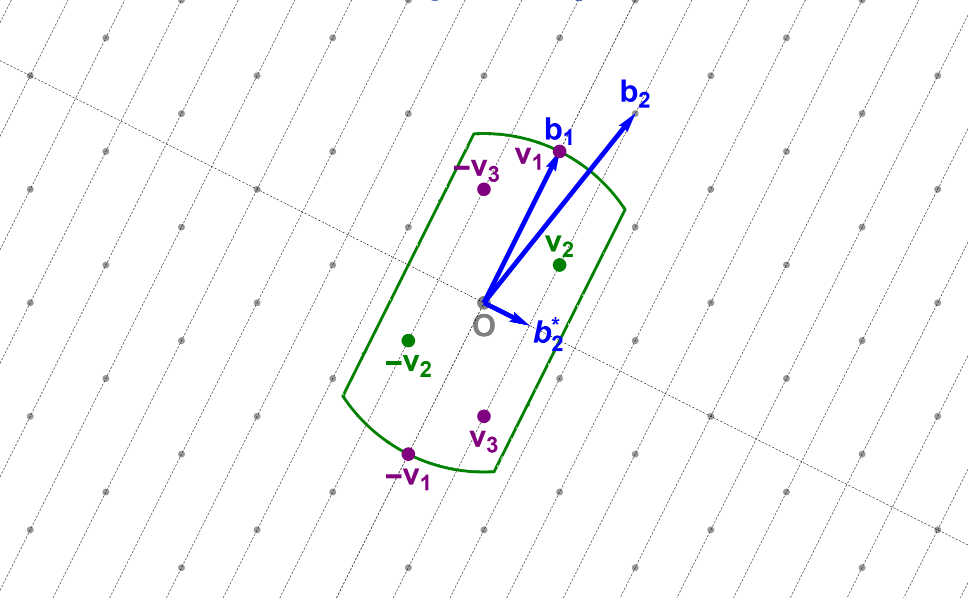

Pruned enumeration

Overview

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Overview

"Well-chosen bounding functions lead asymptotically to an exponential speedup of about $2^{n/4}$ over basic enumeration, maintaining a success probability $\geq 95\%$."

— Gama–Nguyen–Regev, *EUROCRYPT'10*

Pruned enumeration

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"With extreme pruning, the probability of finding the desired vector is actually rather low (say, 0.1%*), but surprisingly, the running time of the enumeration is reduced by a much more significant factor (say, much more than* 1000*)."*

— Gama–Nguyen–Regev, *EUROCRYPT'10*

Pruned enumeration

Overview

- Pruning framework: Enumerate $\mathscr{L} \cap \mathscr{B} \cap P$ for well-chosen P
- Continuous pruning [GNR10]: P is a cylinder intersection.
- Discrete pruning [AN17]: P is a union of boxes.

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- Continuous pruning [GNR10]: P is a cylinder intersection.
- Discrete pruning [AN17]: P is a union of boxes.

"We now know continuous pruning and discrete pruning [...] but a theoretical asymptotical comparison is not easy. Can a combination of both, or another form of pruning be more efficient?"

— Aono–Nguyen, *EUROCRYPT'17*

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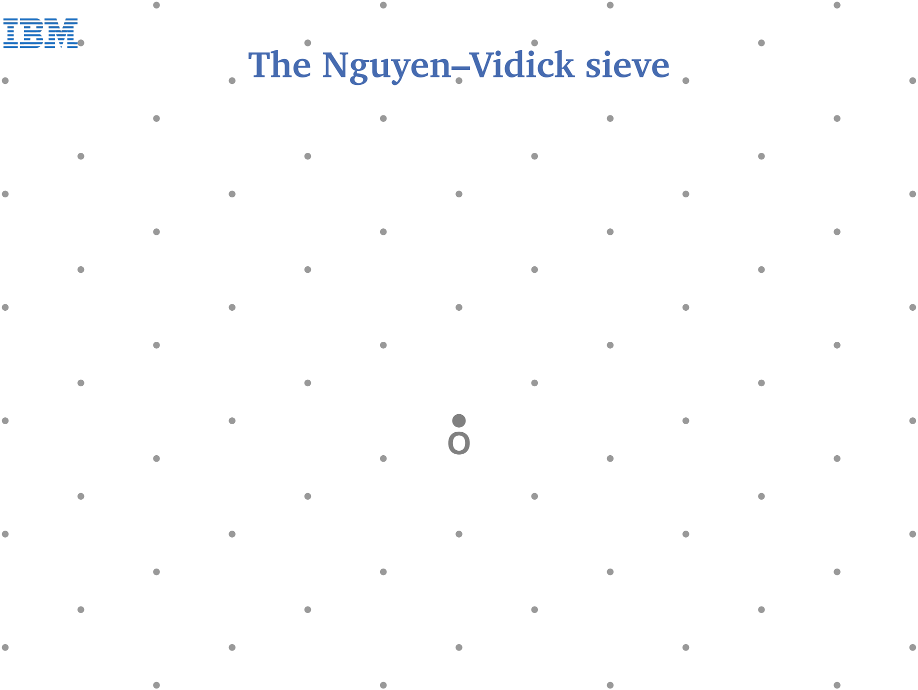
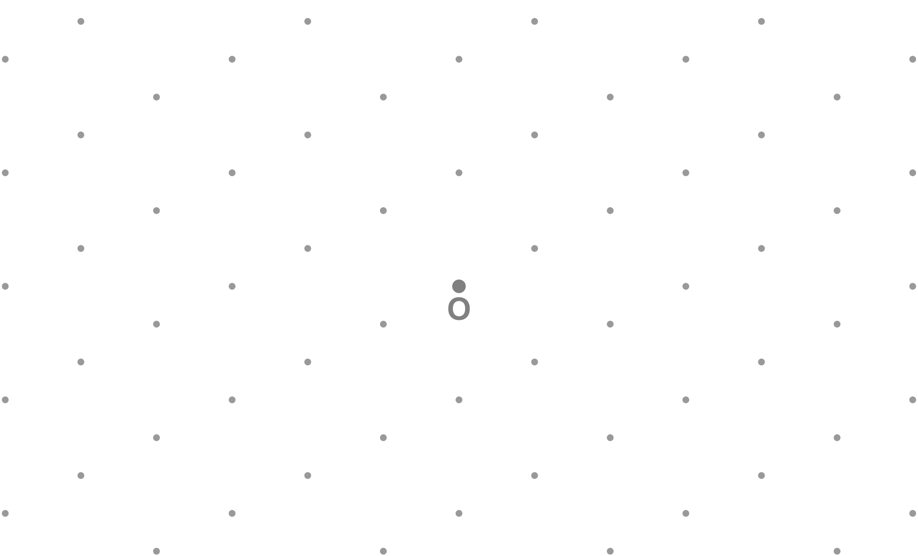
The Nguyen–Vidick sieve



The Nguyen–Vidick sieve

Sample a list L of random lattice vectors



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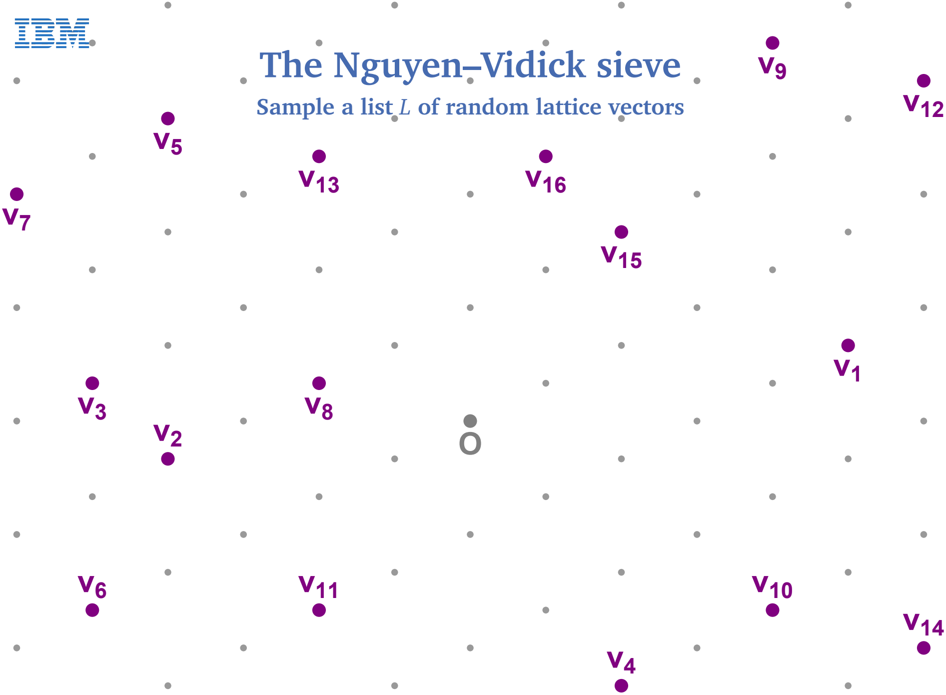
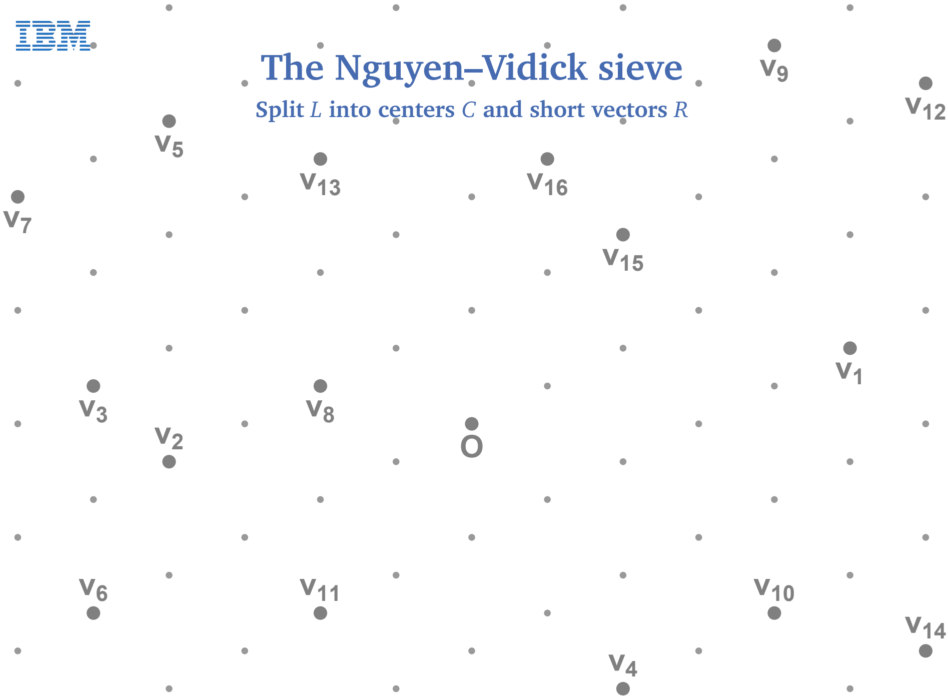


The Nguyen–Vidick sieve

Split L into centers C and short vectors R



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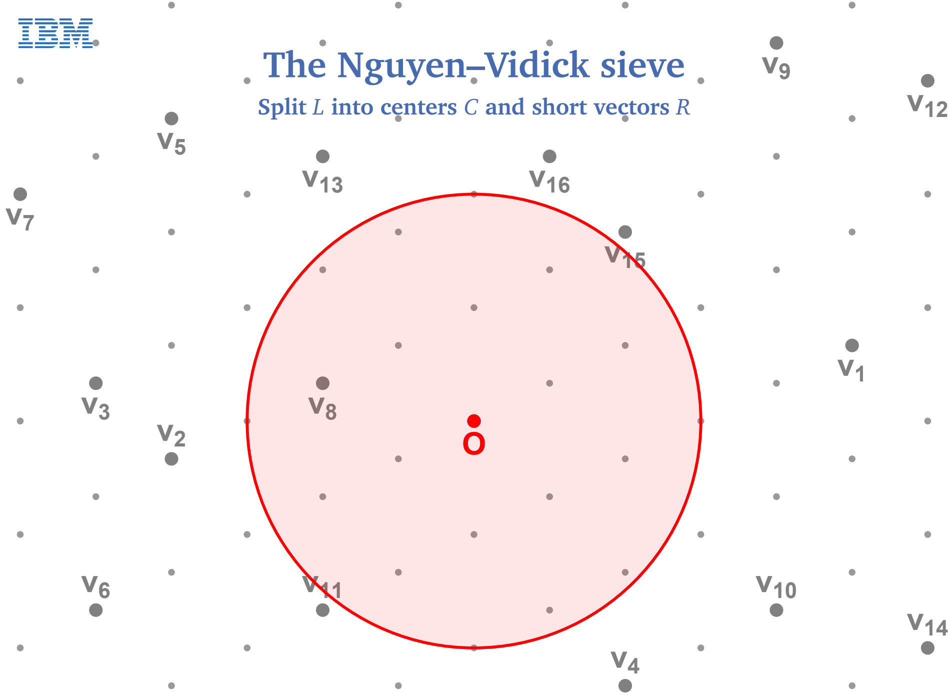

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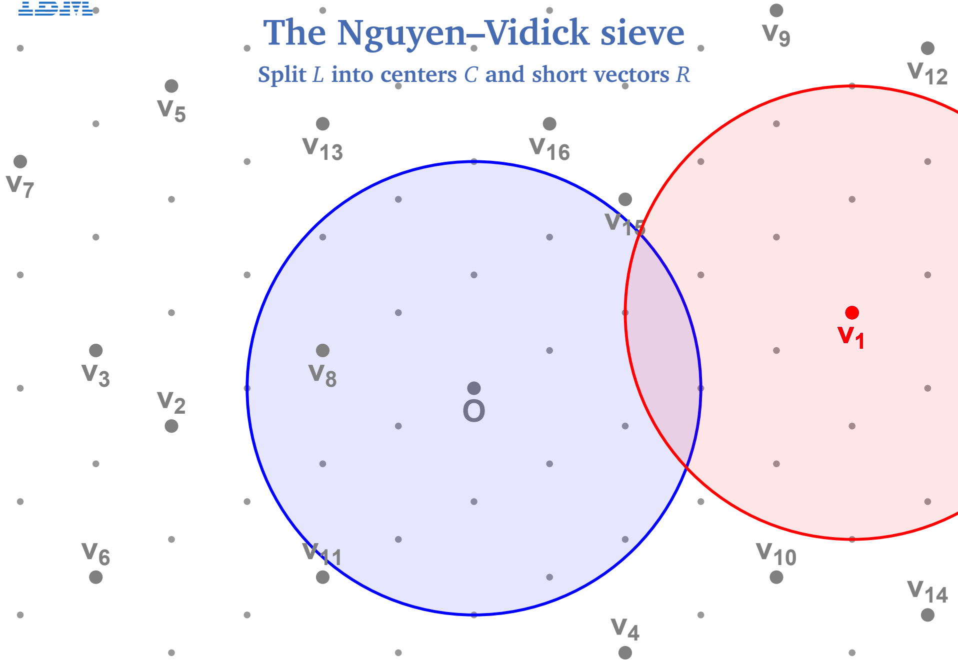

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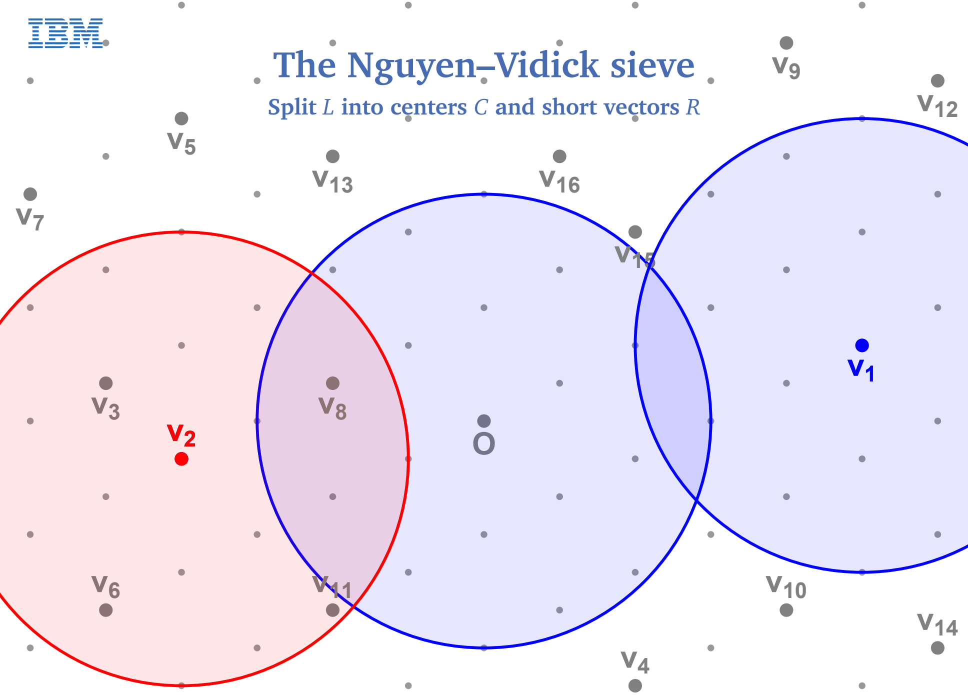
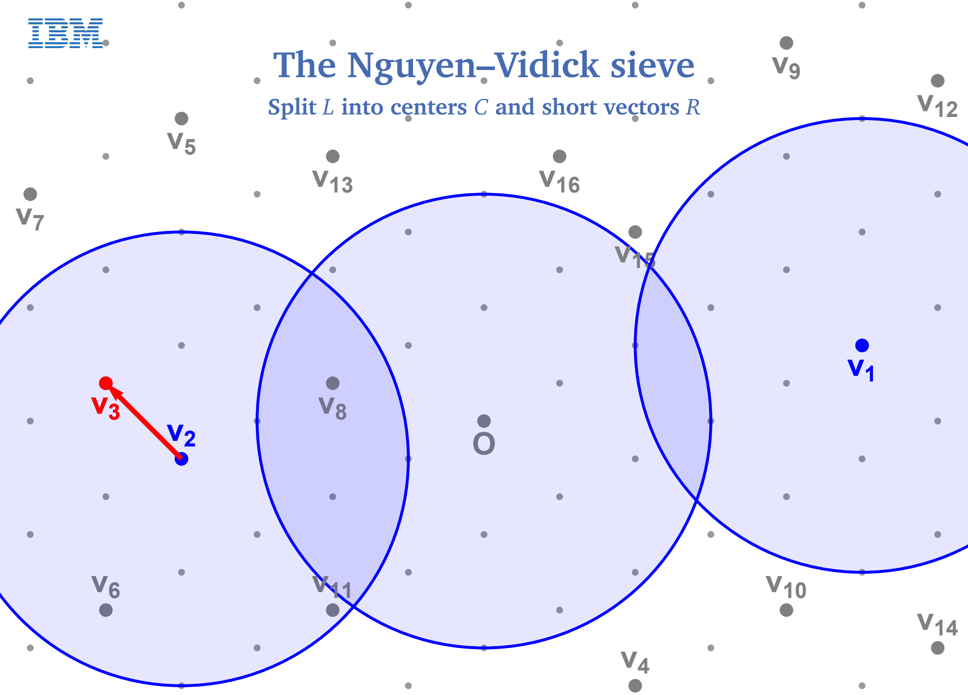
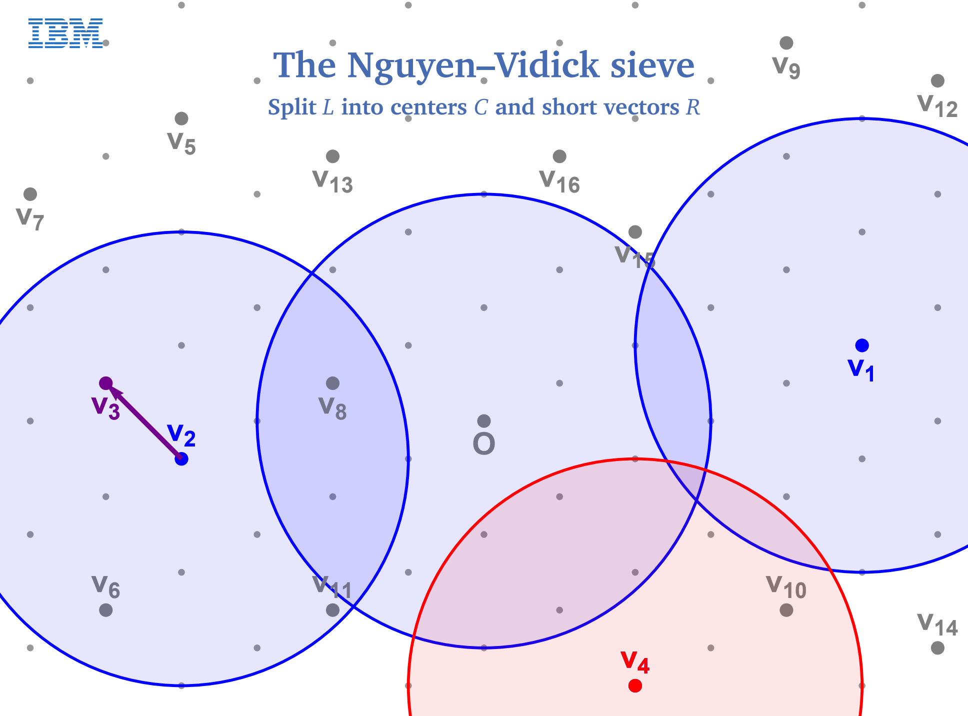


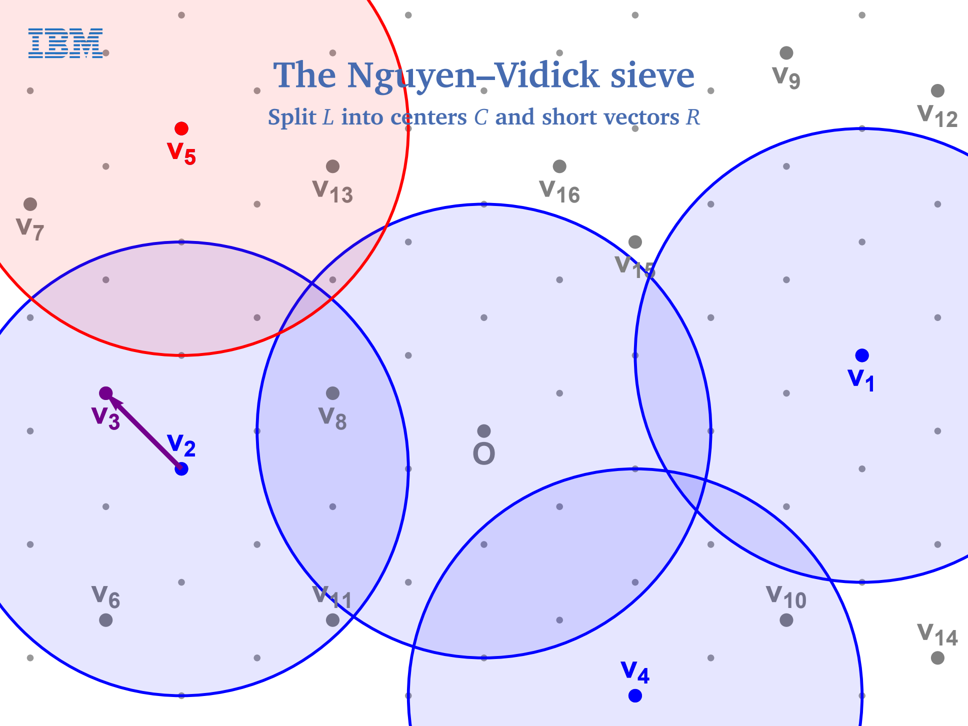


The Nguyen–Vidick sieve
Split L into centers C and short vectors R
v_1
v_2
v_3
v_4
v_5
v_6
v_7
v_8
v_9
v_{10}
v_{11}
v_{12}
v_{13}
v_{14}
v_{15}
v_{16}
O

The Nguyen–Vidick sieve

Split L into centers C and short vectors R

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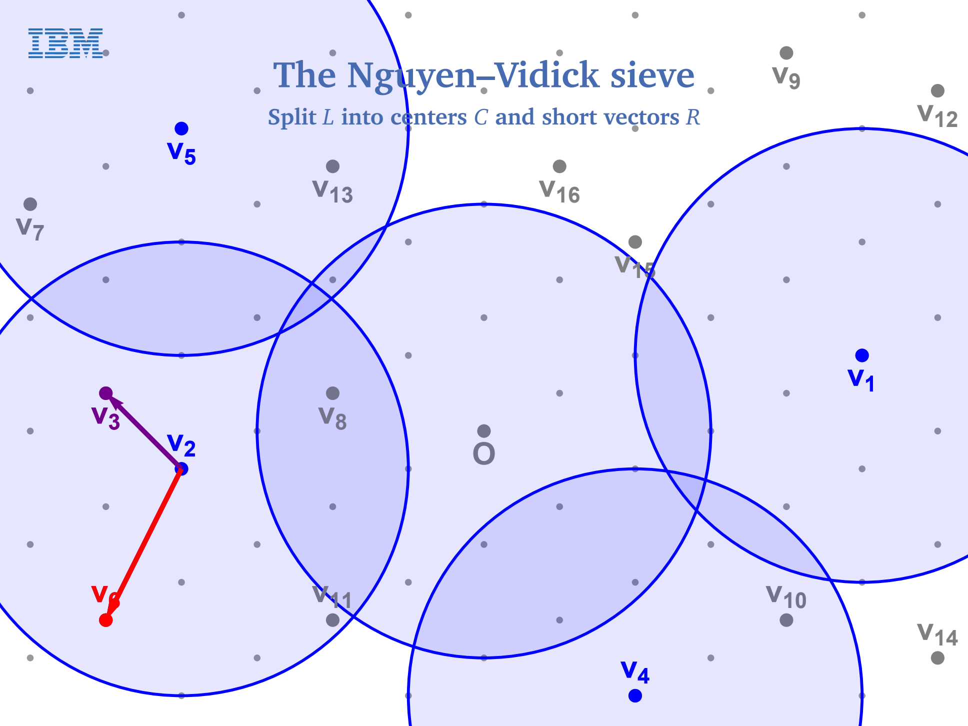
v5
v13
v16
v9
v12
v7
v15
v1
v3
v8
v2
O
v6
v11
v10
v14
v4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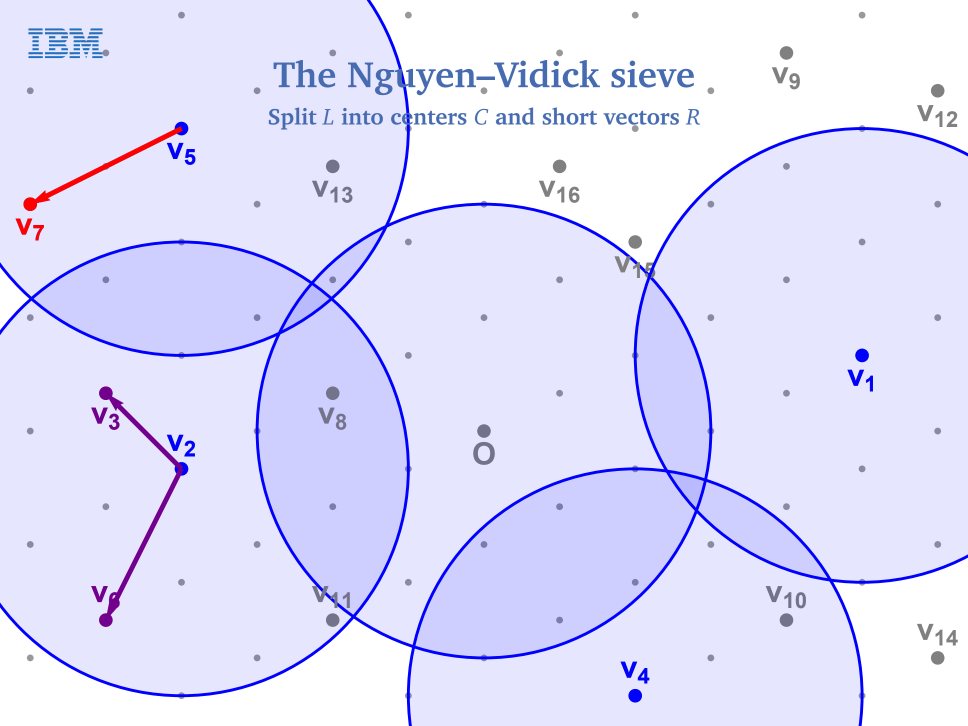

The Nguyen–Vidick sieve

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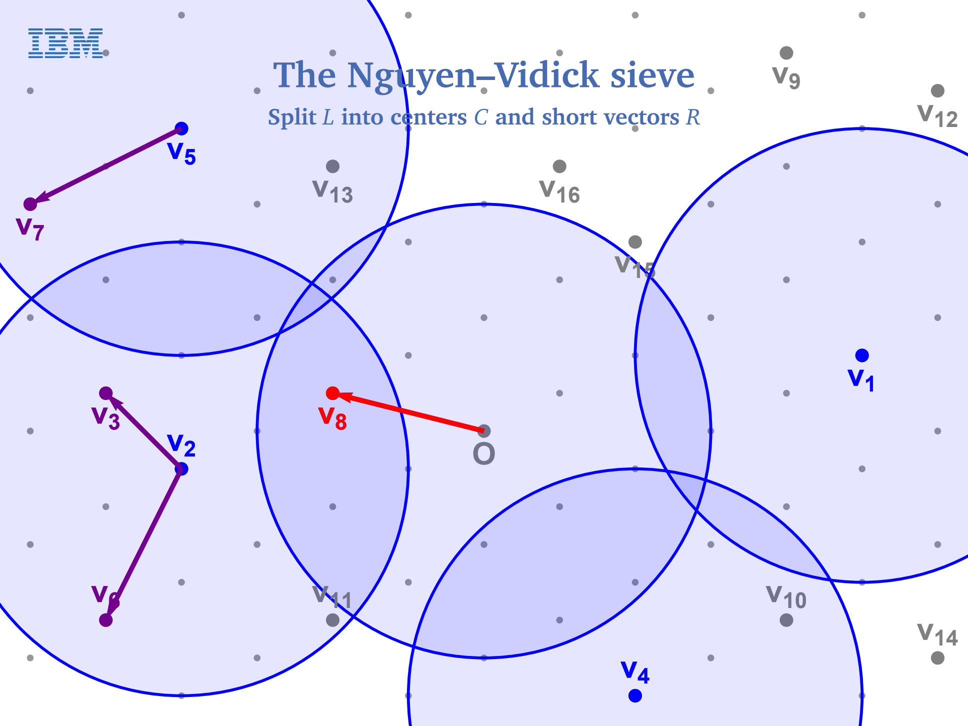

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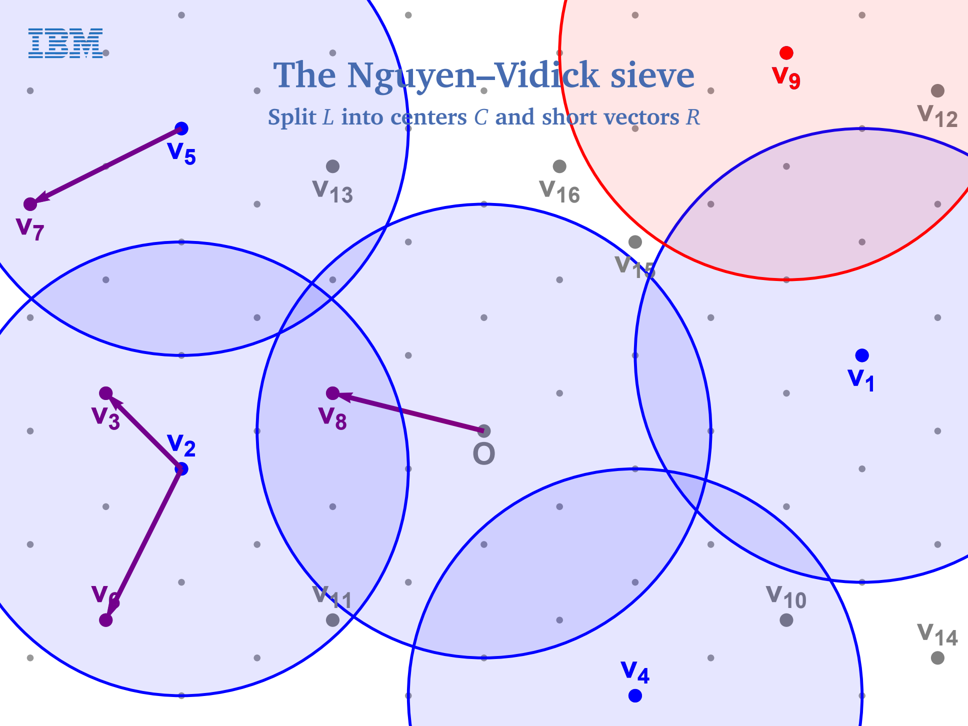

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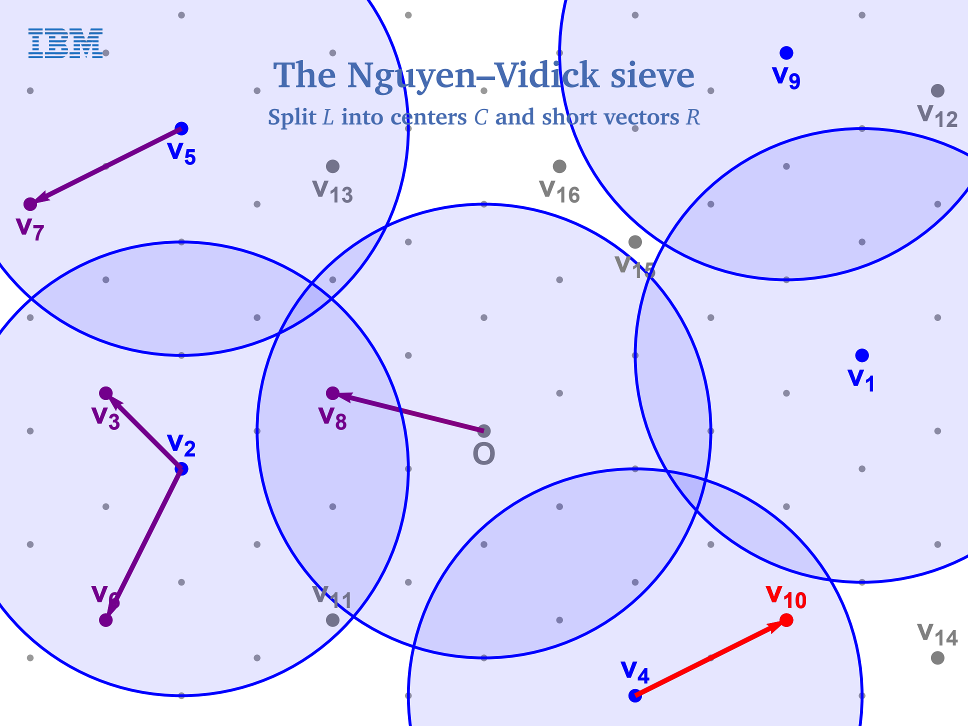

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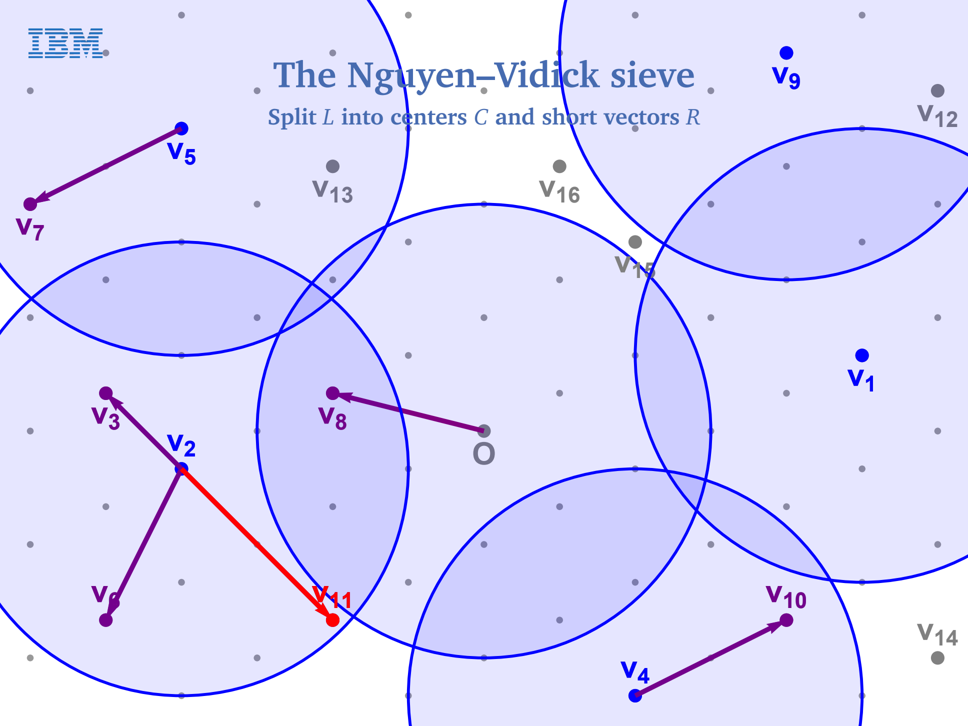

