Algorithms for hard lattice problems

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Lattices

What is a lattice?

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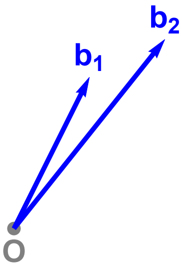

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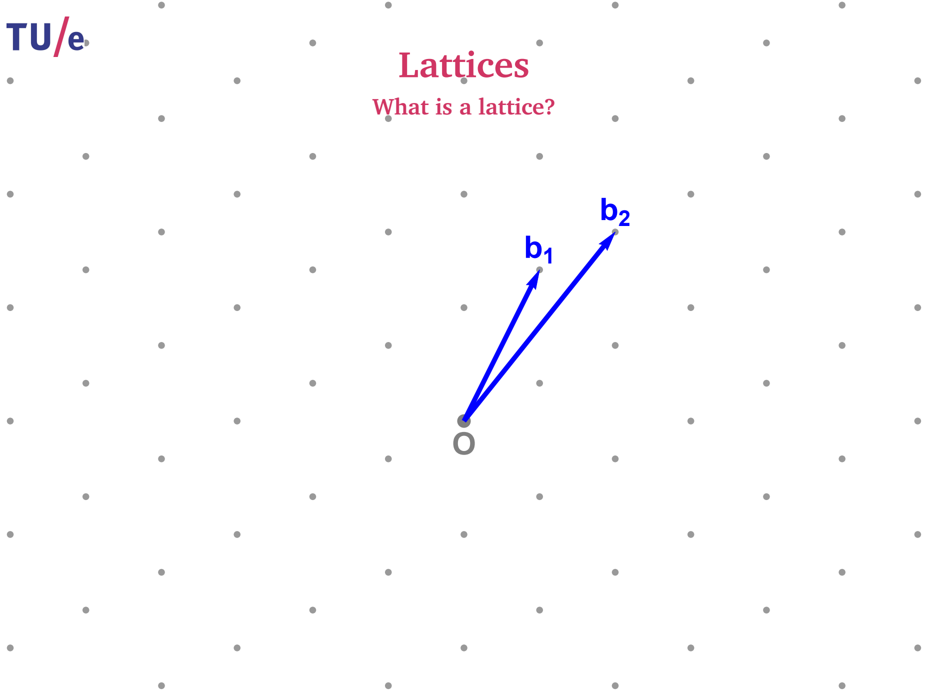
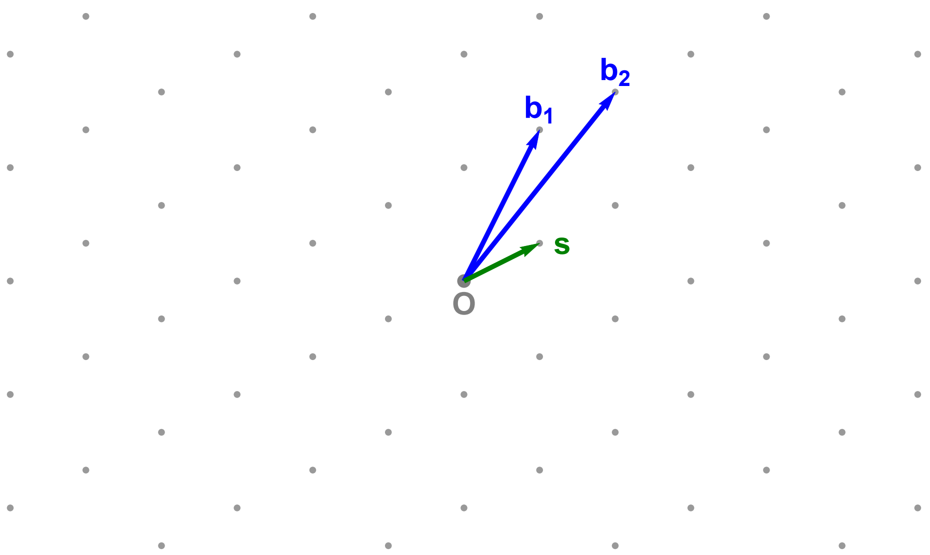


Lattices

Shortest Vector Problem (SVP)



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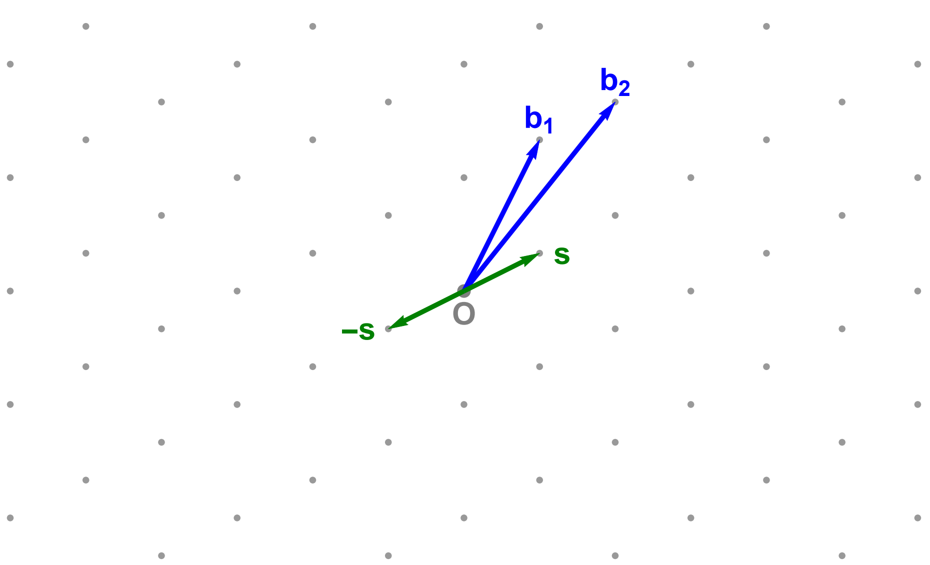

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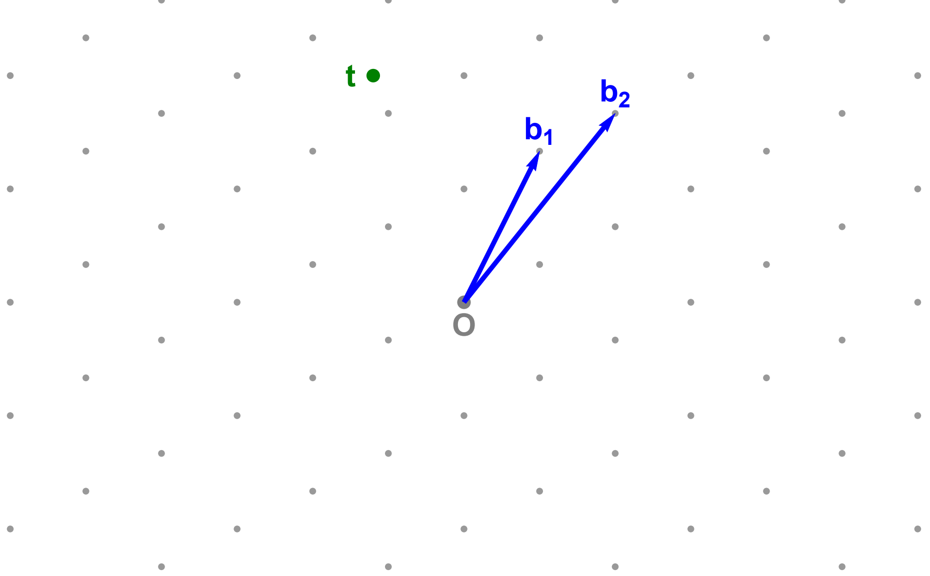

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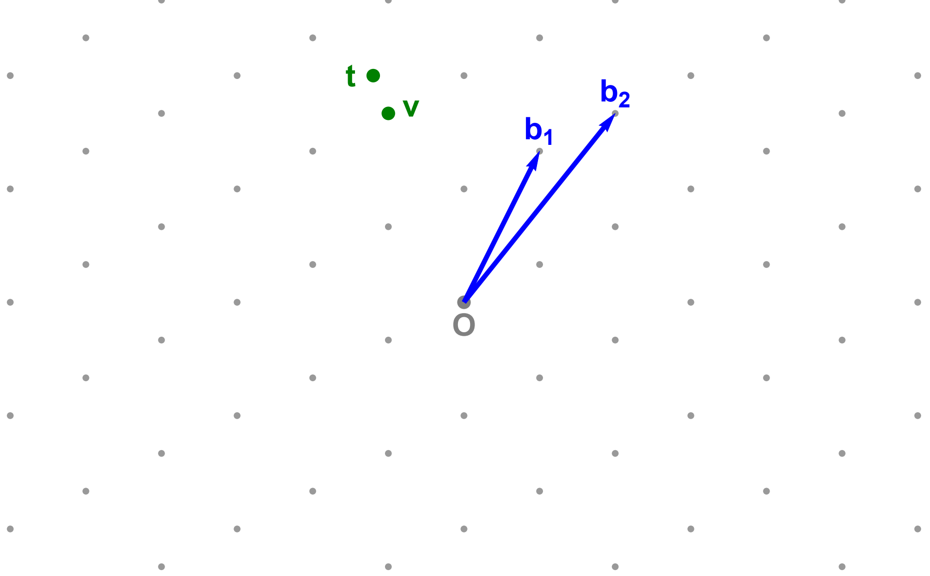
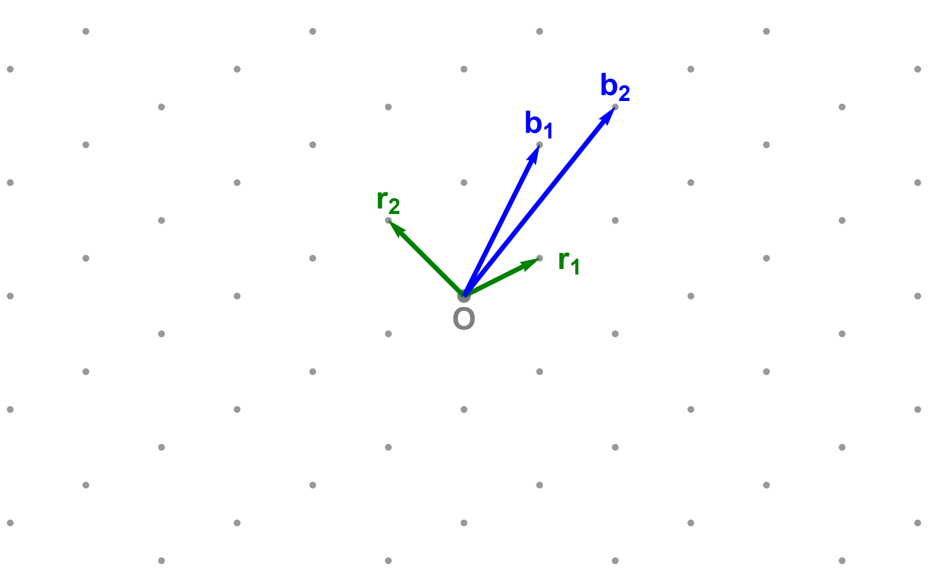


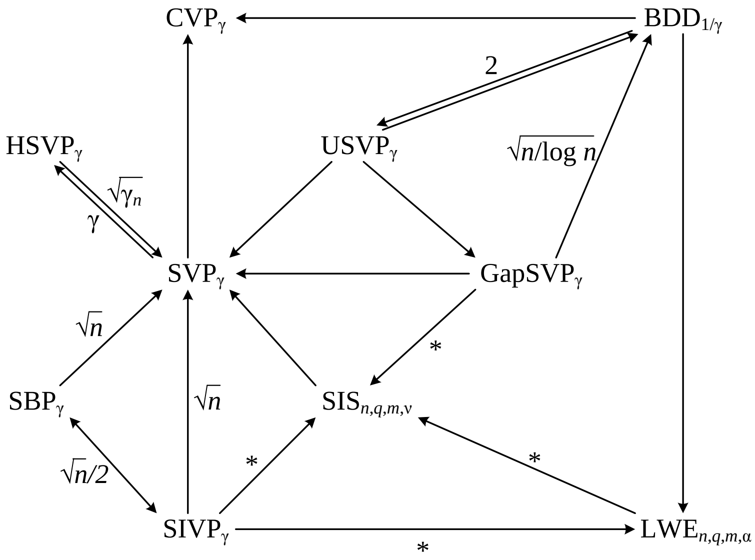
Lattices

Lattice basis reduction



Lattices

Hard lattice problems [LvdPdW12]



Lattices

Lattice-based cryptography

Problem: Security of lattice-based cryptographic primitives

- Lattice-based crypto relies on hardness of lattice problems
- Most lattice problems reducible to (approximate) SVP
- State-of-the-art: BKZ basis reduction [Sch87, SE94, ...]
  - BKZ uses exact SVP algorithm as subroutine
  - Complexity of BKZ dominated by *exact* SVP calls

SVP costs $\Longrightarrow$ BKZ costs $\Longrightarrow$ Security estimates $\Longrightarrow$ Parameters

Problem: How hard is SVP in high dimensions?

Outline

SVP algorithms

- Enumeration
- Sieving

SVP hardness

- Theory
- Practice
- NIST submissions

Conclusion

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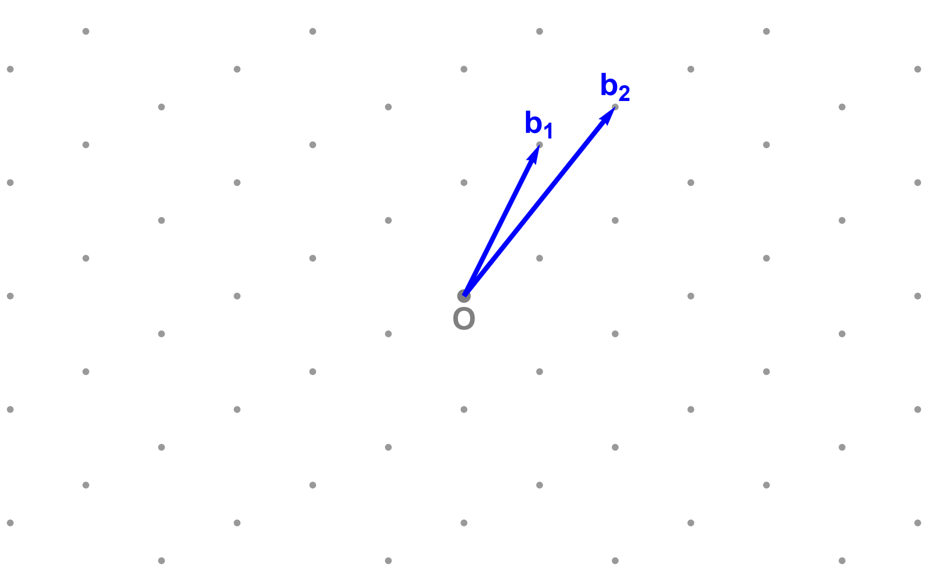
Enumeration

1. Determine possible coefficients of b_2



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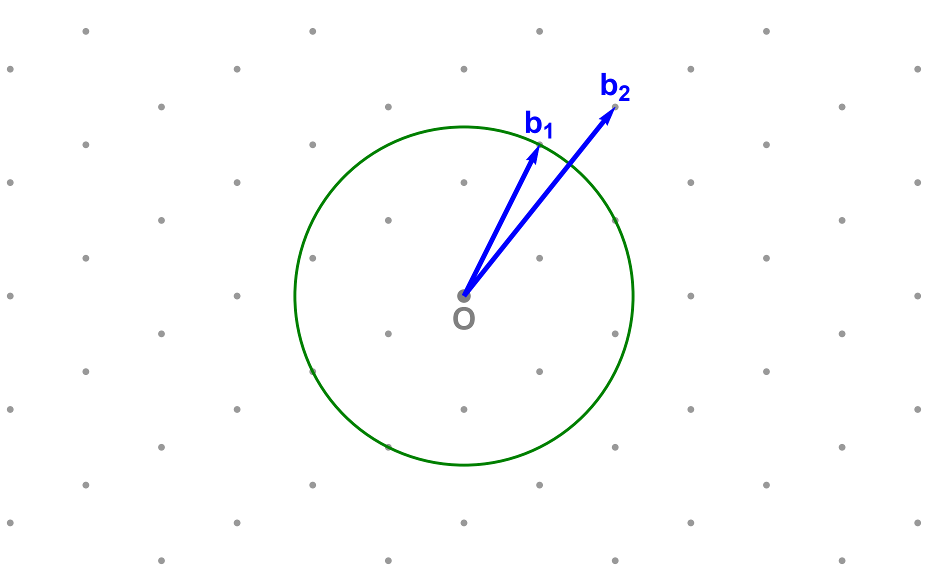

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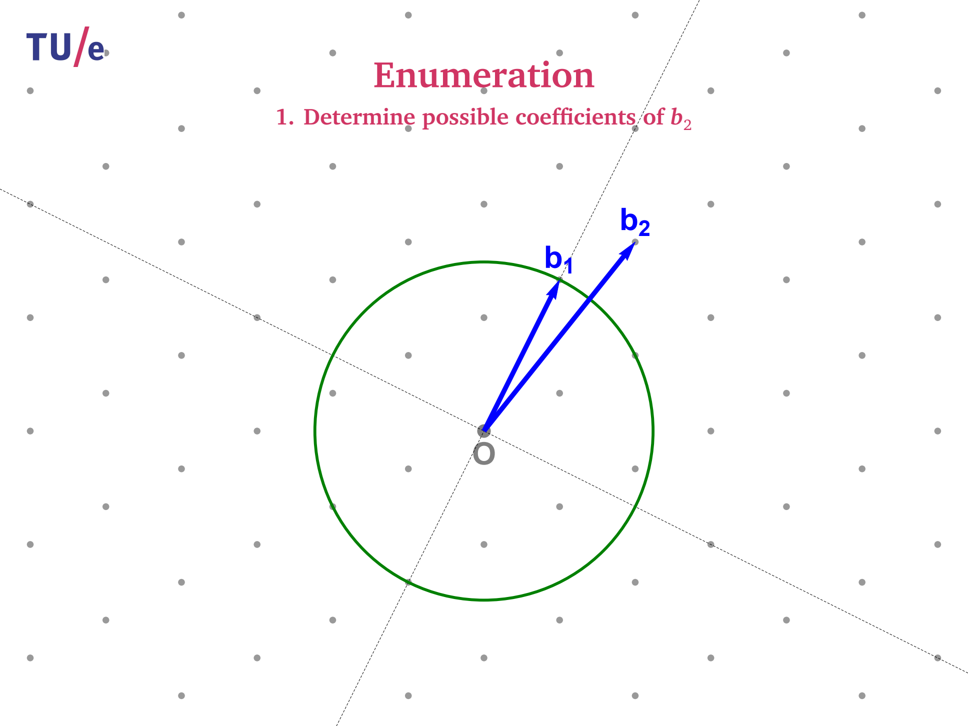

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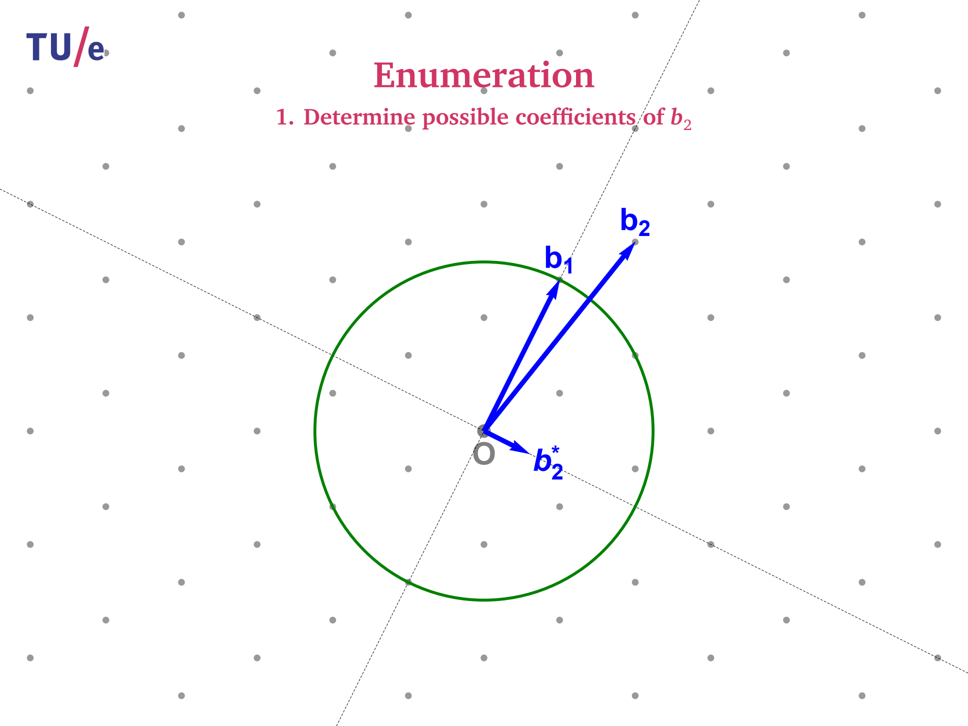

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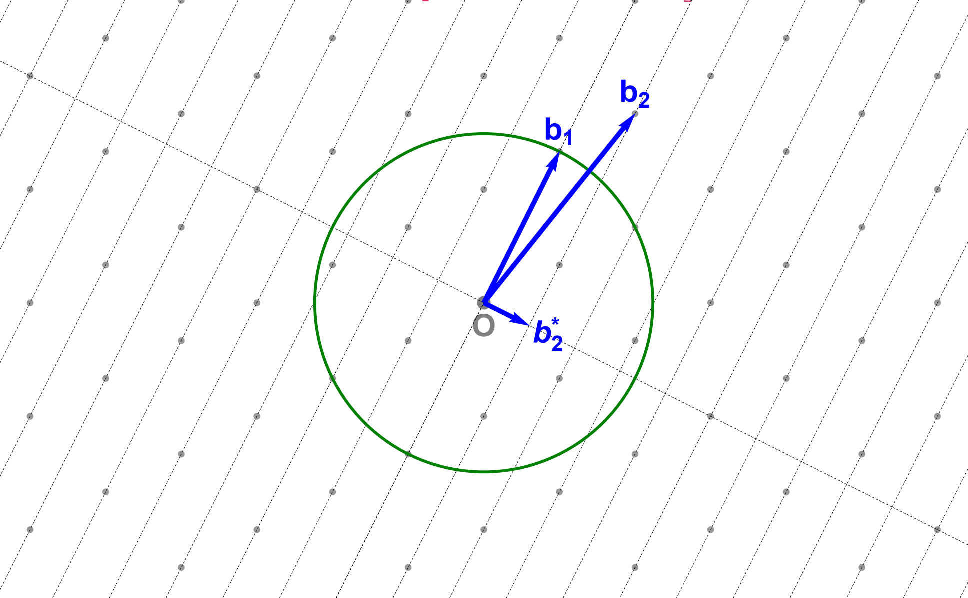

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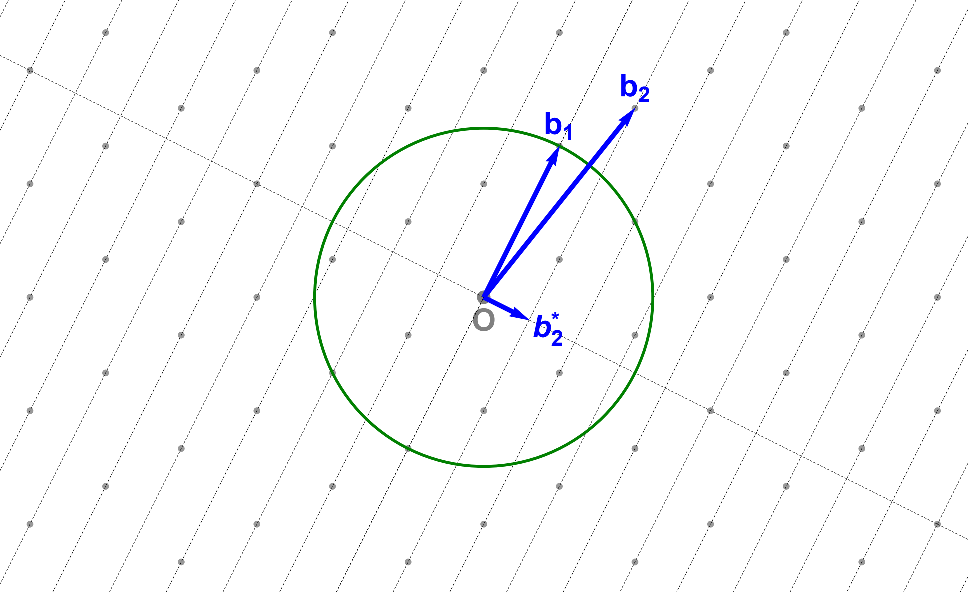
2. Find short vectors for each coefficient of b_2



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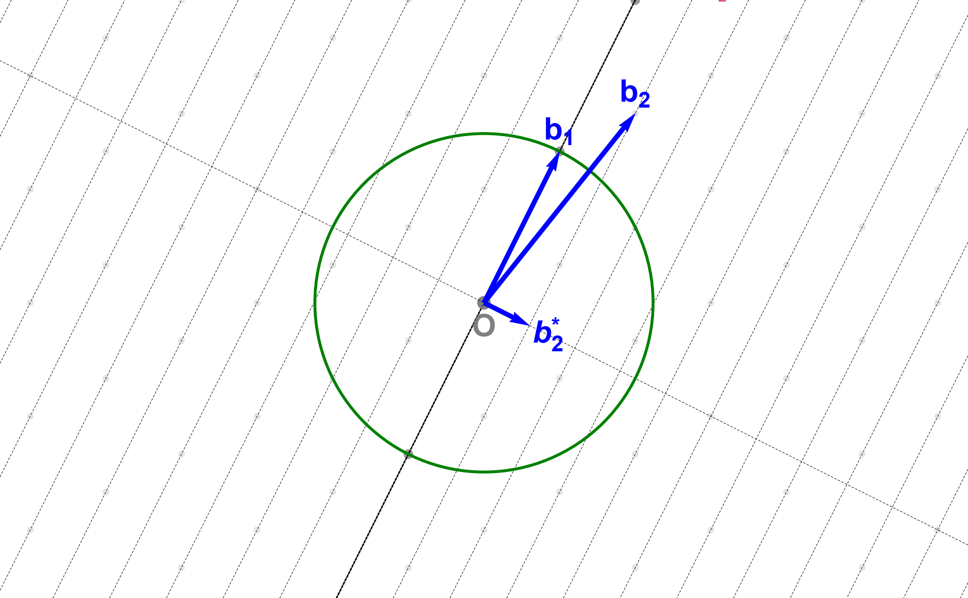

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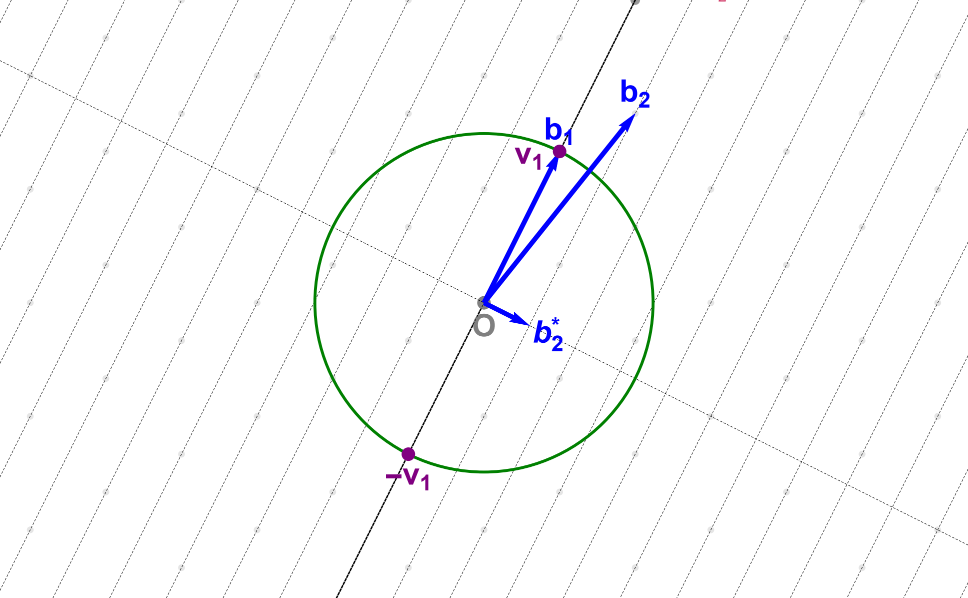

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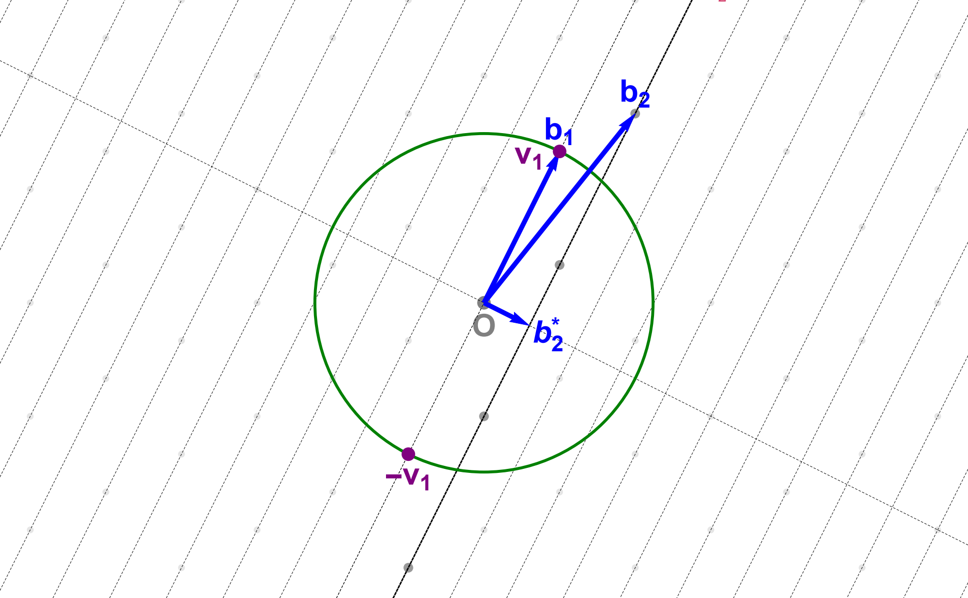