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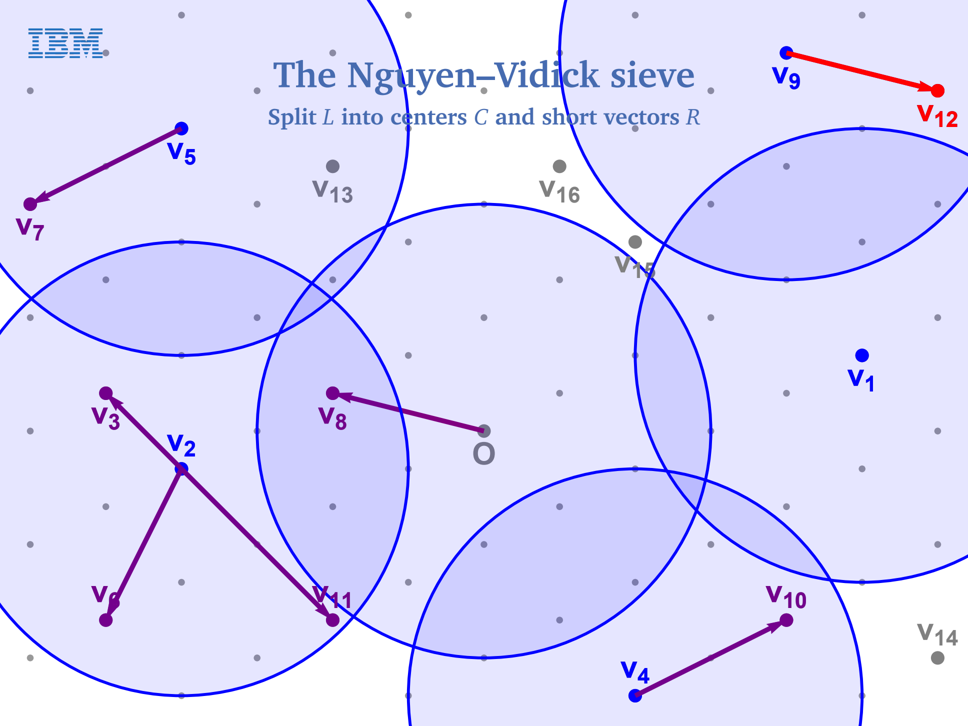
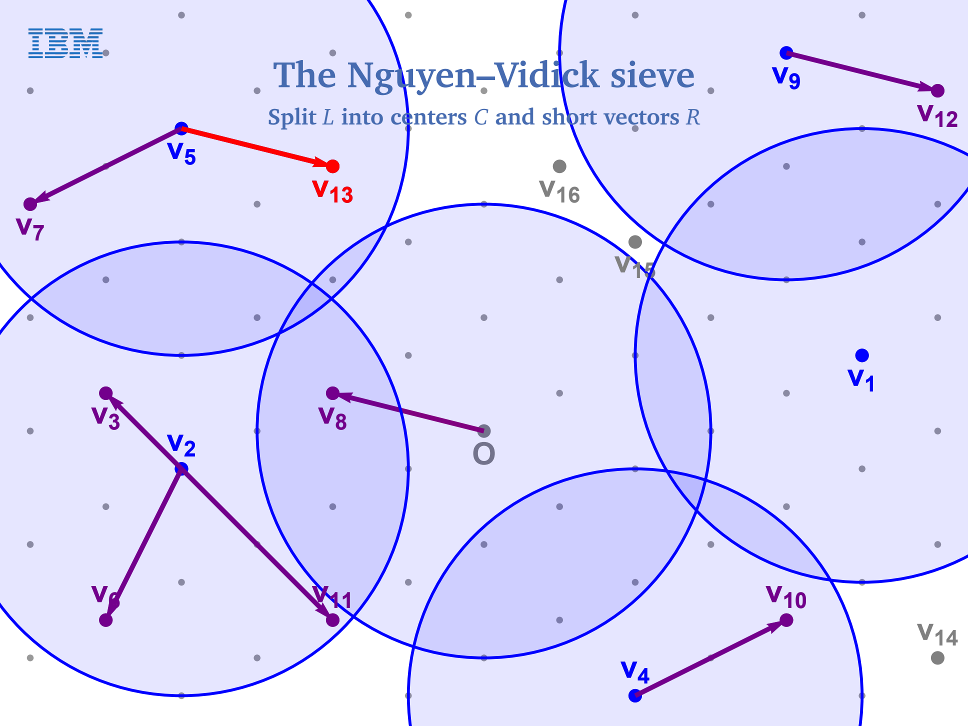
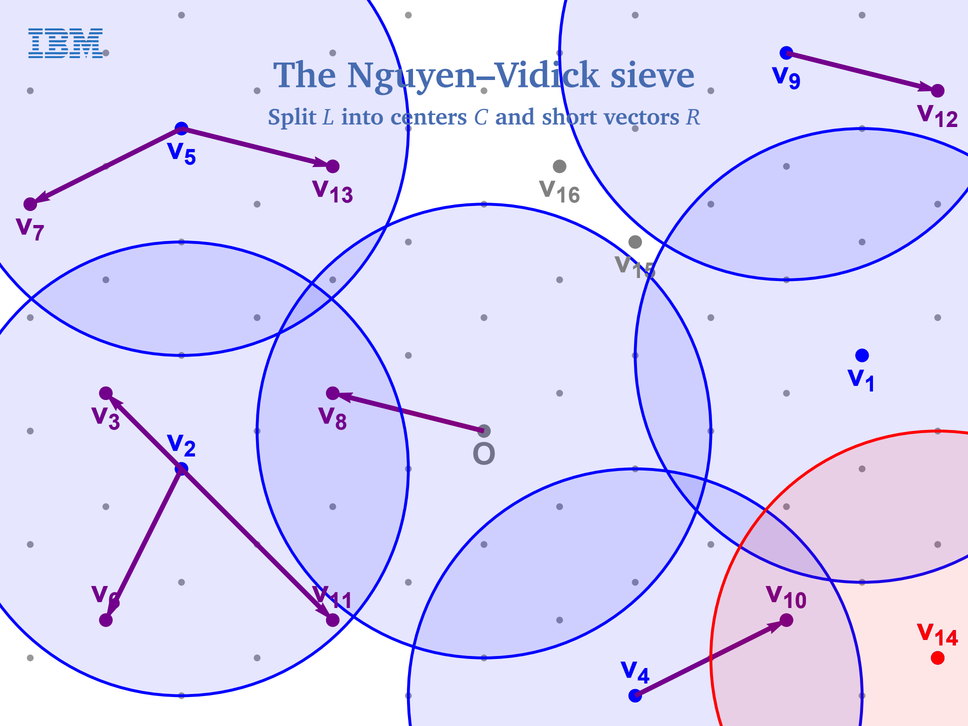




The Nguyen–Vidick sieve
Split L into centers C and short vectors R
v_1
v_2
v_3
v_4
v_5
v_7
v_8
v_9
v_{10}
v_{11}
v_{12}
v_{13}
v_{14}
v_{15}
v_{16}
O


IBM
The Nguyen–Vidick sieve
Split L into centers C and short vectors R
v9
v12
v5
v13
v16
v7
v15
v1
v3
v8
v2
O
v11
v10
v14
v4

The Nguyen–Vidick sieve

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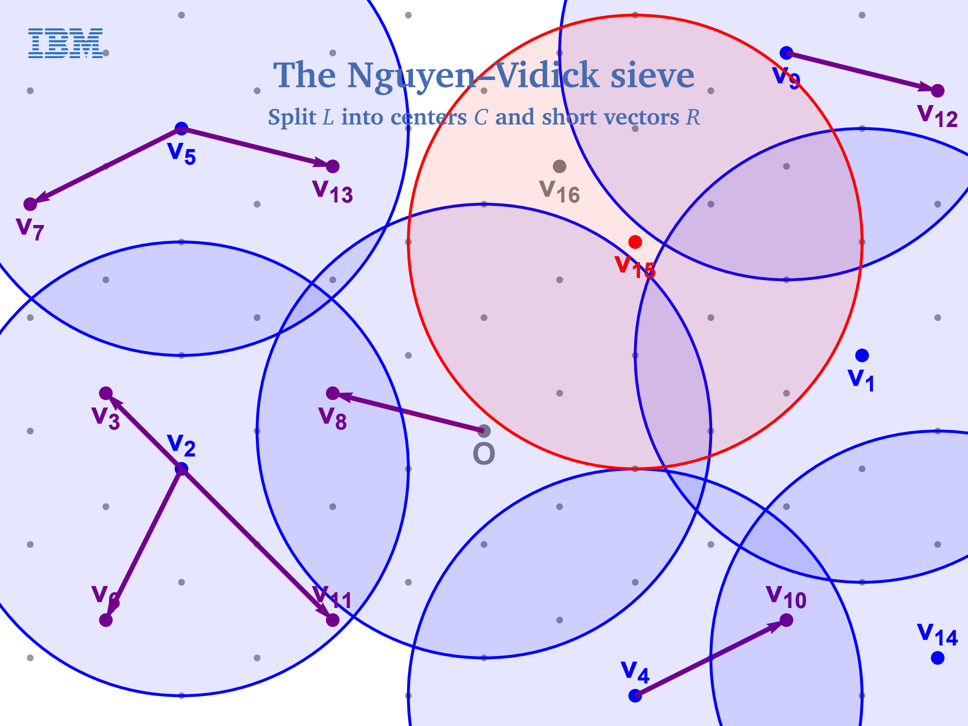

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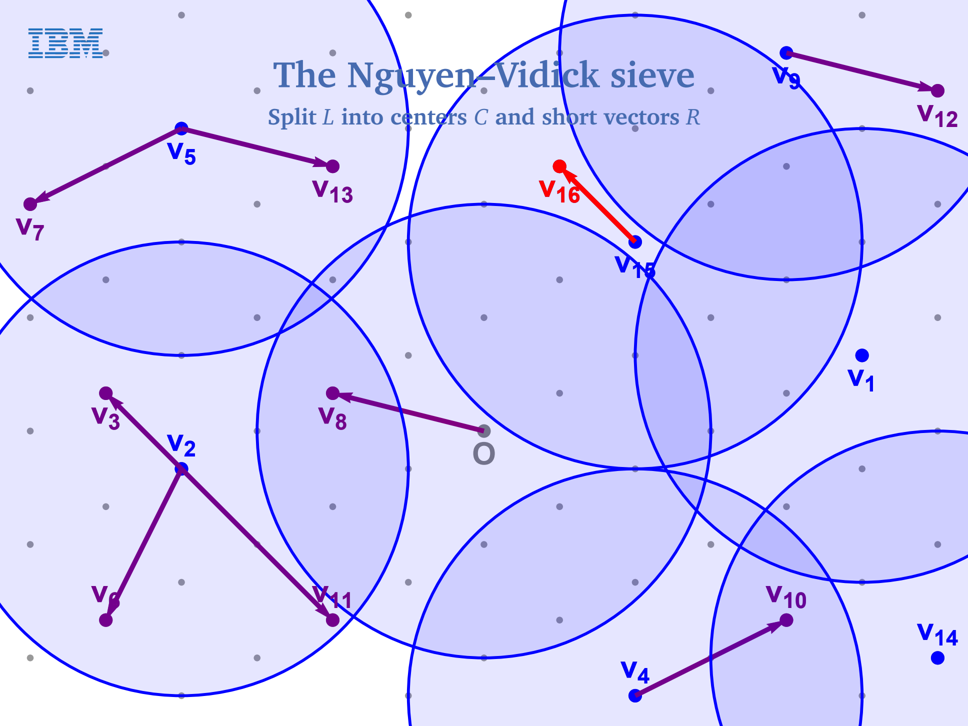

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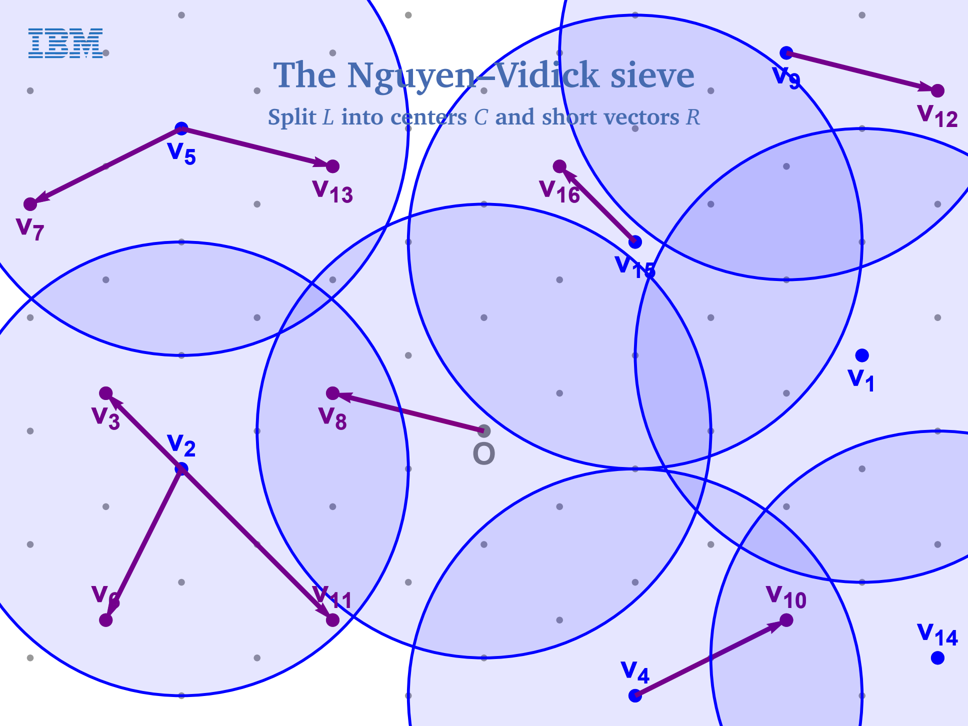

The Nguyen–Vidick sieve

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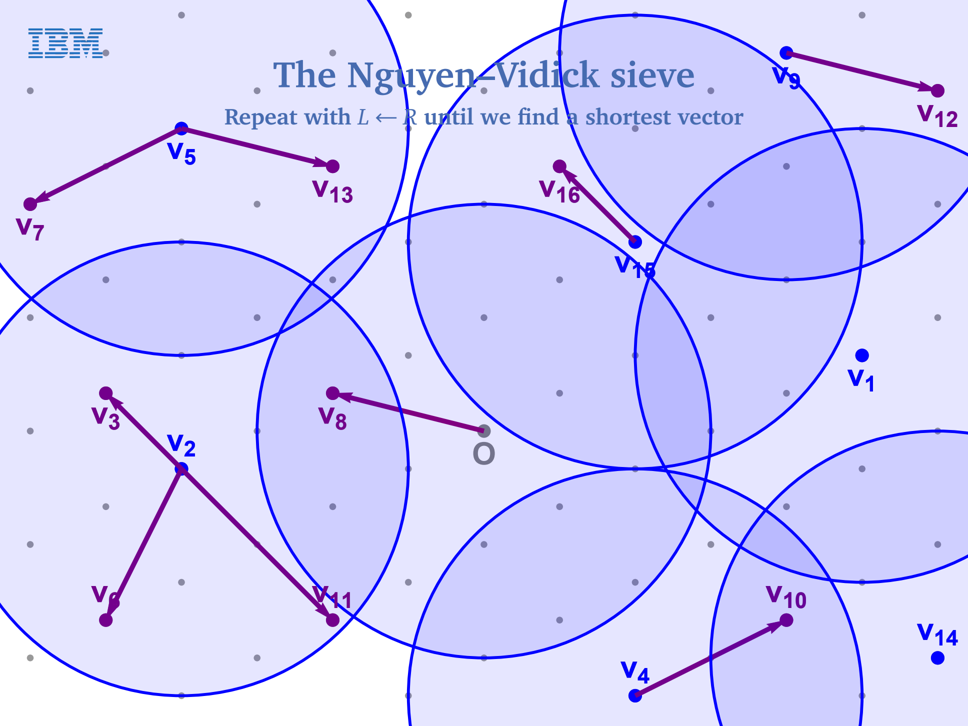

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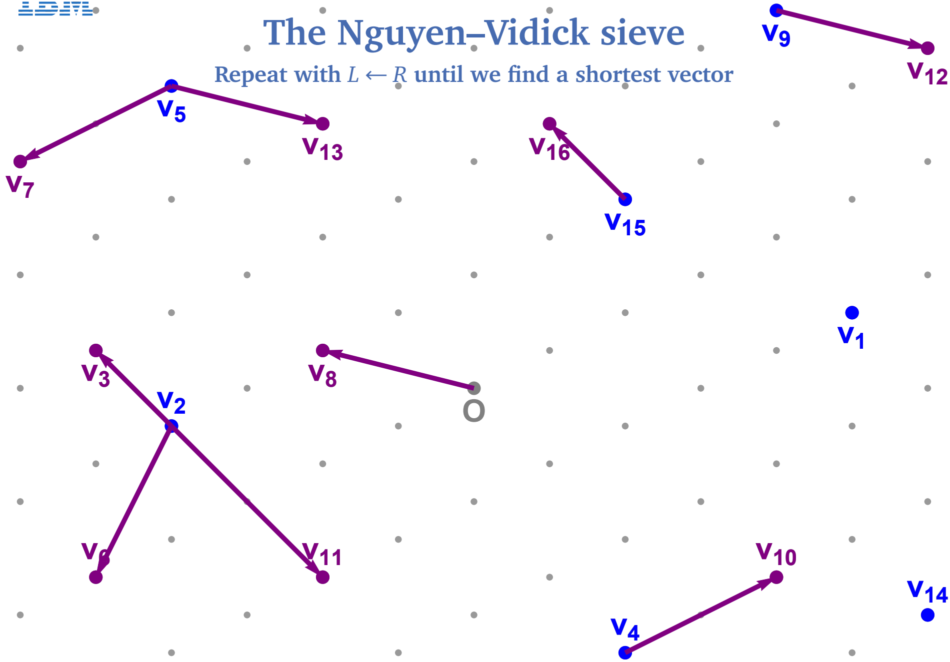

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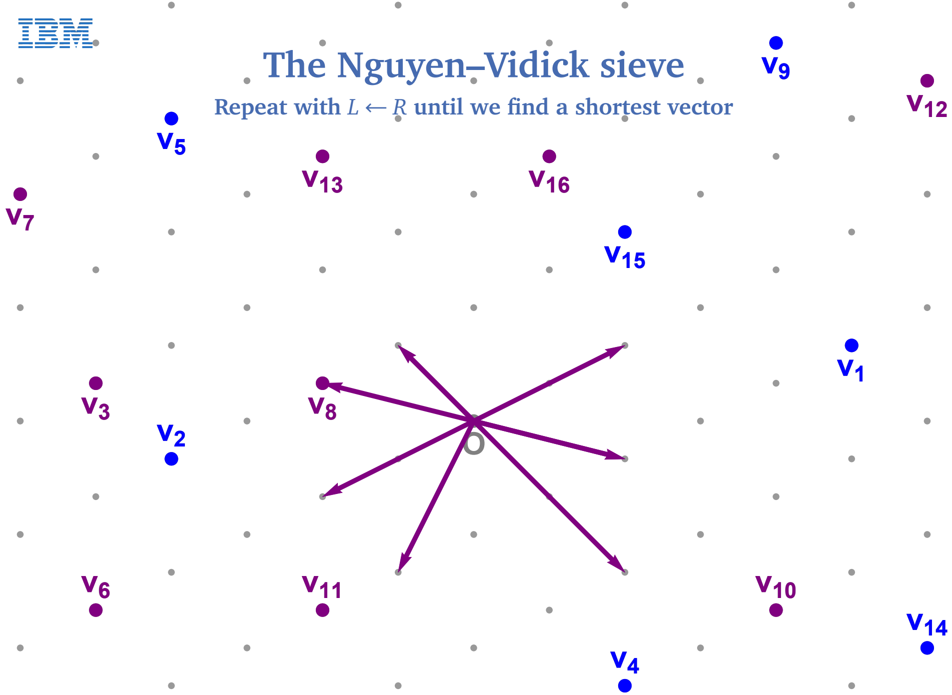

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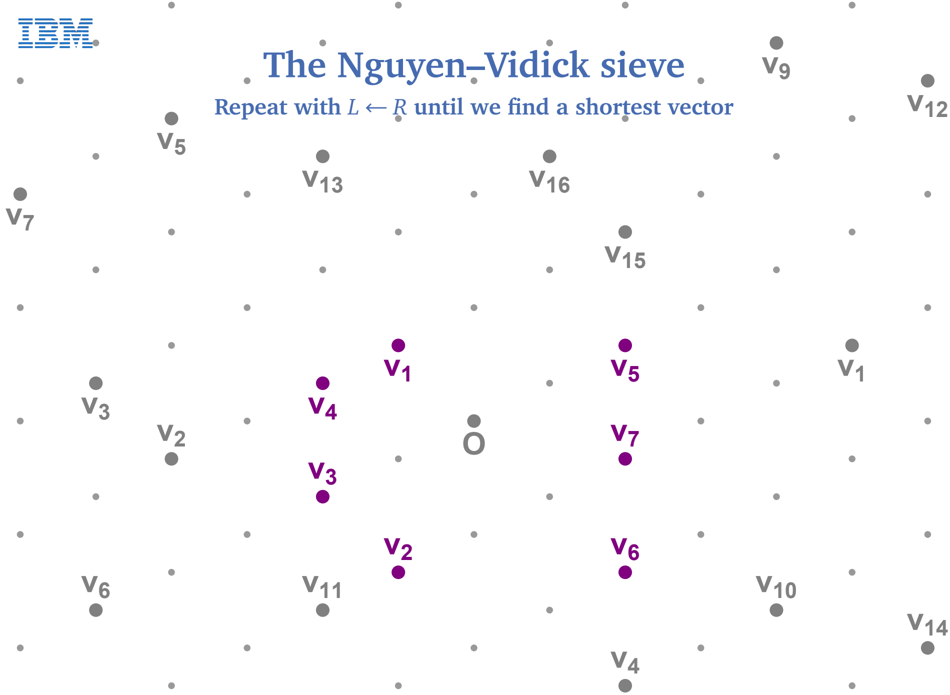
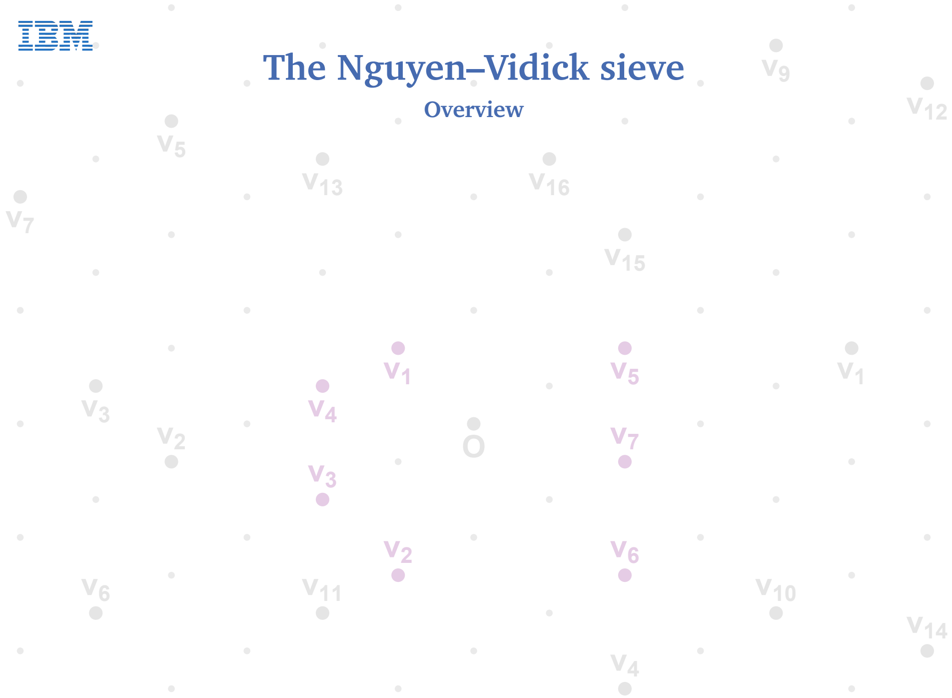


The Nguyen–Vidick sieve

Overview



The Nguyen–Vidick sieve

Overview

- Space complexity: $\sqrt{4/3}^n \approx 2^{0.21n+o(n)}$ vectors
  - Need $\sqrt{4/3}^n$ vectors to cover all corners of $\mathbb{R}^n$

The Nguyen–Vidick sieve

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  - Comparing a target vector to all centers: $2^{0.21n+o(n)}$
  - Repeating this for each list vector: $2^{0.21n+o(n)}$
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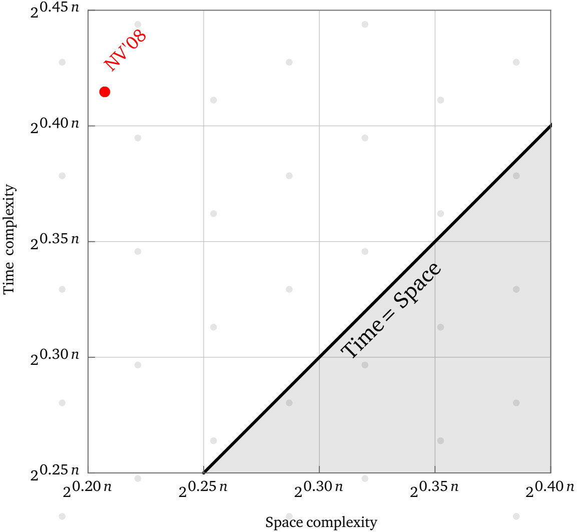
Heuristic result (Nguyen–Vidick, J. Math. Crypt. '08)

The NV-sieve runs in time $2^{0.42n+o(n)}$ and space $2^{0.21n+o(n)}$.

The Nguyen–Vidick sieve

Space/time trade-off

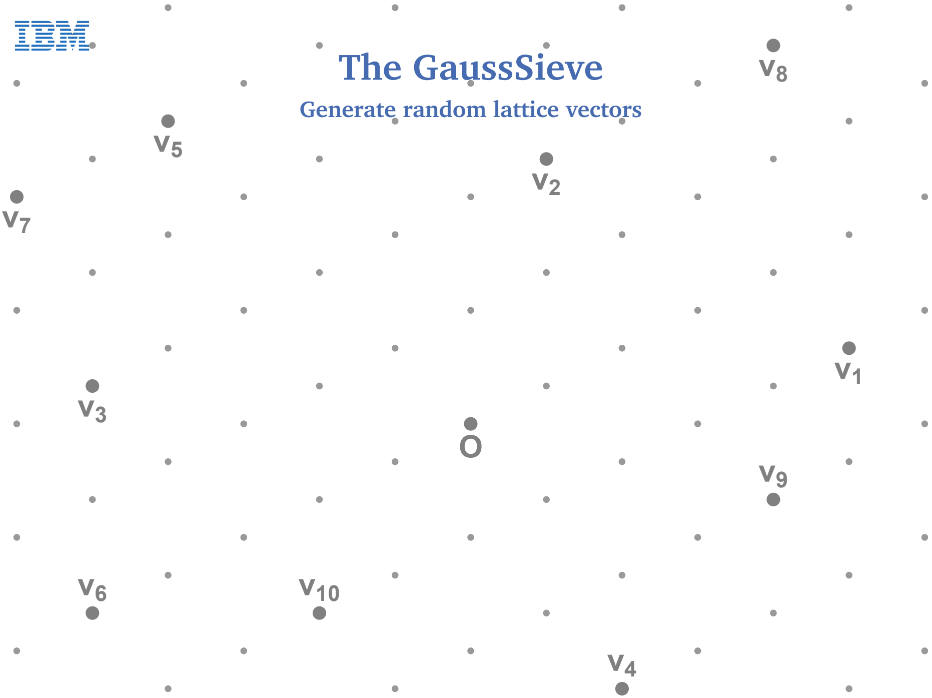


The GaussSieve

Generate random lattice vectors

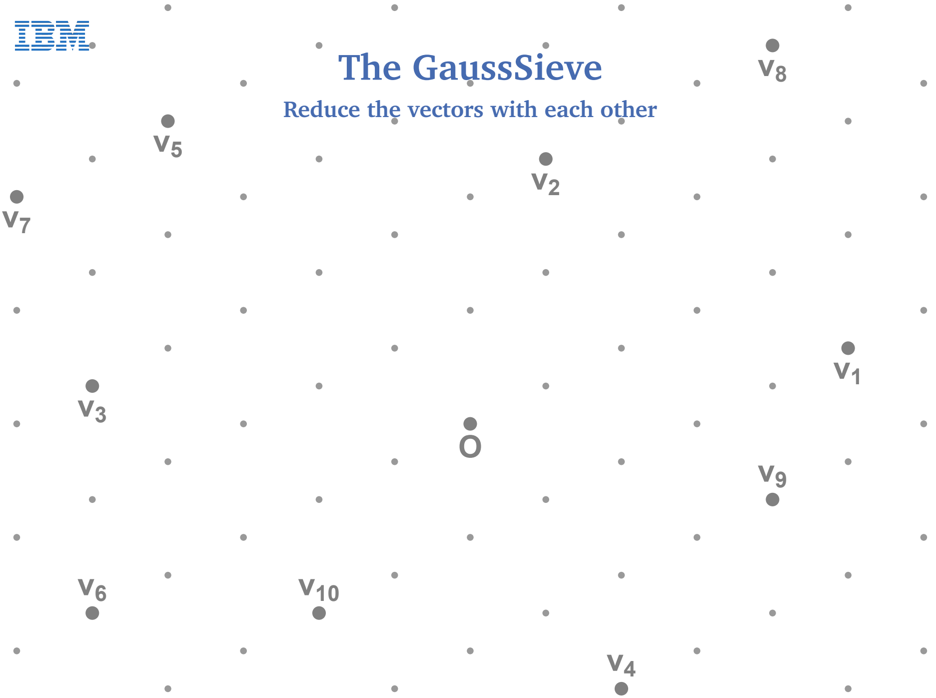
The GaussSieve

Generate random lattice vectors



The GaussSieve

Reduce the vectors with each other



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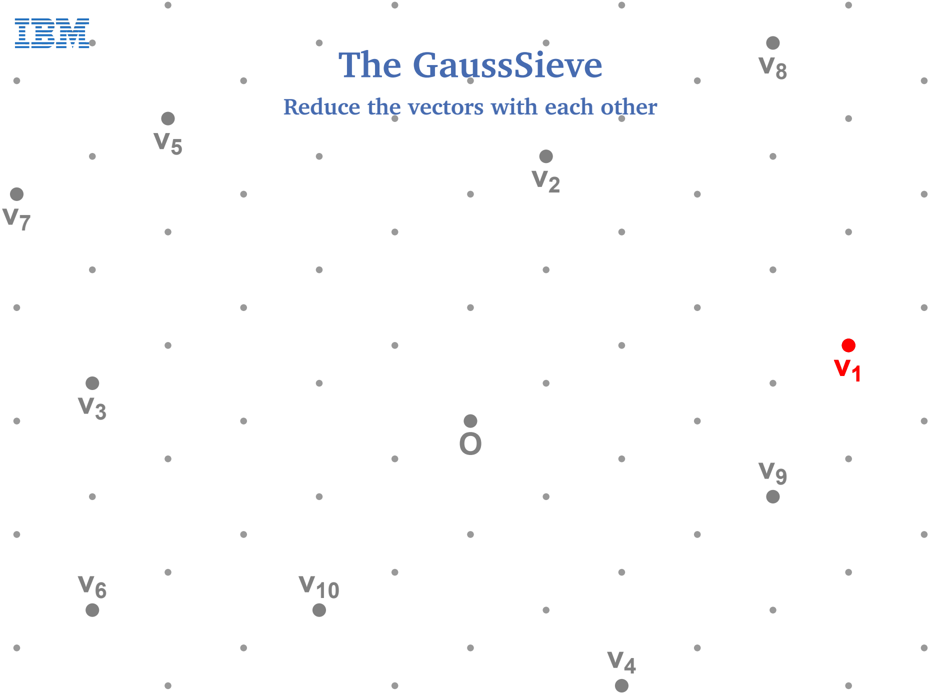

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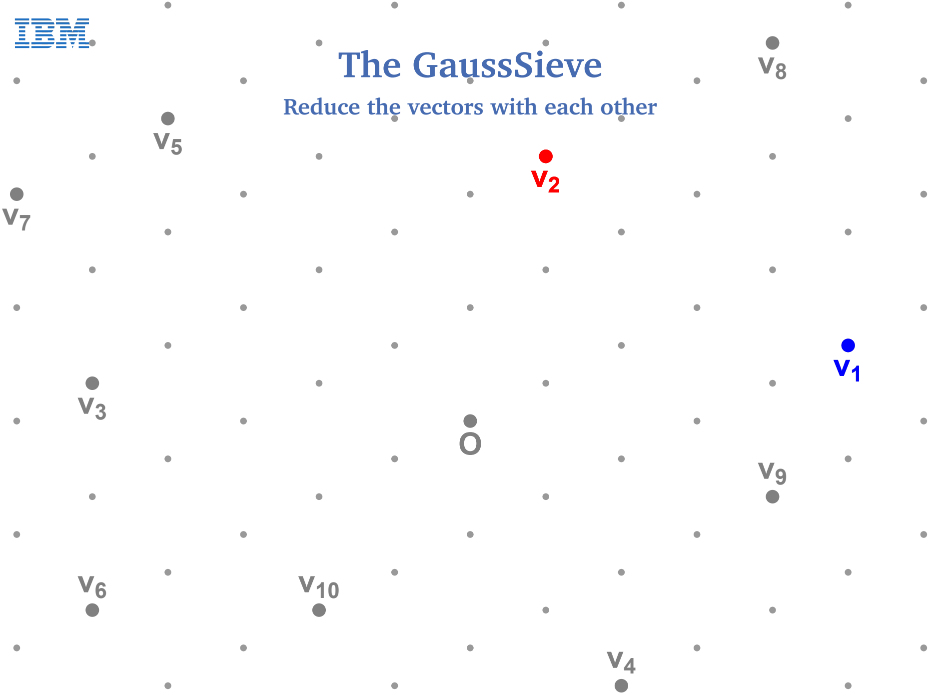

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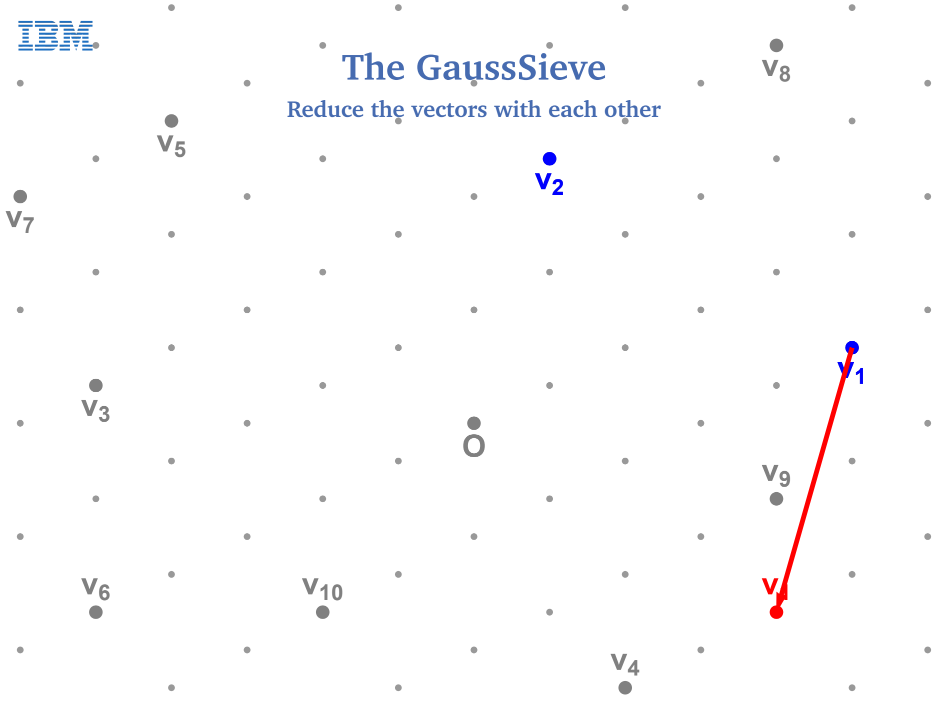

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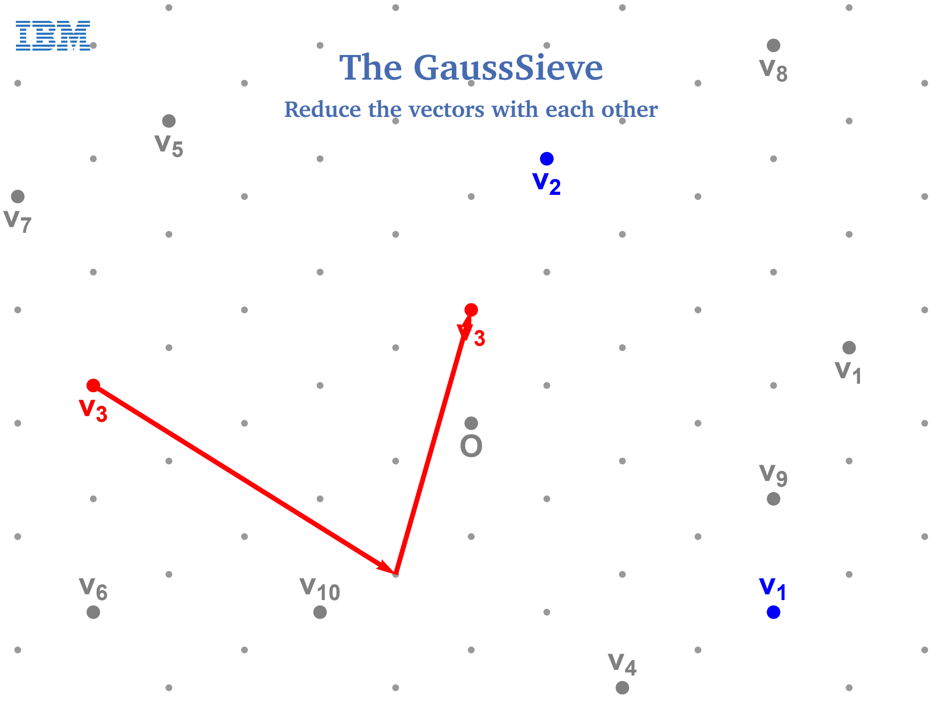

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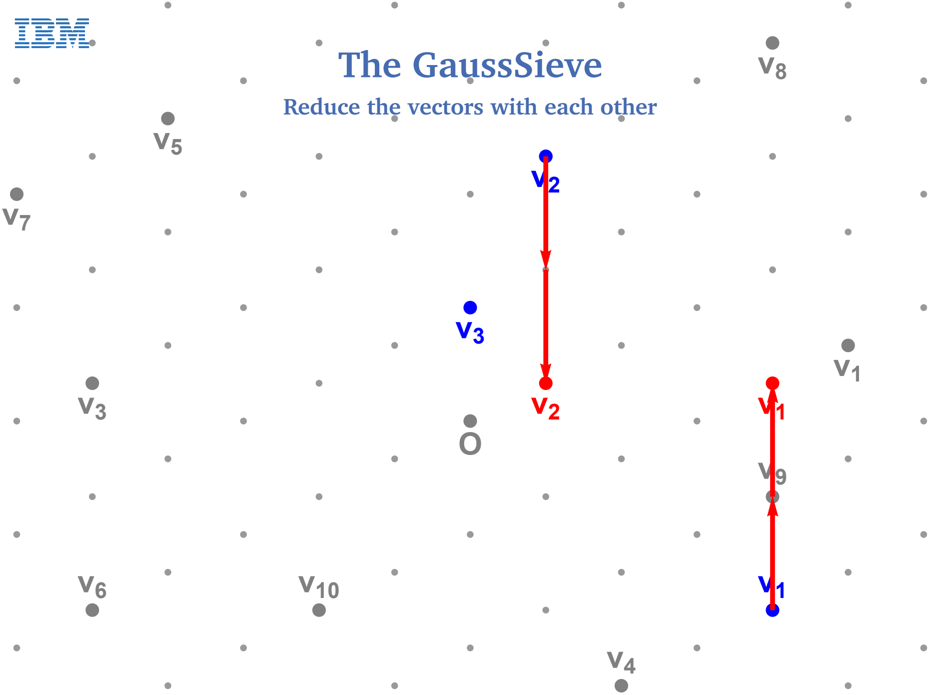

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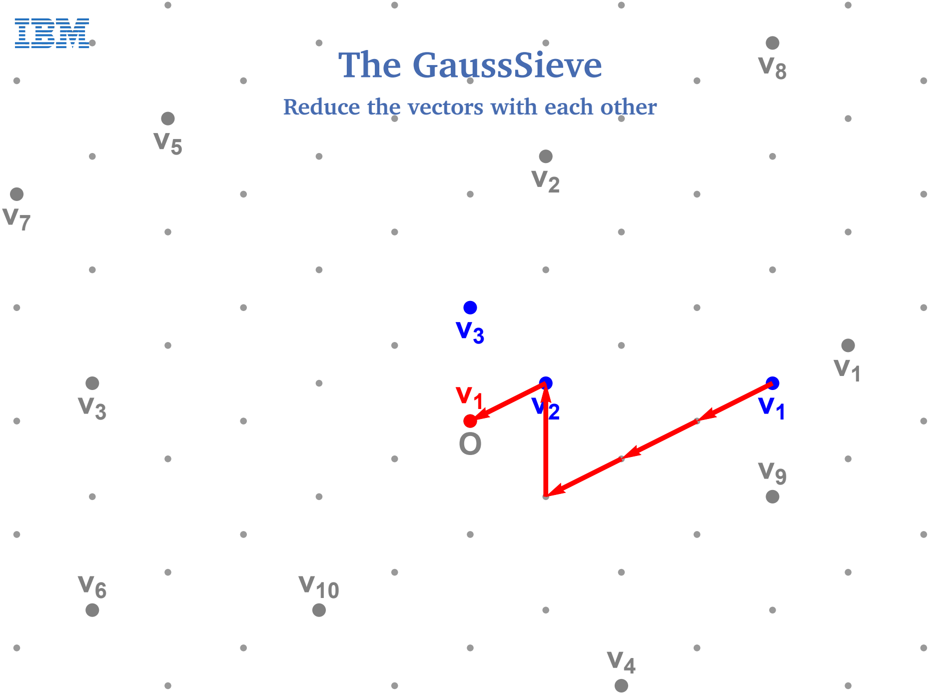