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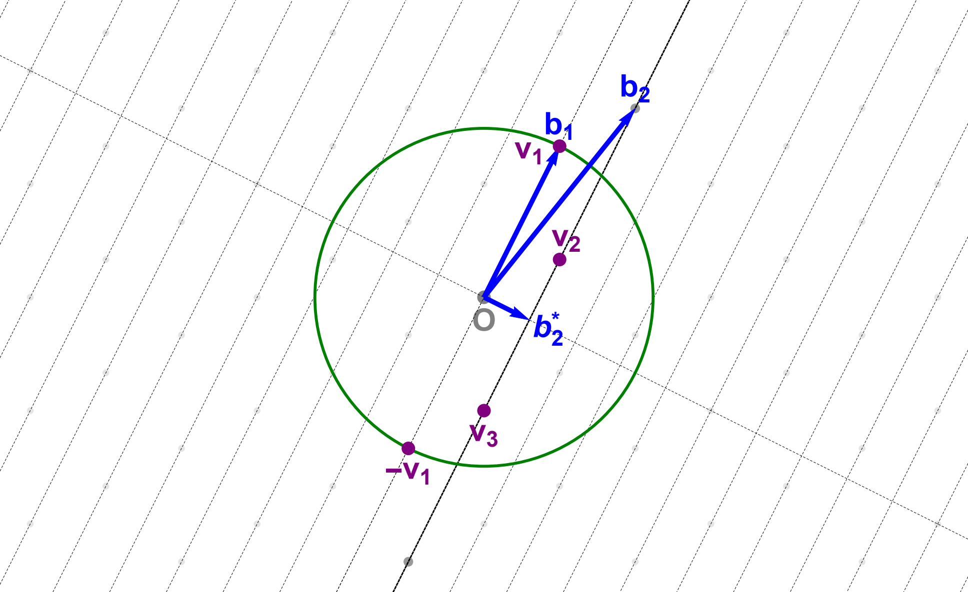

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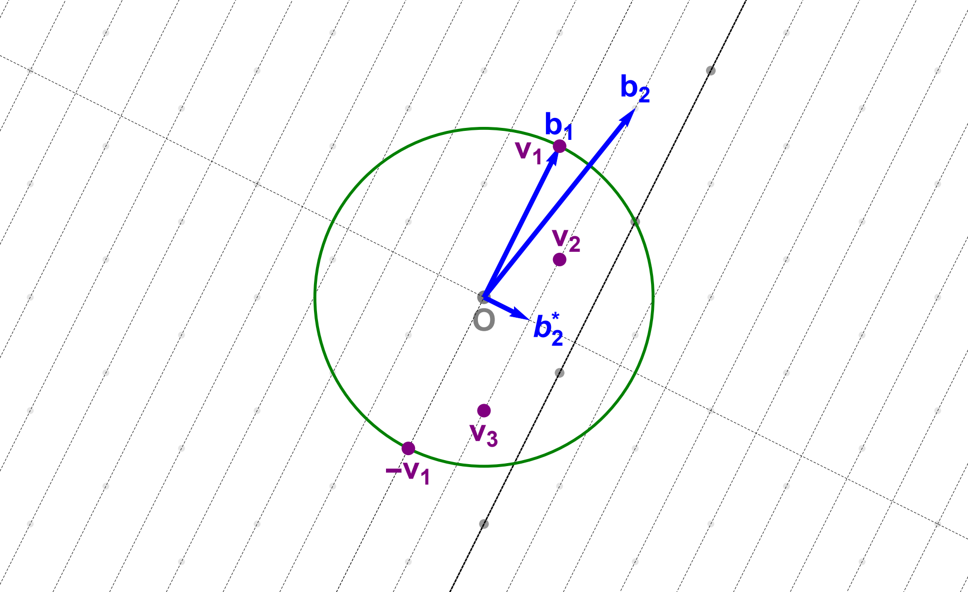

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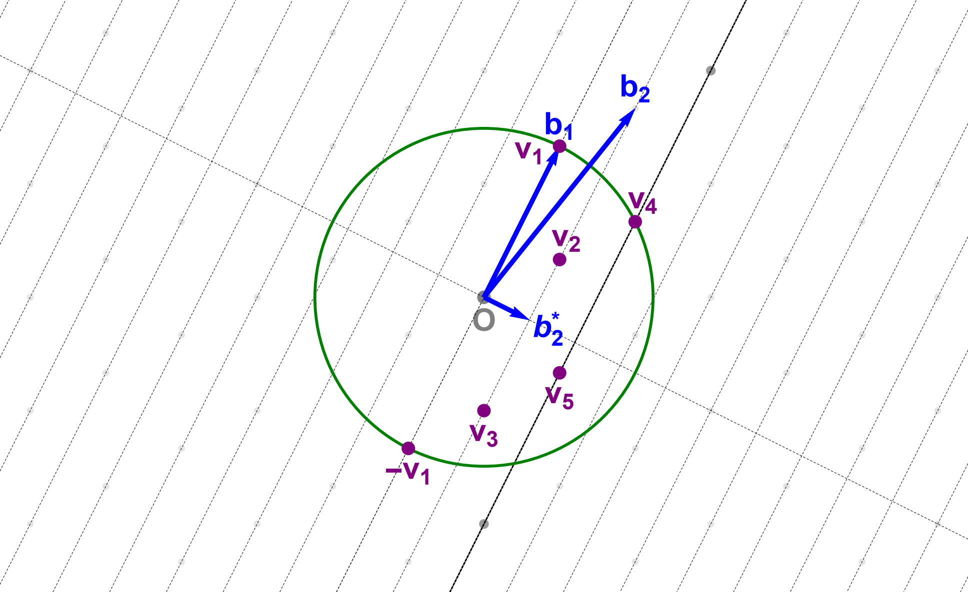

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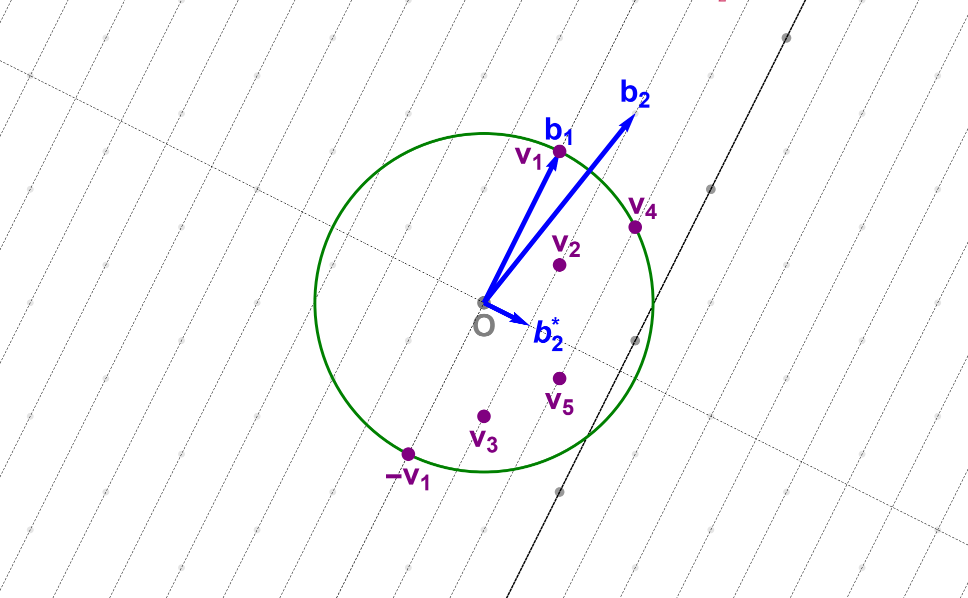

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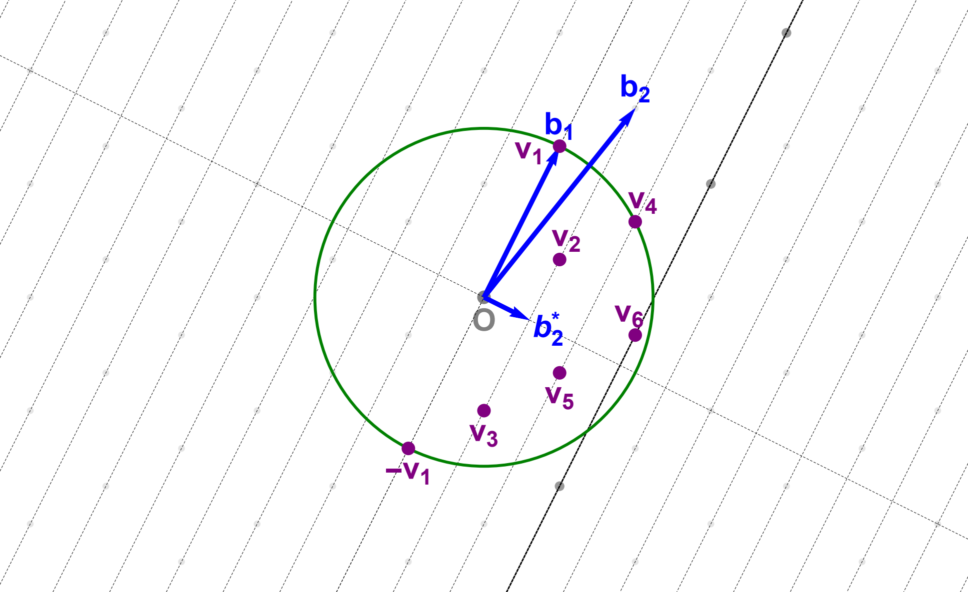

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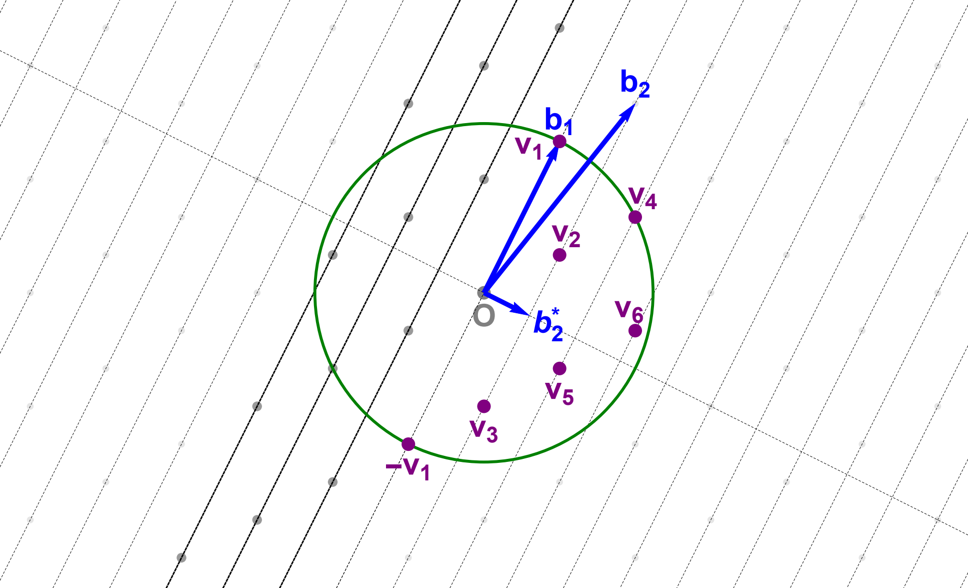

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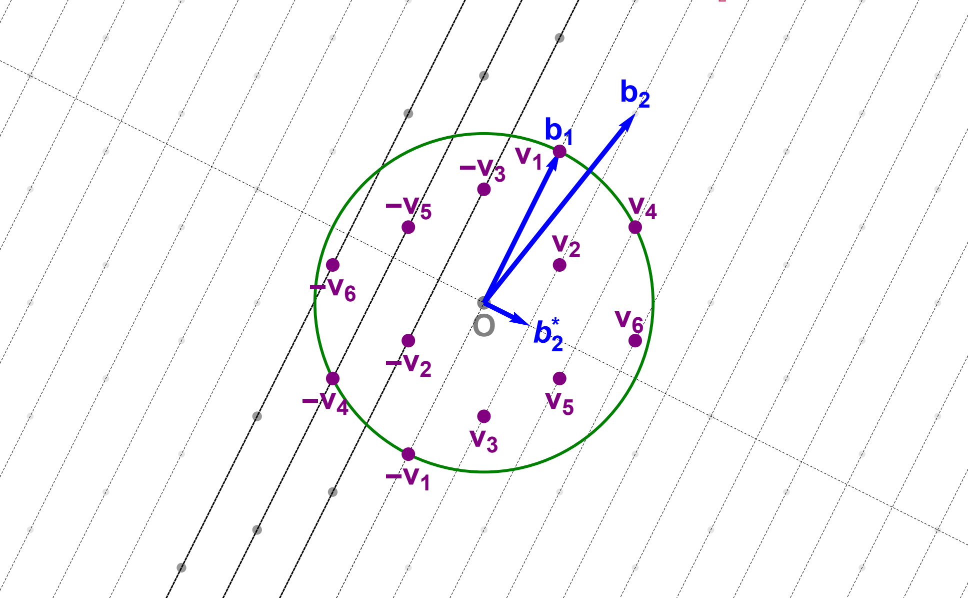

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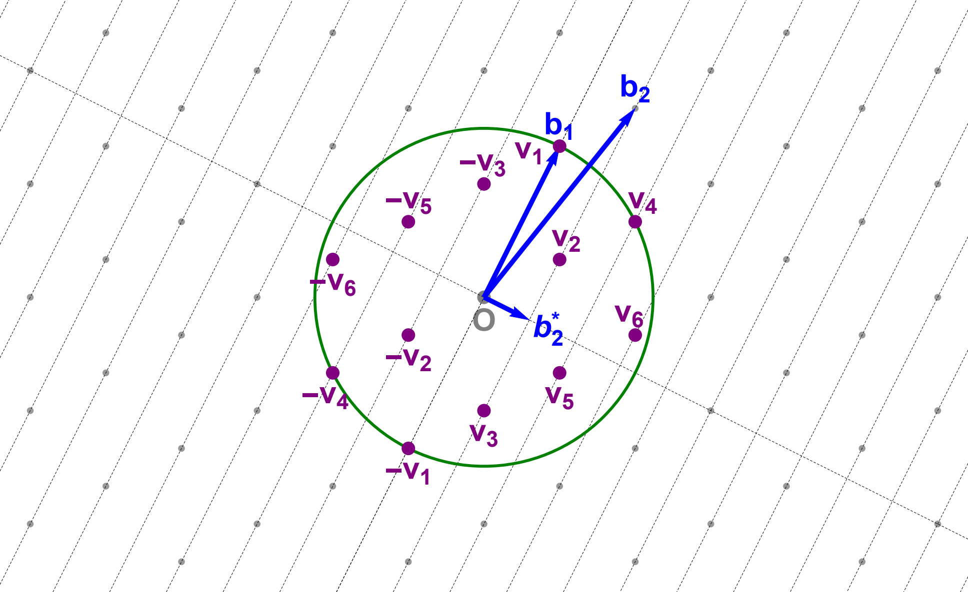