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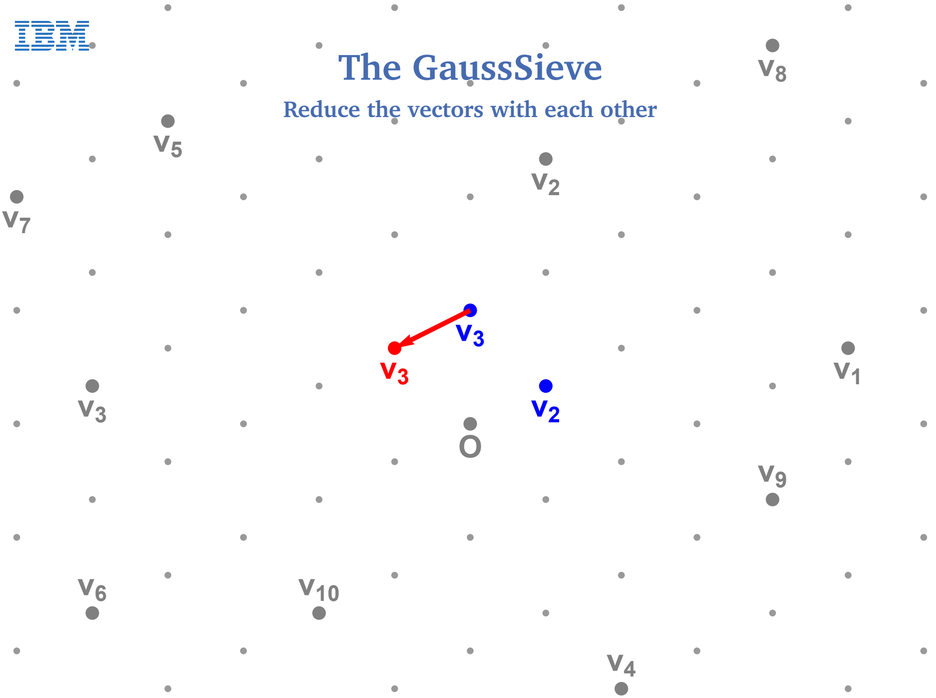

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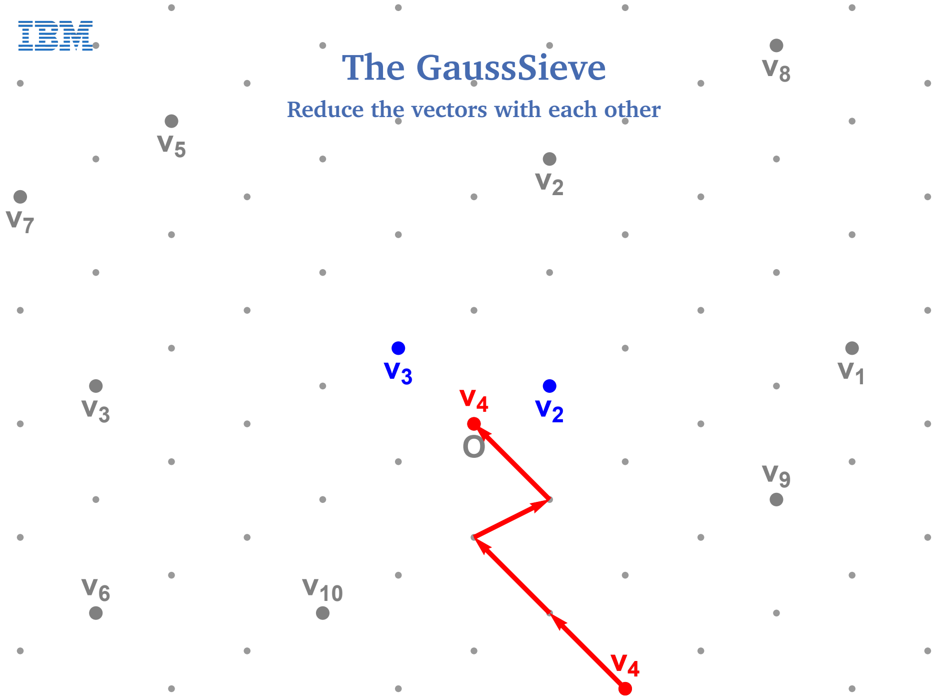

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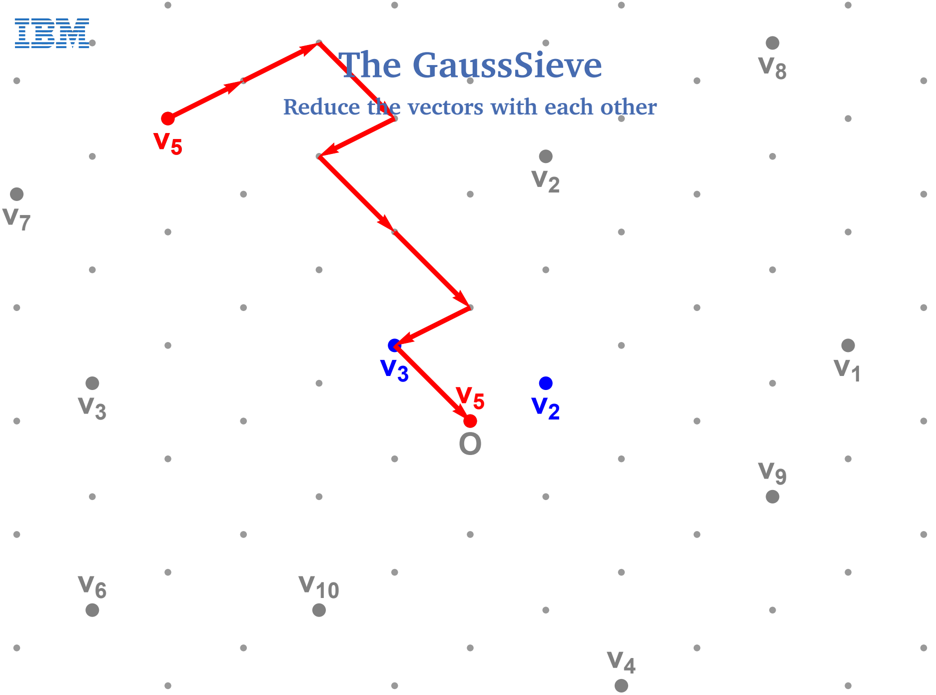

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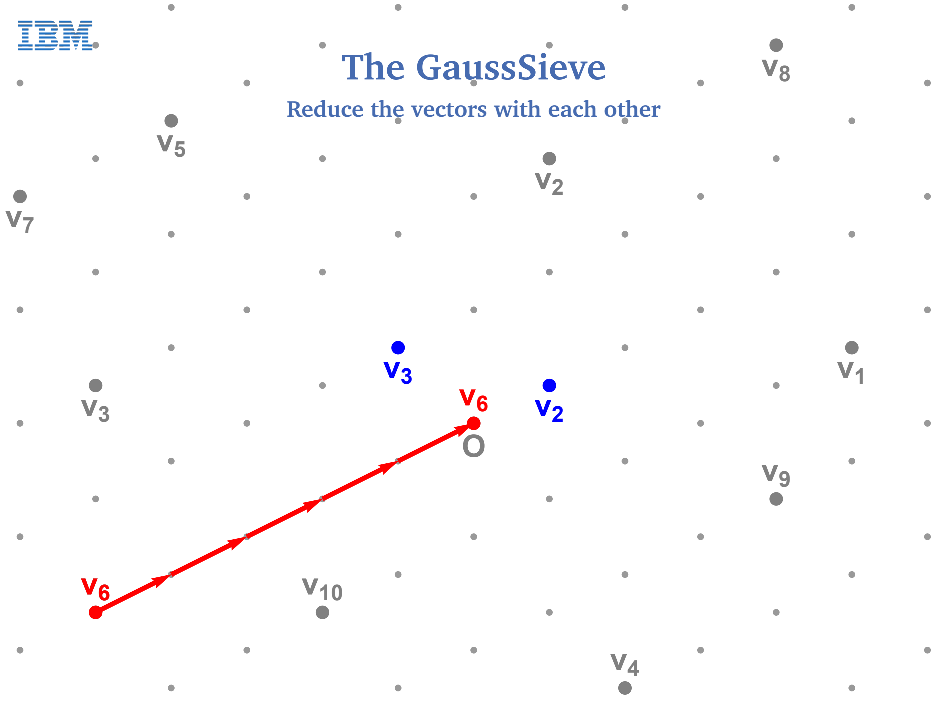

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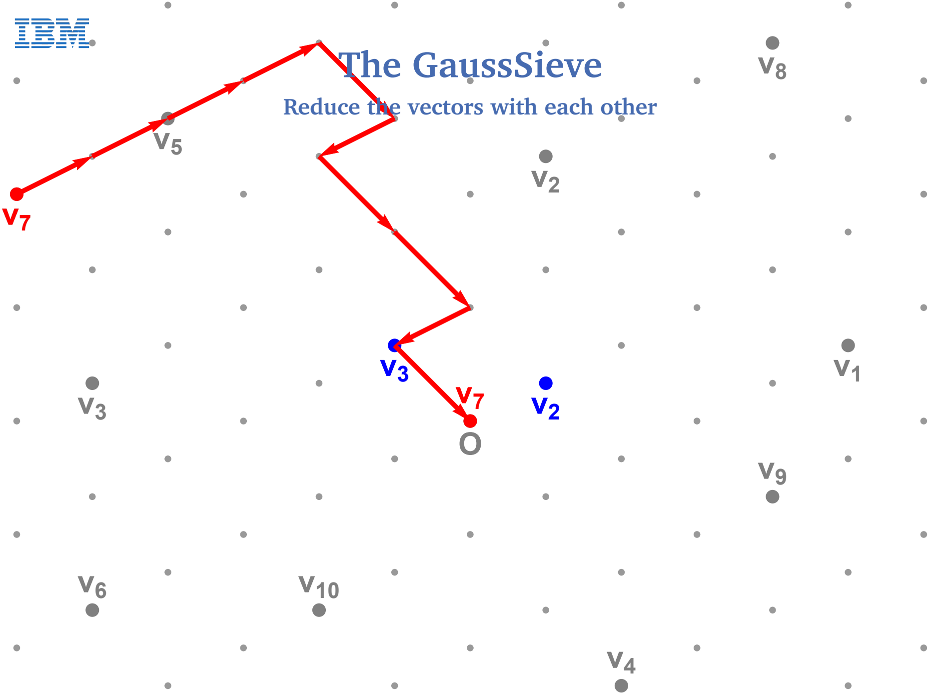

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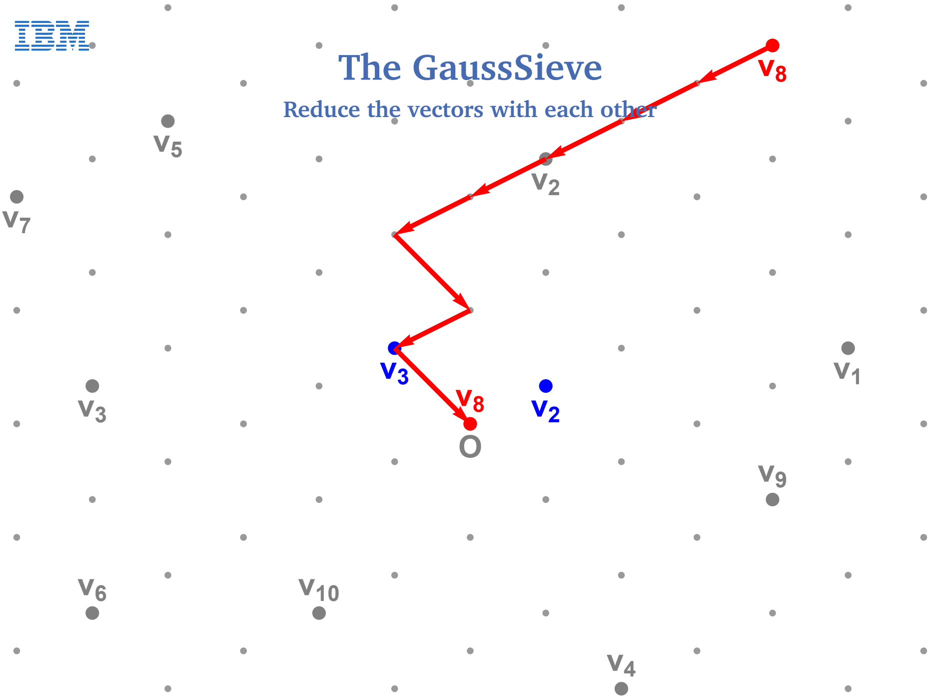

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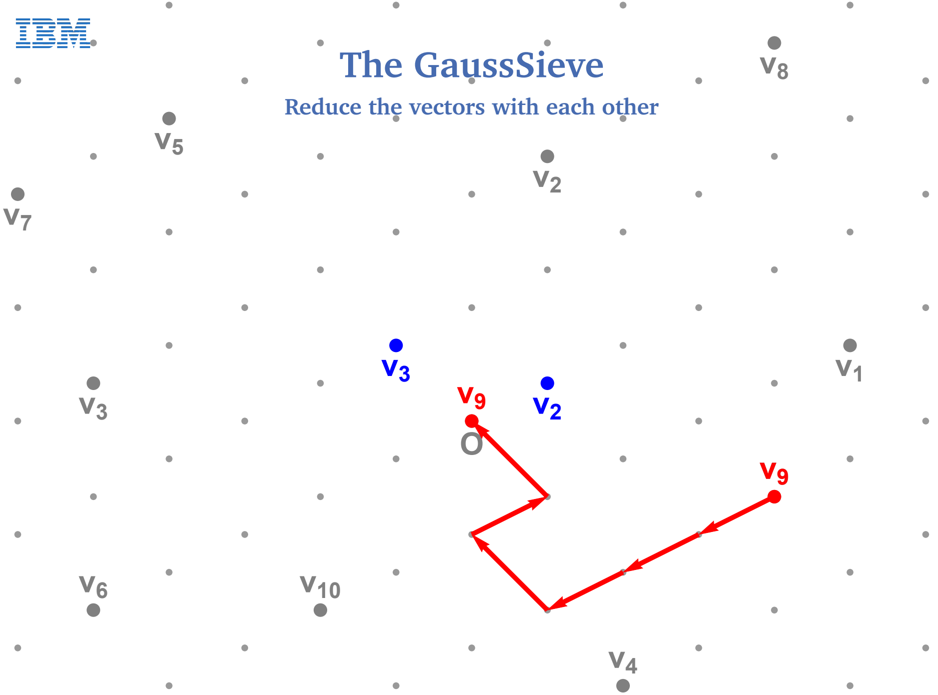

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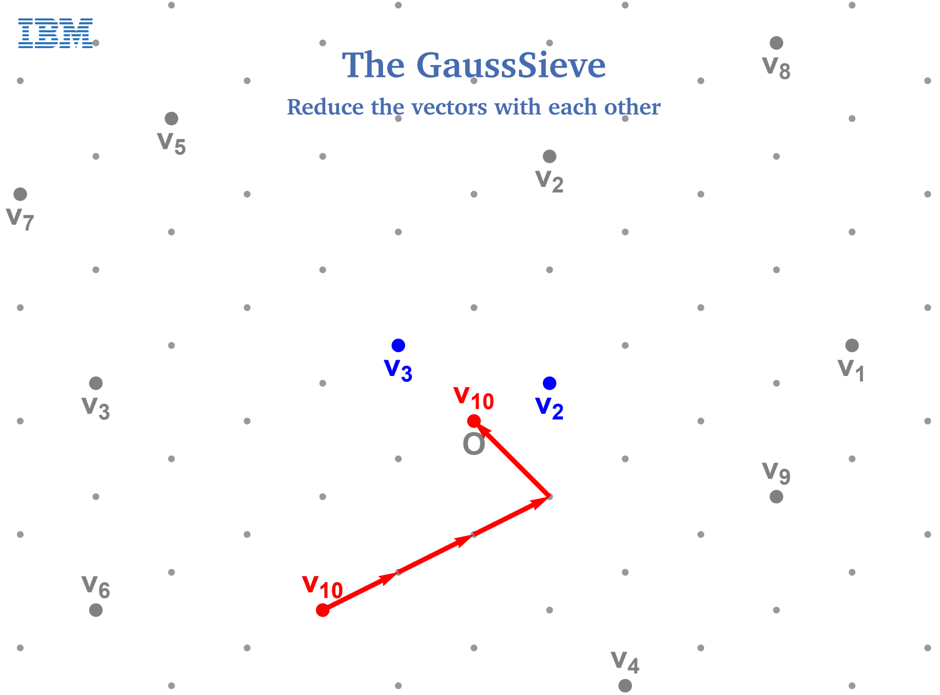

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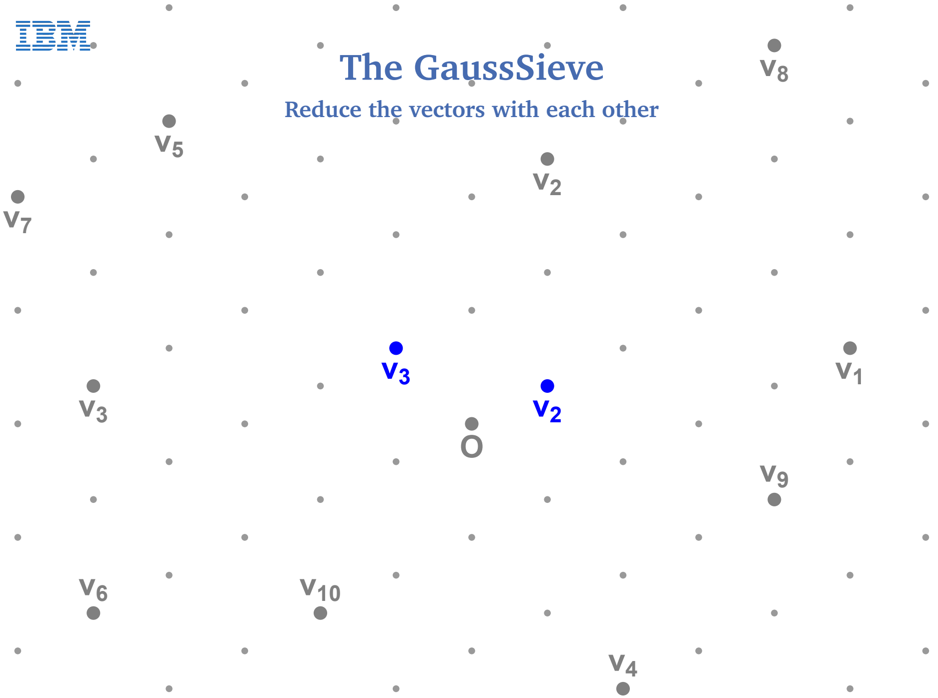

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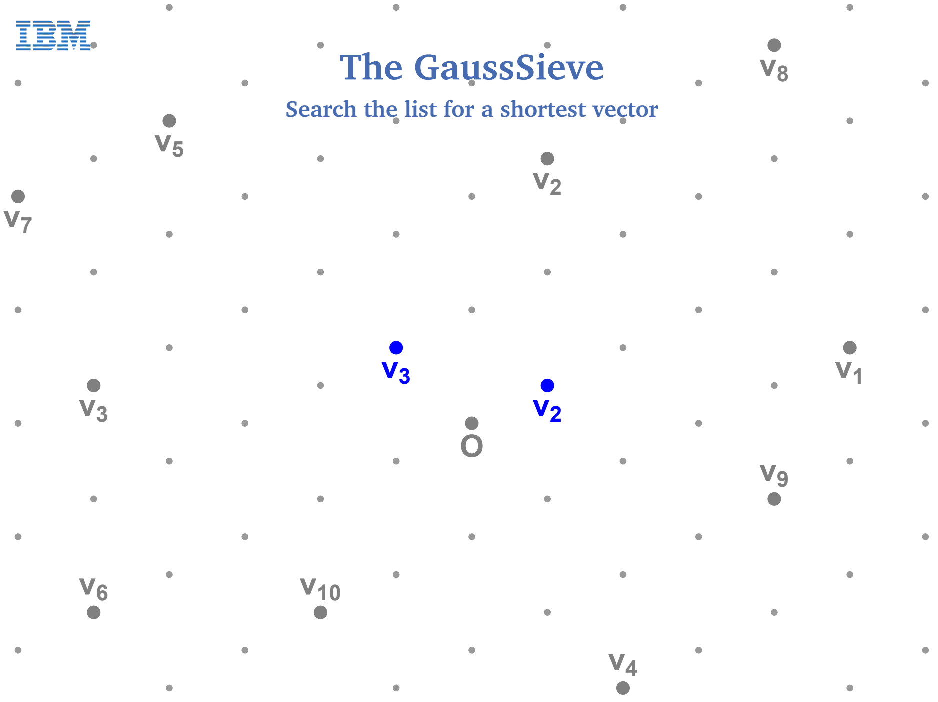
The GaussSieve

Search the list for a shortest vector

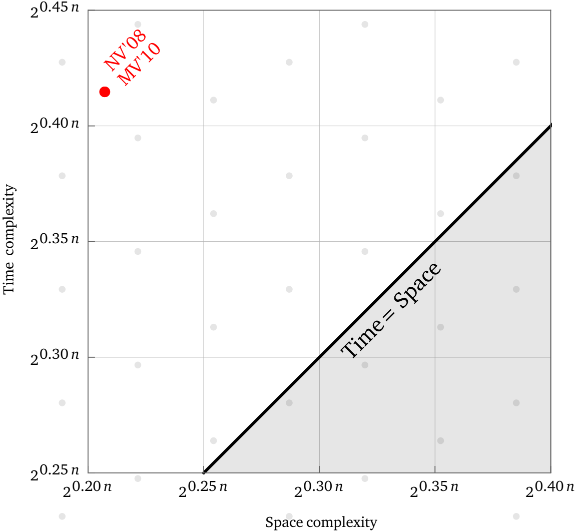


The GaussSieve

Search the list for a shortest vector

The GaussSieve

Space/time trade-off



Two-level sieve

Sample a list L of random lattice vectors

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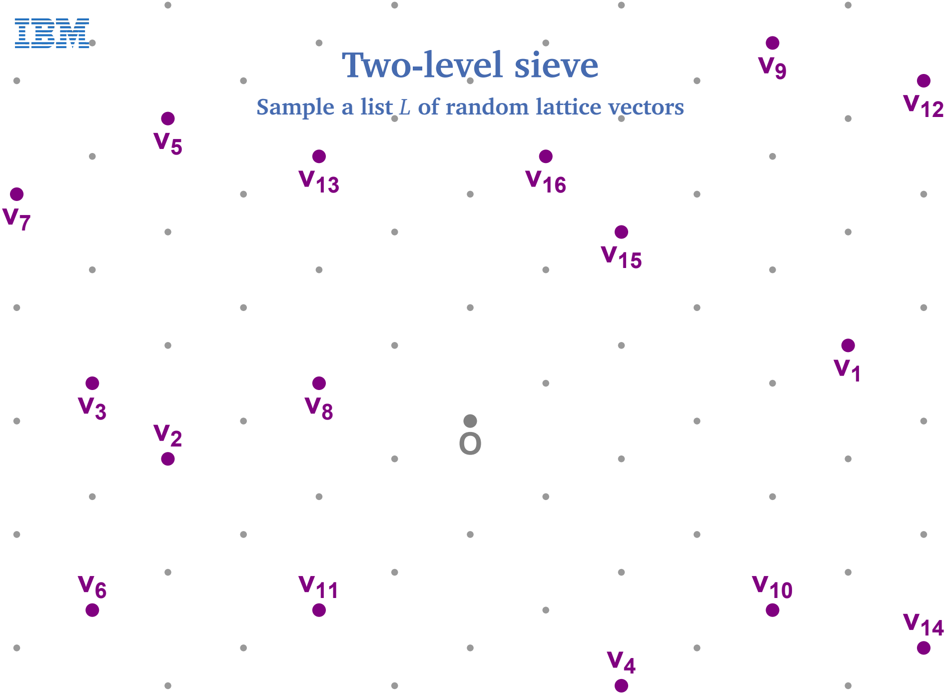
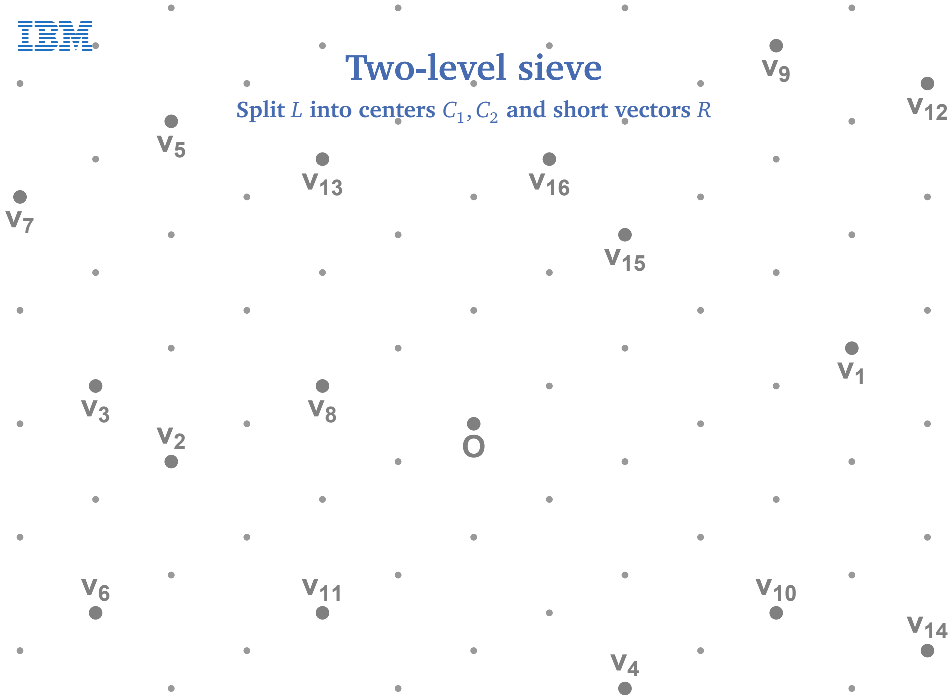


Two-level sieve

Split L into centers C_1, C_2 and short vectors R



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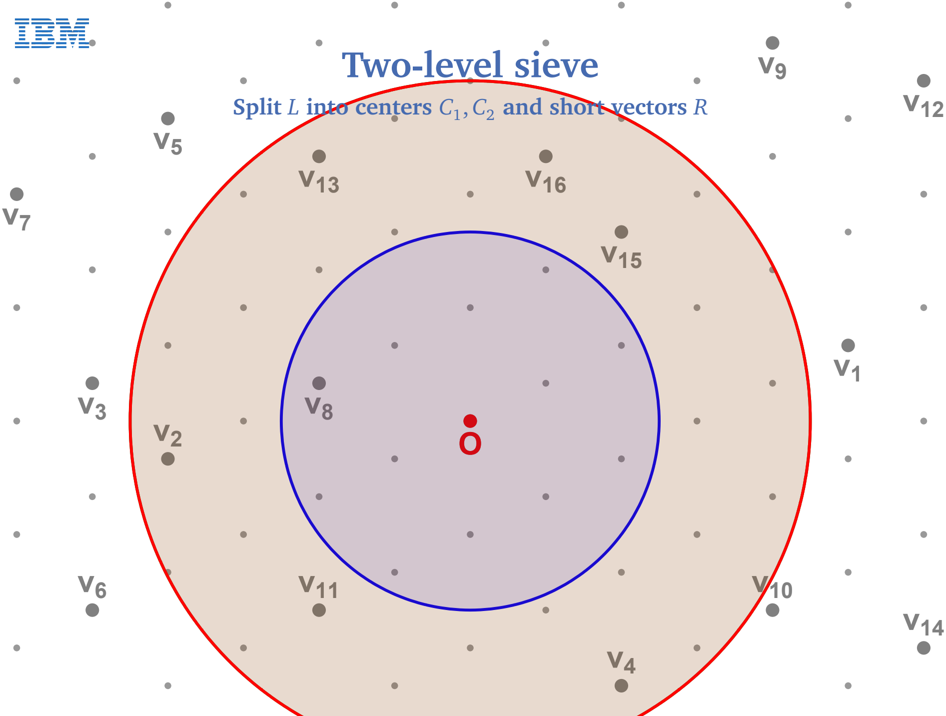

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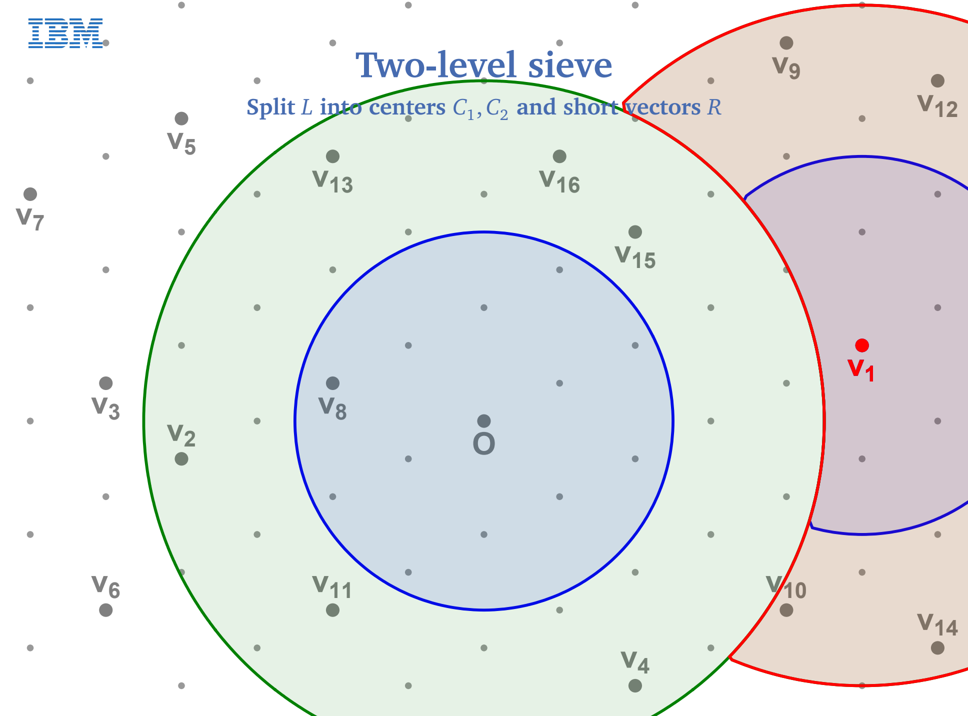

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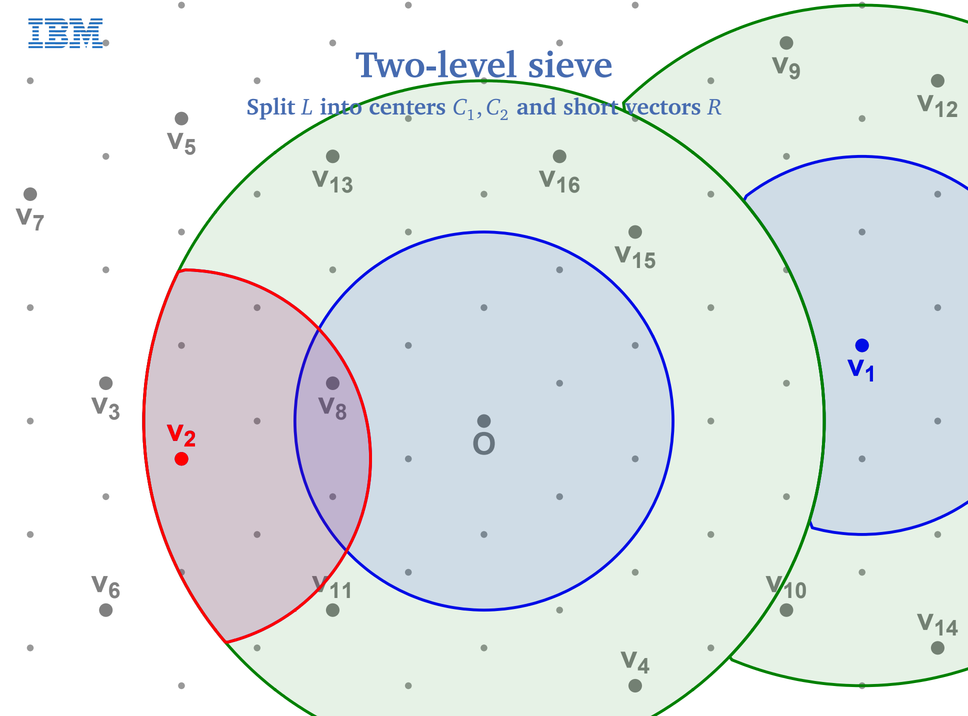

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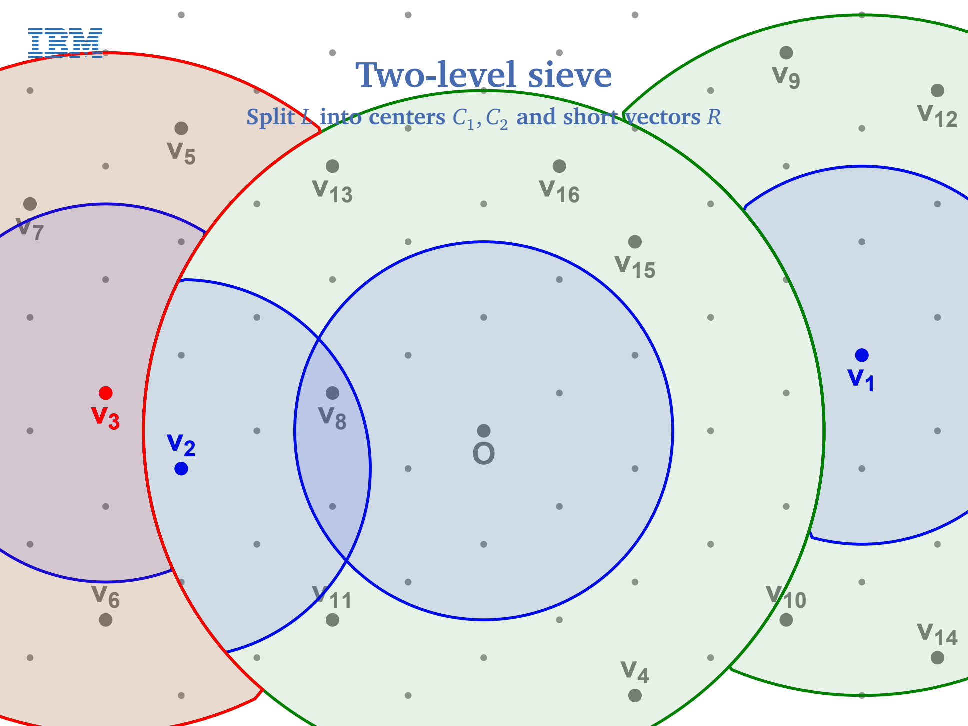

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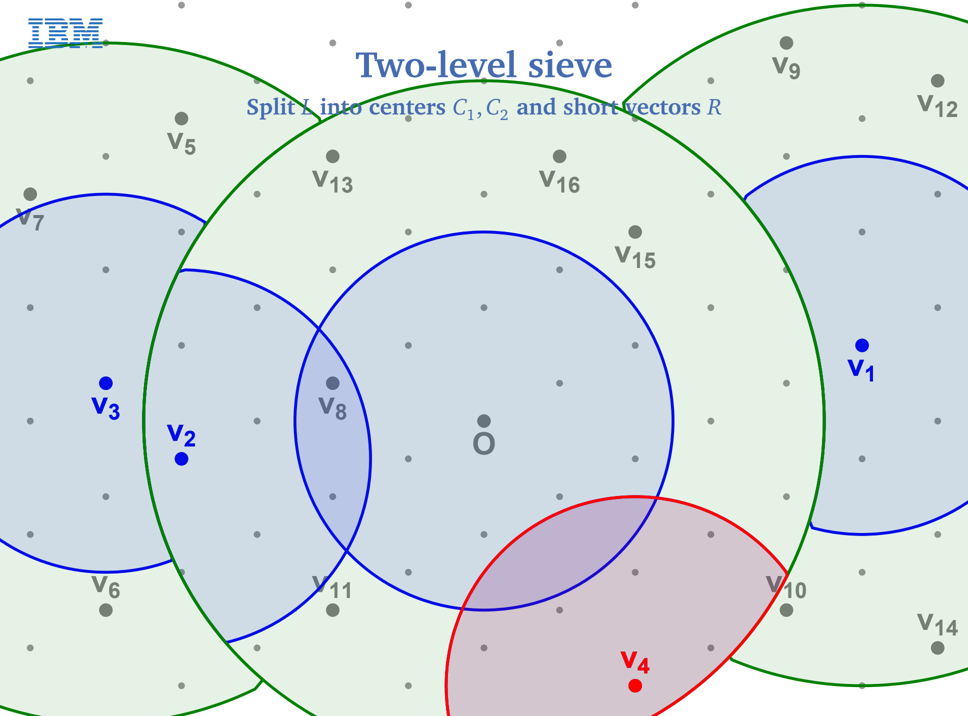

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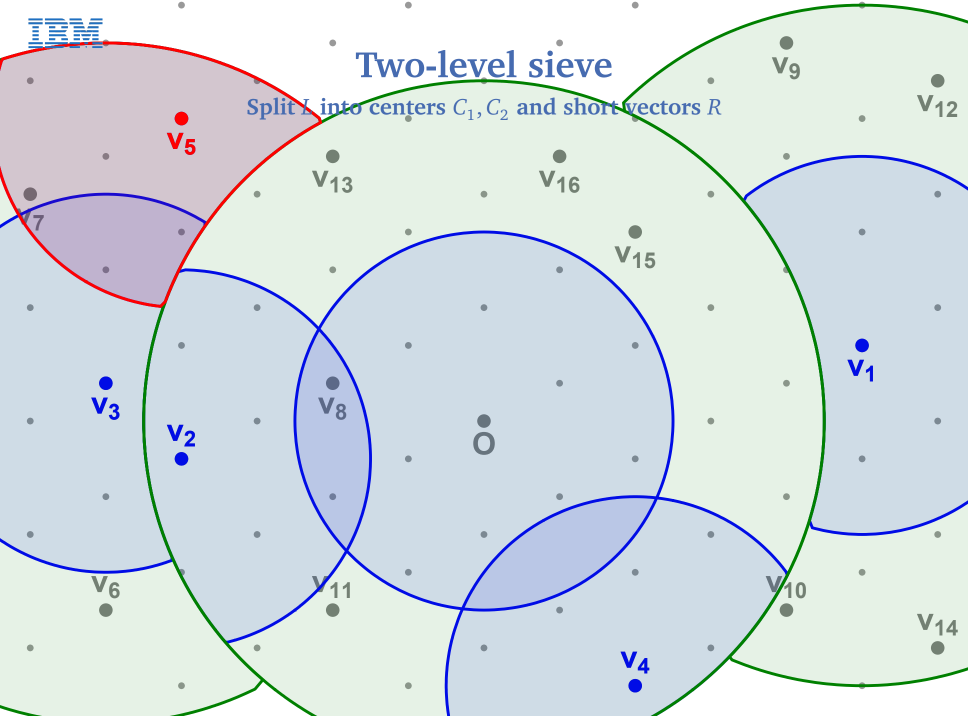

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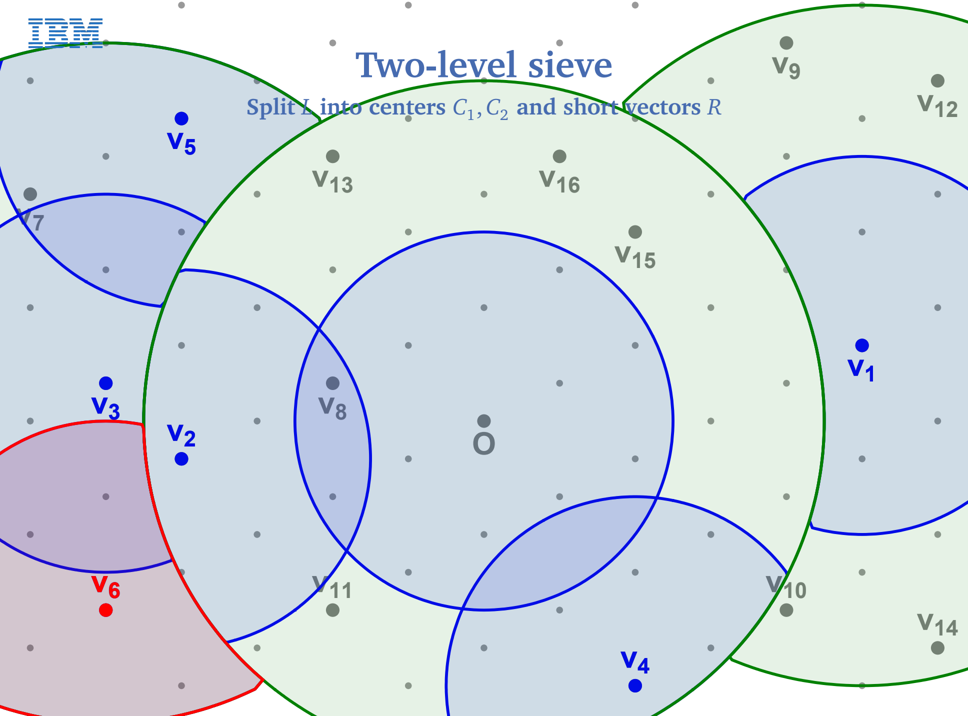