Enumeration

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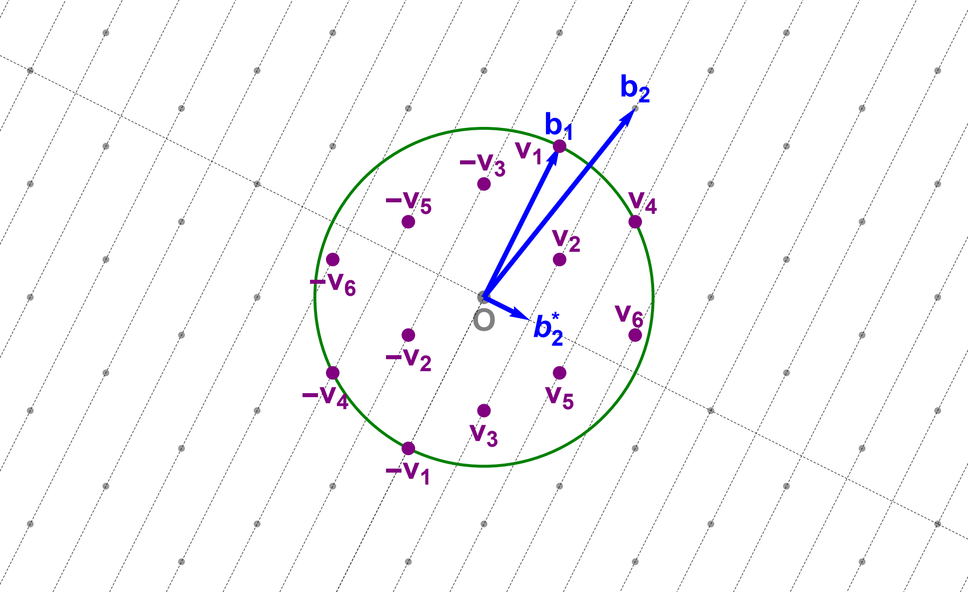

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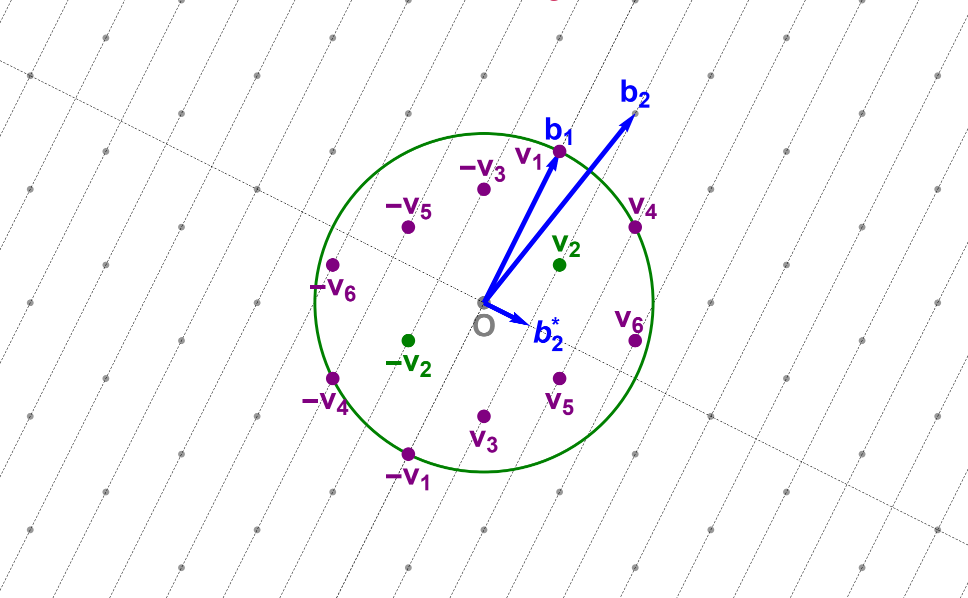

Enumeration

Overview

Enumeration

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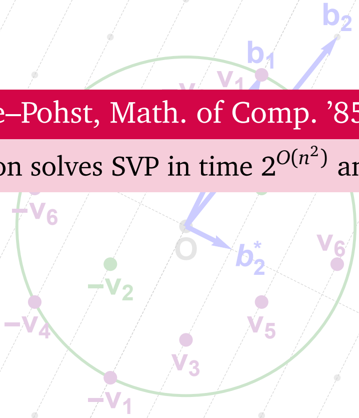

Theorem (Fincke–Pohst, Math. of Comp. ’85)

Lattice enumeration solves SVP in time $2^{O(n^2)}$ and space poly(n).

Enumeration

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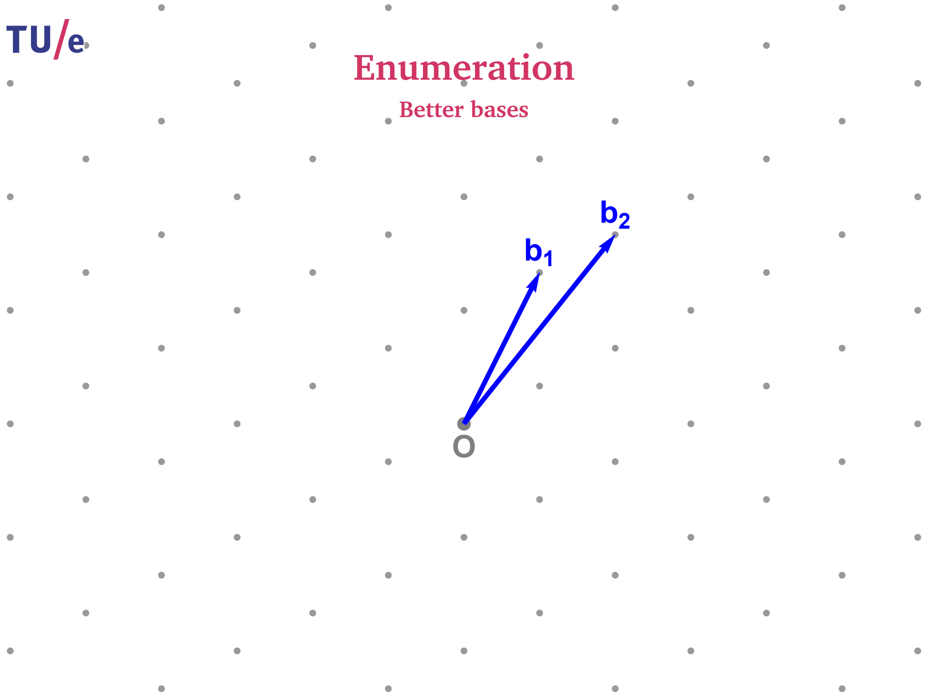
Lattice enumeration solves SVP in time $2^{O(n^2)}$ and space poly(n).

Essentially reduces SVP_n (CVP_n) to $2^{O(n)}$ instances of CVP_{n-1}.

Enumeration

Better bases



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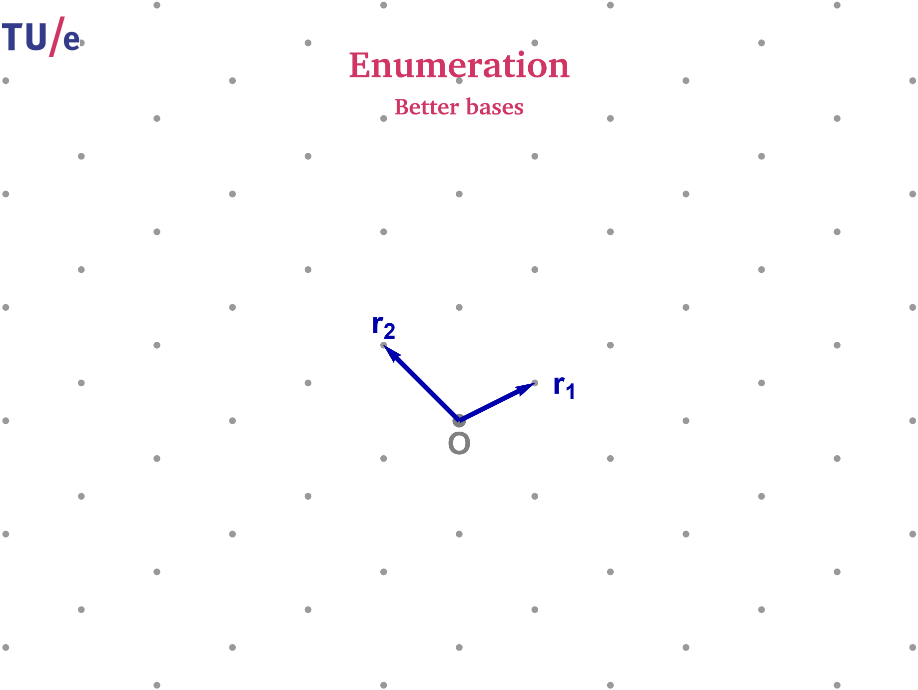

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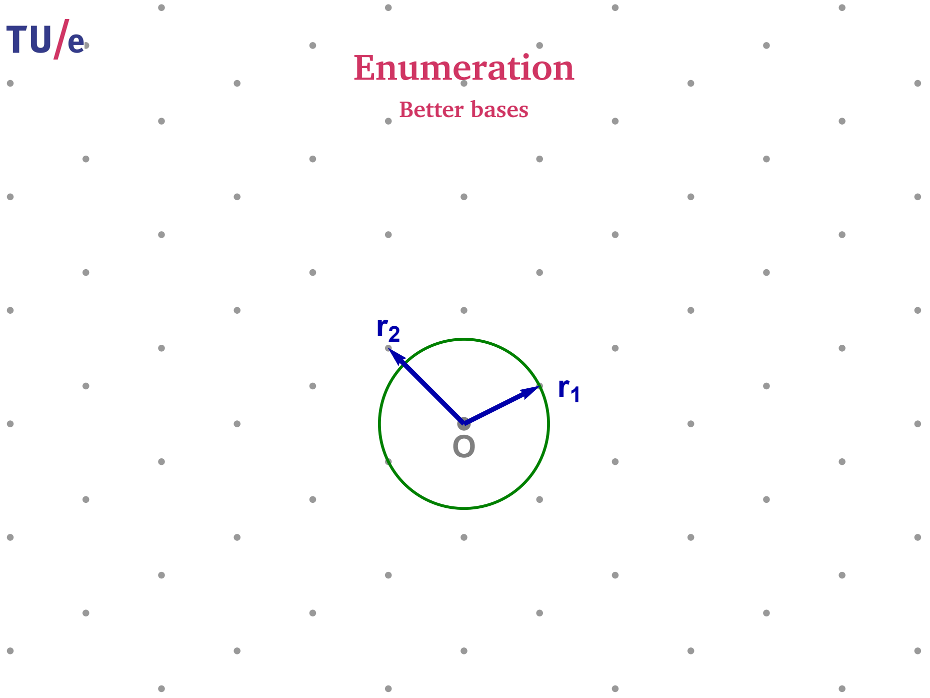

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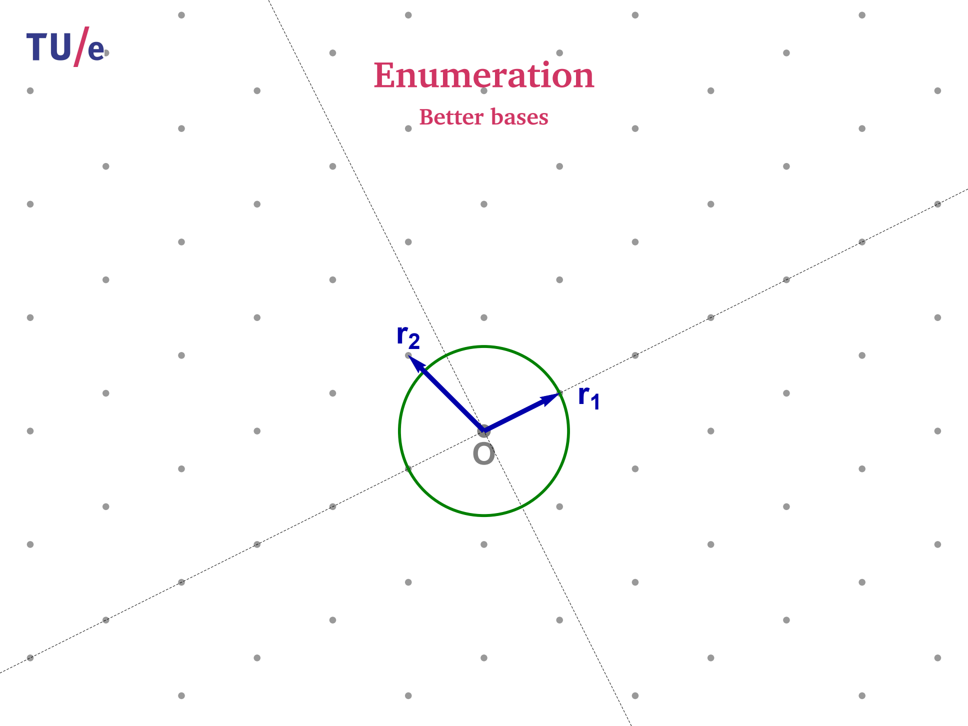

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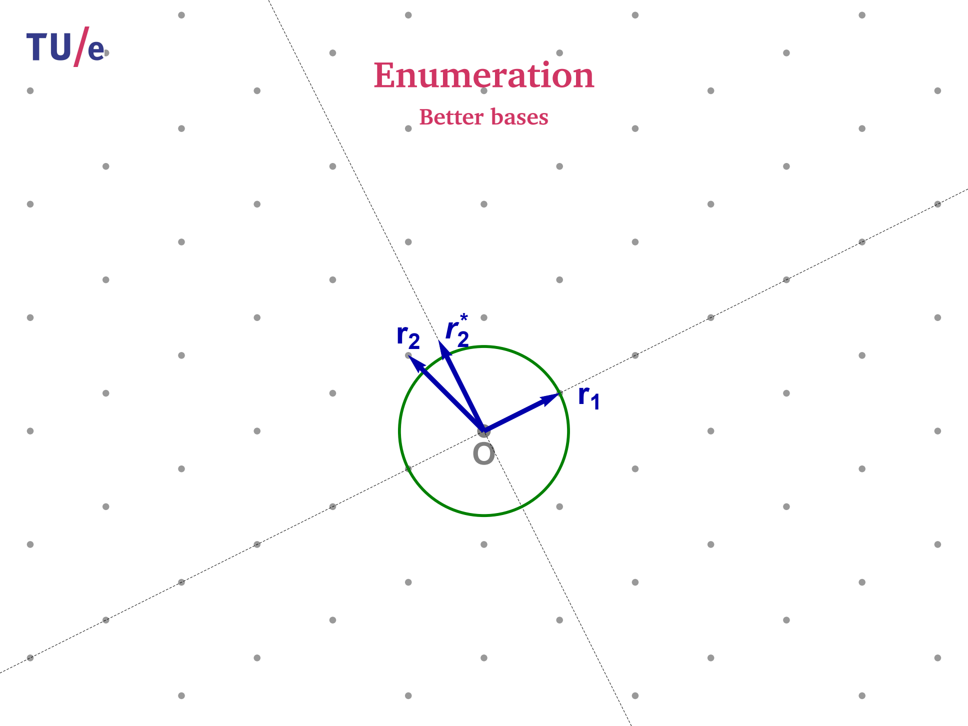