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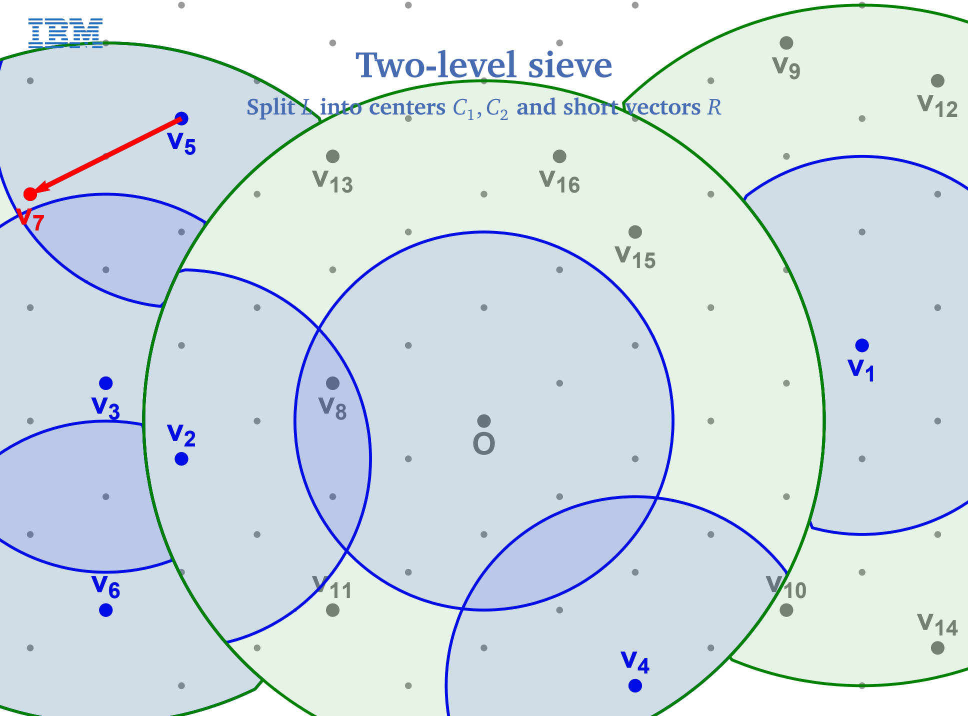

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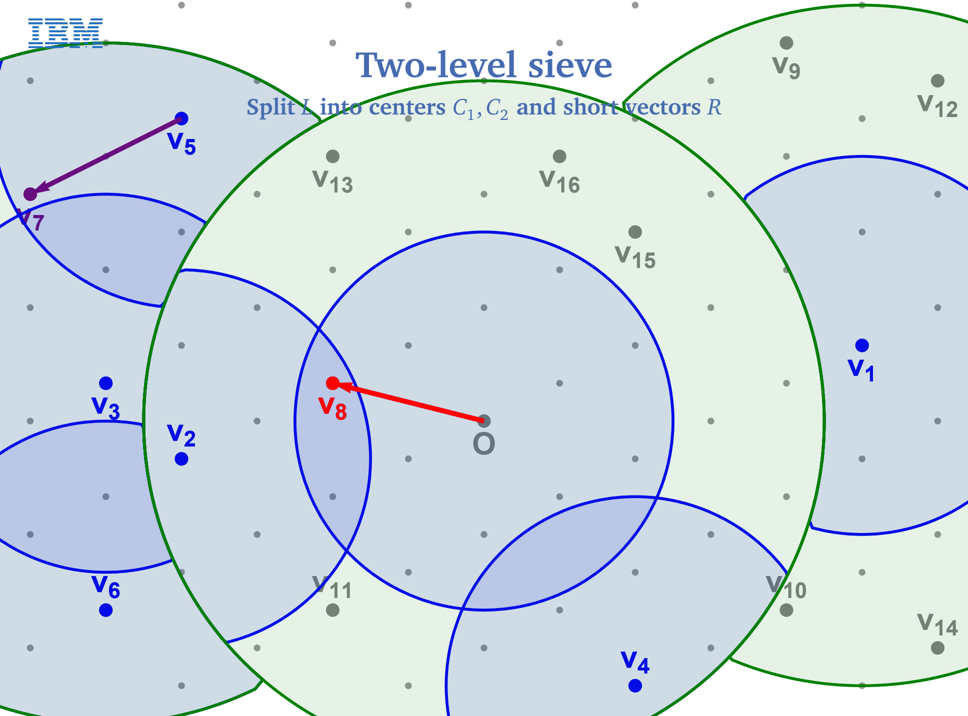

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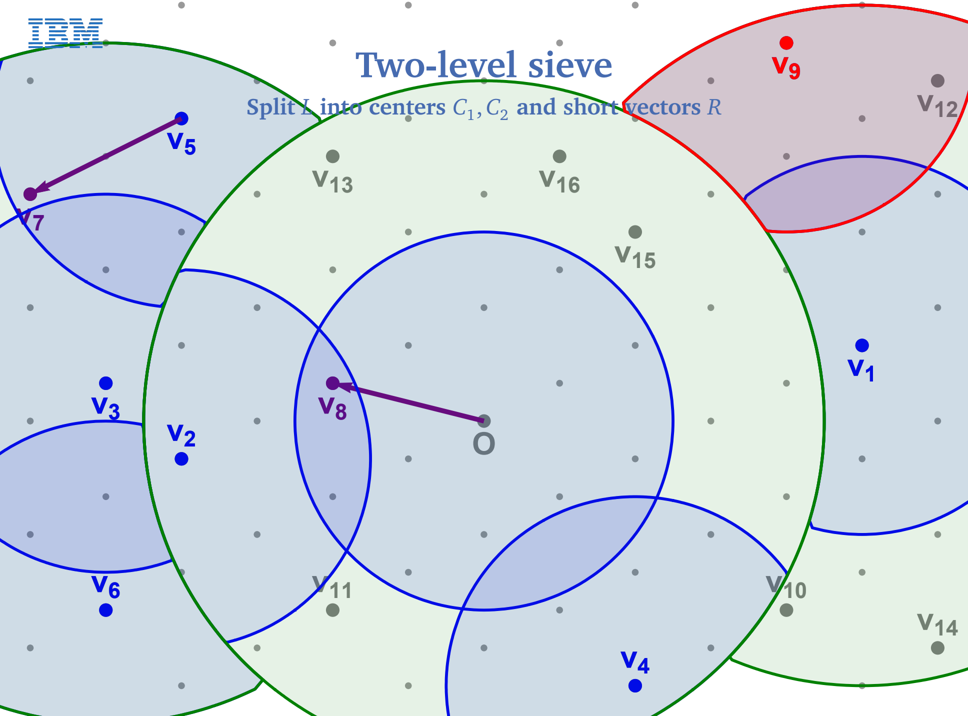

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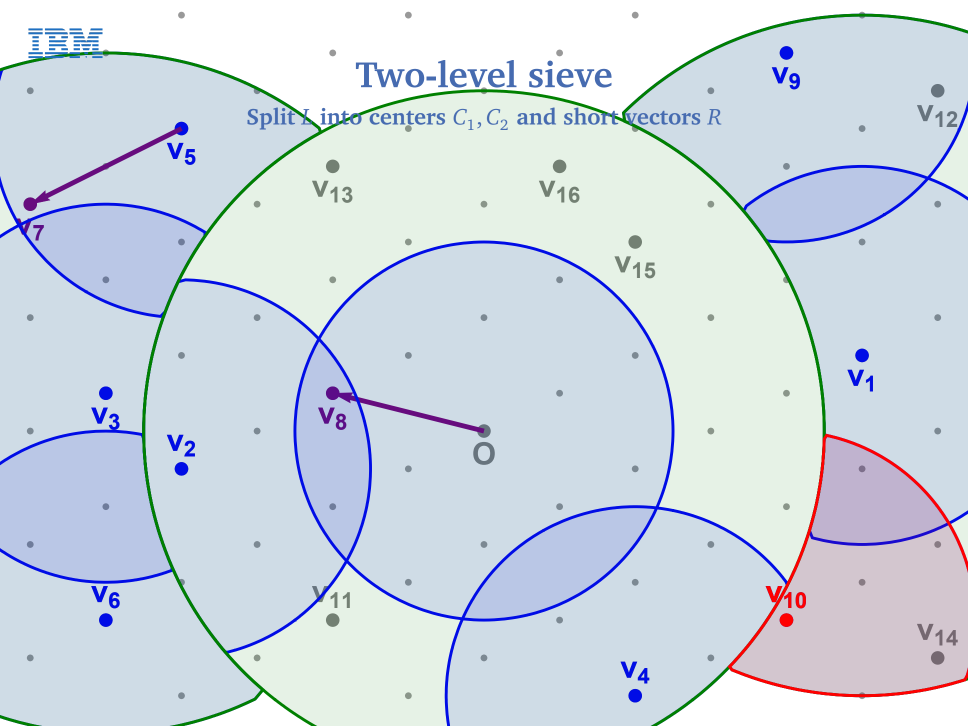

IBM

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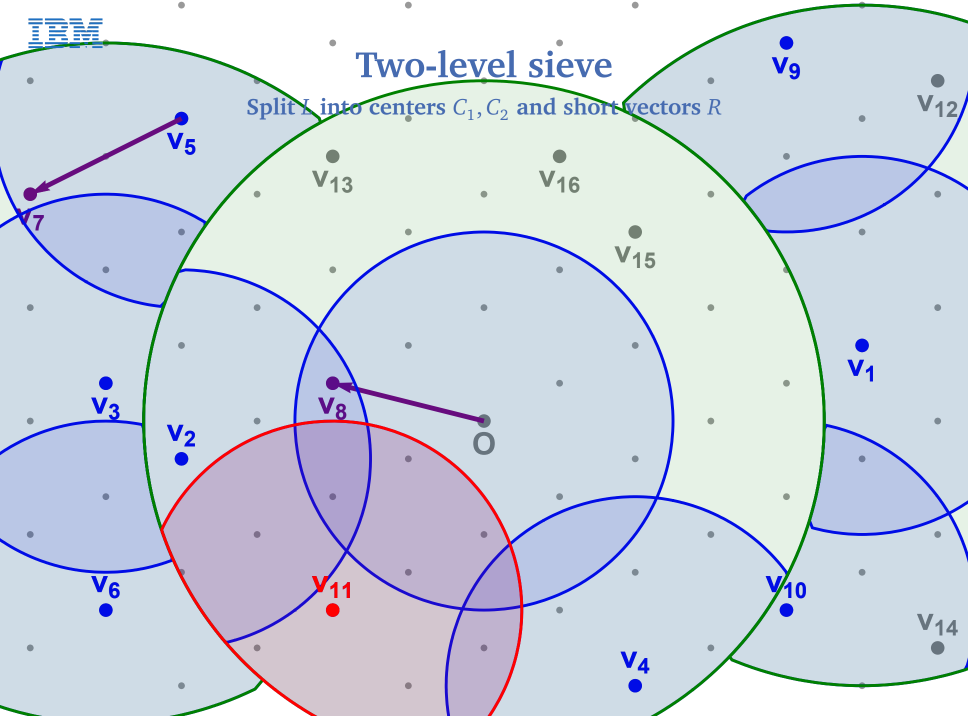

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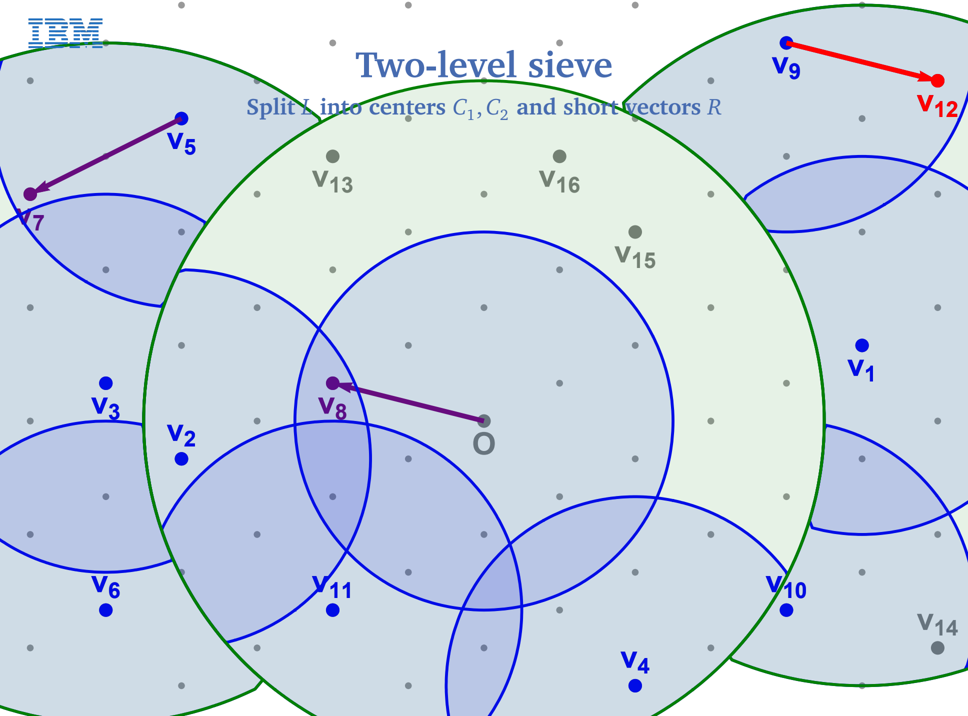

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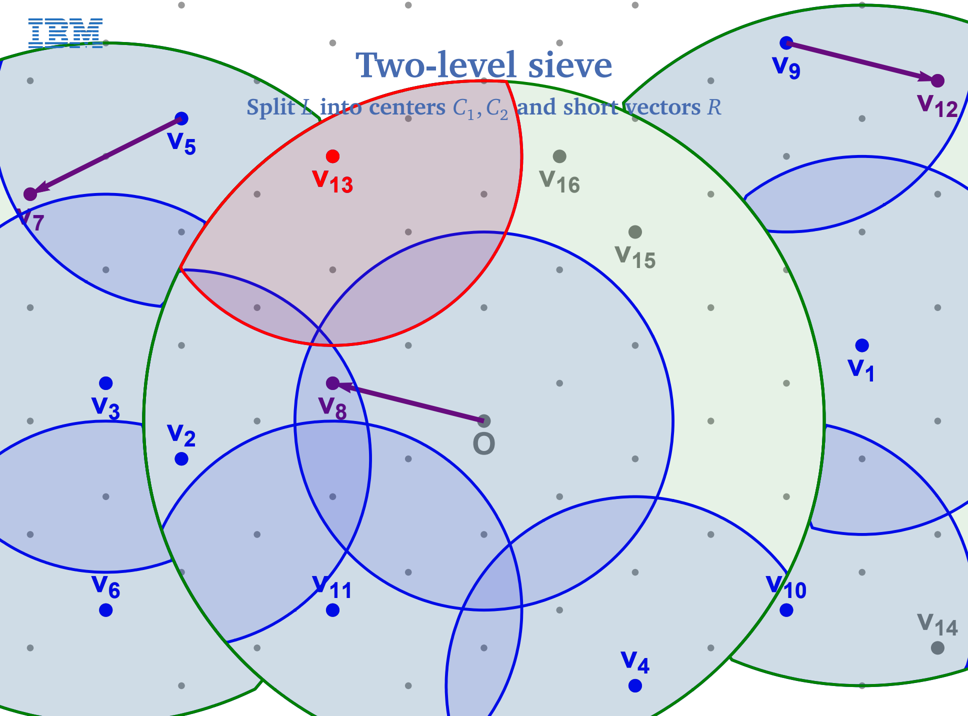

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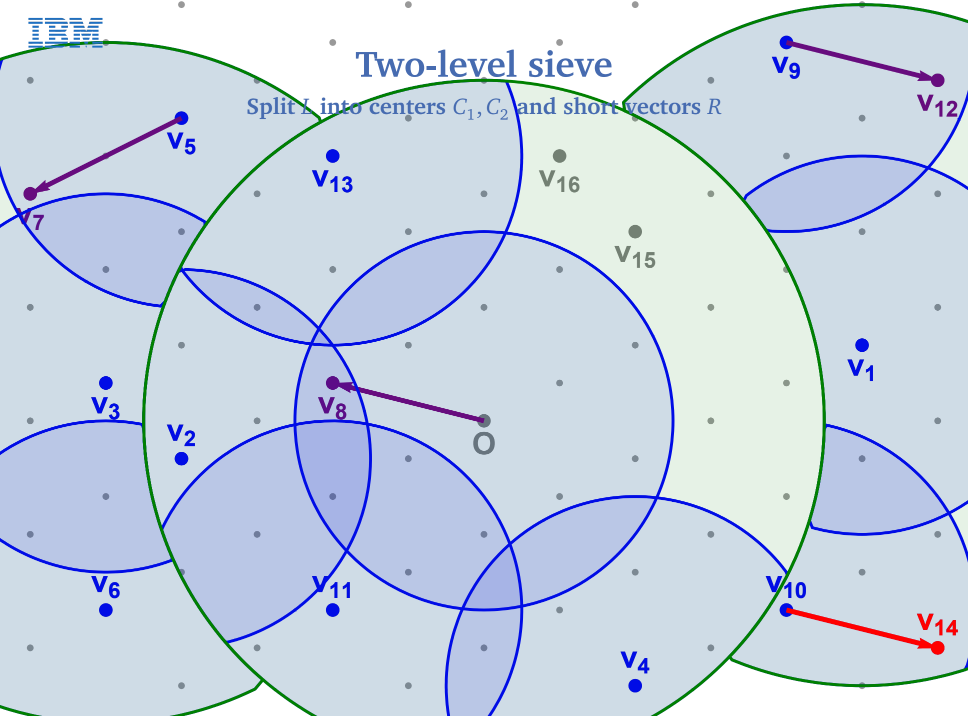

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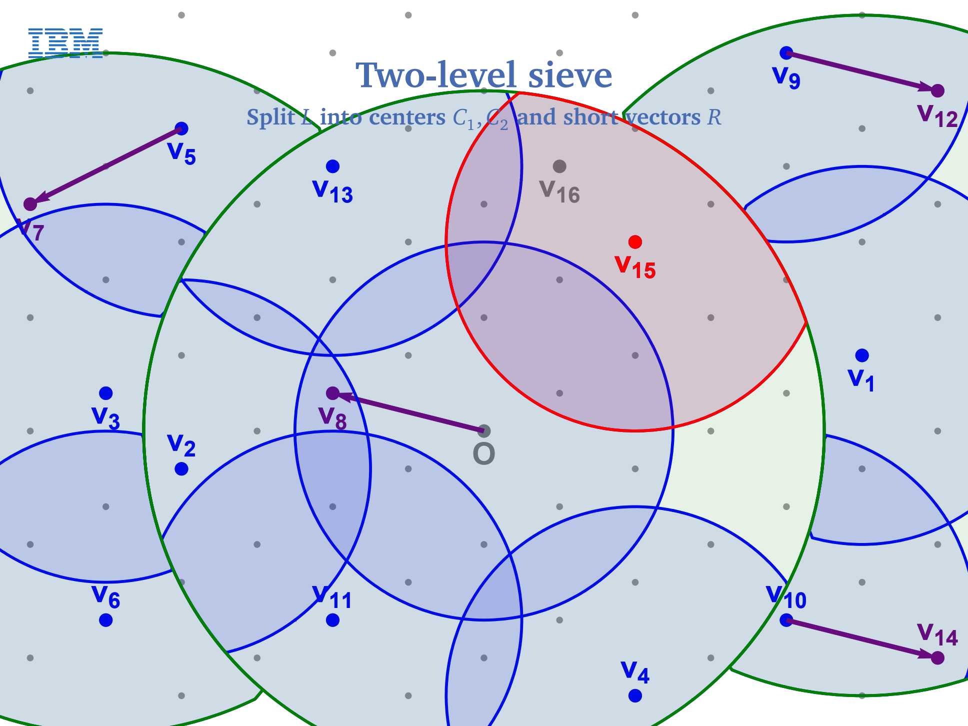

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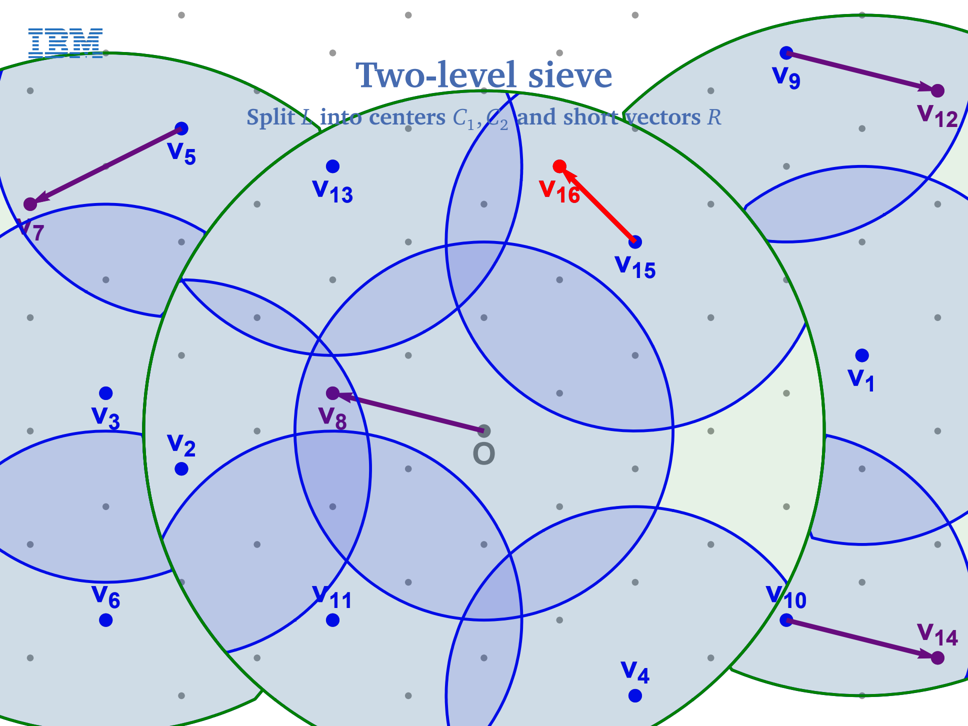

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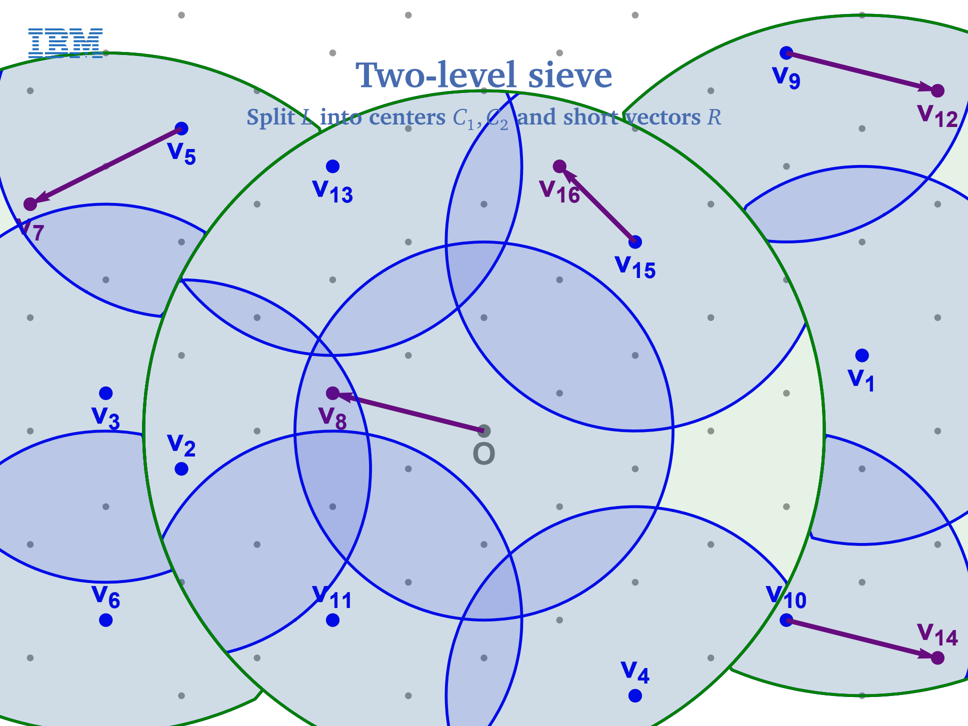

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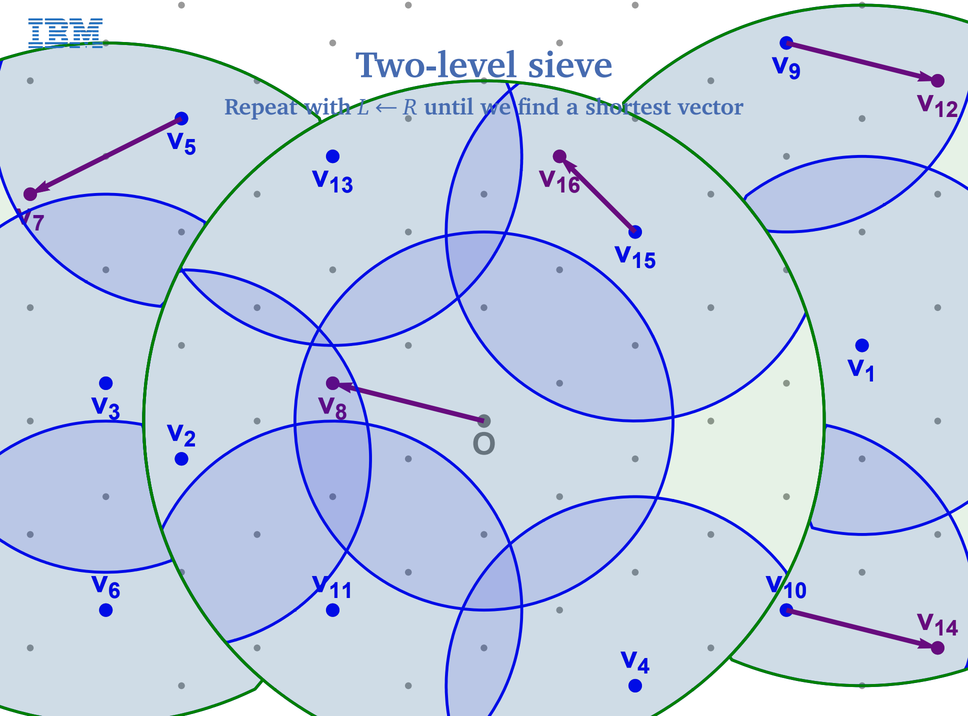

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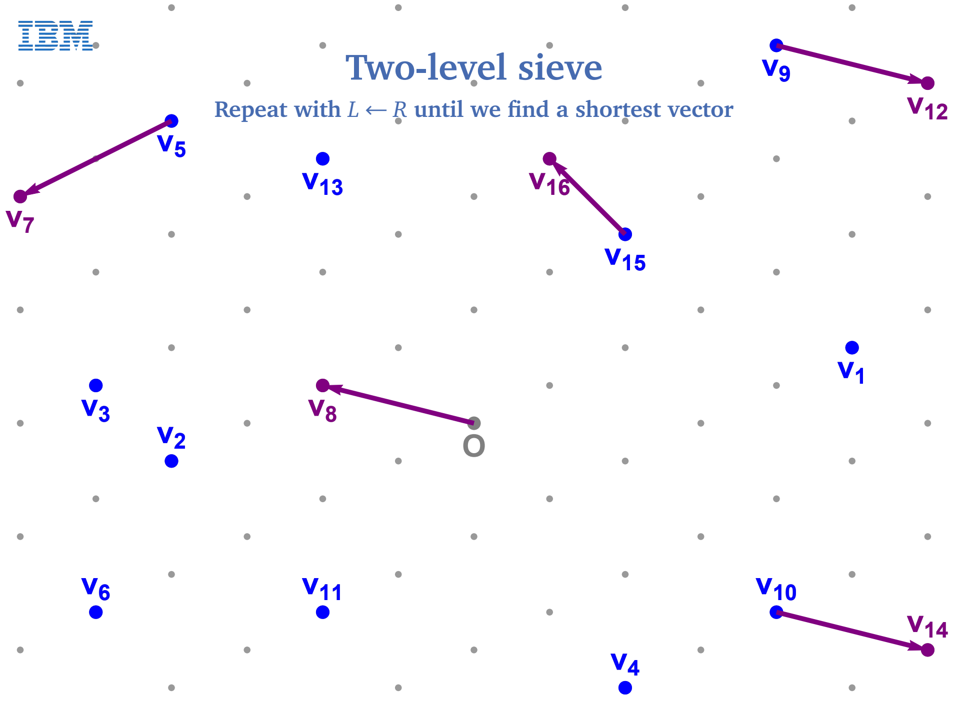

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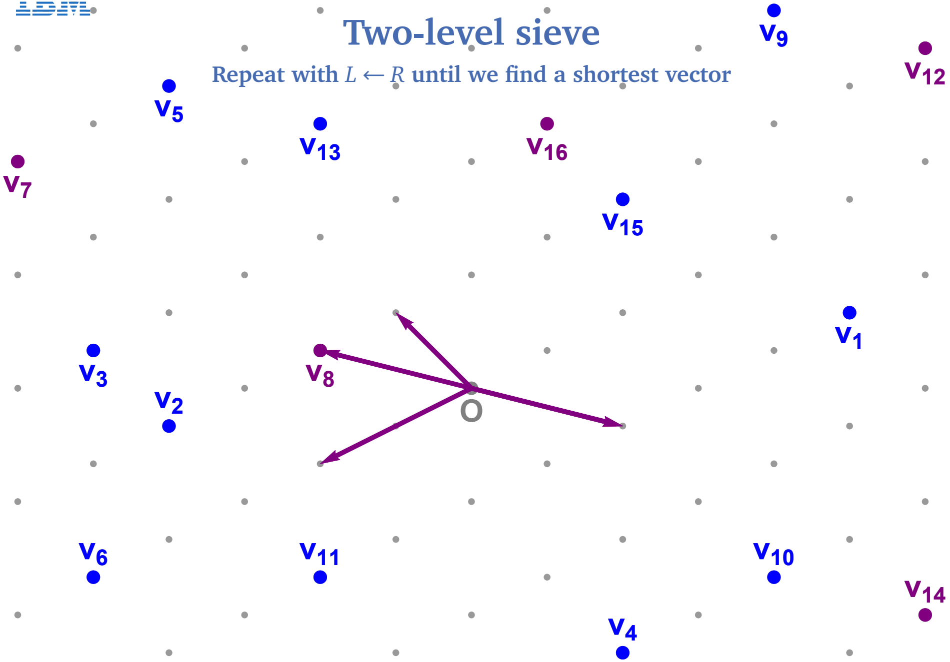

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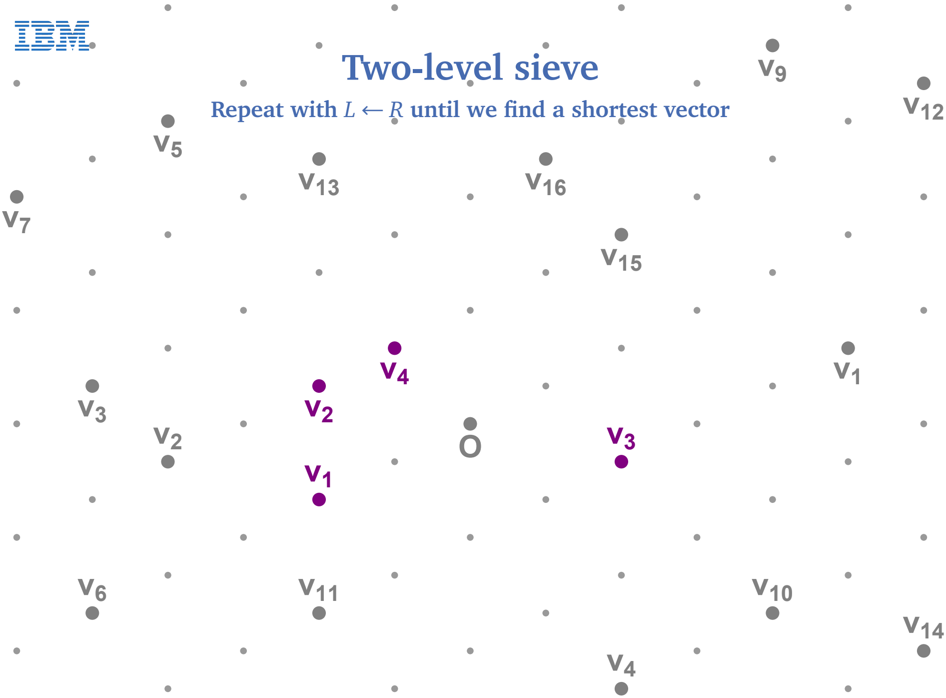
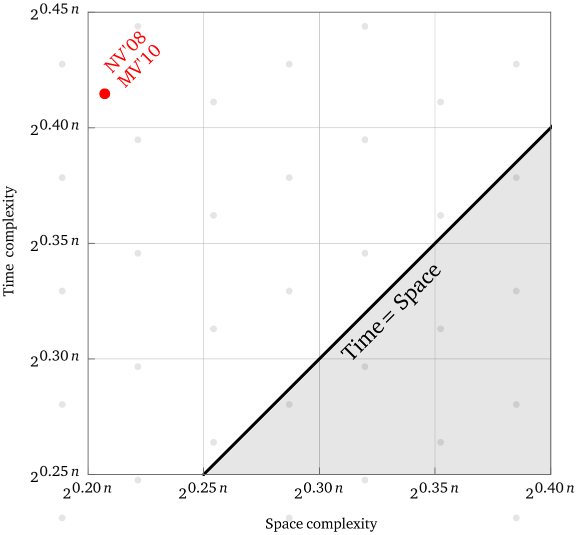


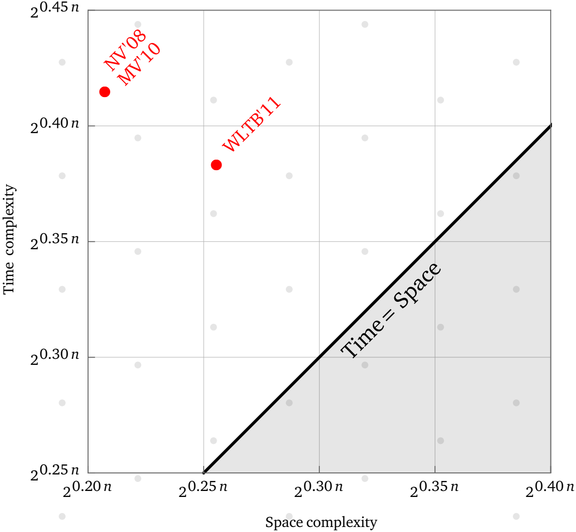
Two-level sieve

Space/time trade-off



Two-level sieve

Space/time trade-off



Three-level sieve

Overview

Heuristic result (Nguyen–Vidick, J. Math. Crypt. ’08)

The one-level sieve runs in time $2^{0.4150n}$ *and space* $2^{0.2075n}$.

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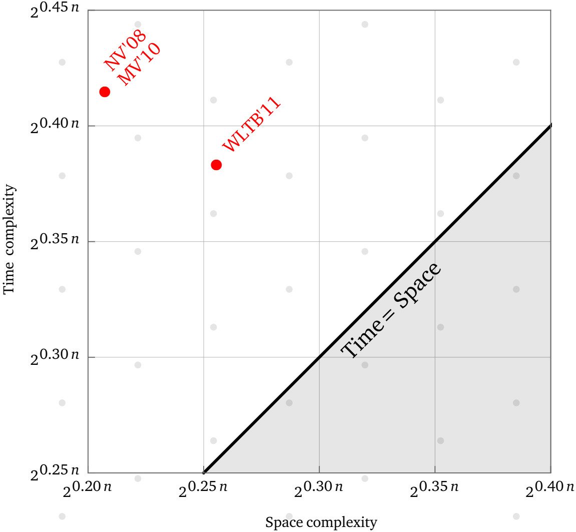
Conjecture
The four-level sieve runs in time $2^{0.3774n}$ *and space* $2^{0.2925n}$*, and higher-level sieves are not faster than this.*

Three-level sieve

Space/time trade-off



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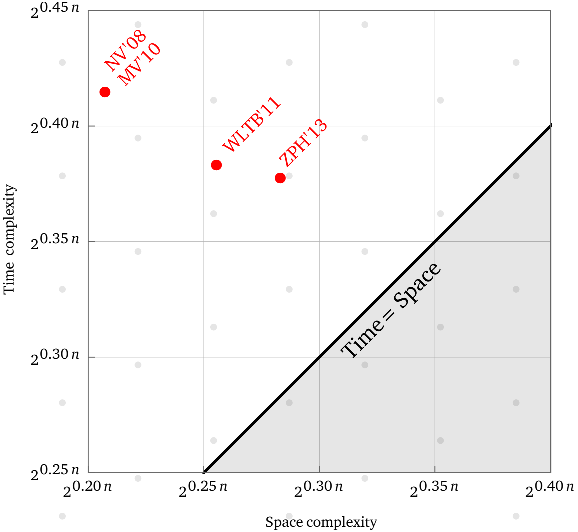
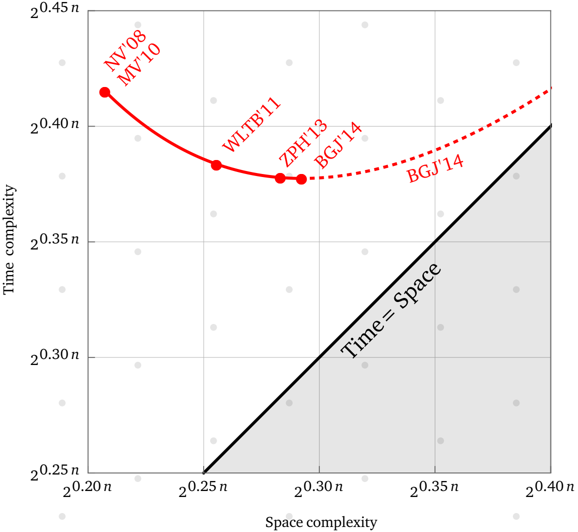


Decomposition approach

Space/time trade-off



Locality-sensitive hashing

Introduction

Problem: Given a high-dimensional data set $D \subset \mathbb{R}^n$, preprocess it such that when later given a target $\boldsymbol{t} \in \mathbb{R}^n$, we can quickly find a nearby vector to $\boldsymbol{t}$ in D.