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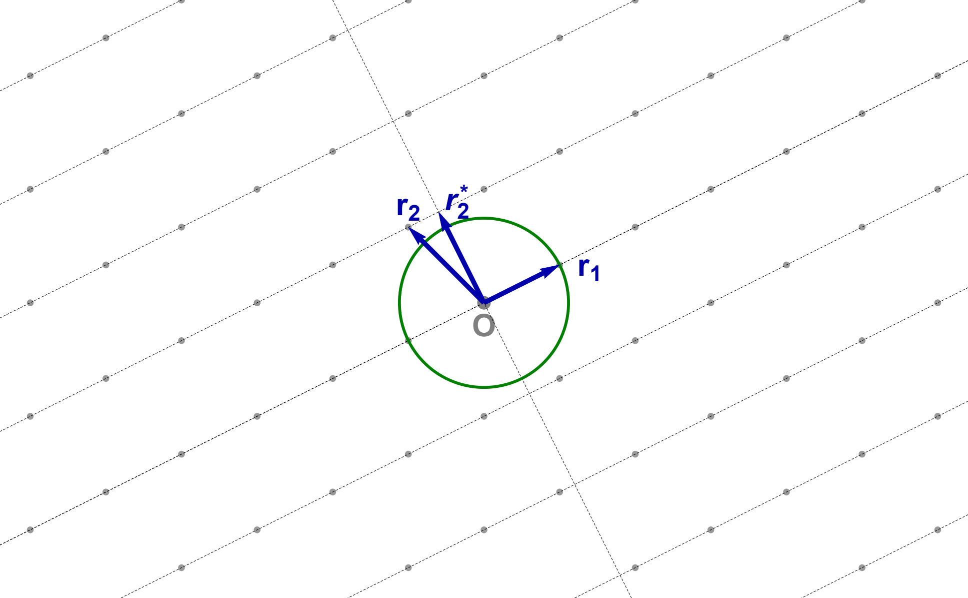

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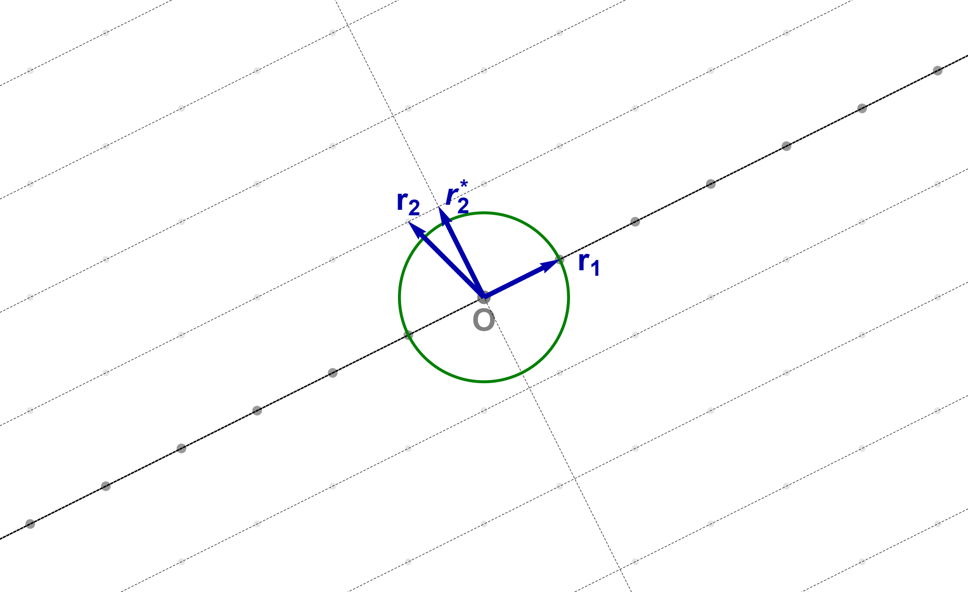

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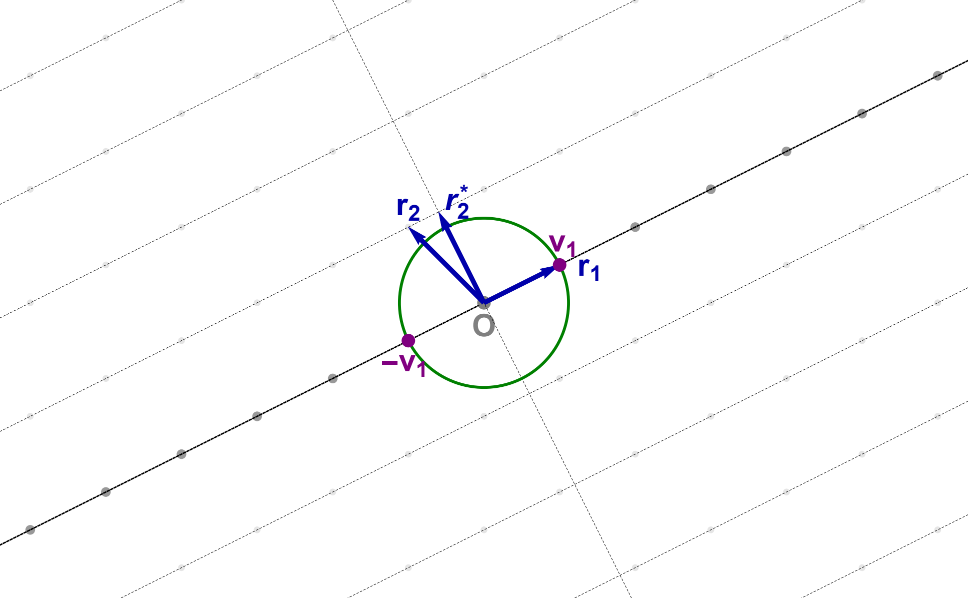

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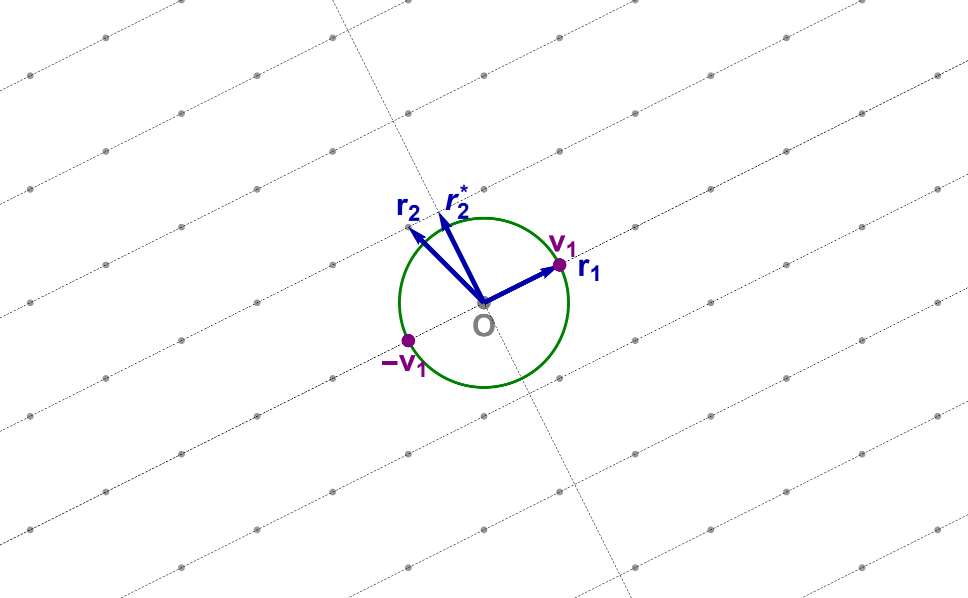

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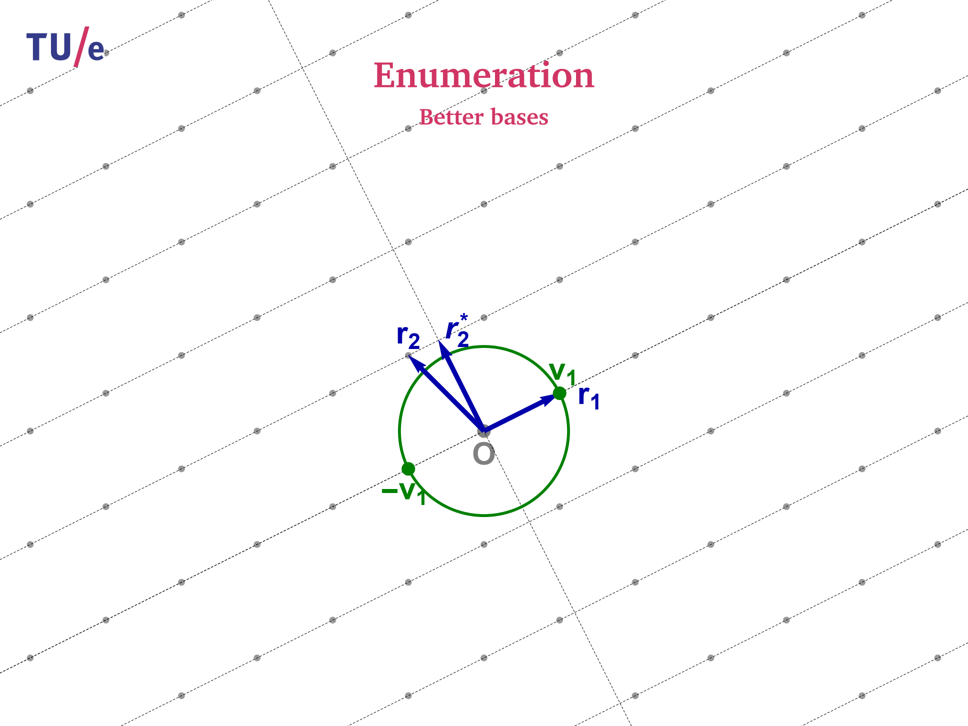

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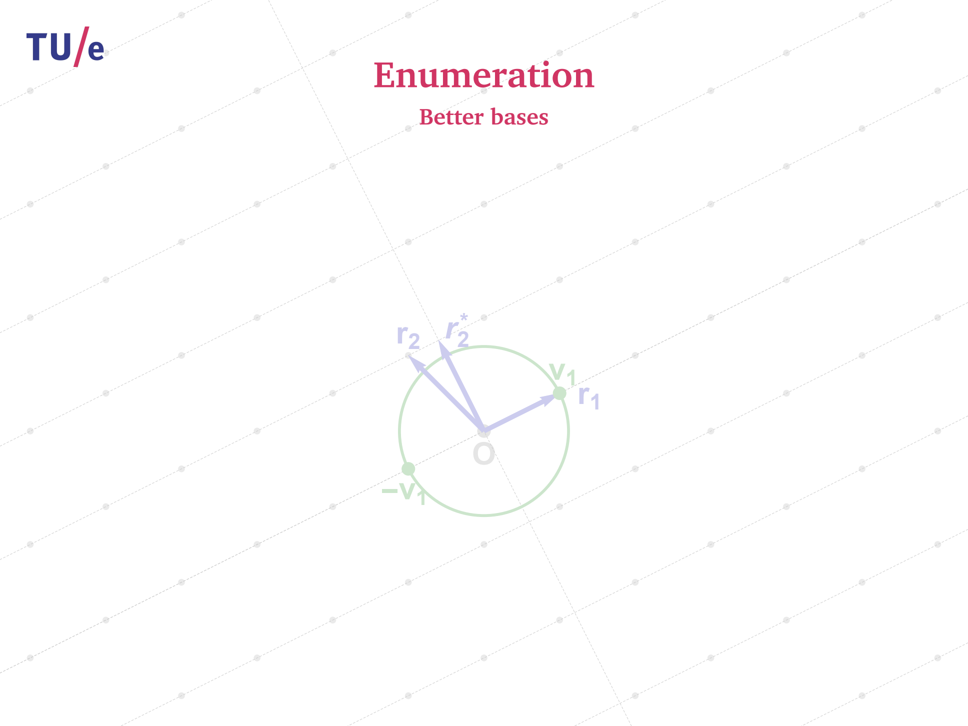

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Theorem (Kannan, STOC'83)

Combining enumeration with stronger basis reduction, one can solve SVP in time $2^{O(n \log n)}$ and space $\mathrm{poly}(n)$.

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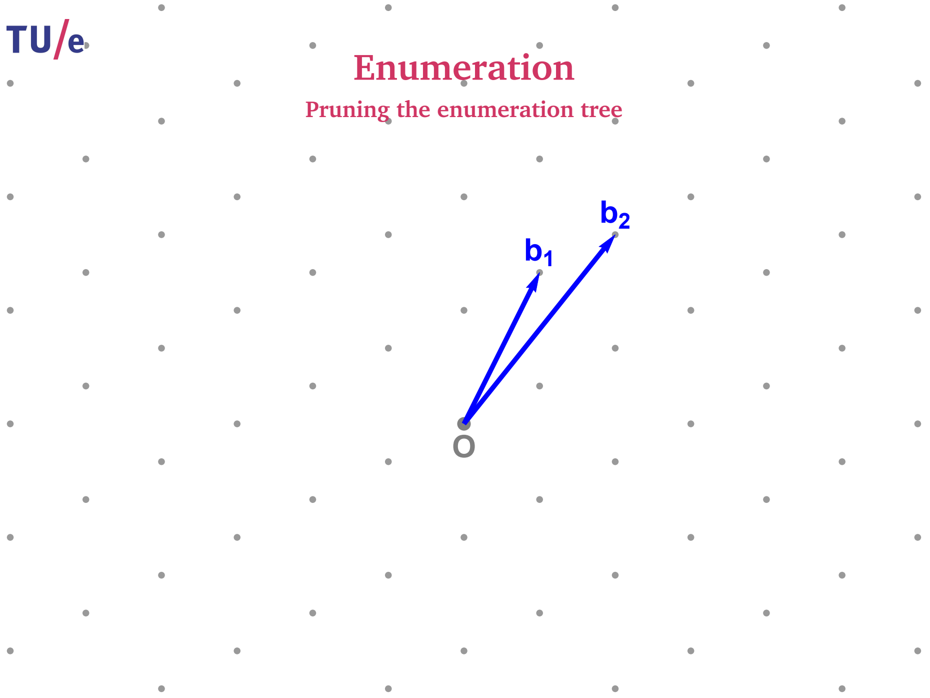
> *"Our algorithm reduces an n-dimensional problem to polynomially many (instead of $2^{O(n)}$) $(n-1)$-dimensional problems. [...] The algorithm we propose, first finds a more orthogonal basis for a lattice in time $2^{O(n \log n)}$."*
>
> – Kannan, *STOC'83*

Enumeration

Pruning the enumeration tree



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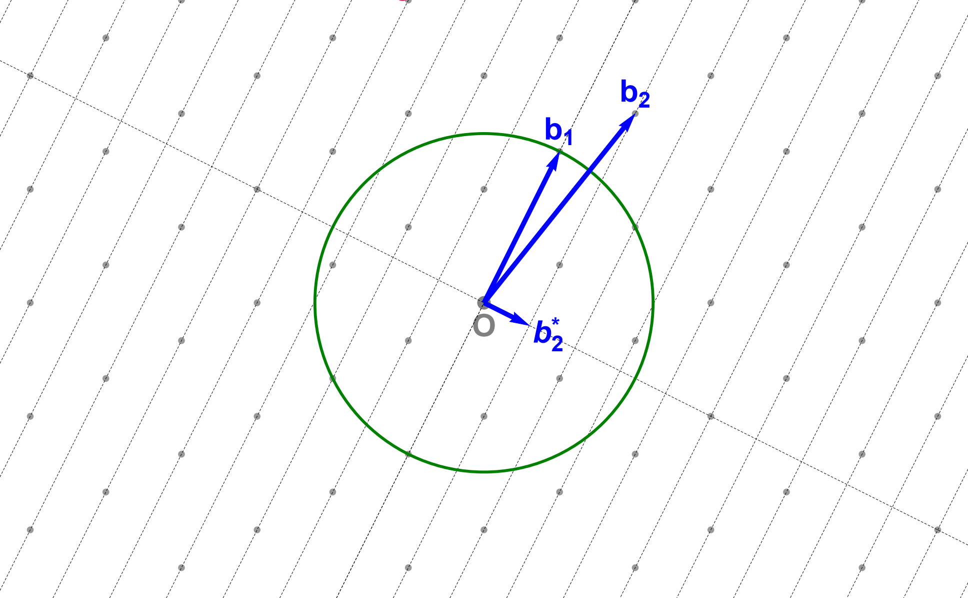