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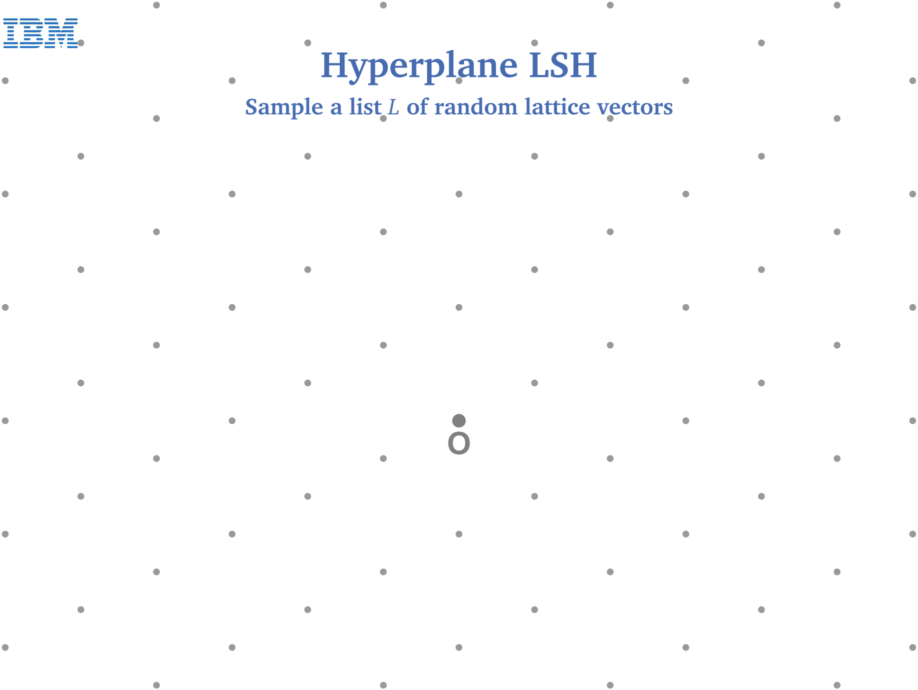
> *"The key idea is to use hash functions such that the probability of collision is much higher for objects that are close to each other than for those that are far apart."*
>
> — Indyk–Motwani, *STOC'98*

Hyperplane LSH

Sample a list L of random lattice vectors



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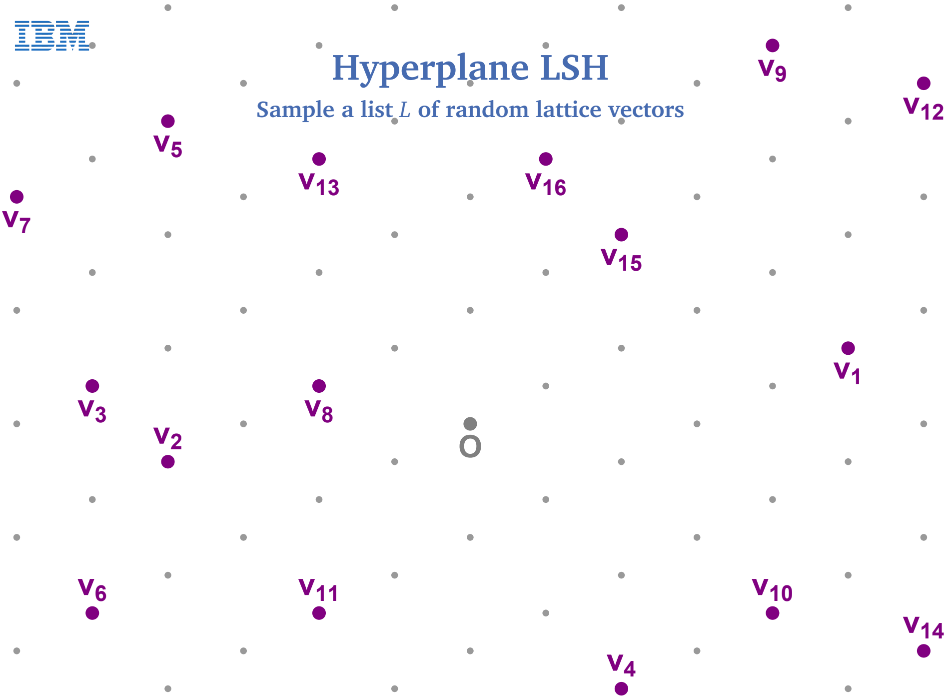
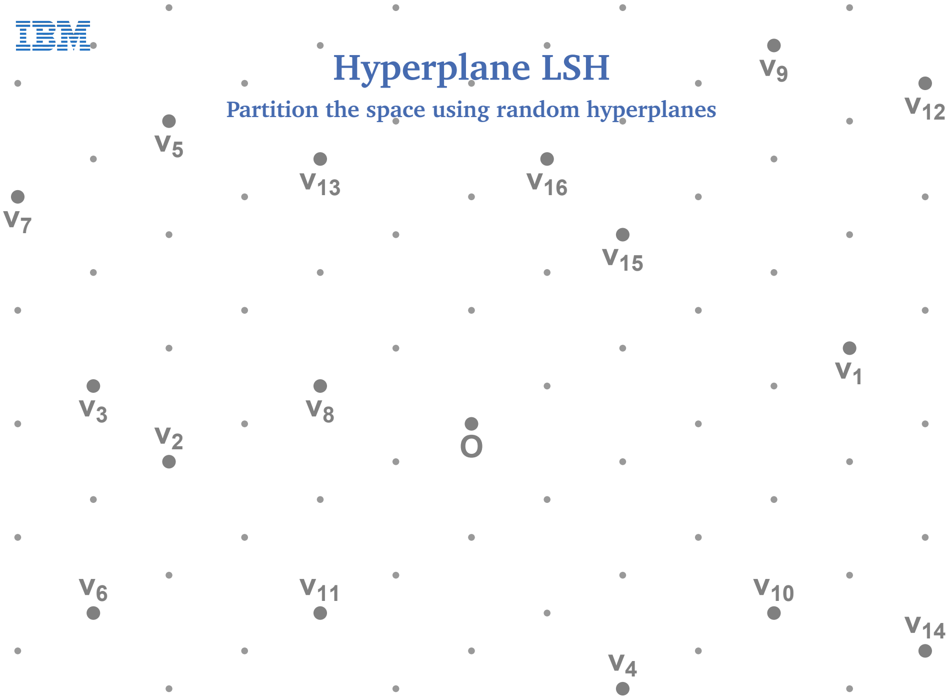


Hyperplane LSH

Partition the space using random hyperplanes



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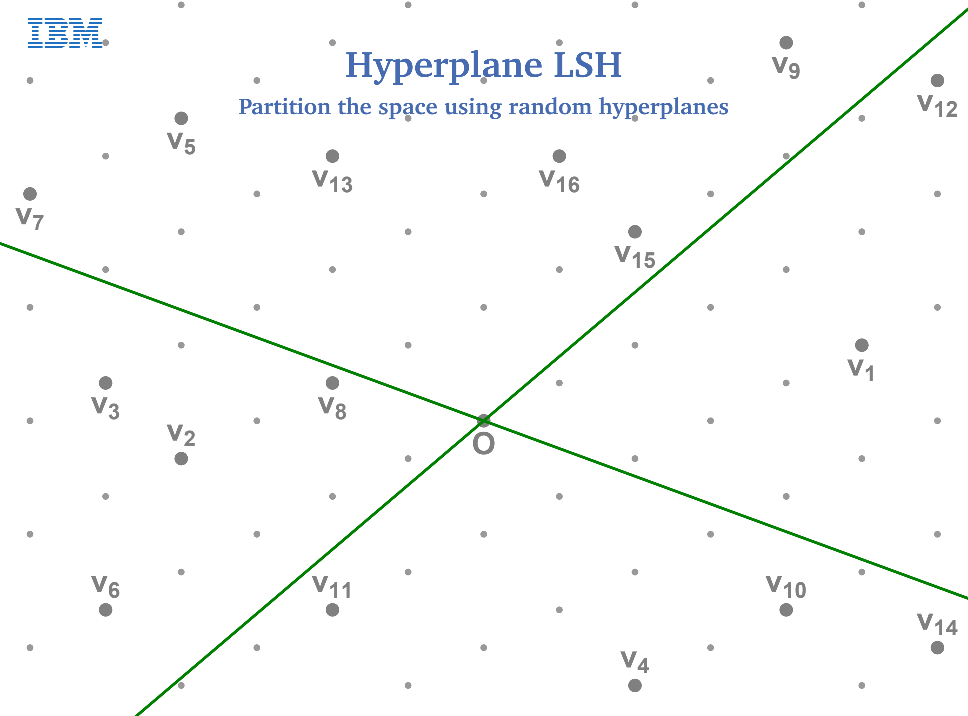
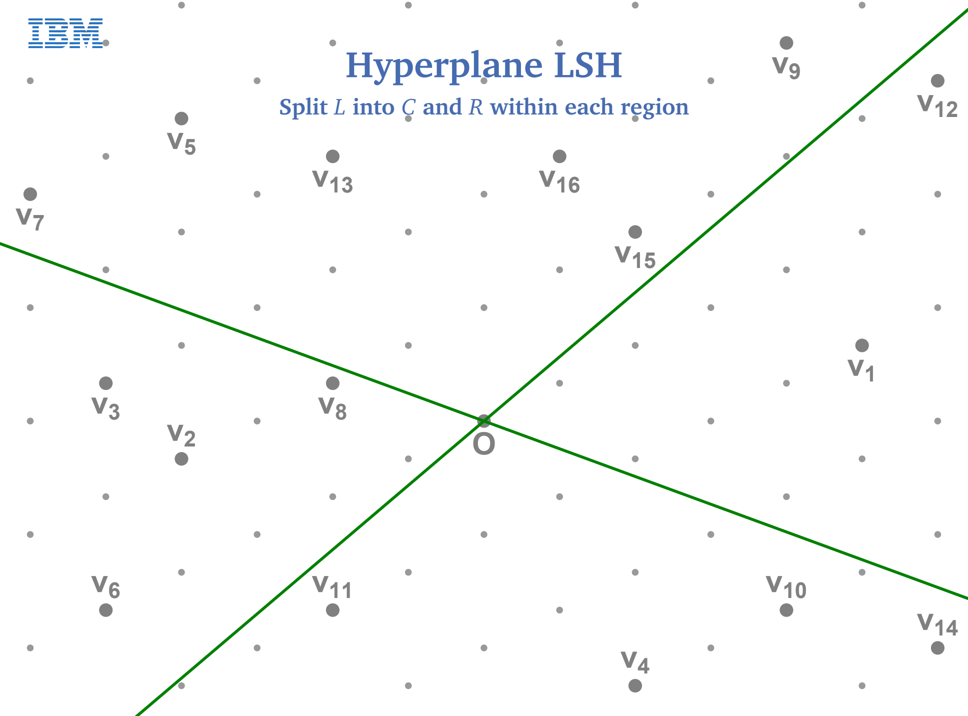


Hyperplane LSH

Split L into C and R within each region



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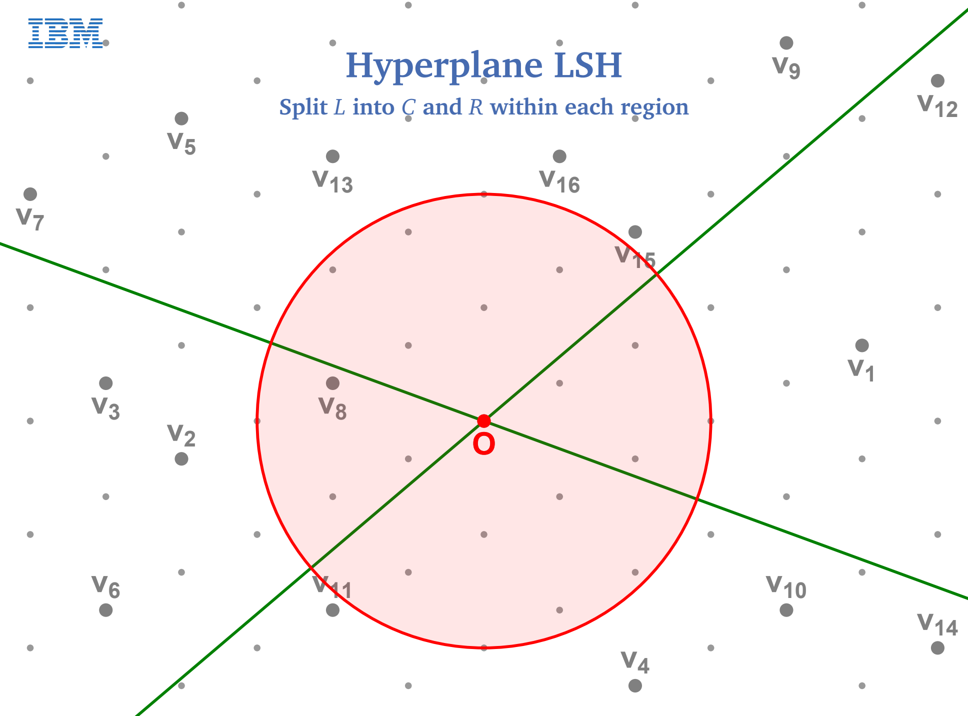

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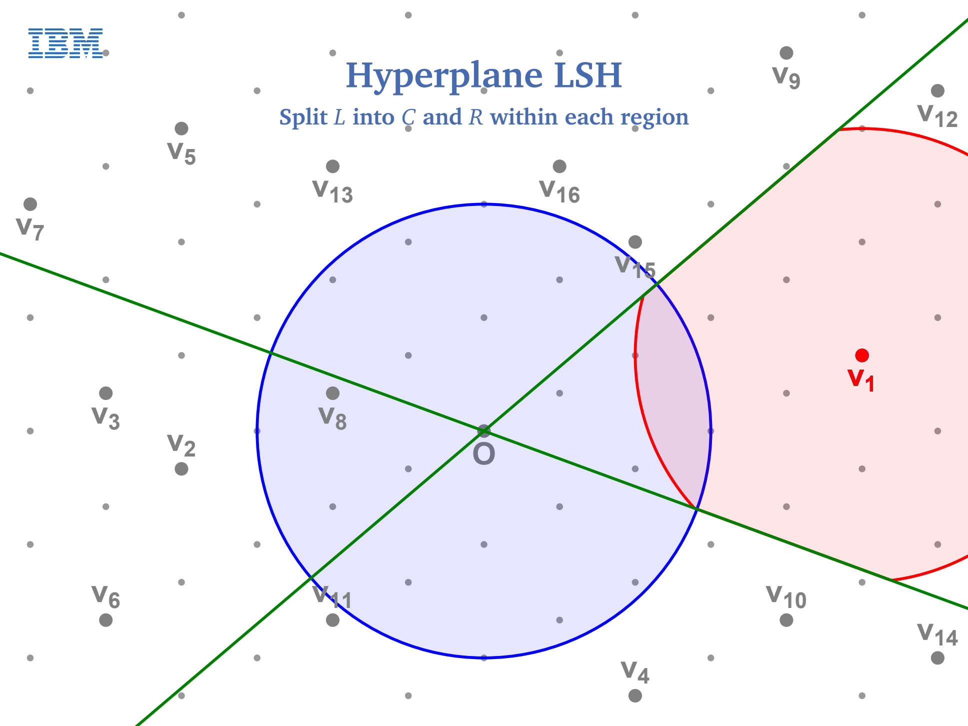

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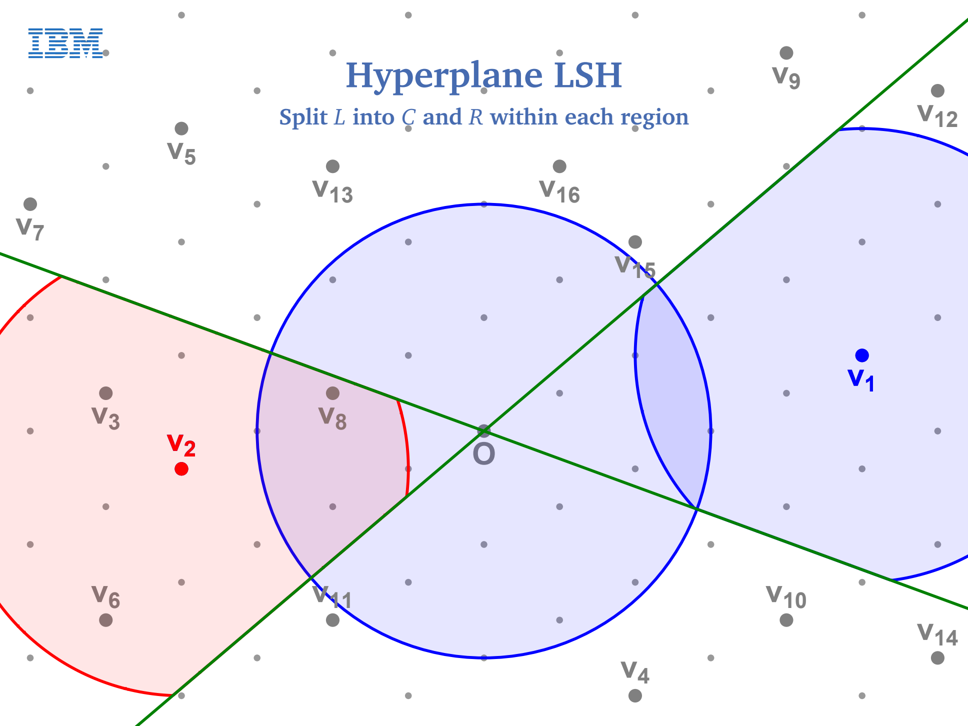

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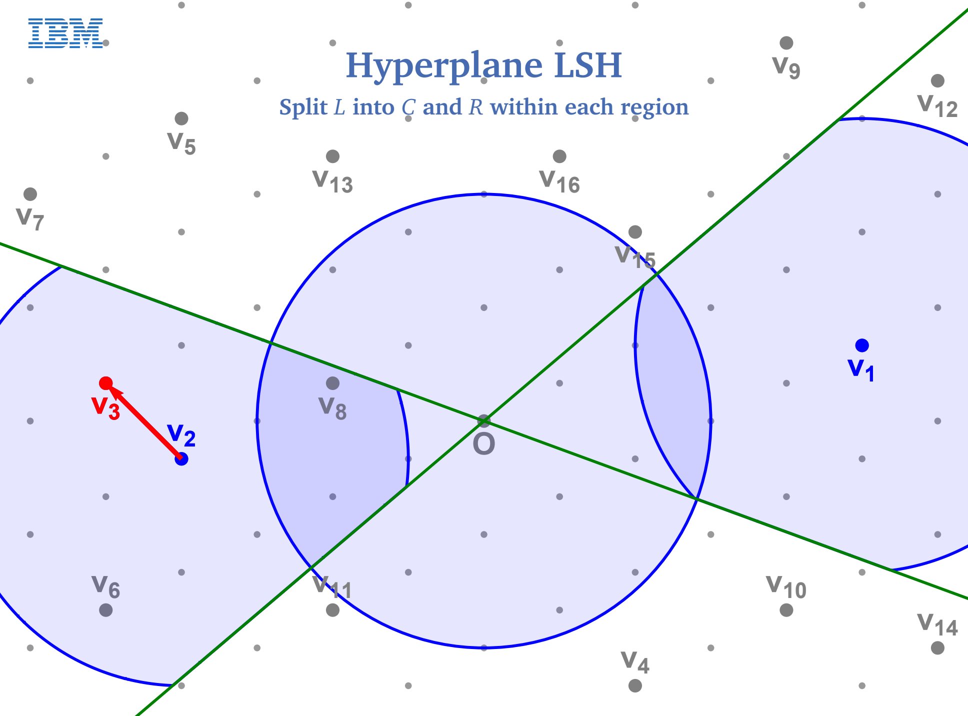

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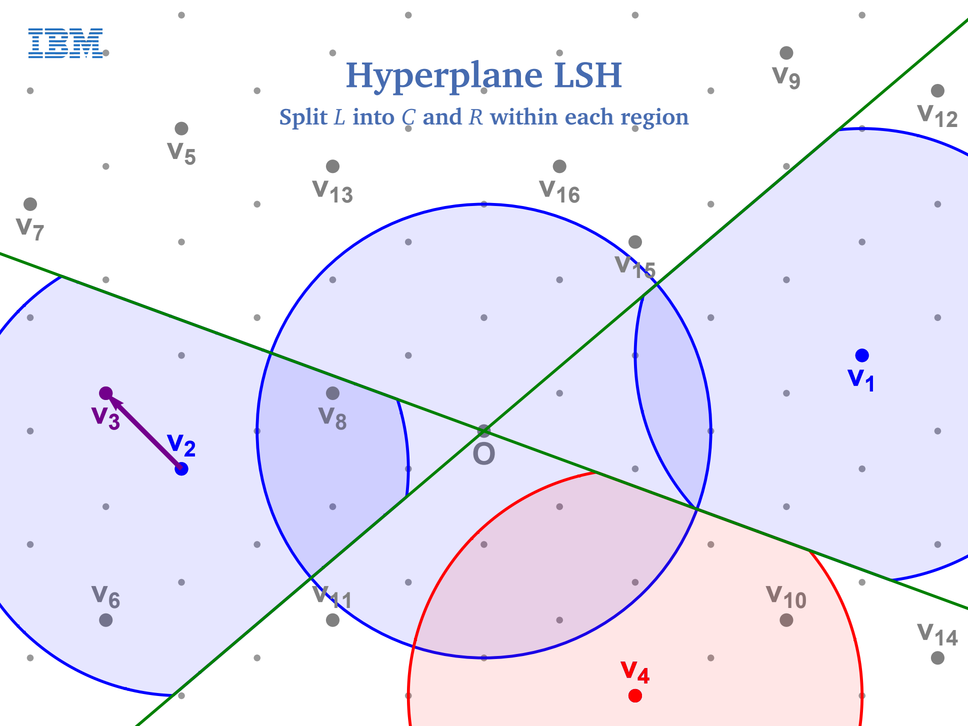

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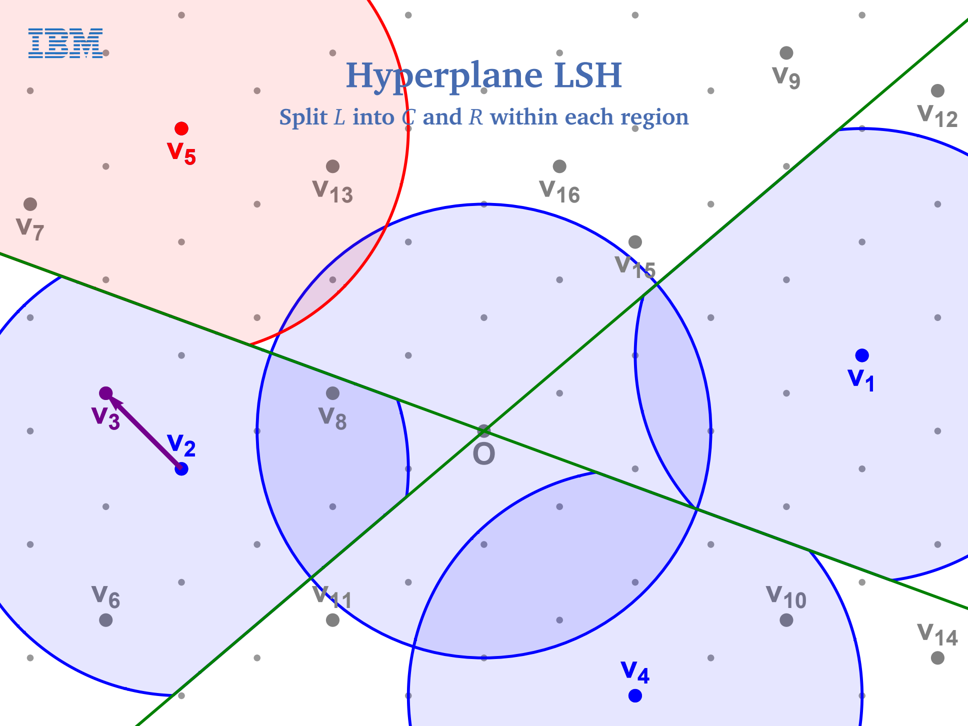

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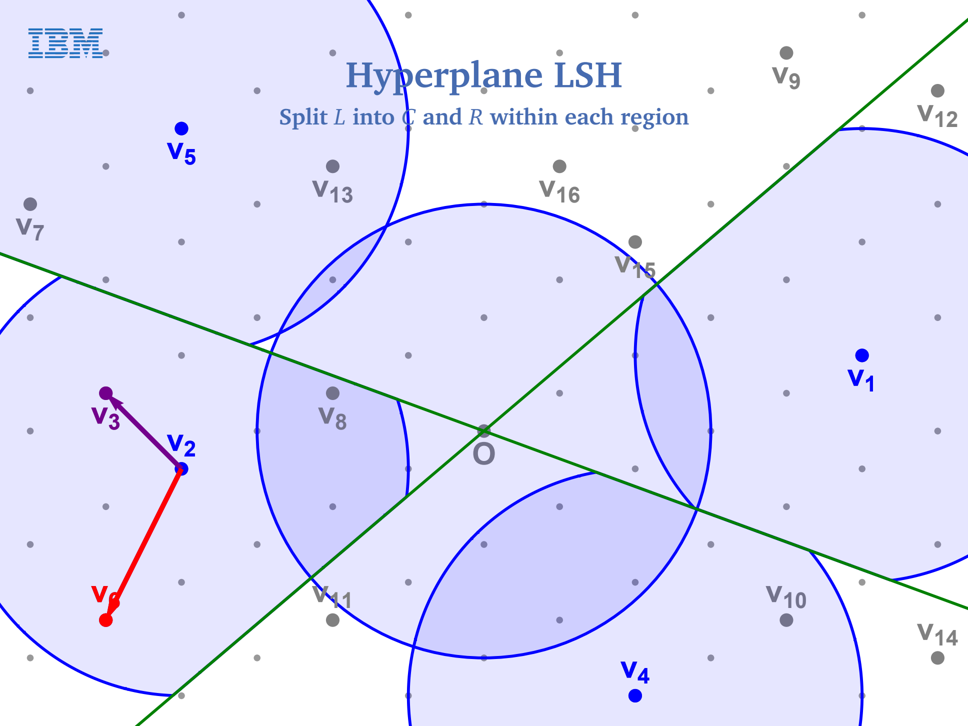

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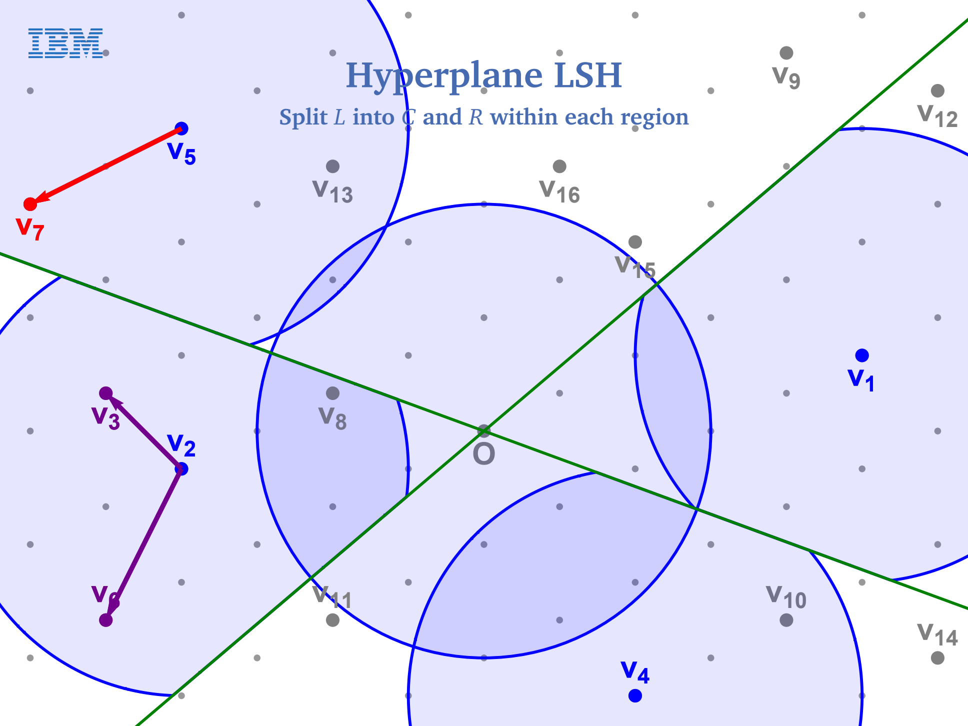

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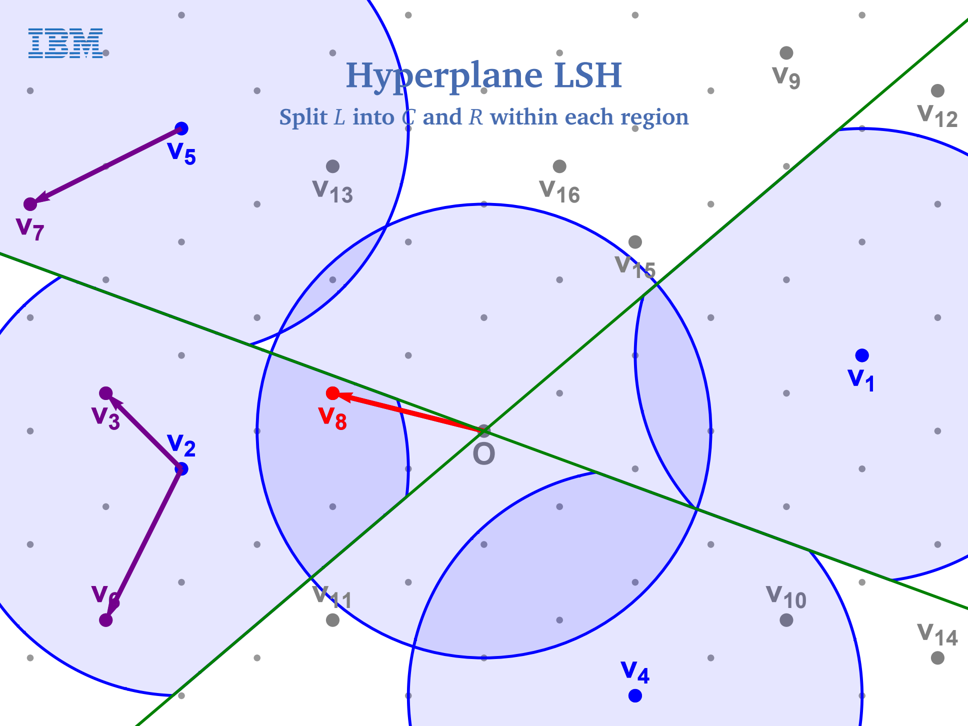

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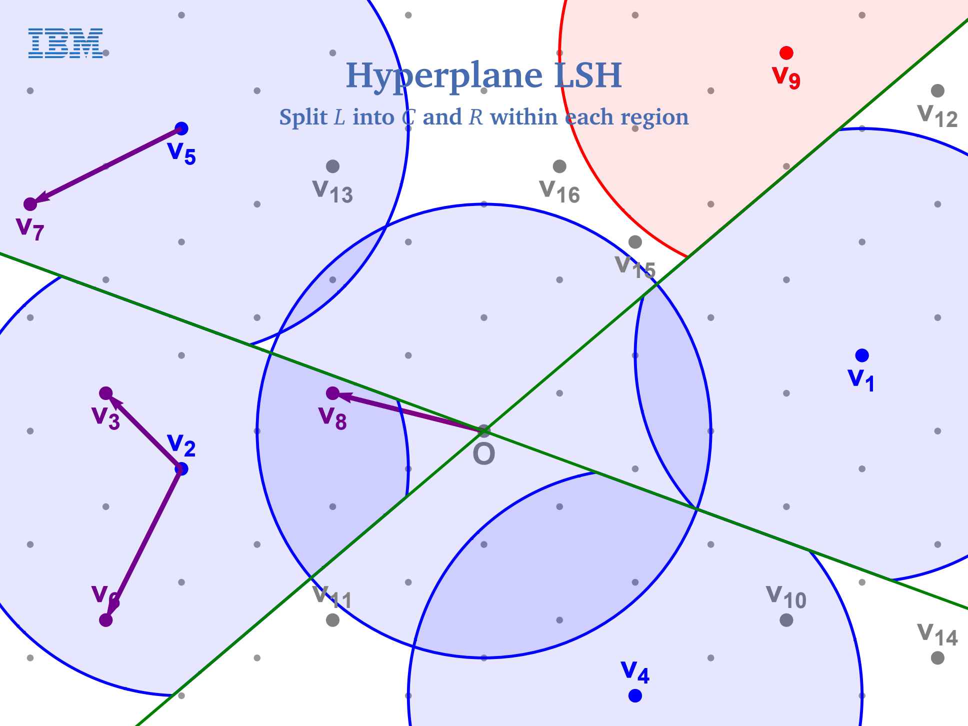

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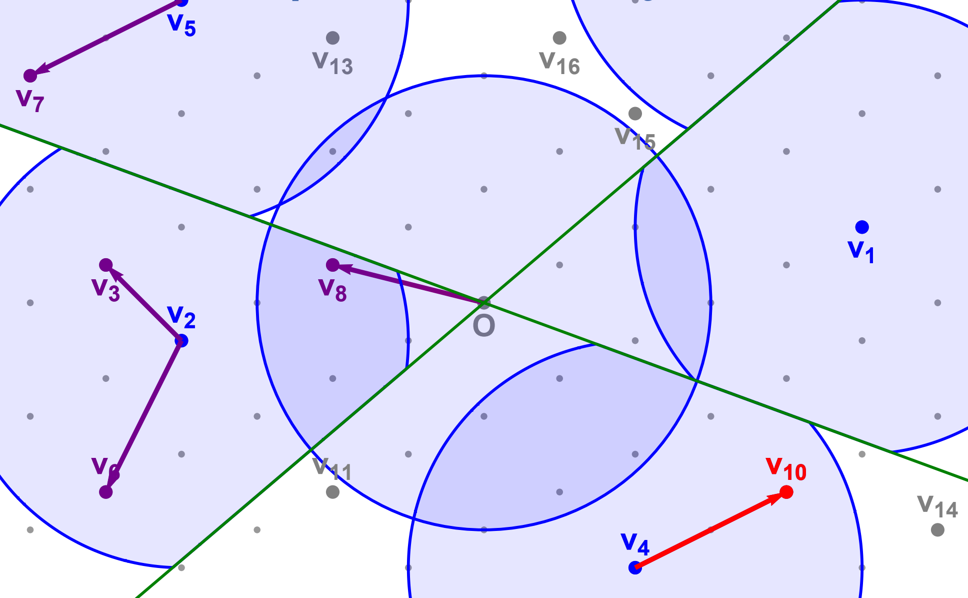

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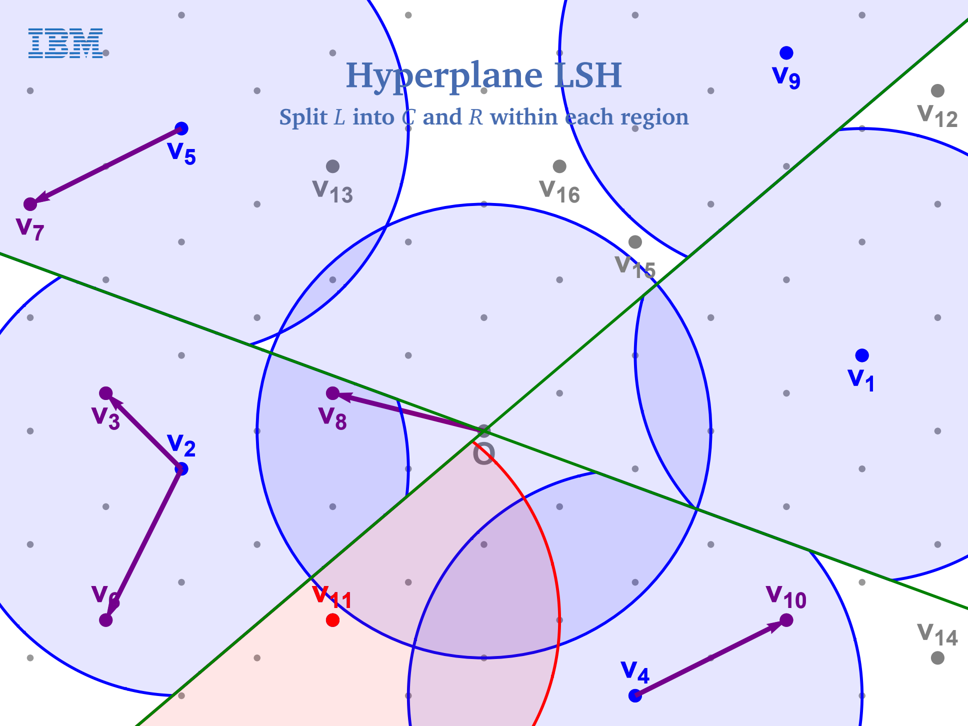

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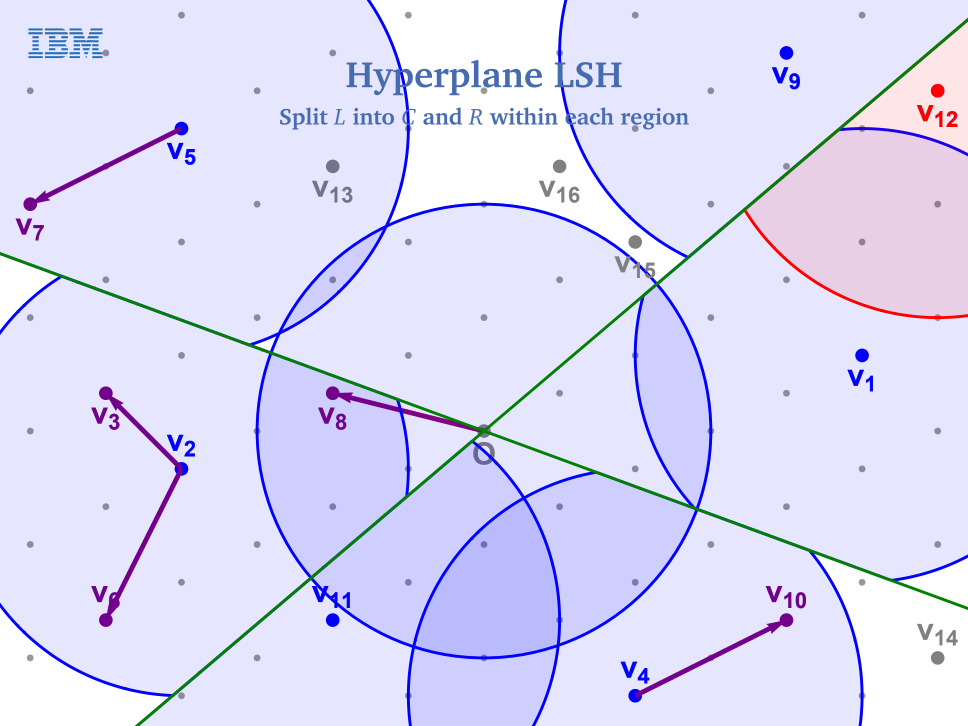

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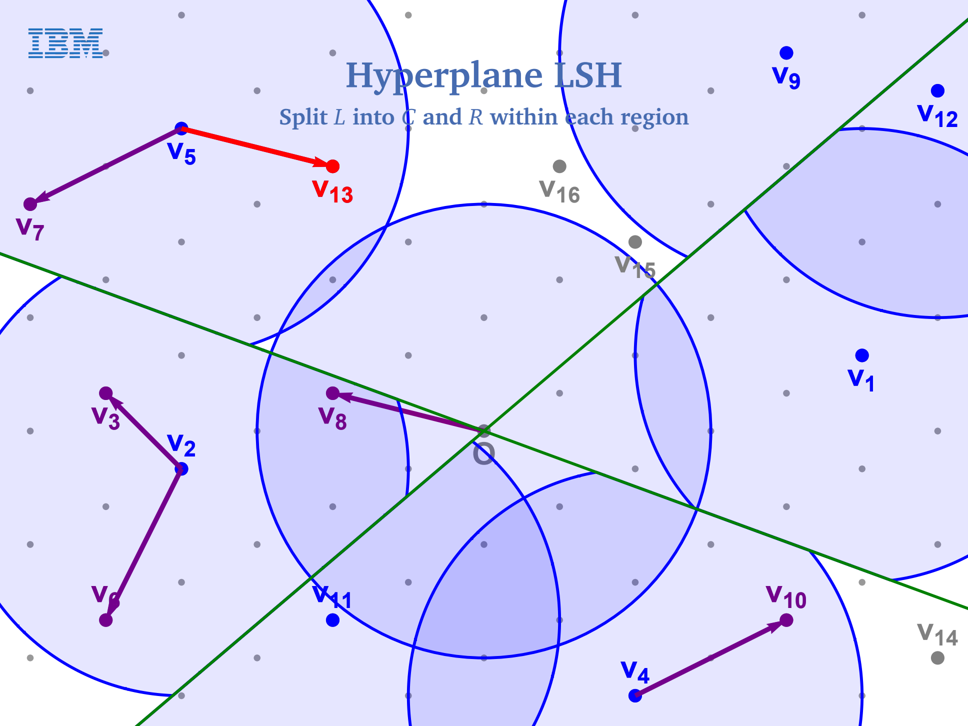

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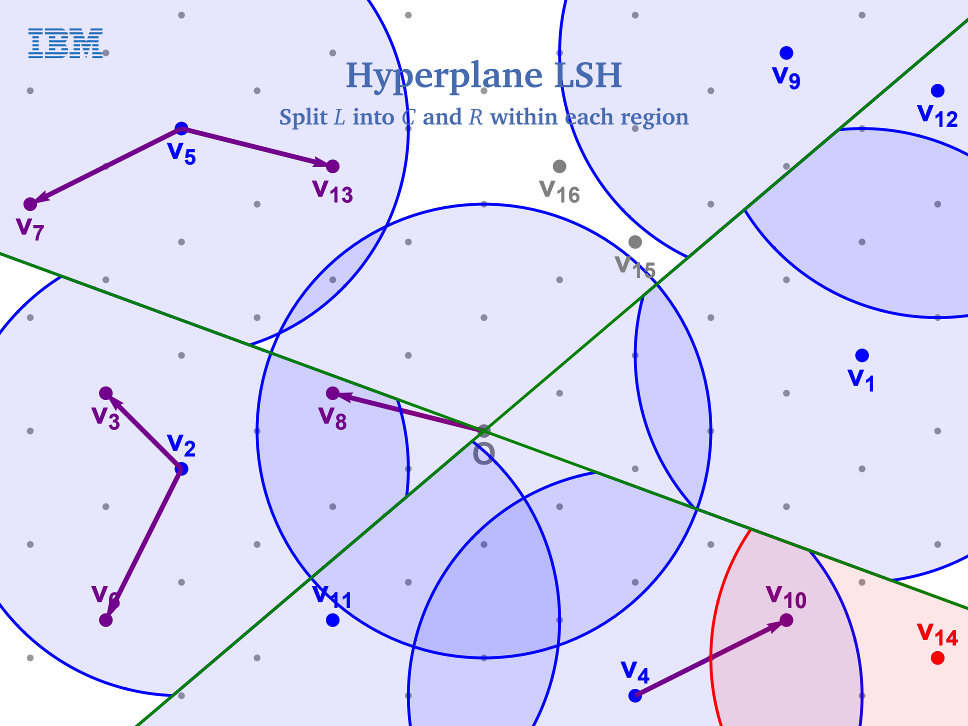

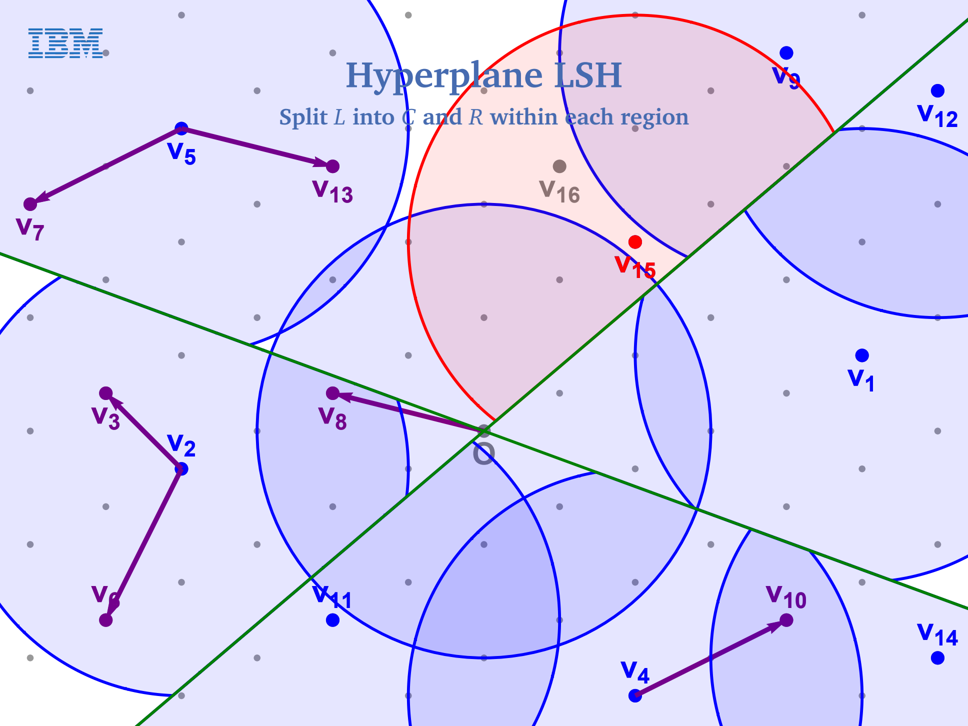
IBM
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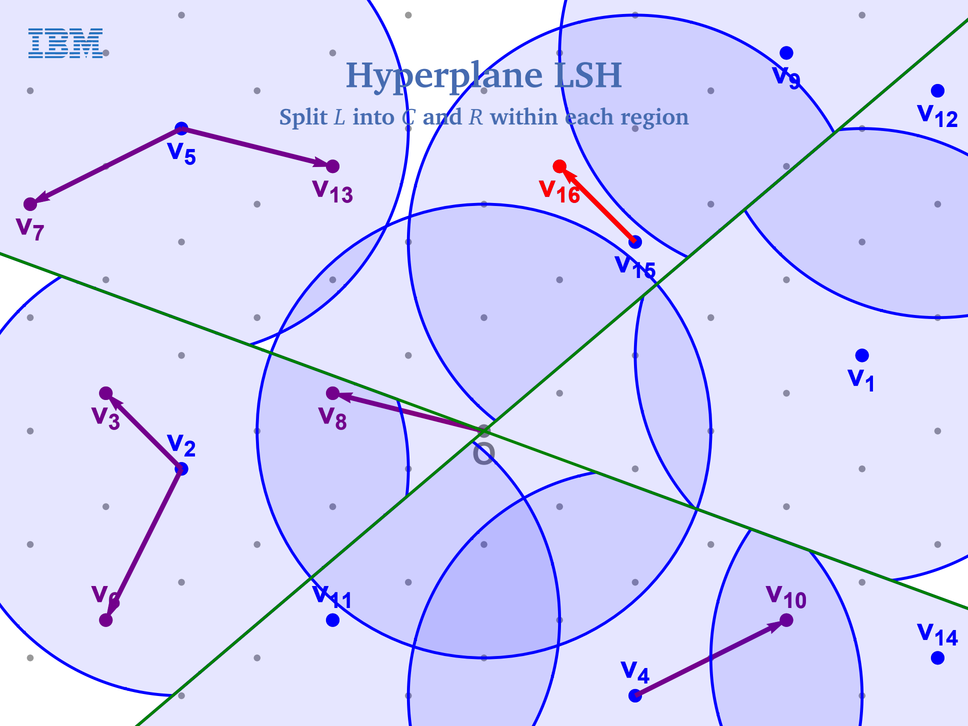

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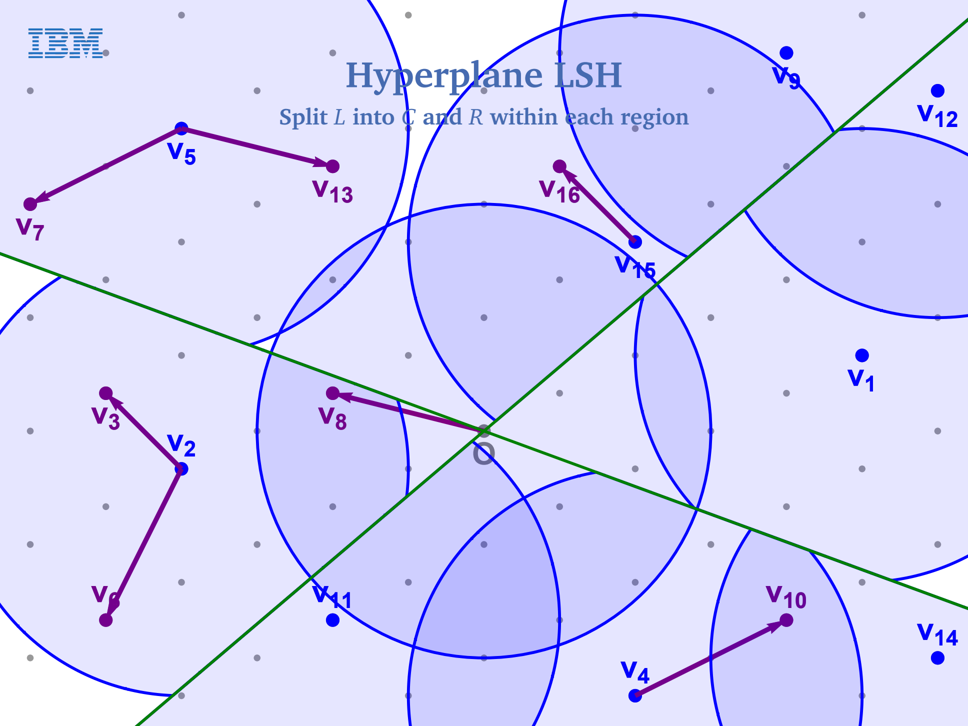

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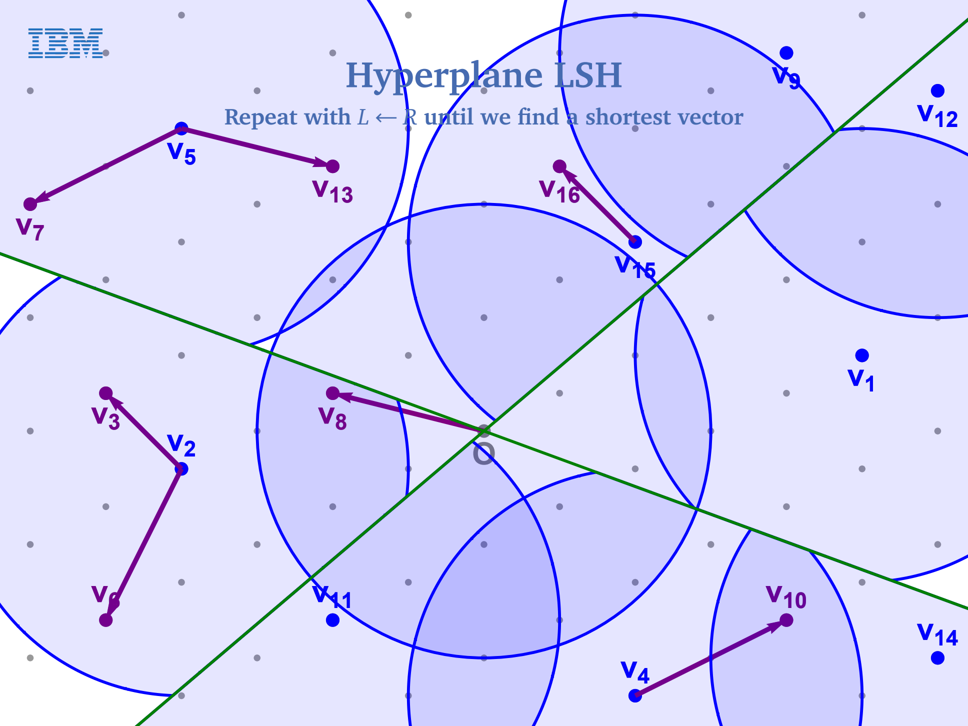

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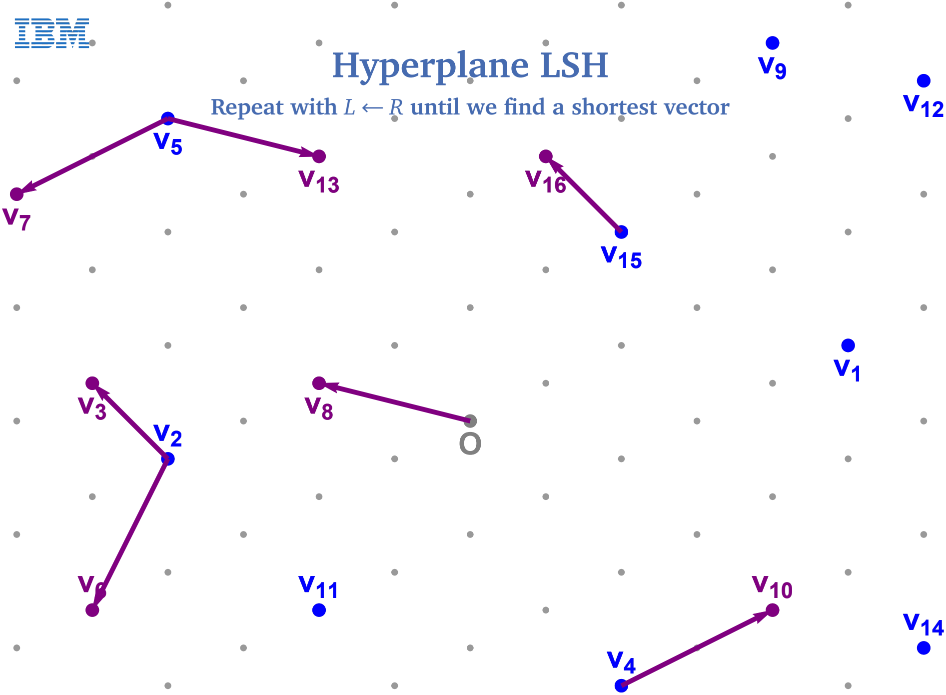

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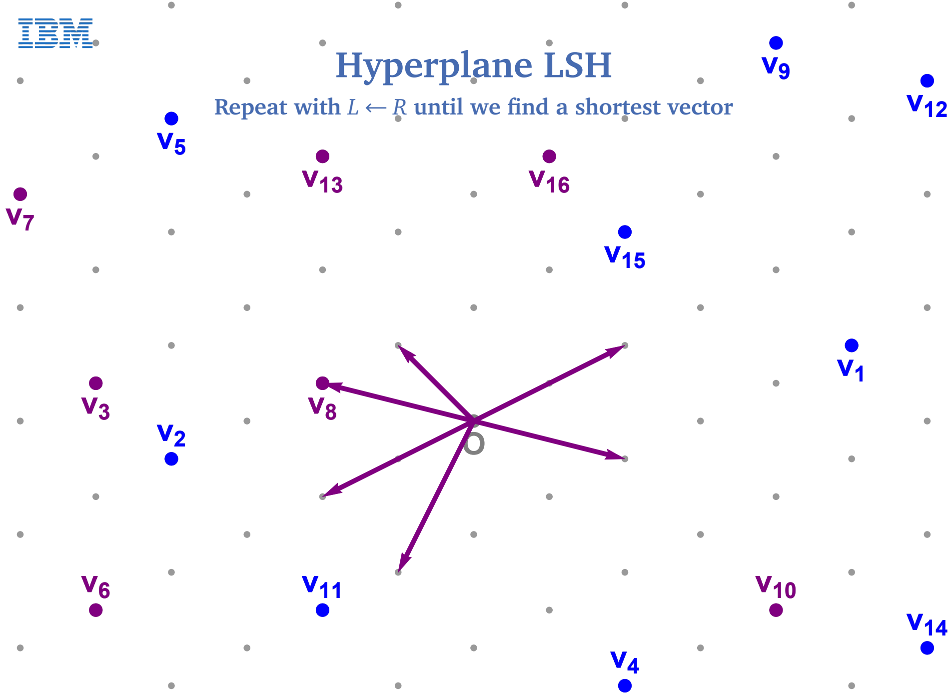

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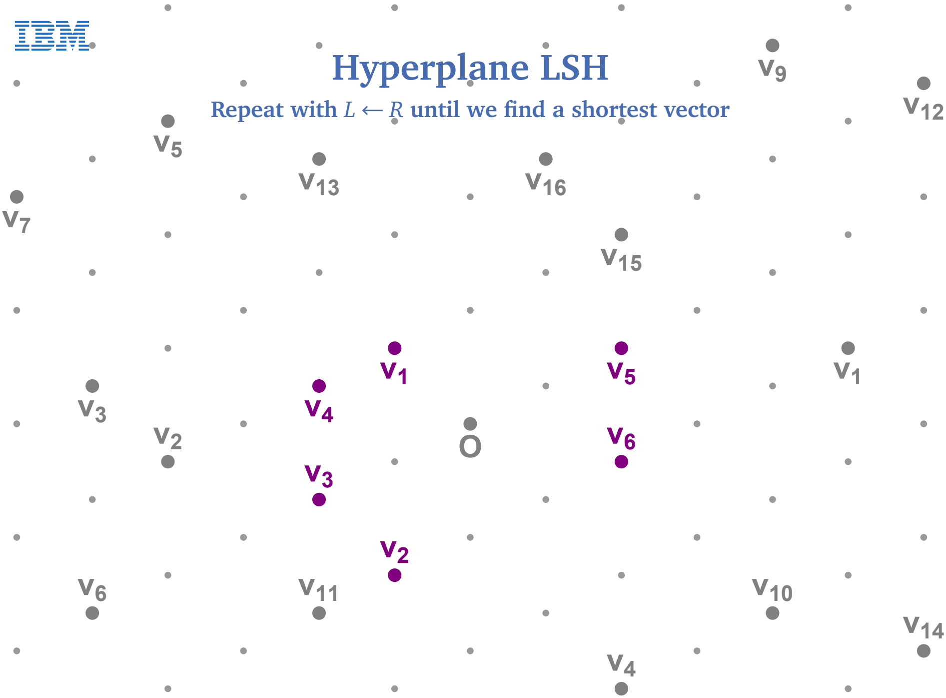

Hyperplane LSH

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- Two parameters to tune
  - $k = O(n)$: Number of hyperplanes, leading to 2^k regions
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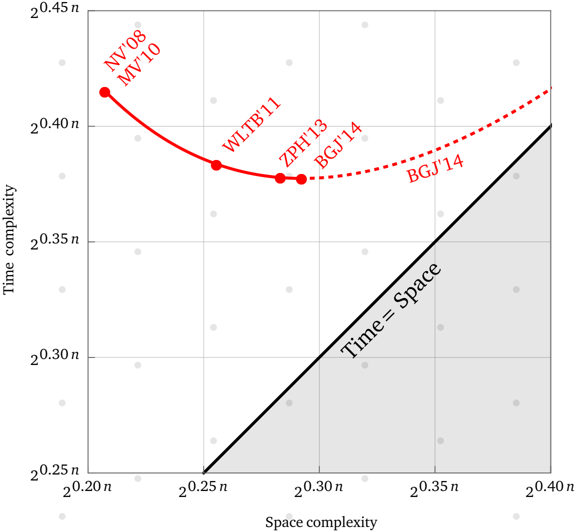
Heuristic result (Laarhoven, CRYPTO'15)

Sieving with hyperplane LSH solves SVP in time $2^{0.337n+o(n)}$.

Hyperplane LSH

Space/time trade-off



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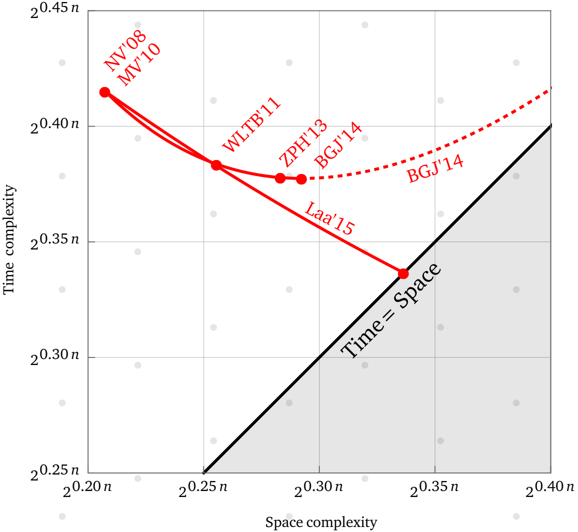

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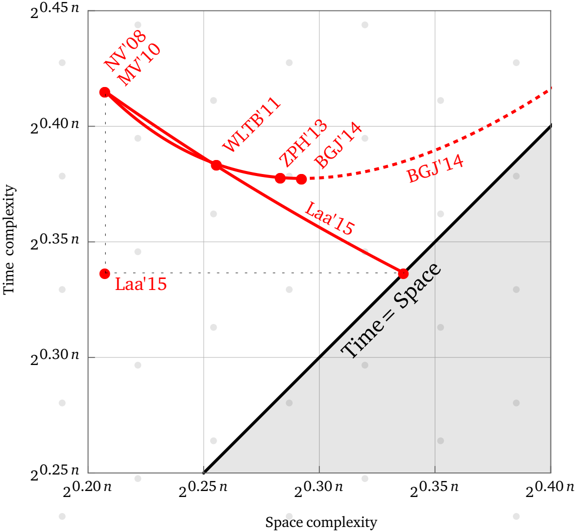

Spherical LSH

Sample a list L of random lattice vectors

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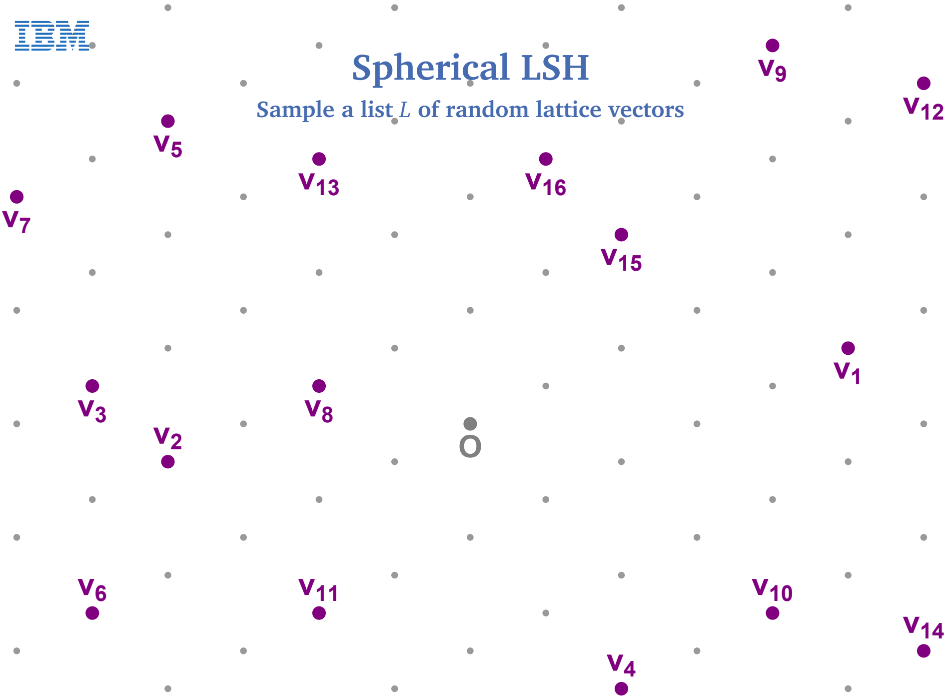
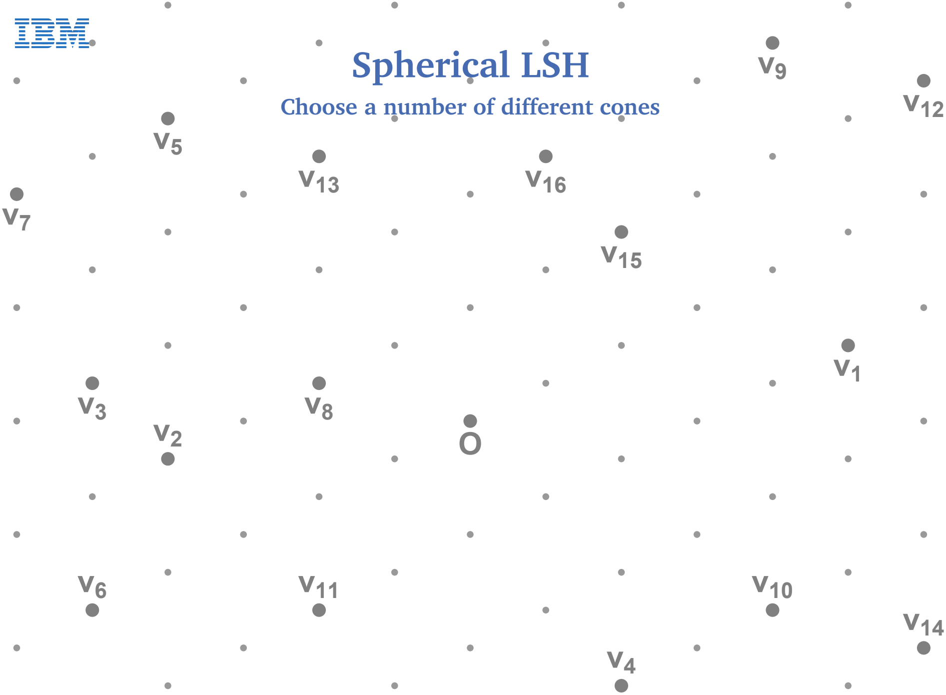


Spherical LSH

Choose a number of different cones



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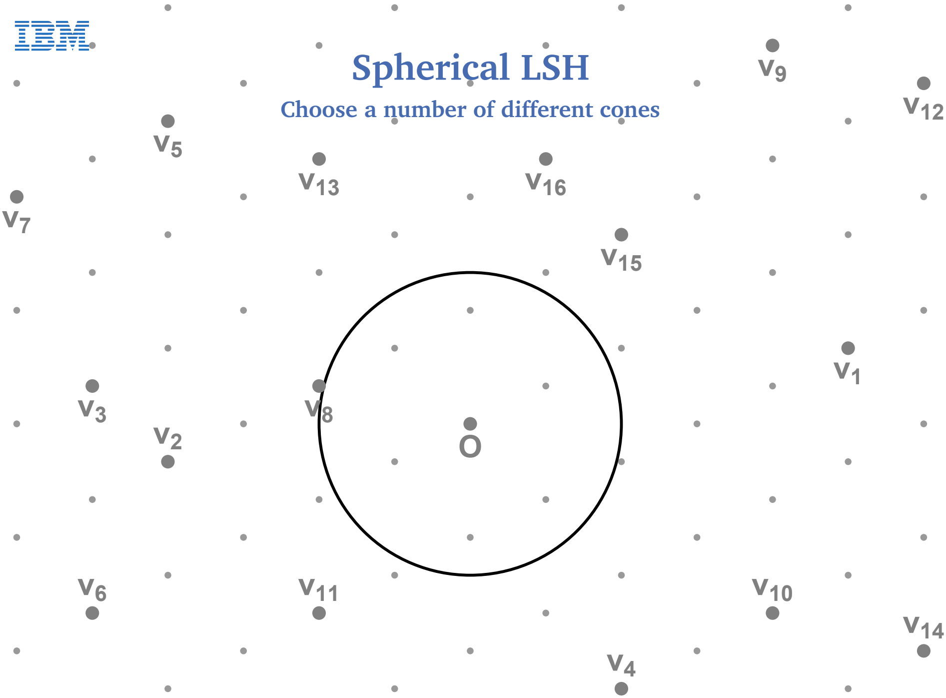

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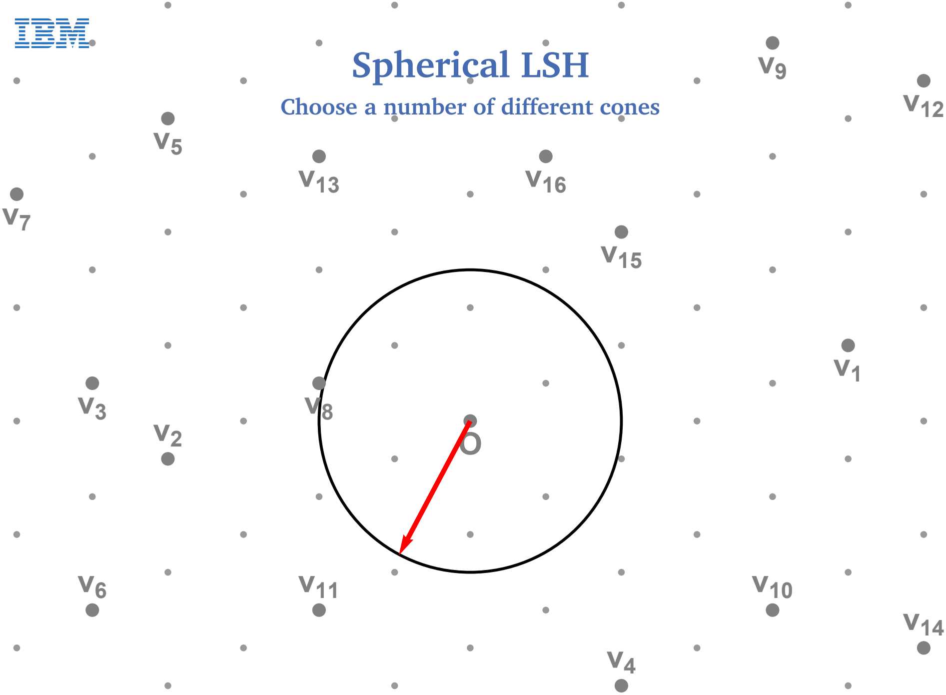

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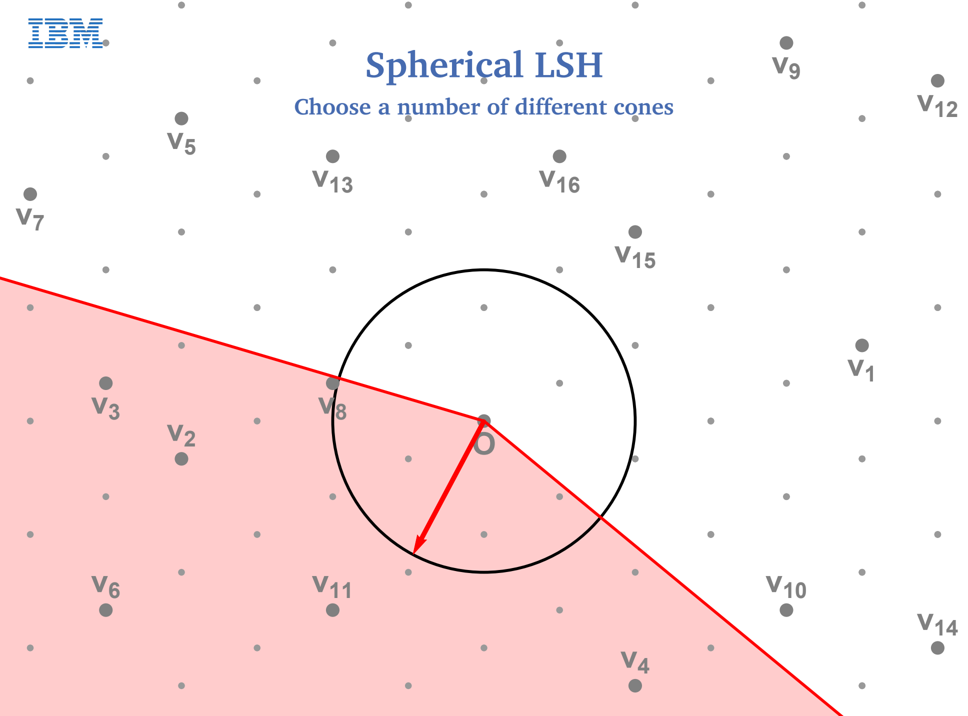

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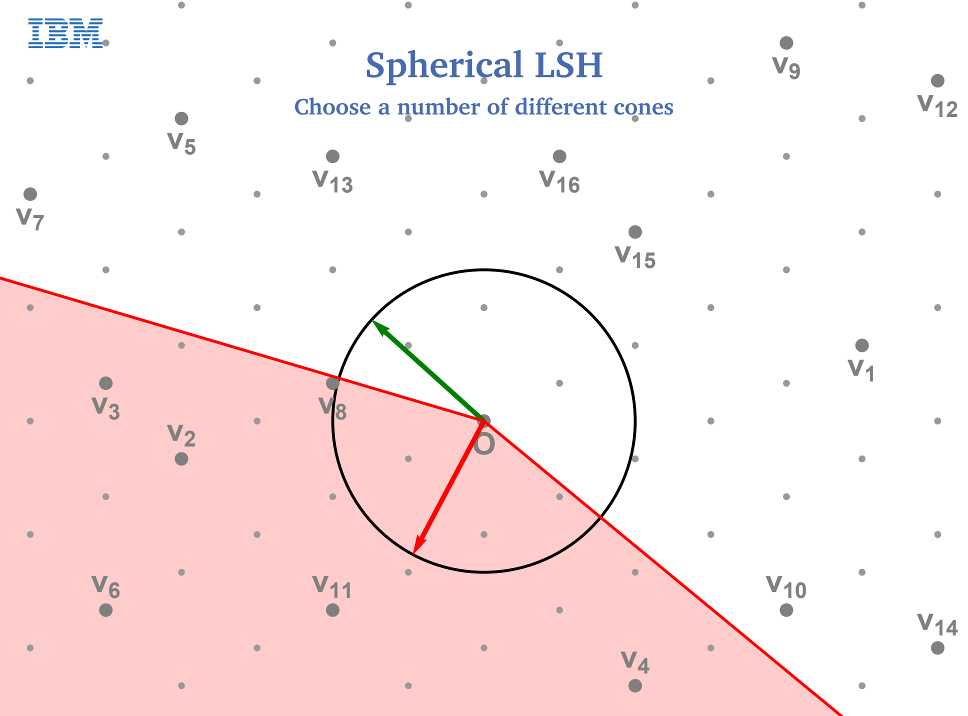

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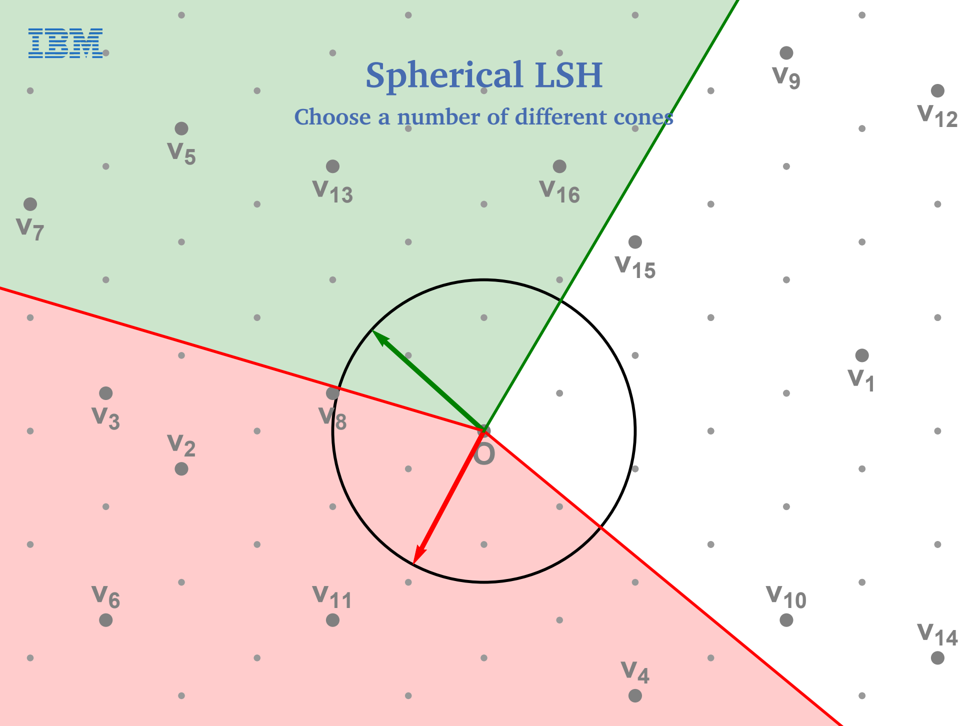

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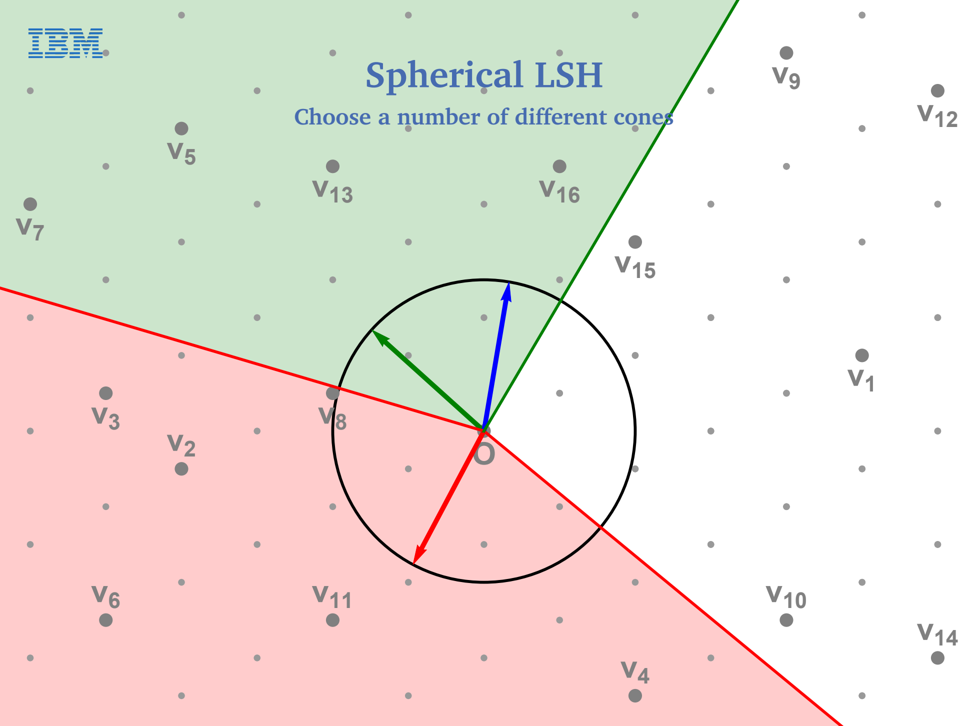

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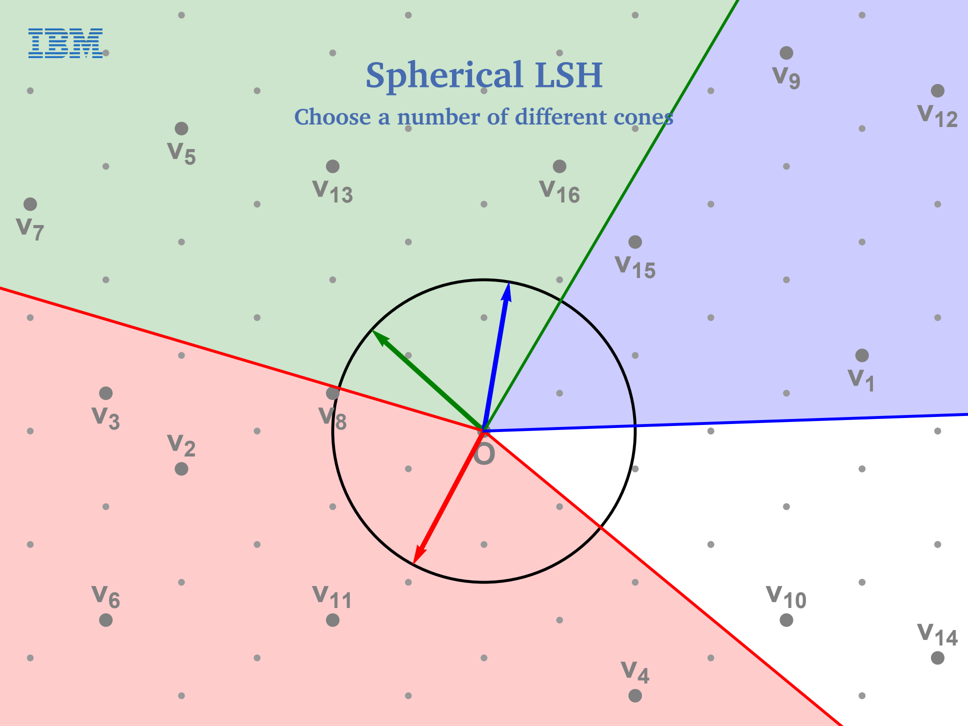