TU/e

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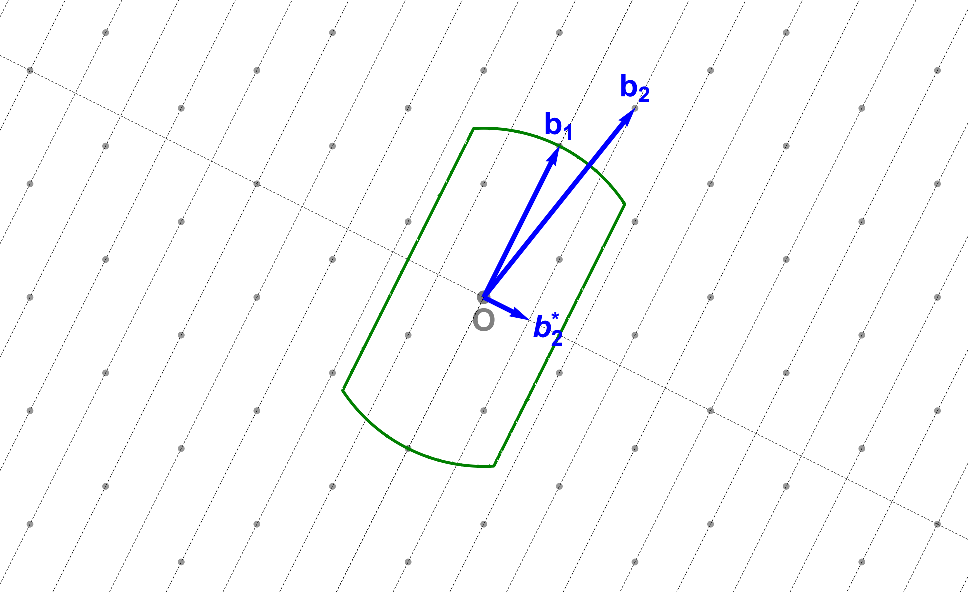

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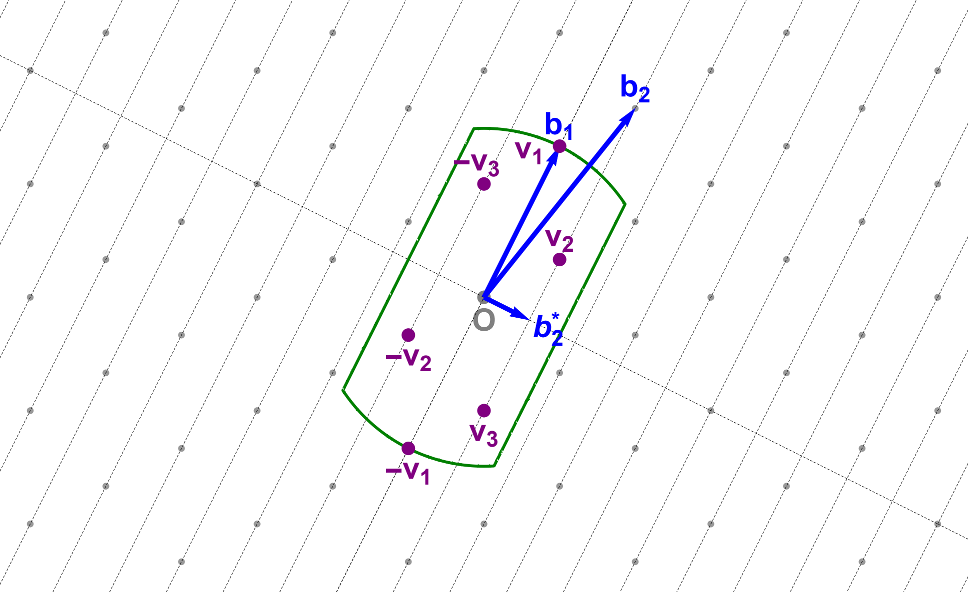

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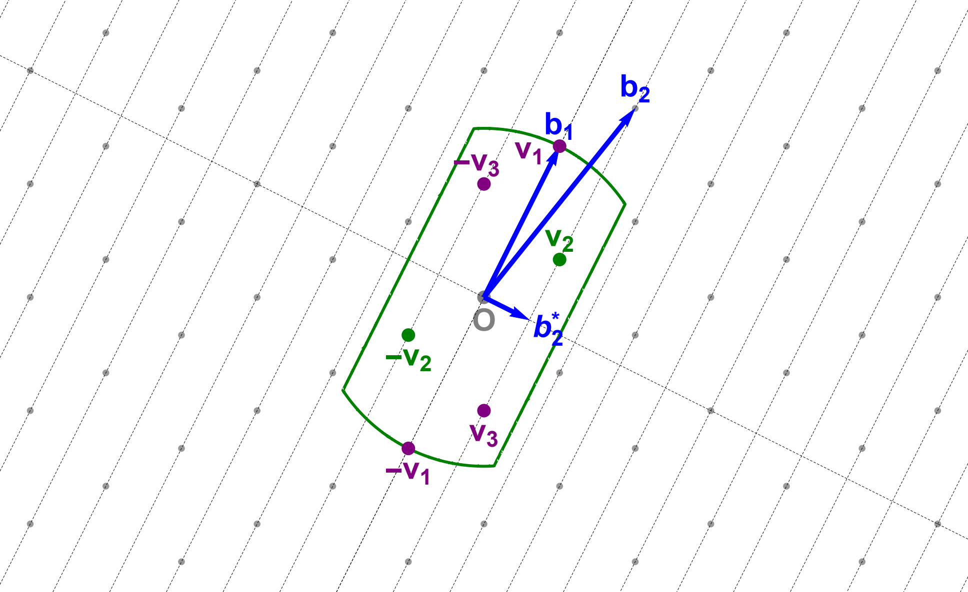

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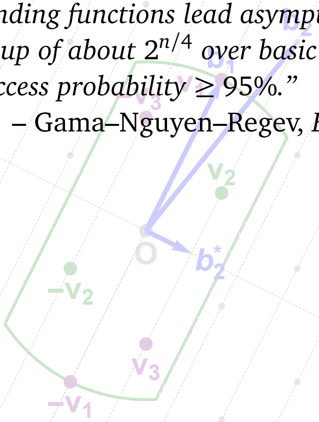
"Well-chosen bounding functions lead asymptotically to an exponential speedup of about $2^{n/4}$ *over basic enumeration, maintaining a success probability* $\geq 95\%$*."*

– Gama–Nguyen–Regev, *EUROCRYPT'10*



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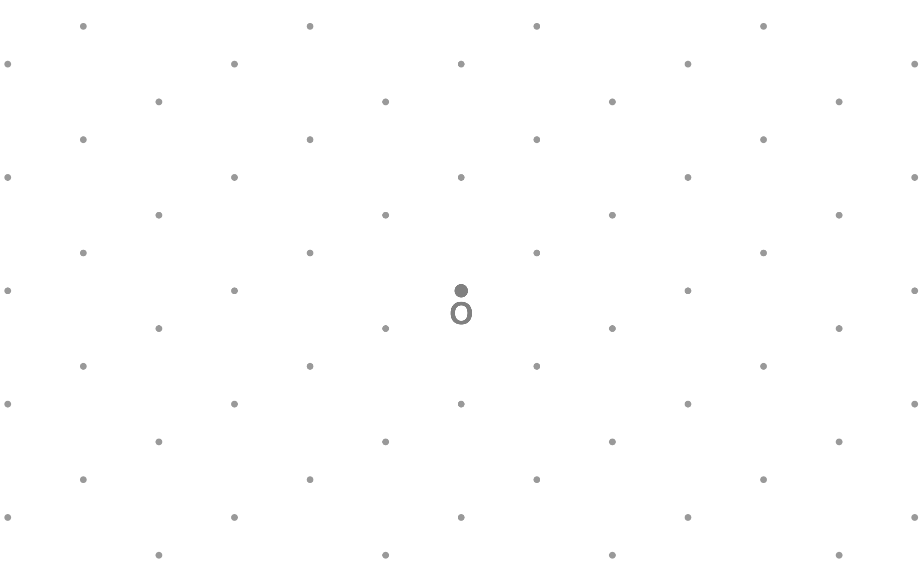
"With extreme pruning, the probability of finding the desired vector is actually rather low (say, 0.1%), but surprisingly, the running time of the enumeration is reduced by a much more significant factor (say, much more than 1000)."

– Gama–Nguyen–Regev, *EUROCRYPT'10*

Sieving

1. Sample a list L of random lattice vectors



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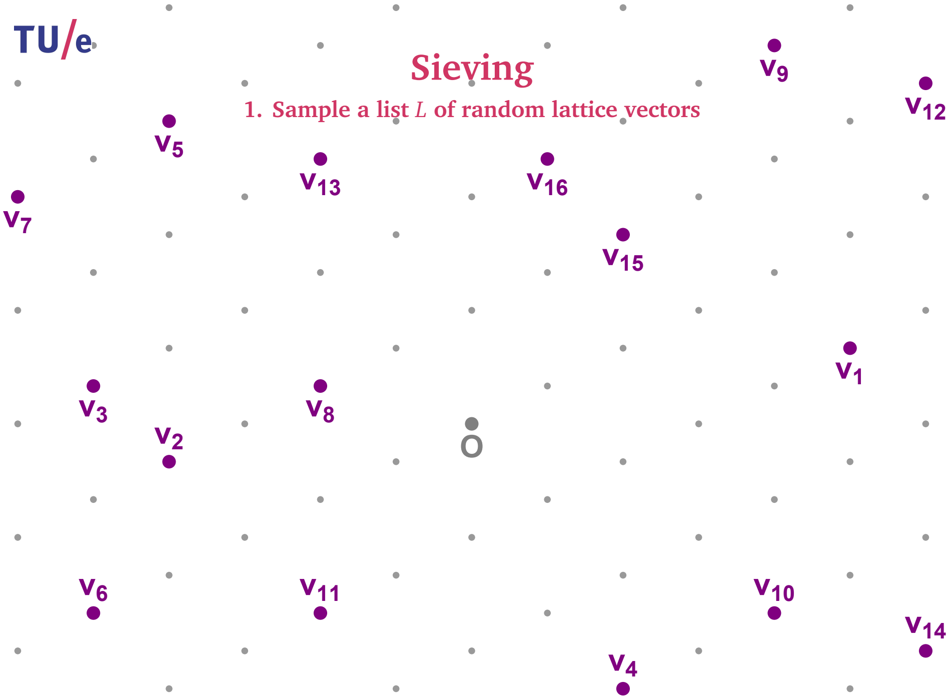
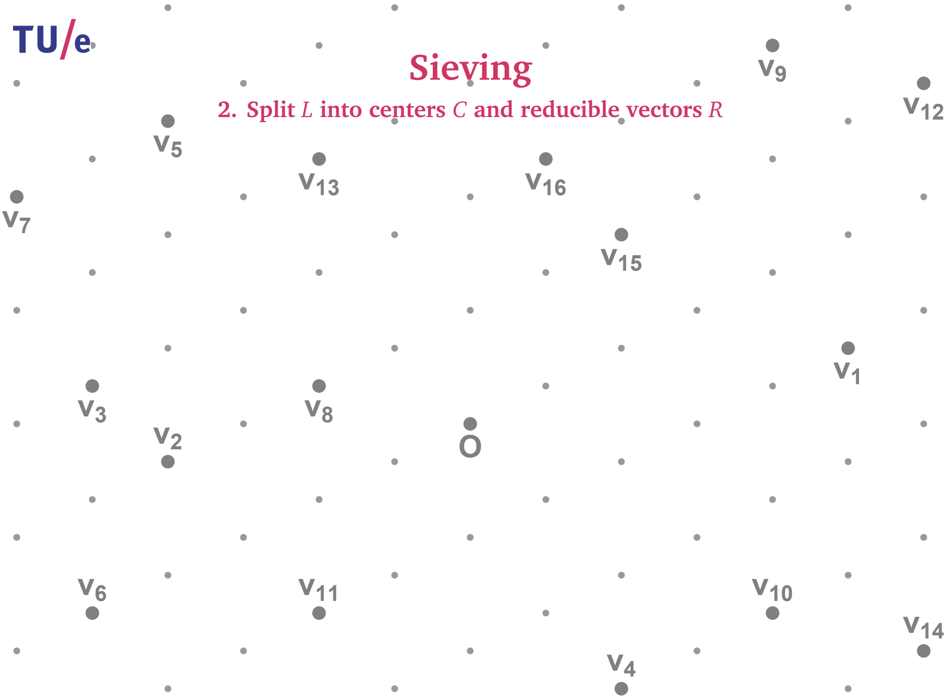


Sieving

2. Split L into centers C and reducible vectors R



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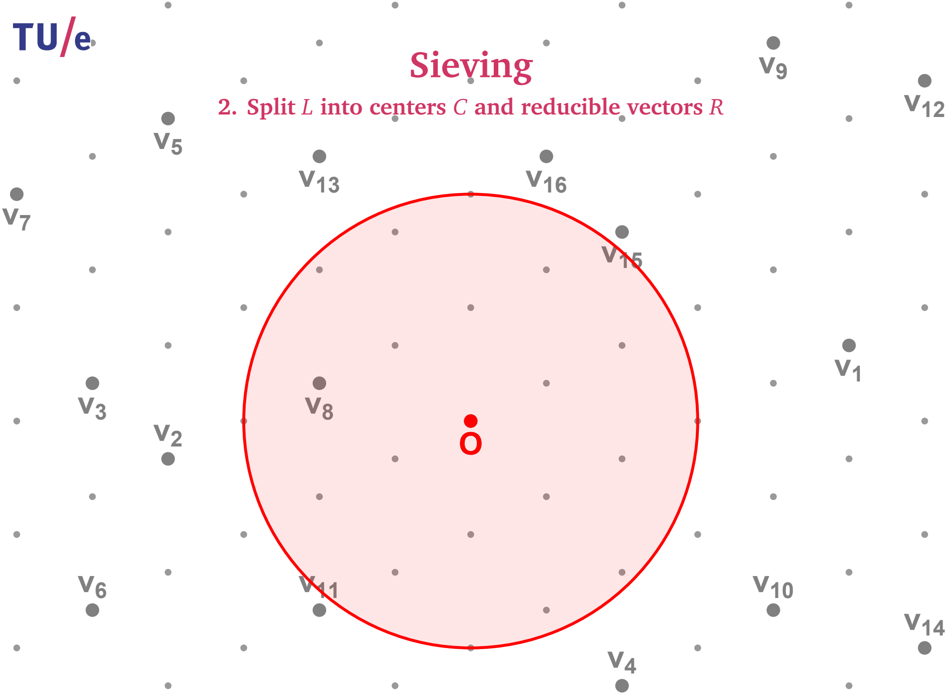