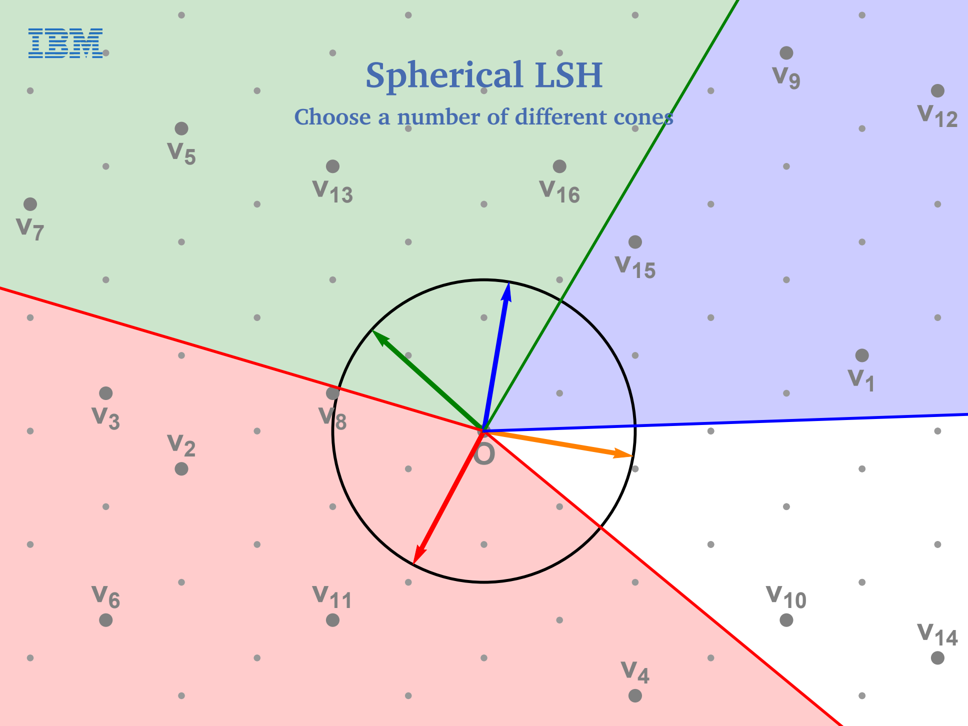

Spherical LSH
Choose a number of different cones
v5
v13
v16
v7
v15
v9
v12
v1
v3
v8
v2
O
v6
v11
v10
v14
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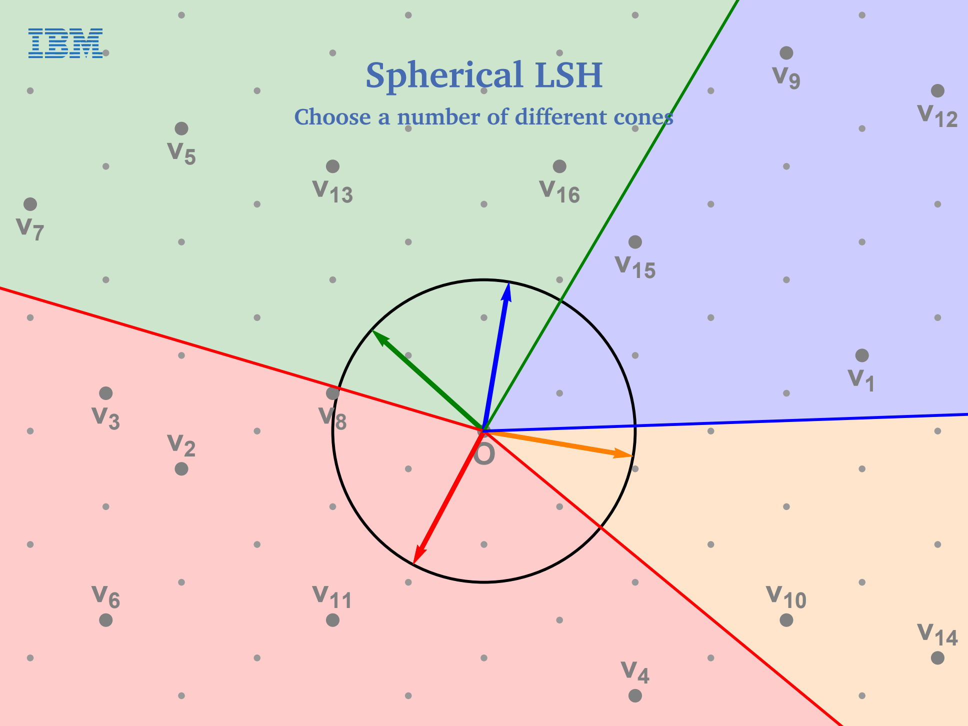
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v9
v12
v5
v13
v16
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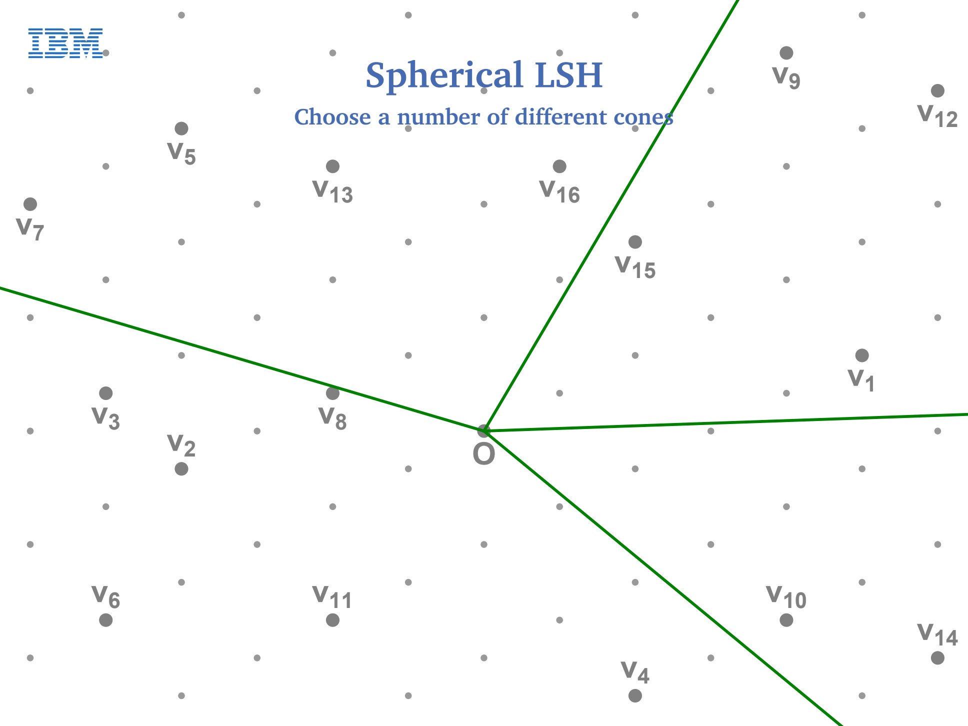
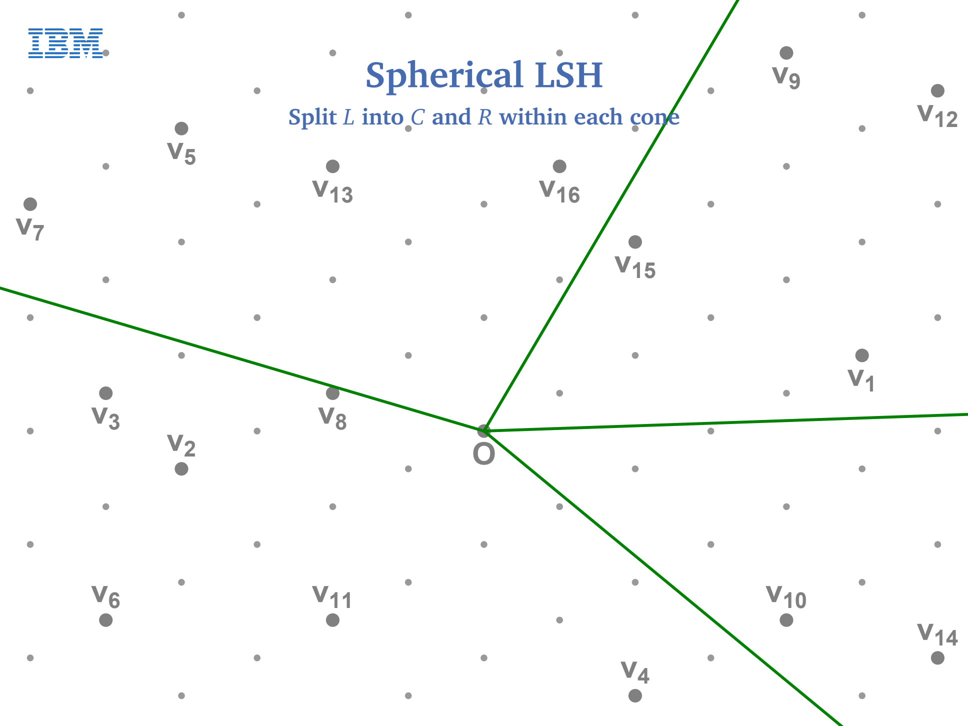


Spherical LSH

Split L into C and R within each cone



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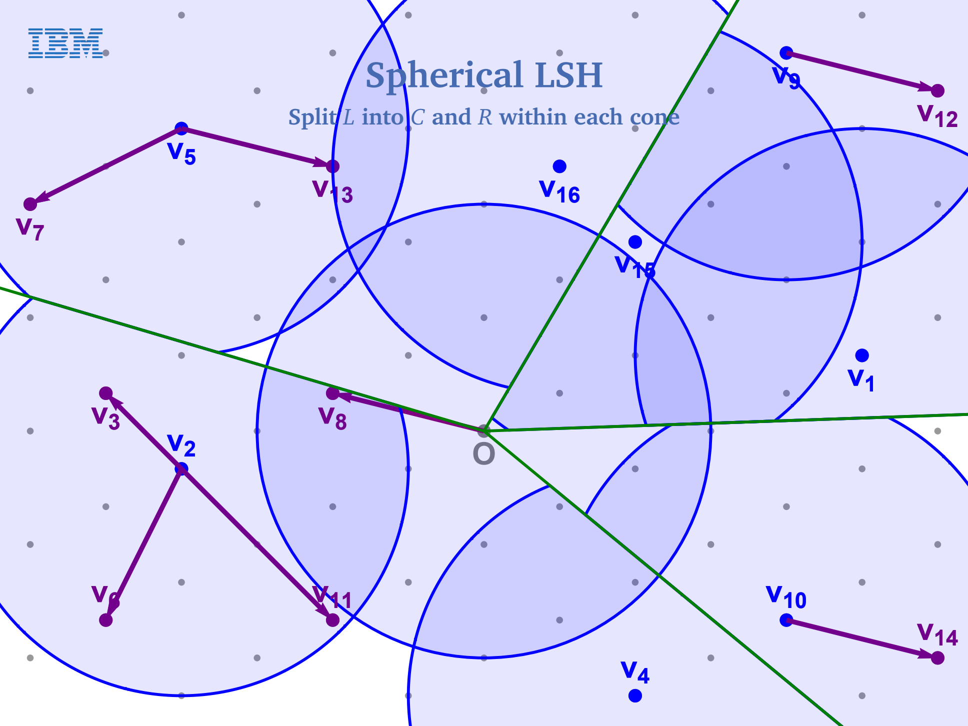
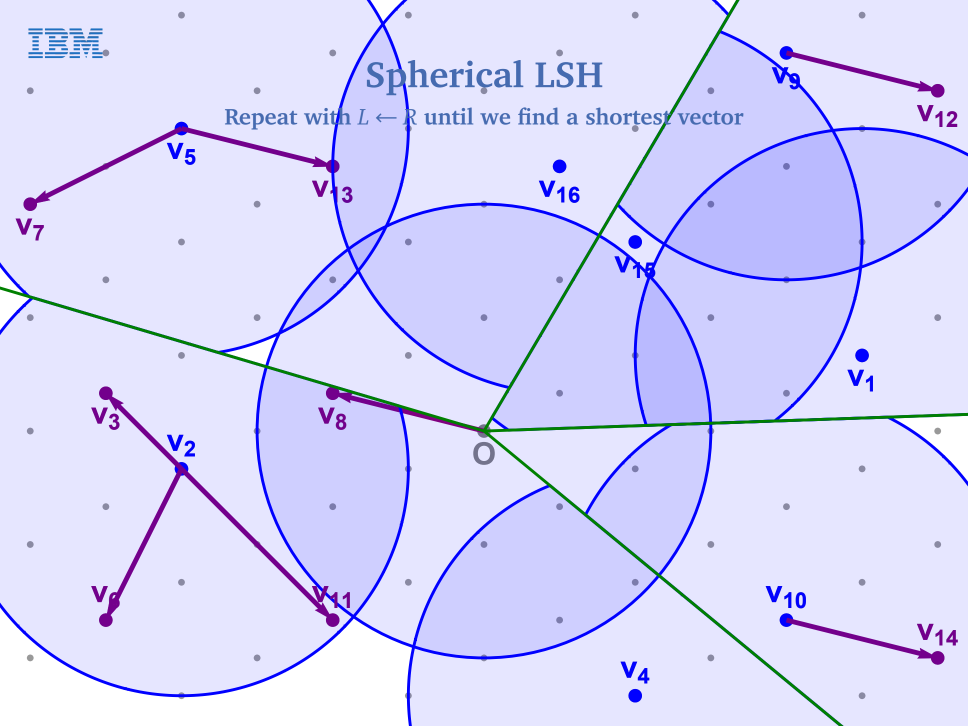


Spherical LSH

Repeat with $L \leftarrow R$ until we find a shortest vector



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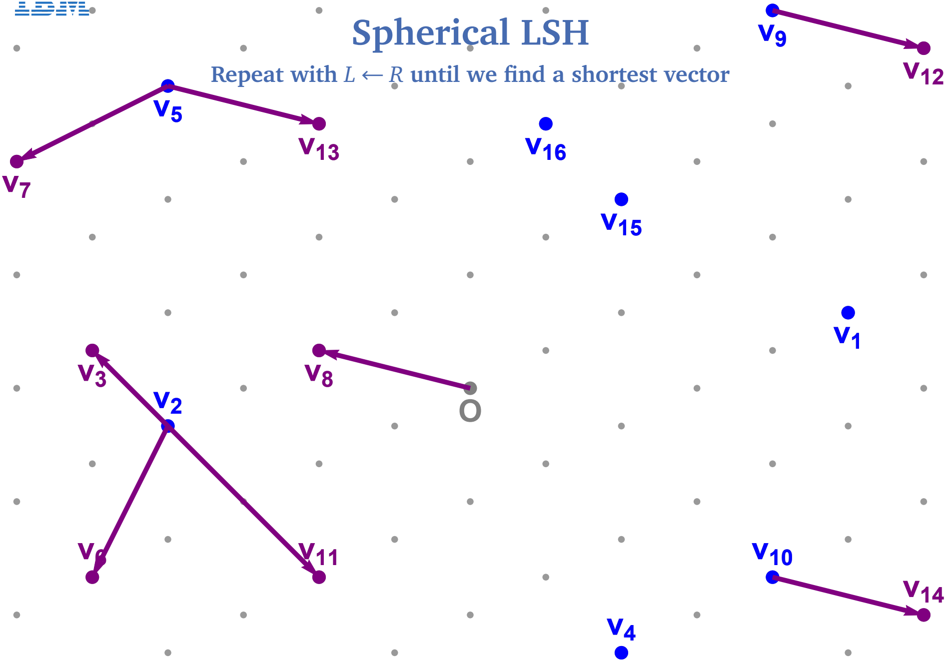

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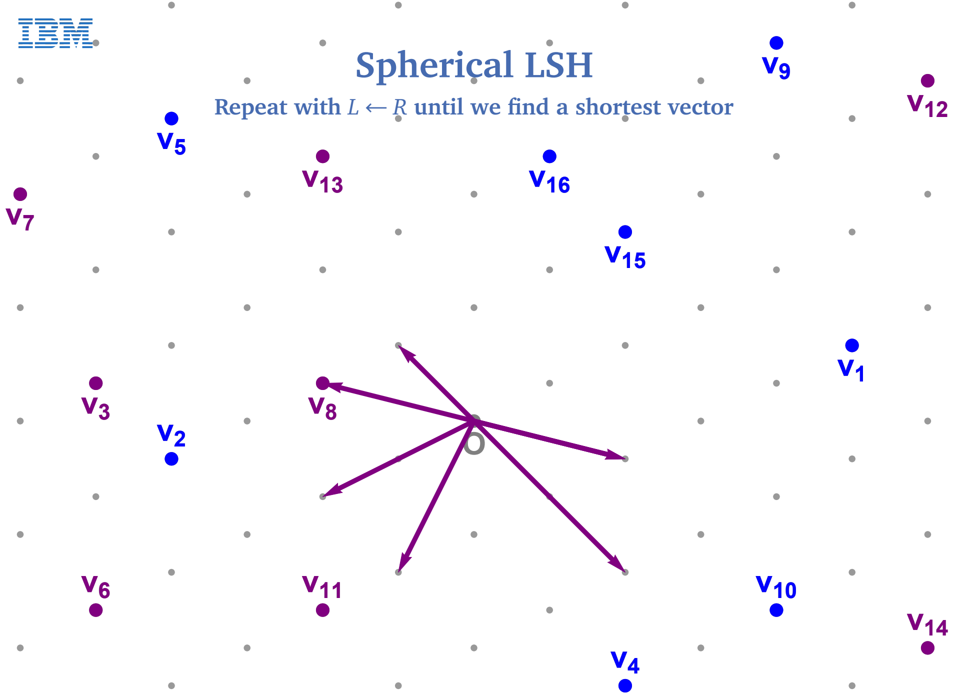

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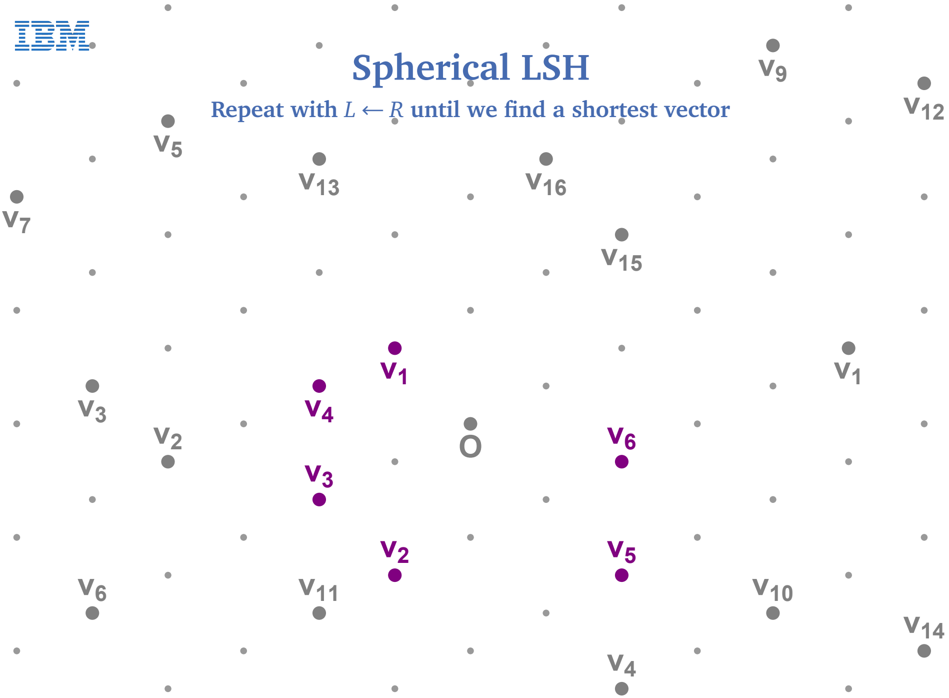

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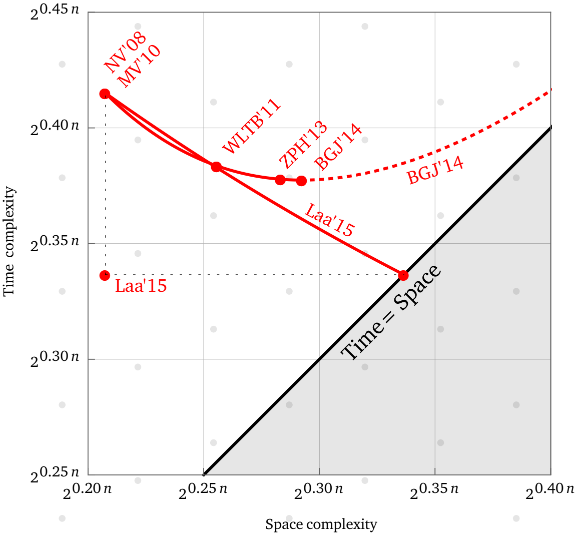
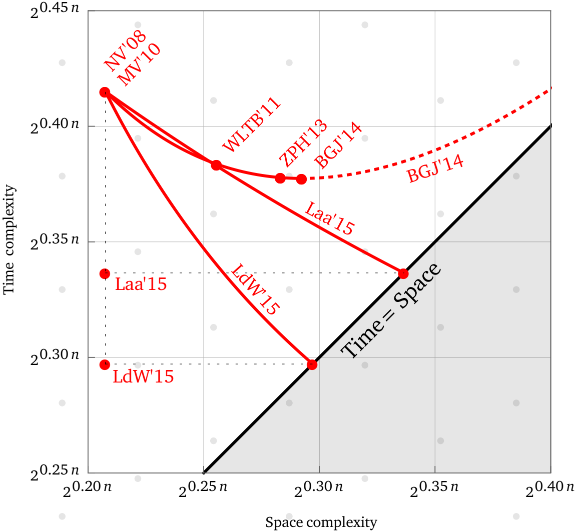
Spherical LSH

Space/time trade-off



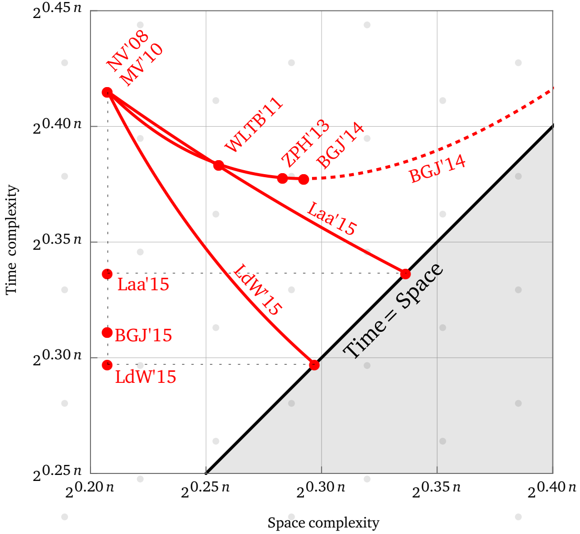
Spherical LSH

Space/time trade-off



May and Ozerov's NNS method

Space/time trade-off



Cross-Polytope LSH

Sample a list L of random lattice vectors

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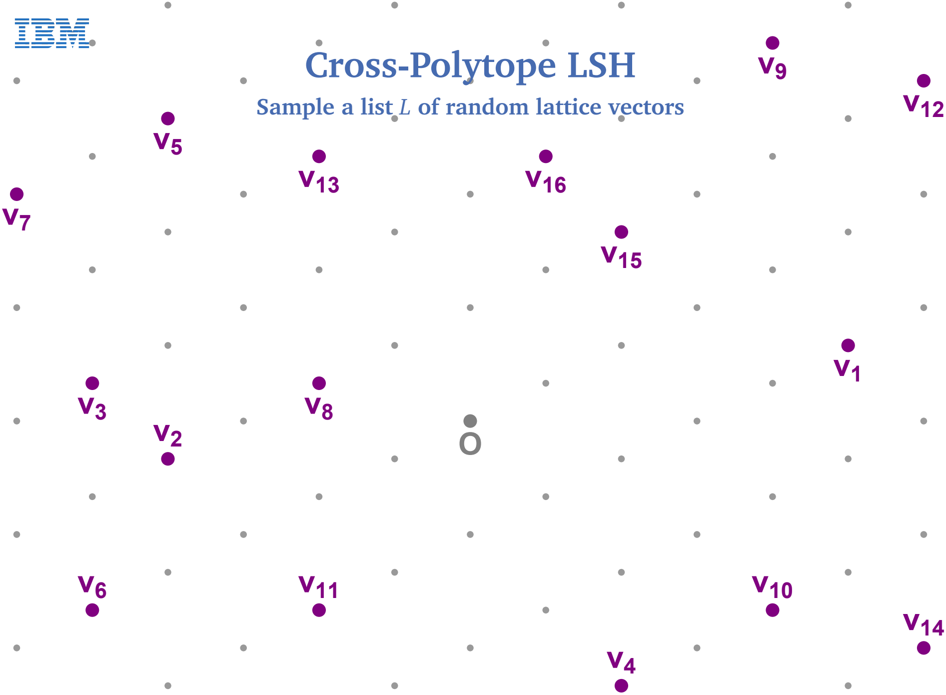
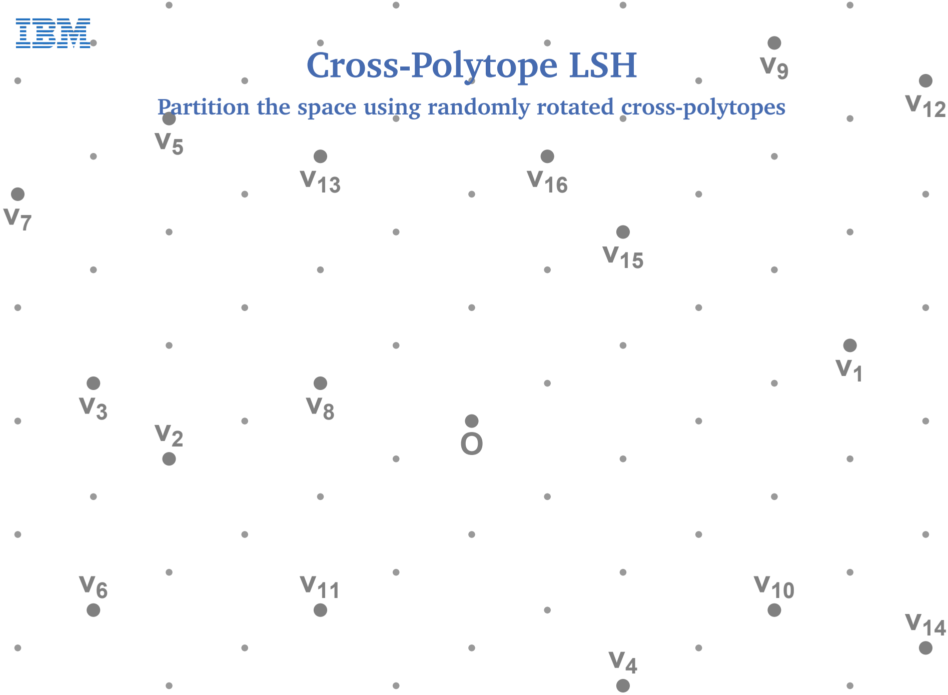


Cross-Polytope LSH

Partition the space using randomly rotated cross-polytopes



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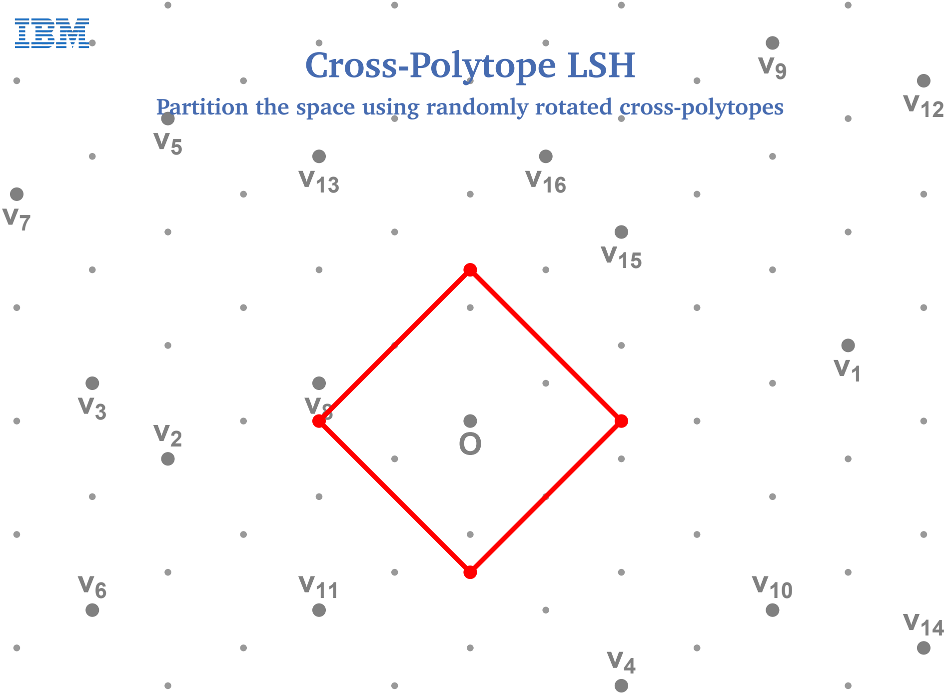

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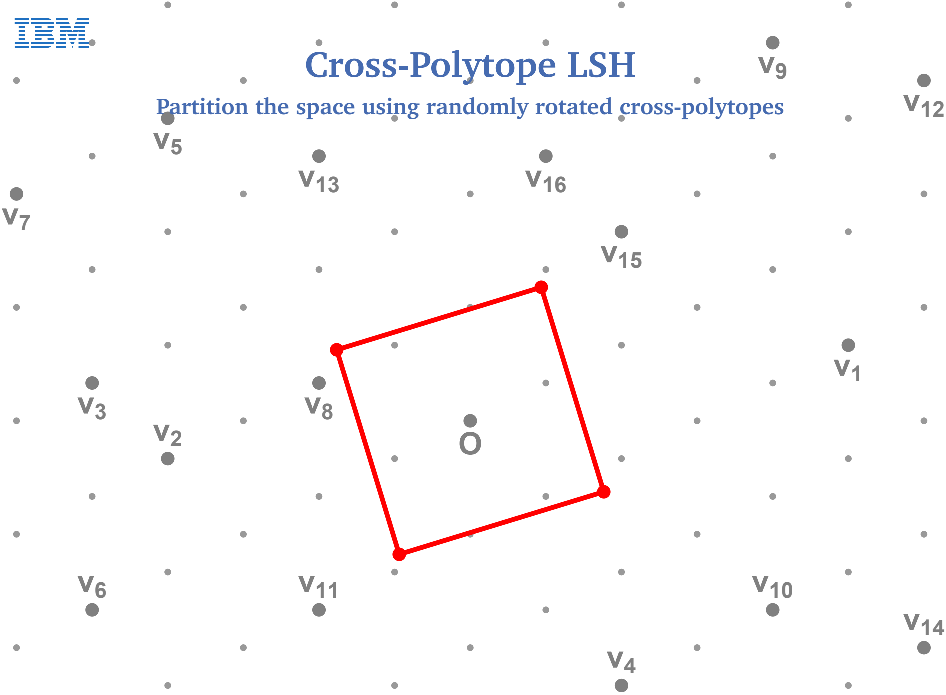

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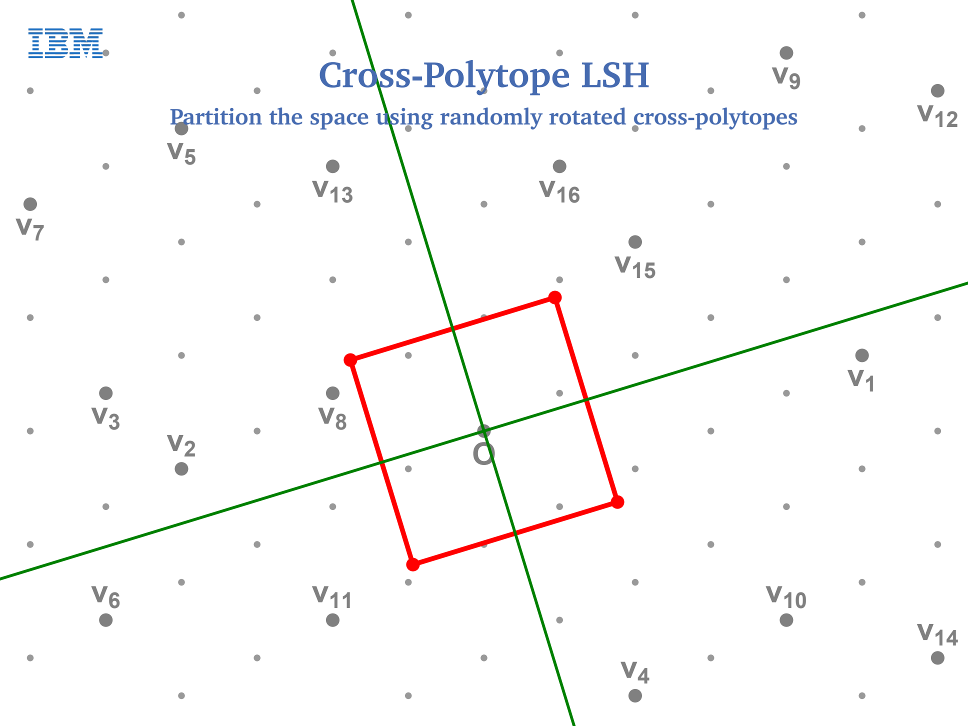

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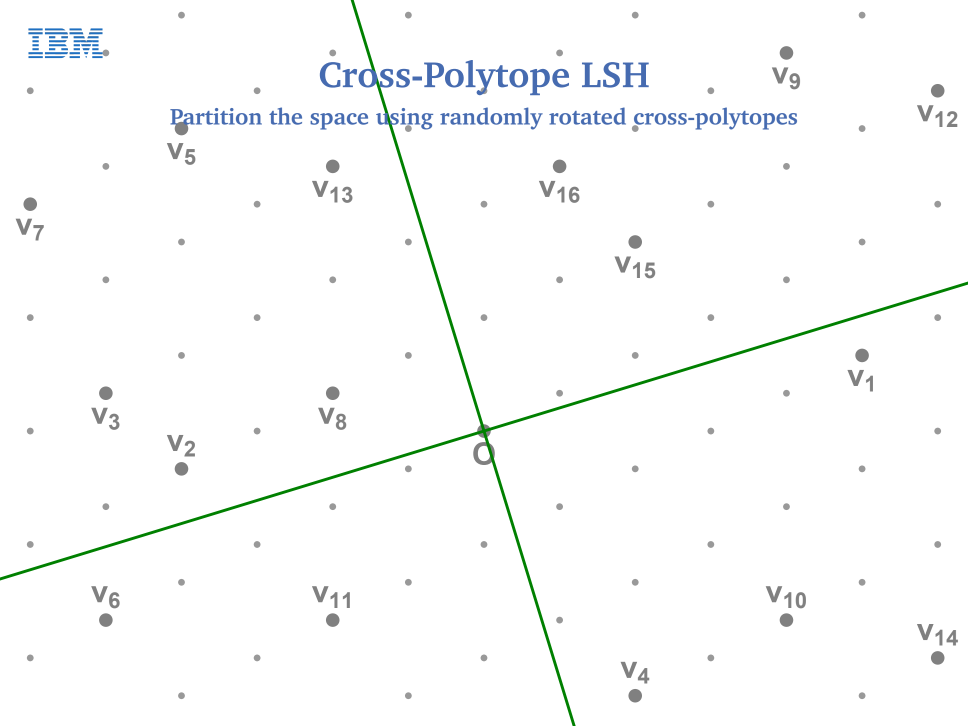
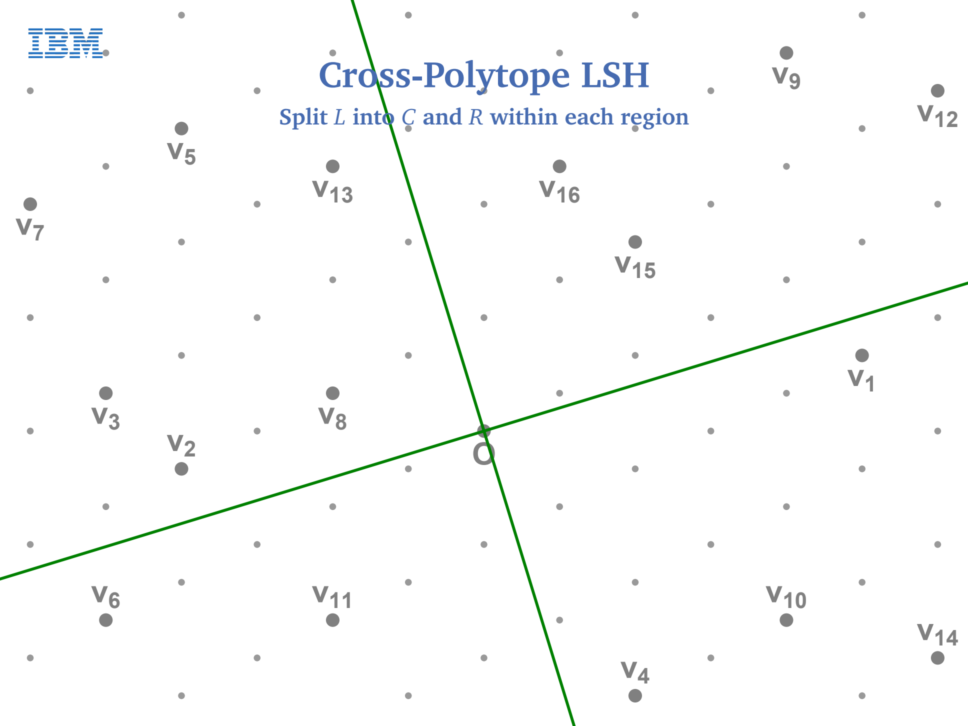


Cross-Polytope LSH

Split L into C and R within each region



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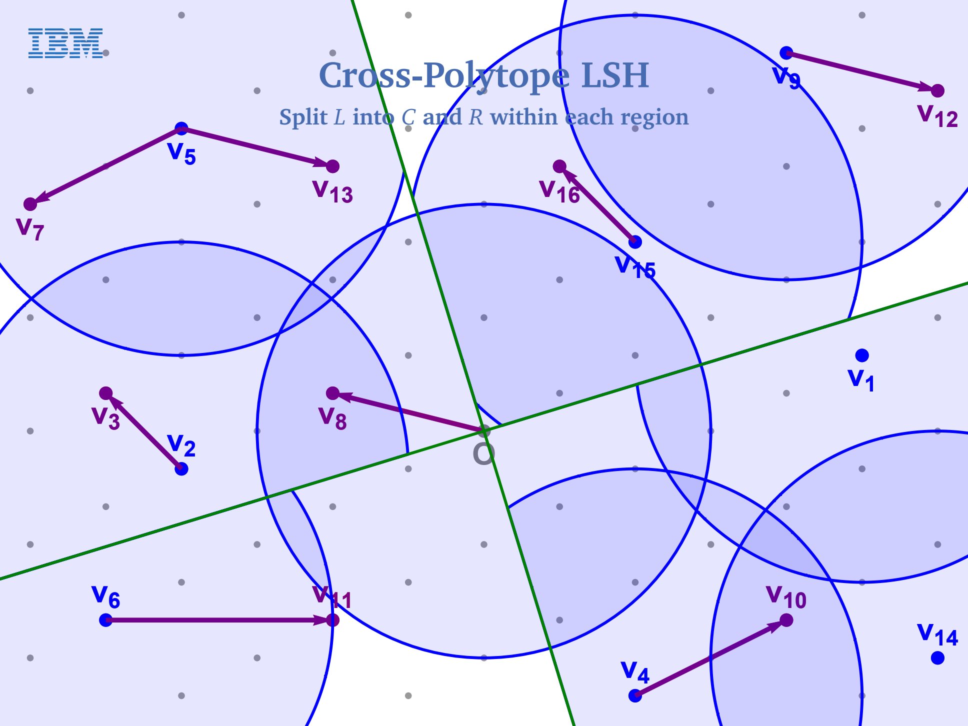
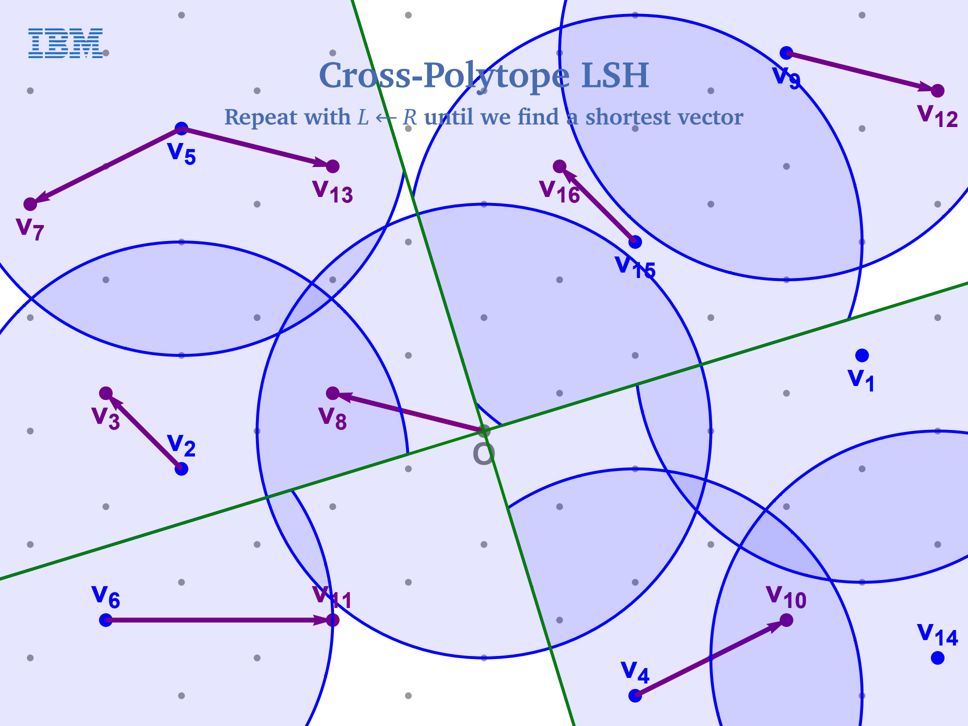