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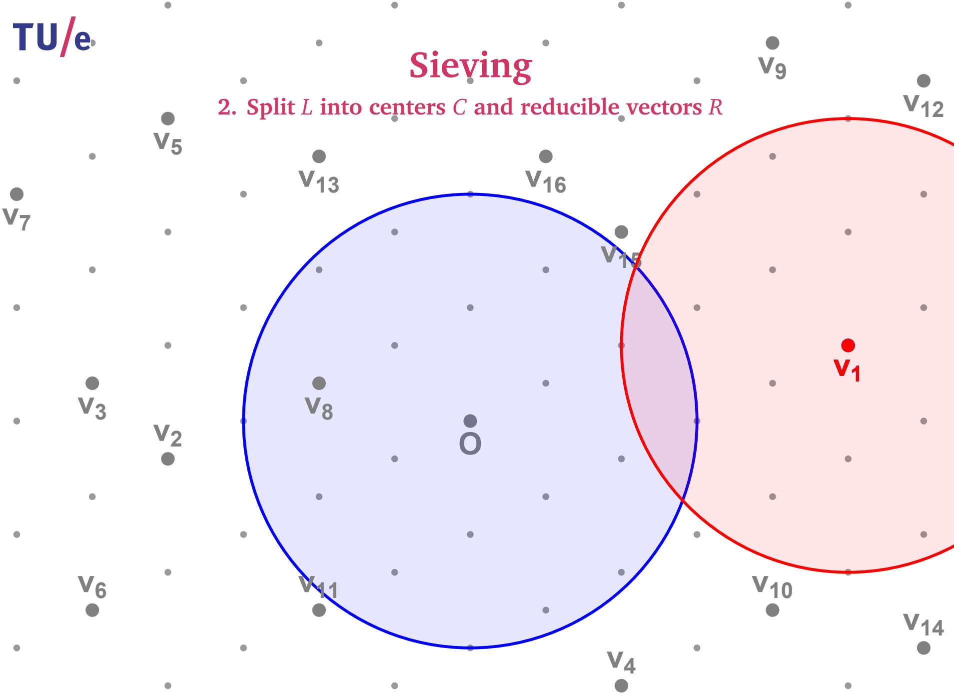

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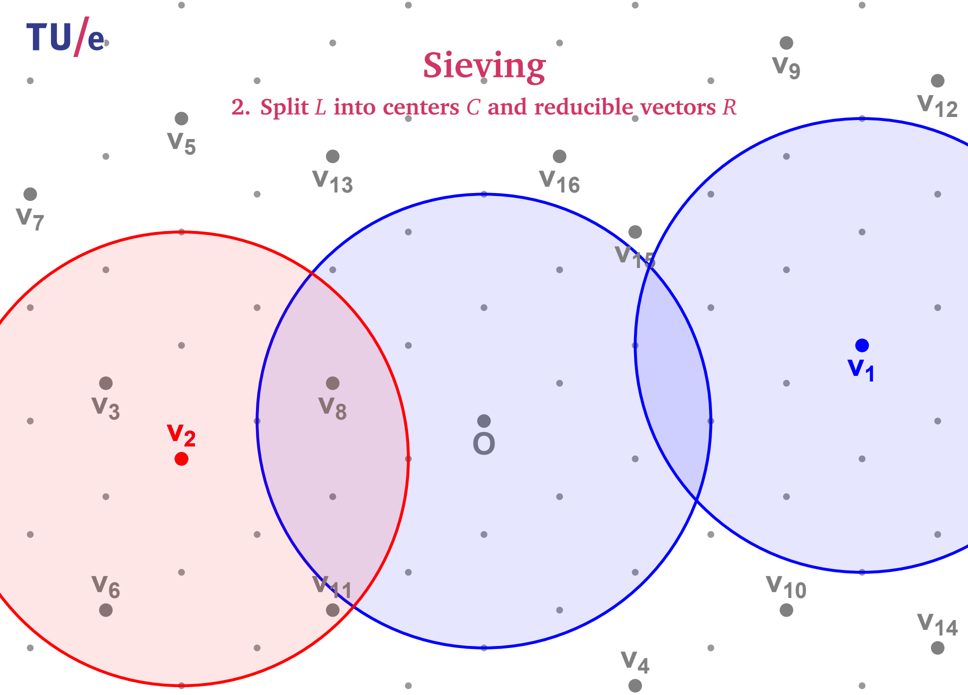

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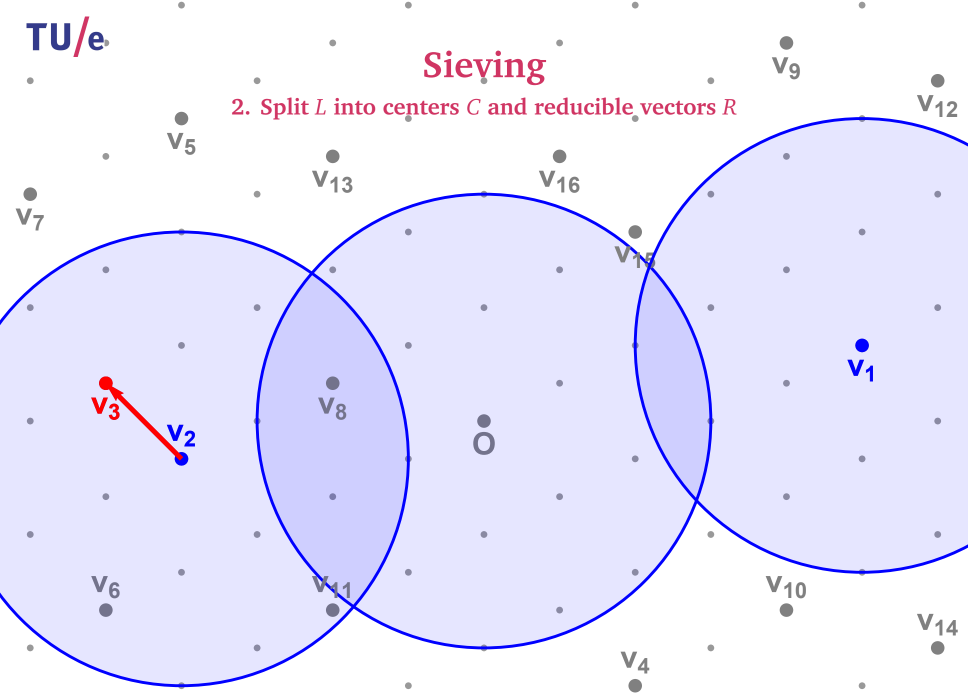

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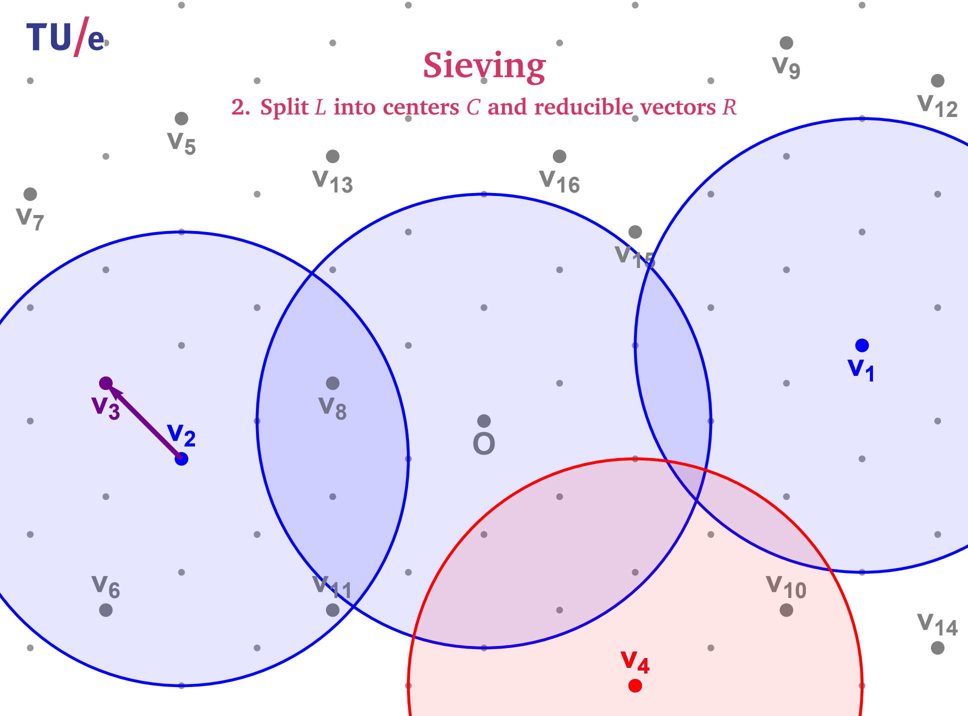

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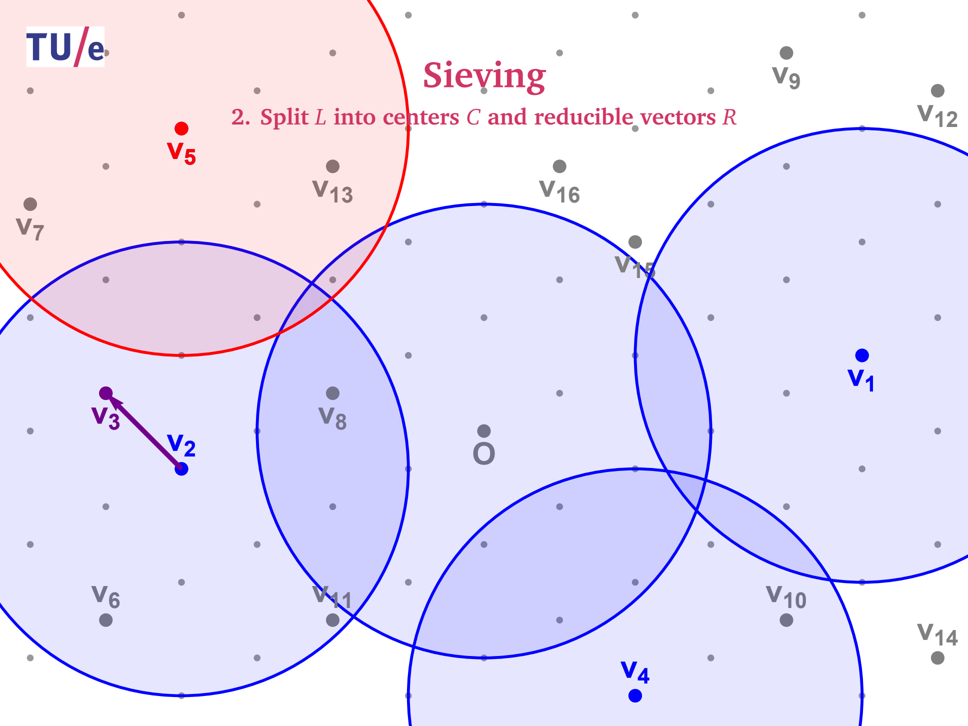

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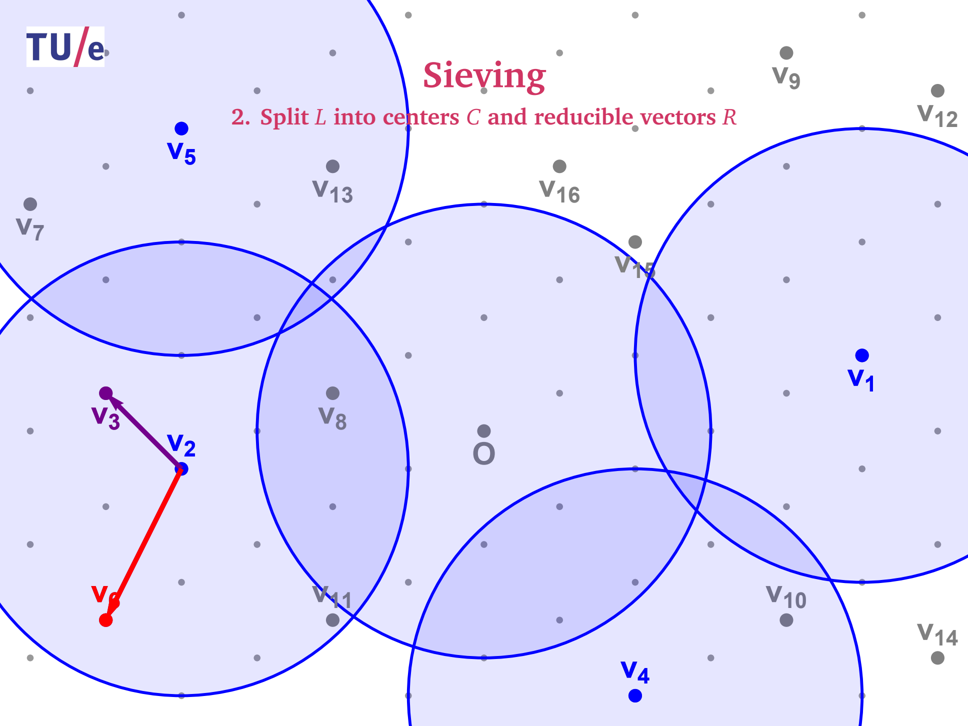

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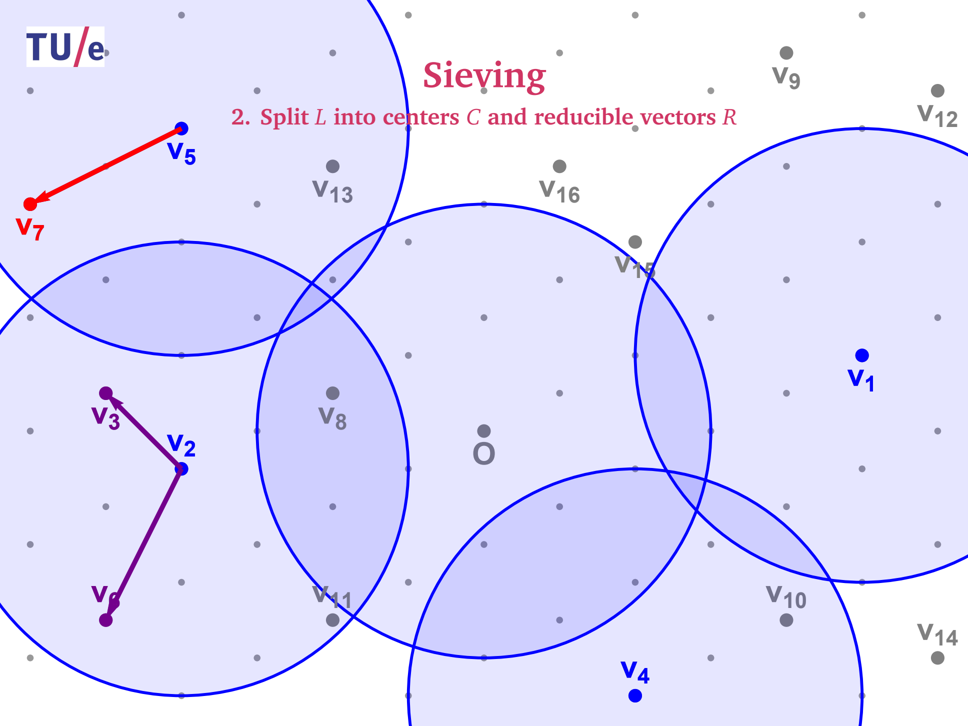