Cross-Polytope LSH

Repeat with $L \leftarrow R$ until we find a shortest vector



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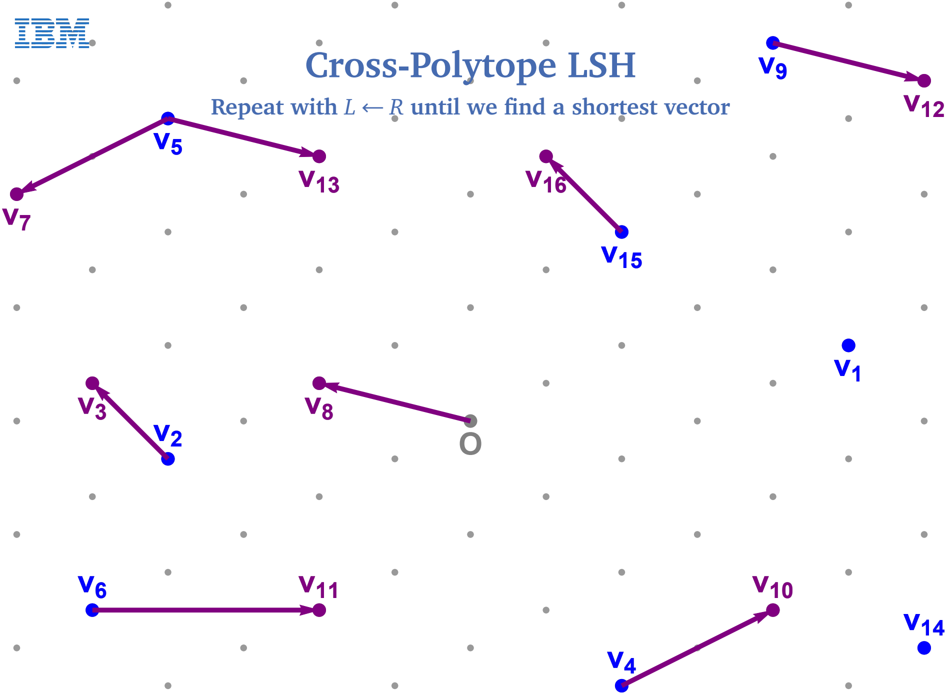

IBM

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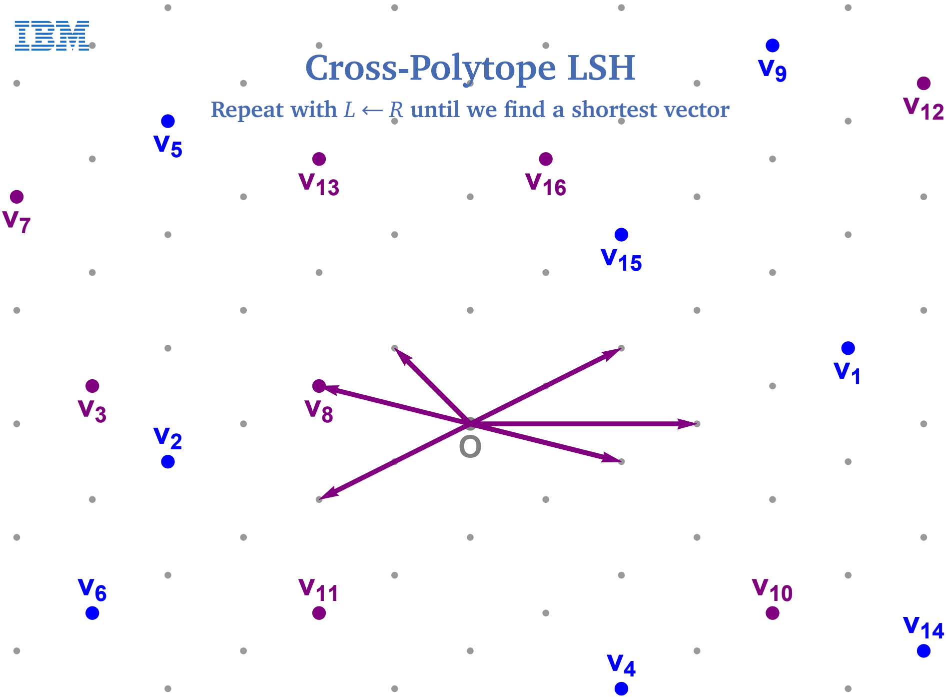

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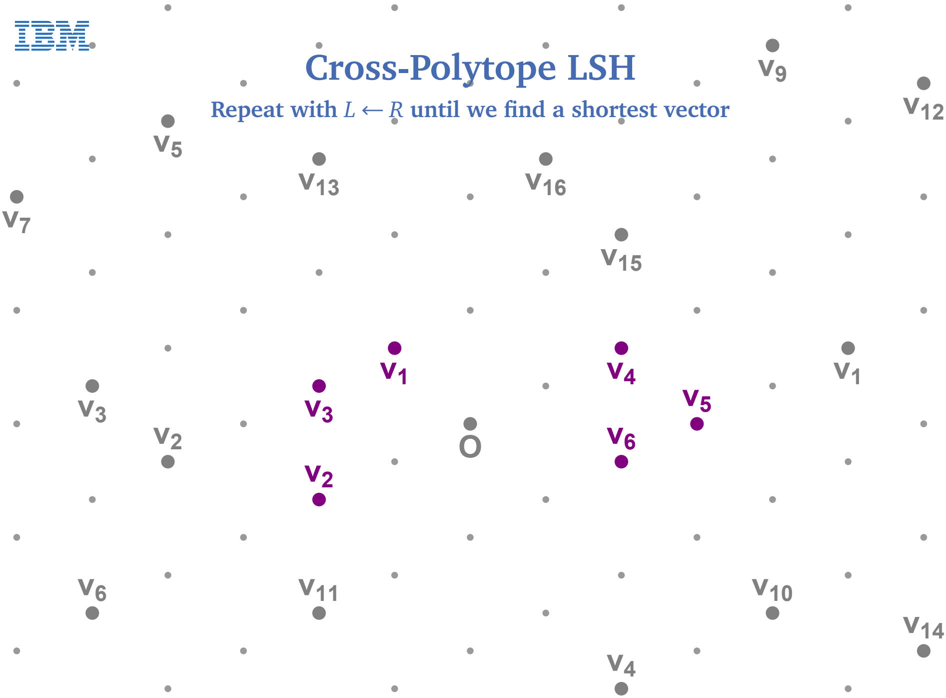

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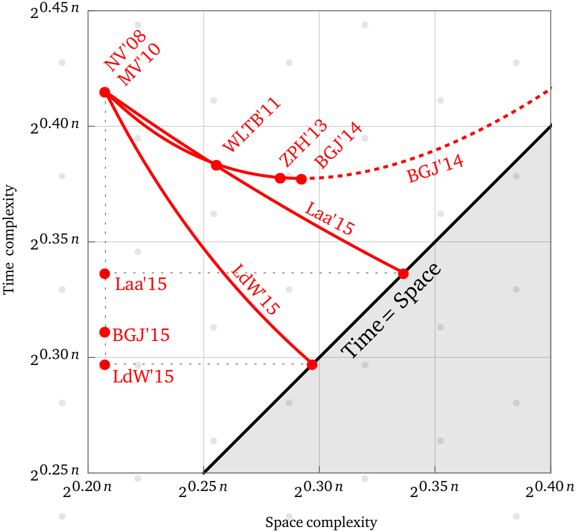

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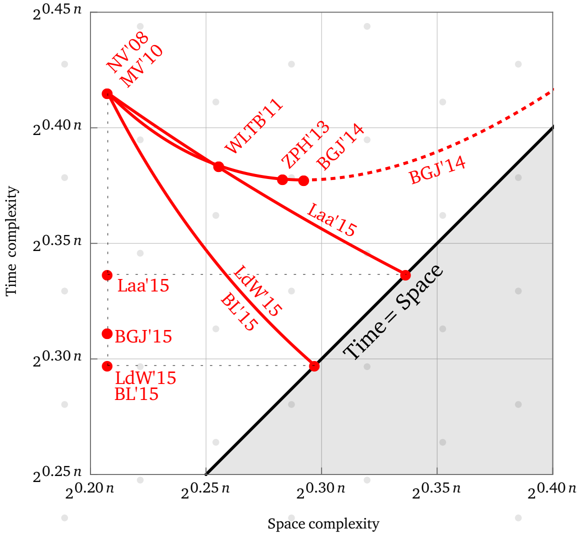

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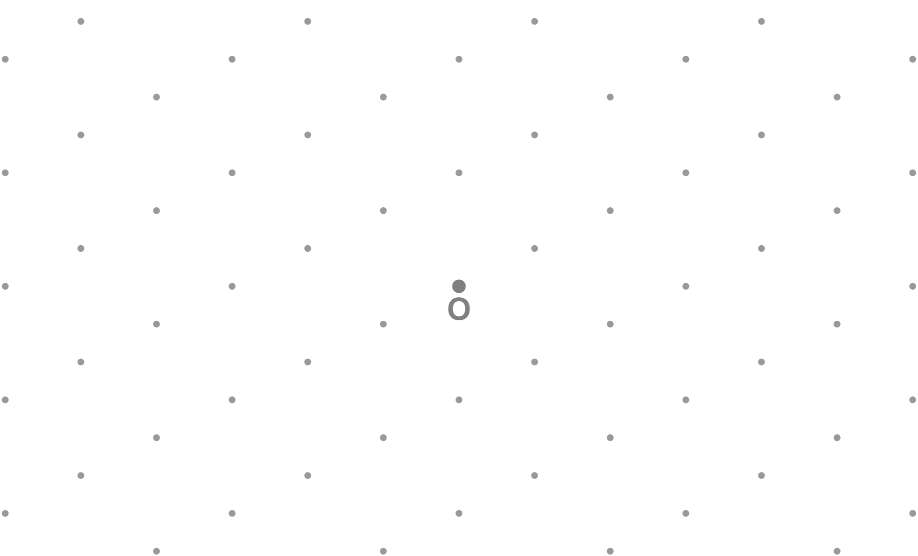

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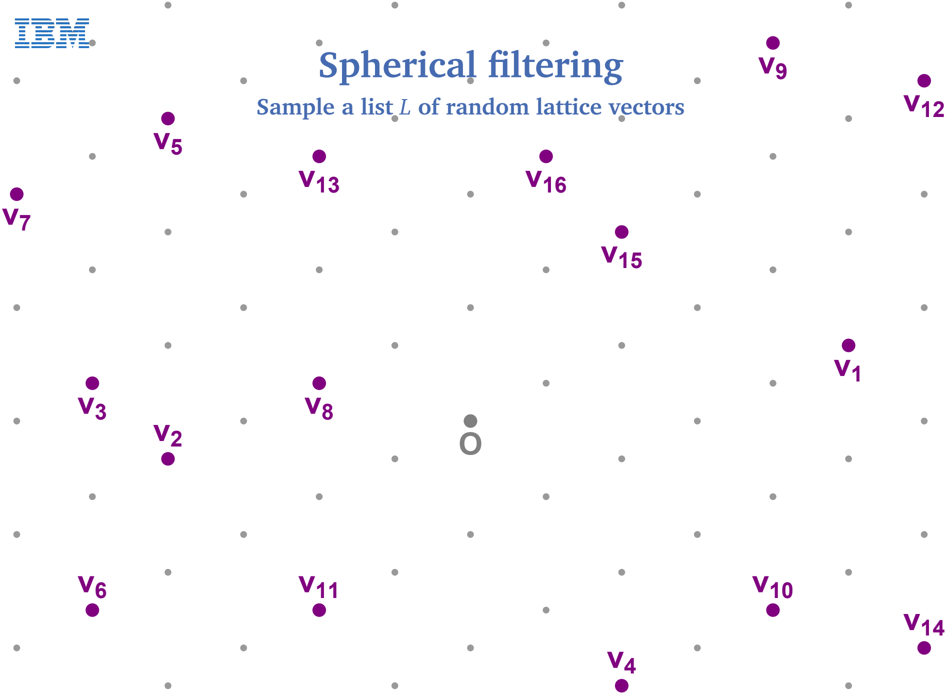
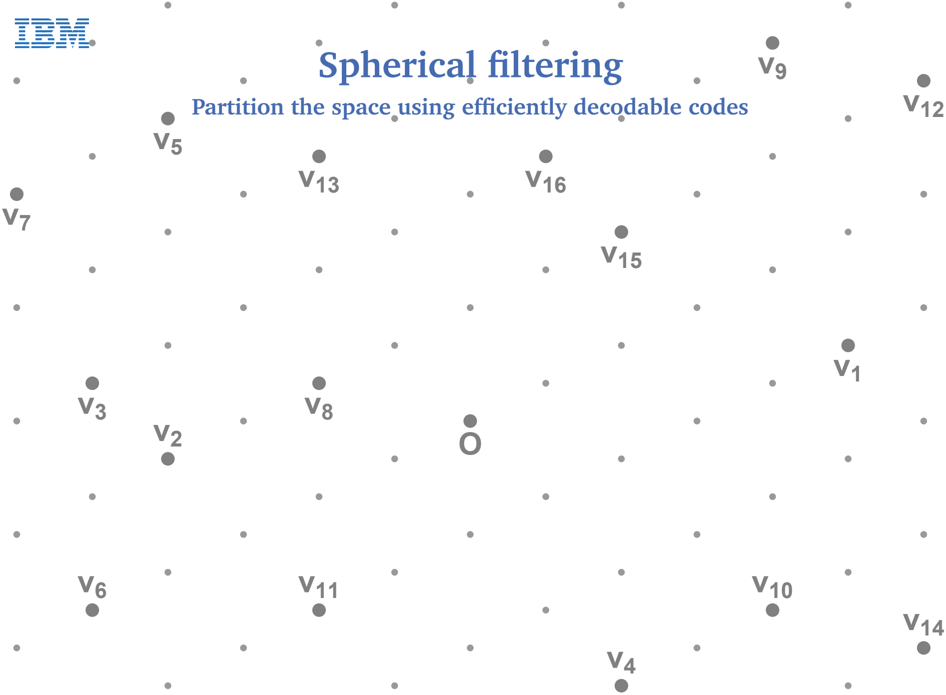


Spherical filtering

Partition the space using efficiently decodable codes



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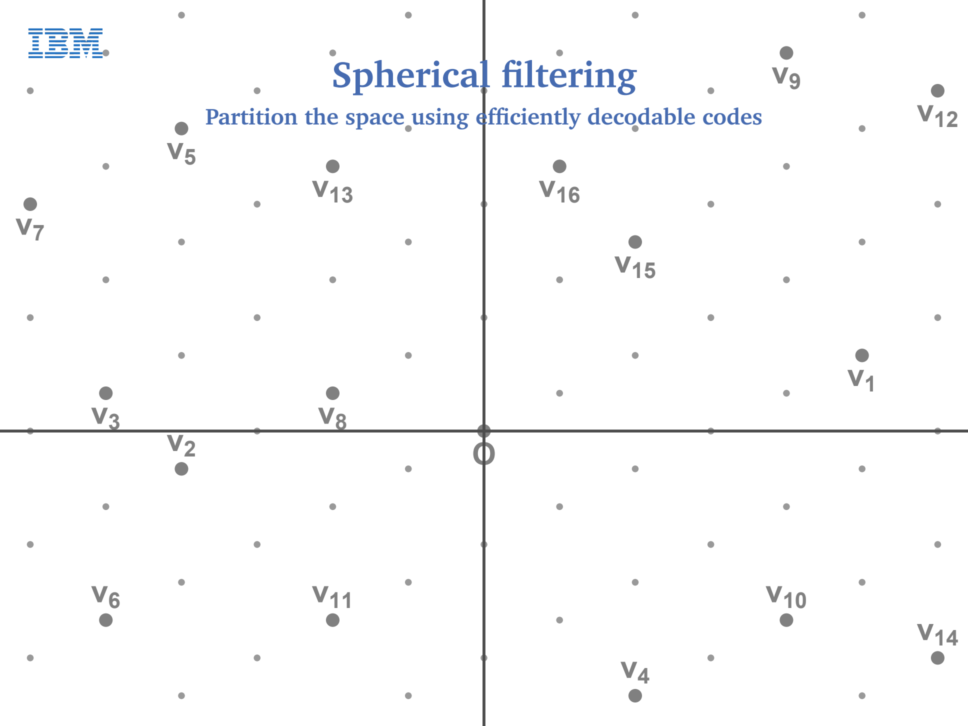

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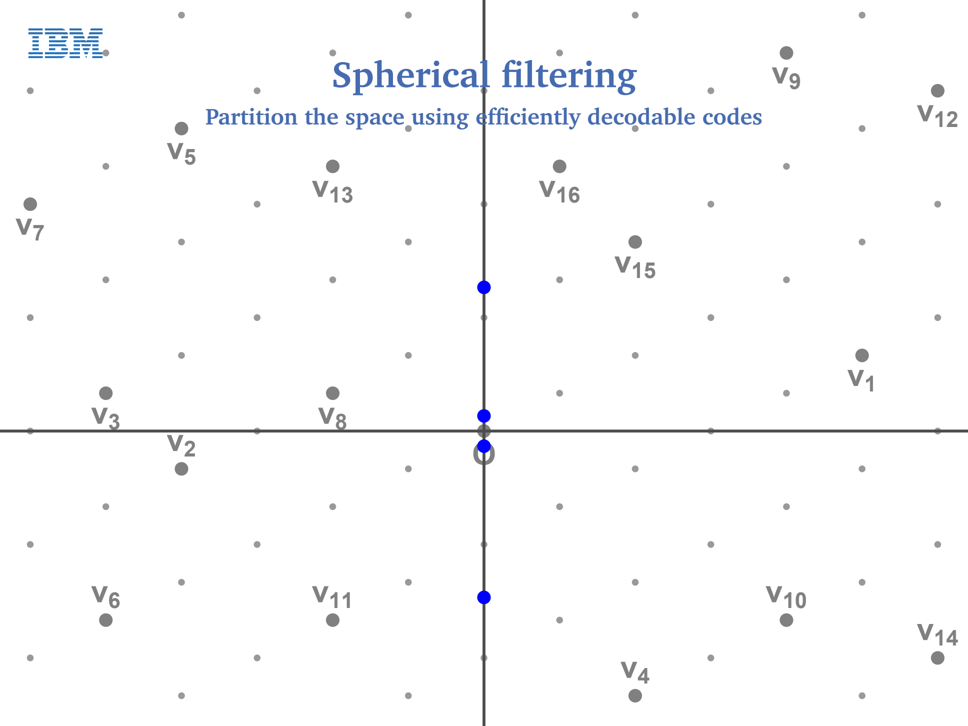

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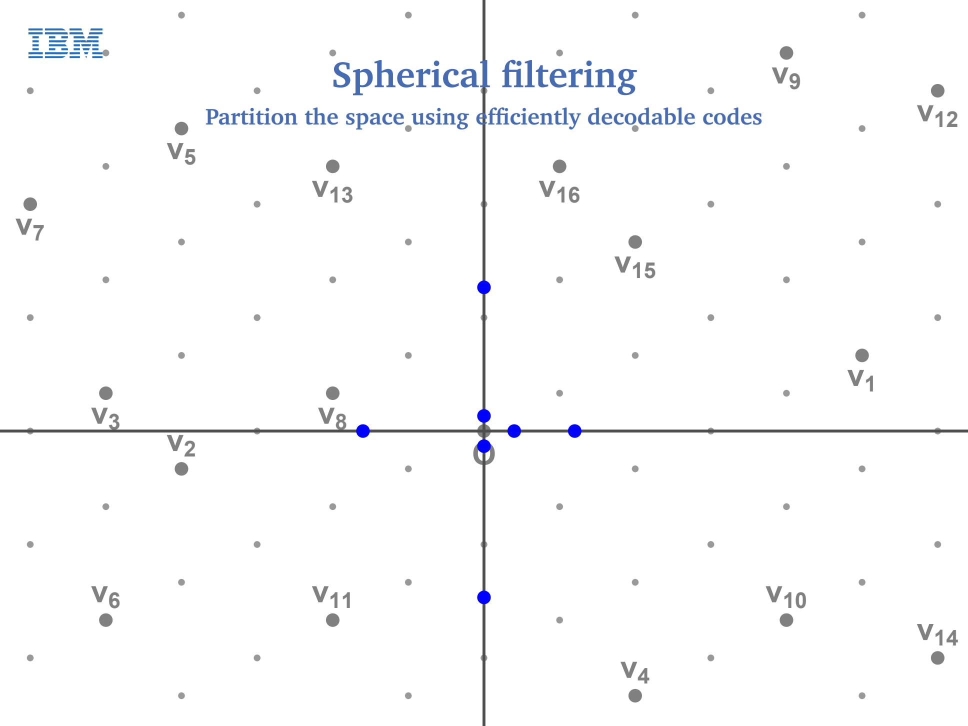

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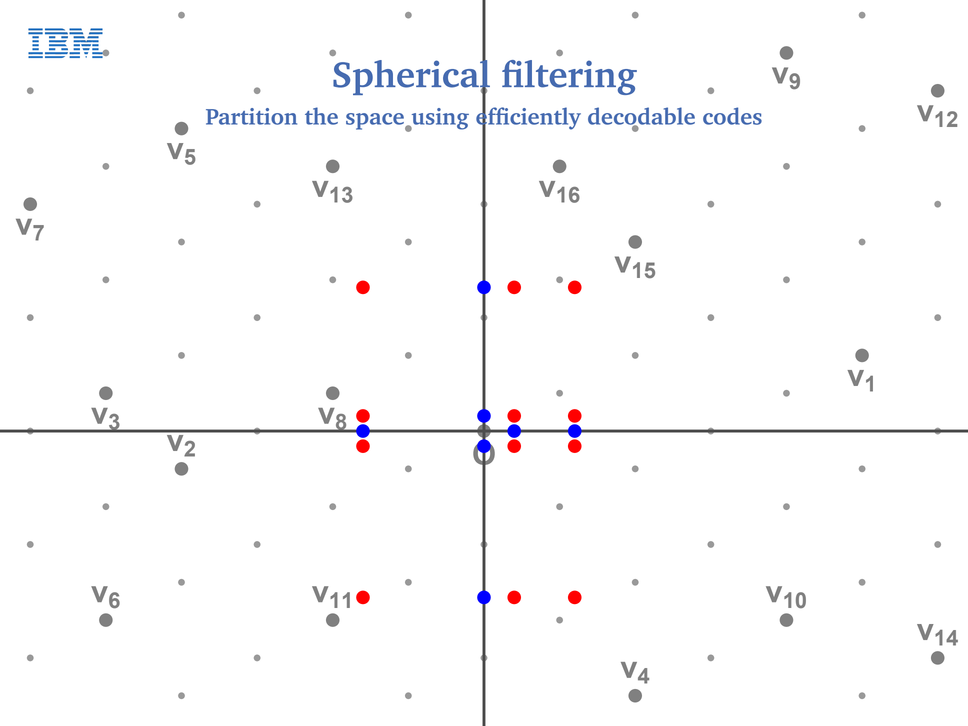

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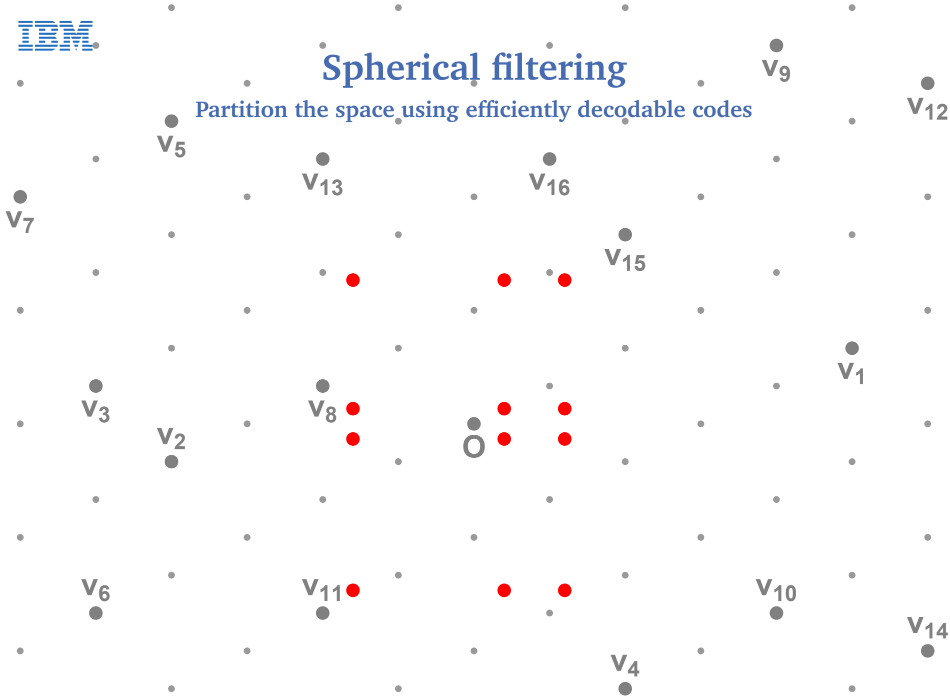

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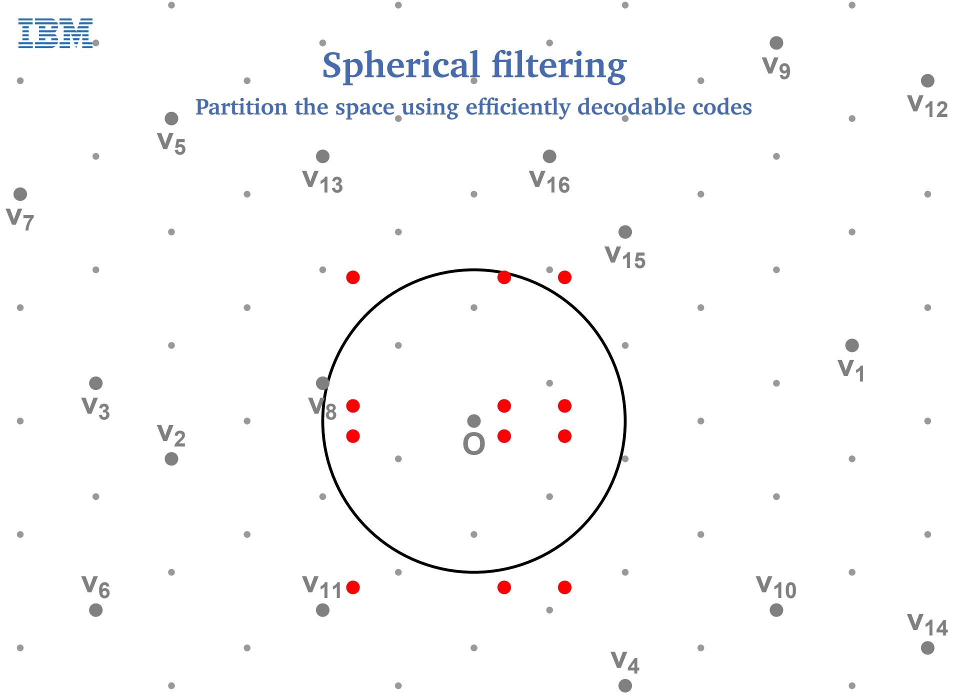

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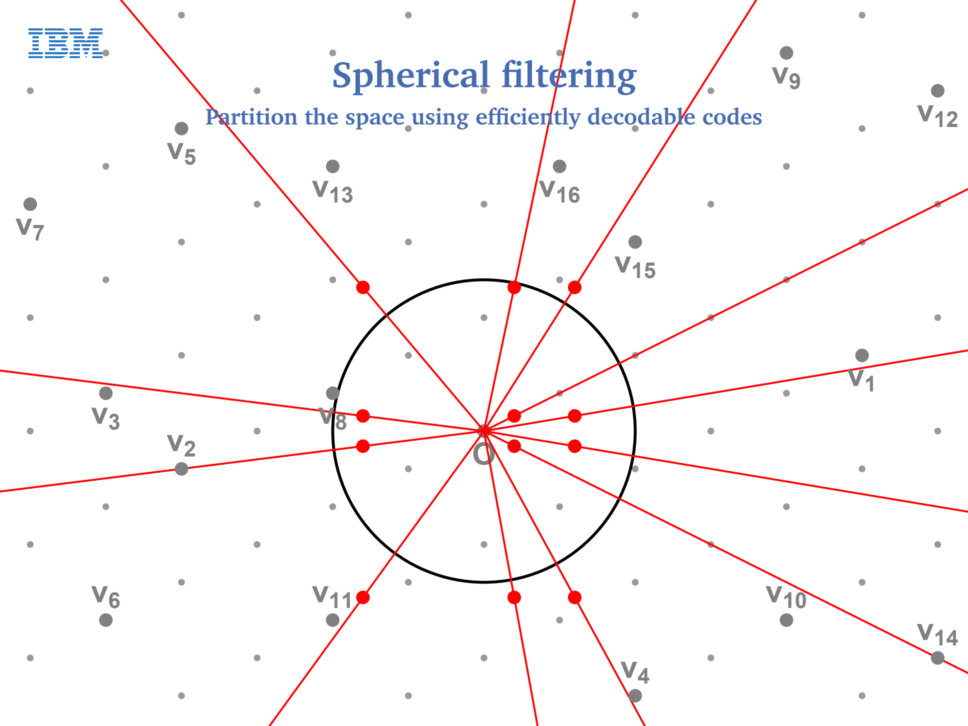

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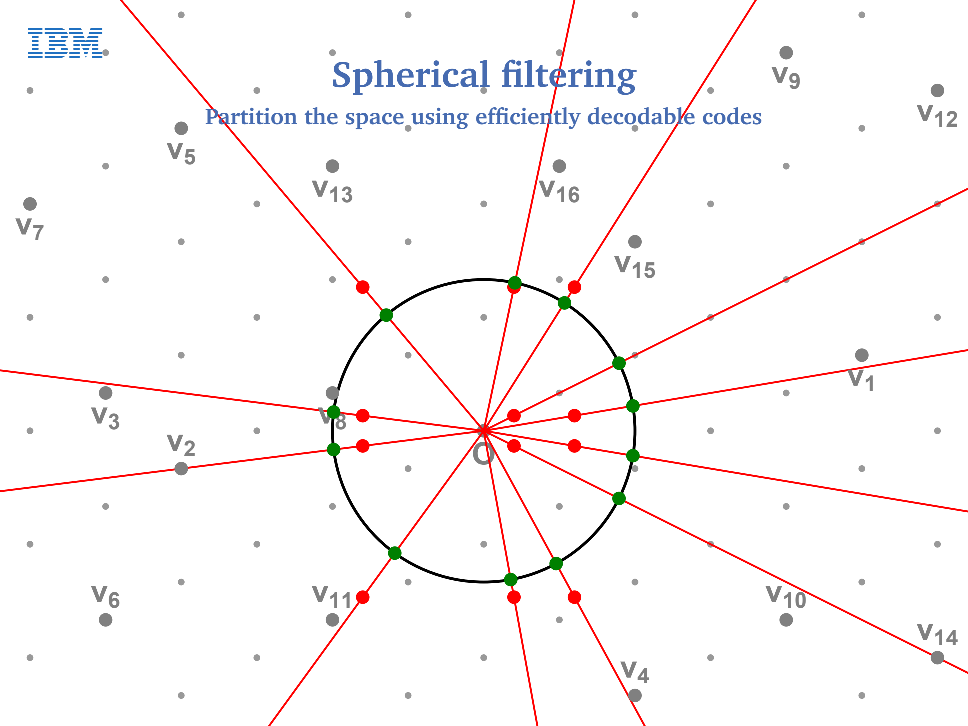

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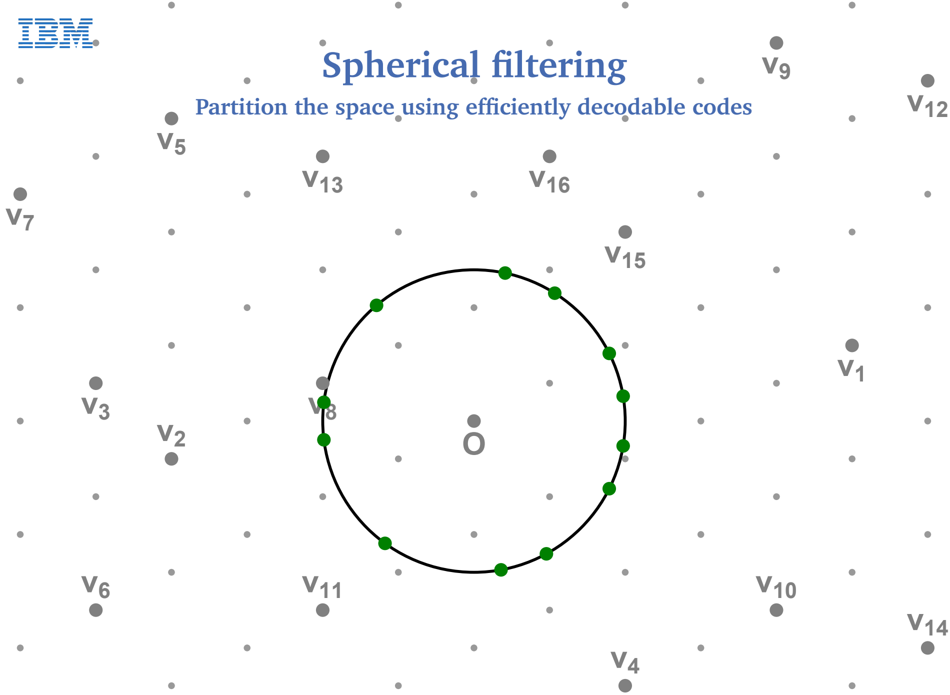

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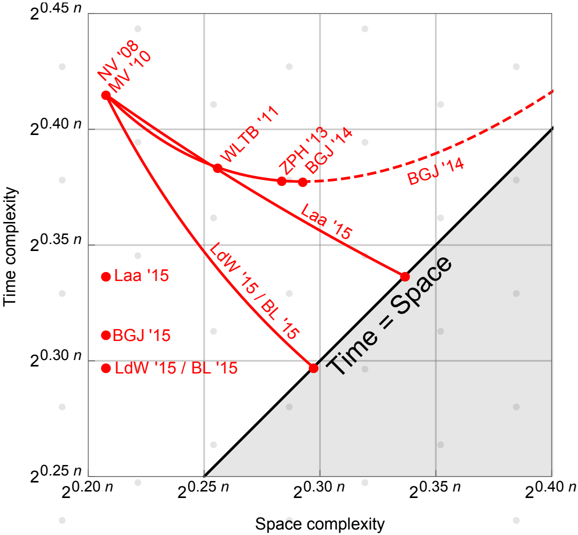

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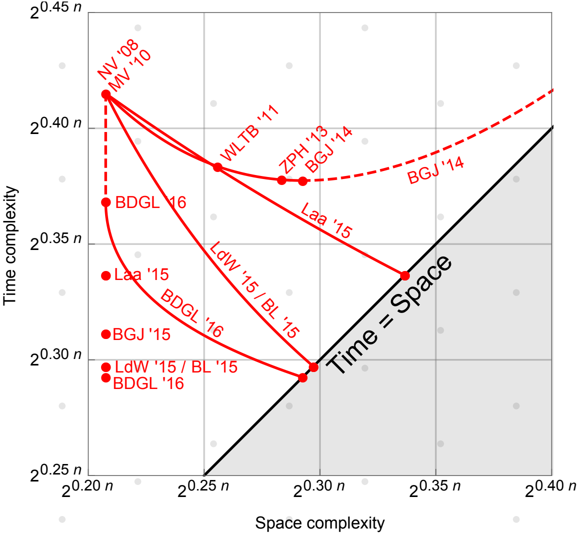

Outline

Lattices
- Basics
- Cryptography

Enumeration algorithms
- Fincke–Pohst enumeration
- Kannan enumeration
- Pruned enumeration

Sieving algorithms
- Basic sieving
- Leveled sieving
- Near neighbor searching

Practical comparison

SVP in practice

"We expect our [enumeration] algorithm to be more efficient than lattice sieving up to dimension $n = 1895$."

— Micciancio–Walter, *SODA'15*

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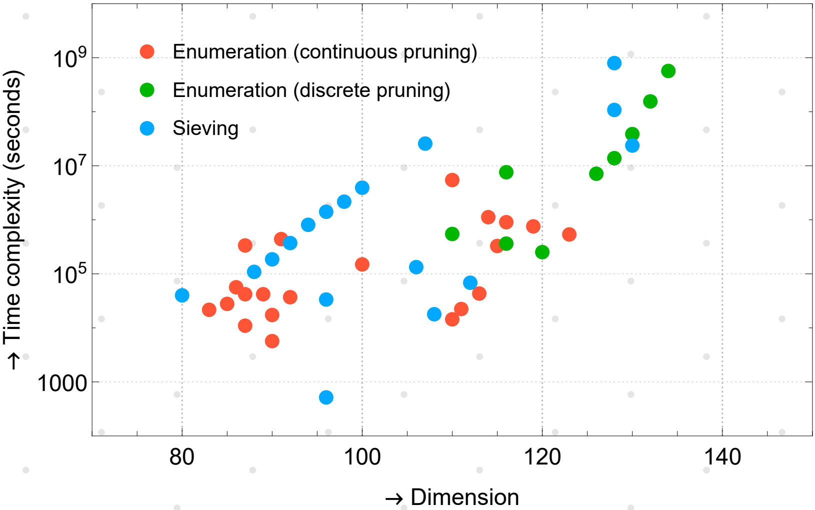
"I compute a cross-over point between enumeration and the HashSieve at dimension $b = 217$."

— Ducas, *Google groups '16*

SVP in practice



Take-home messages

- Lattice-based crypto relies on hardness of finding short bases
- State-of-the-art basis reduction: BKZ with fast SVP subroutine
- Enumeration for SVP:
  - Memory-efficient
  - Best in low dimensions
  - Fast pruning heuristics
- Sieving for SVP:
  - Large memory requirement
  - Fastest in high dimensions
  - Practical near neighbor speedups
- Enumeration still leading, but sieving is catching up!

Questions?