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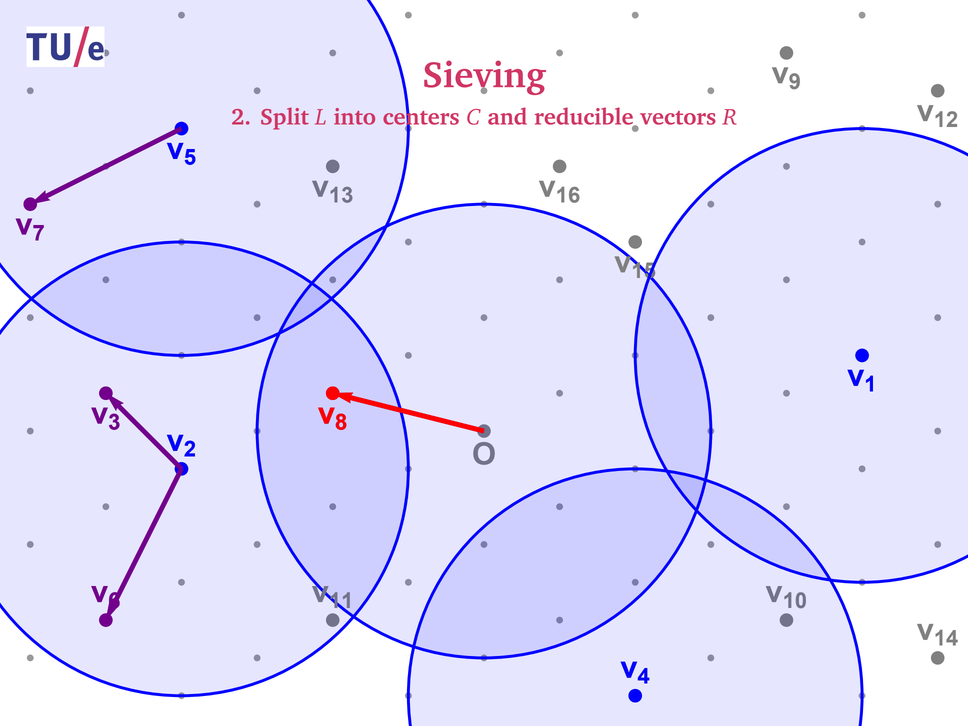

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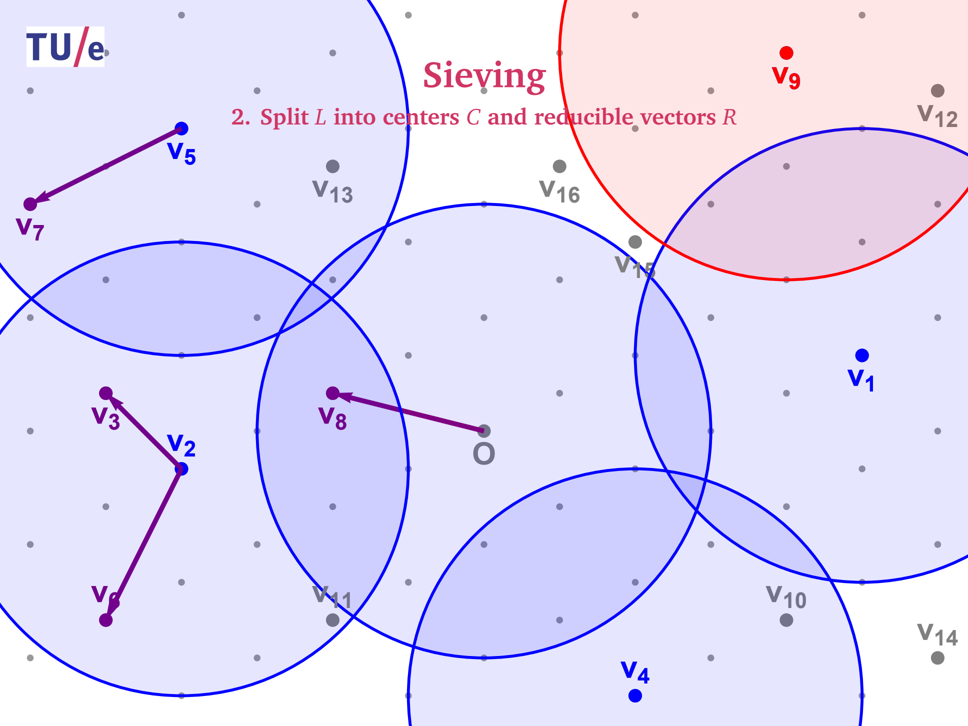

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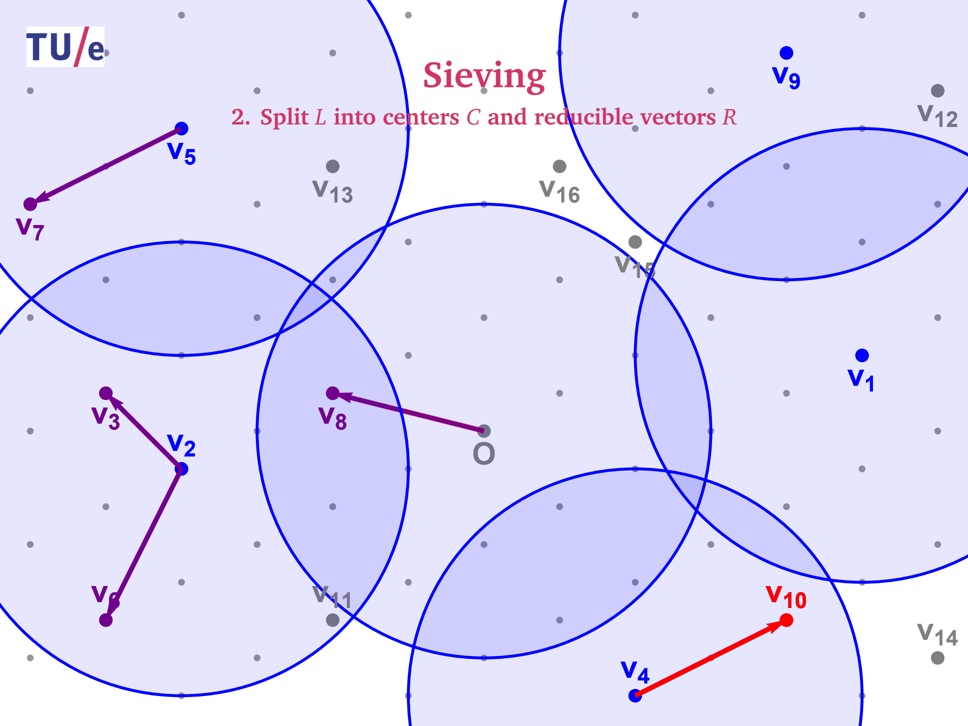

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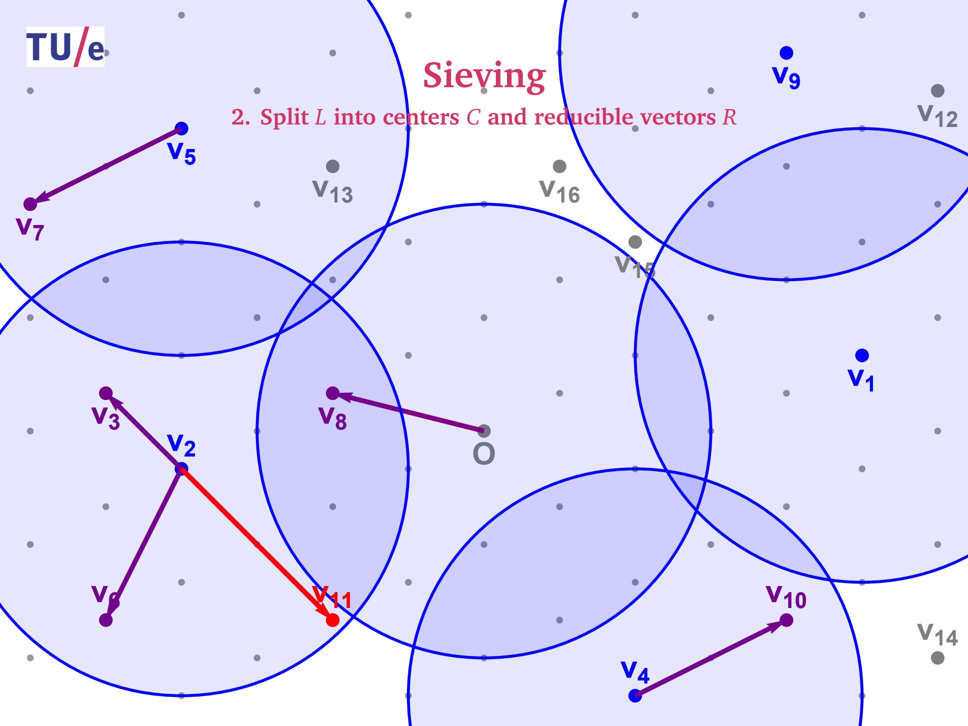

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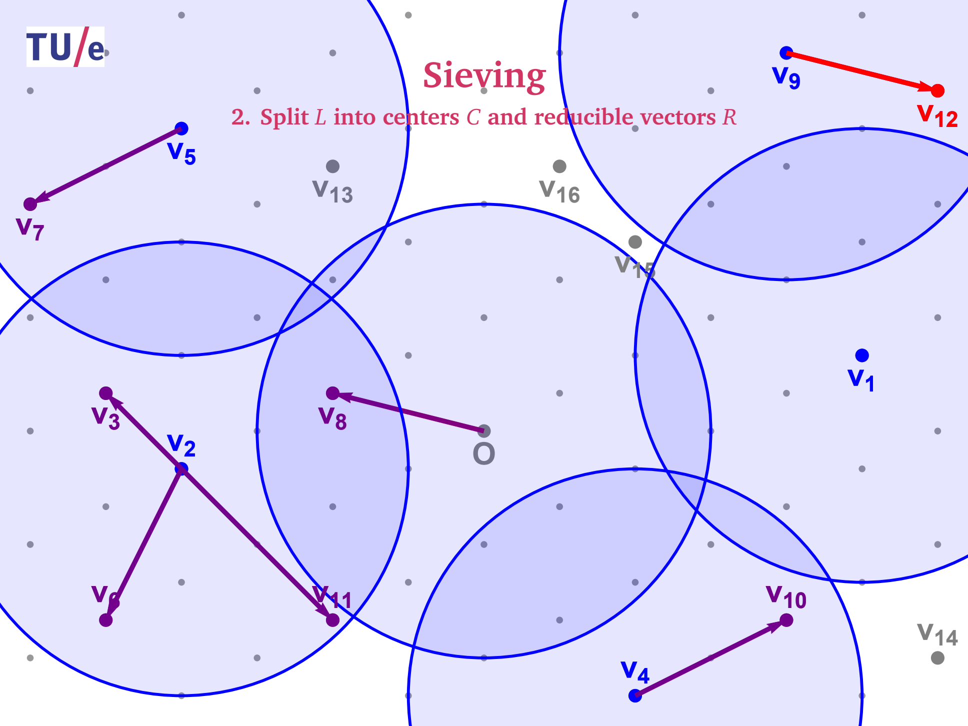

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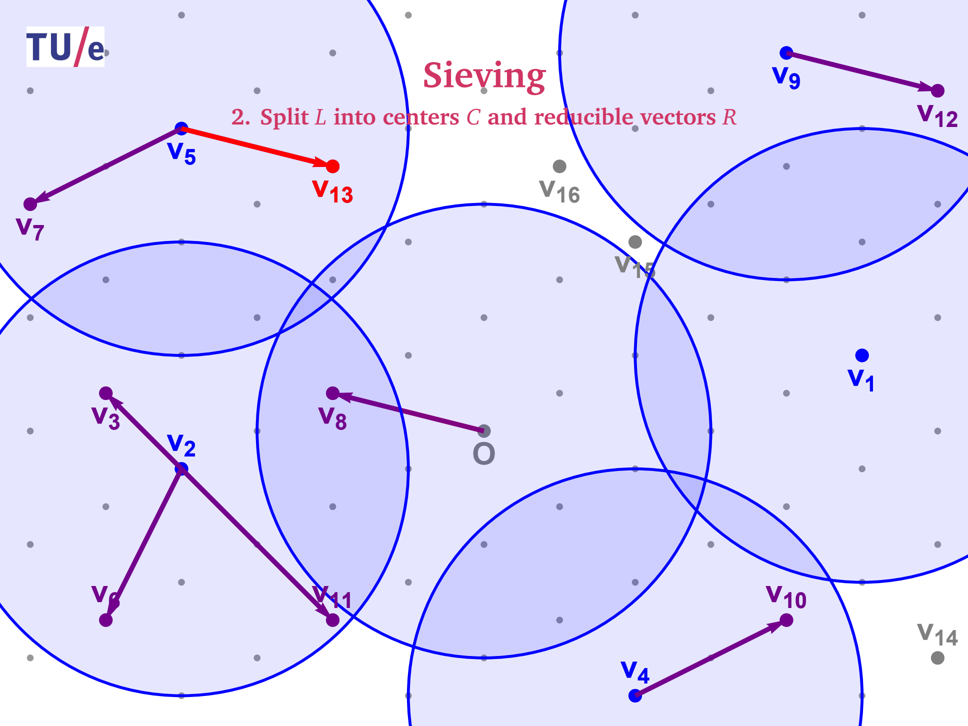

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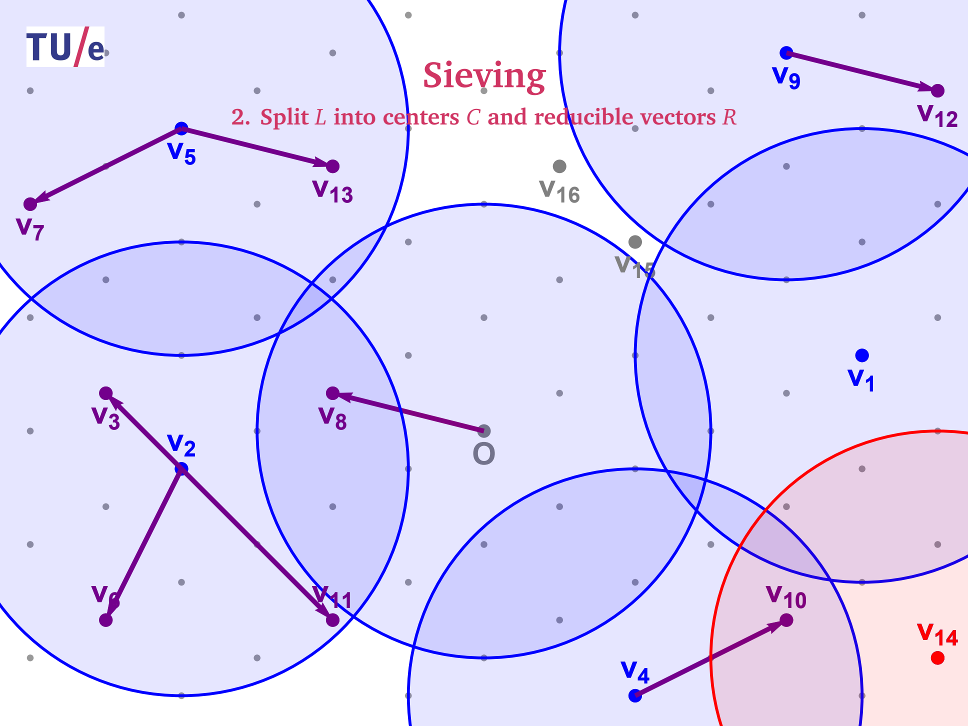

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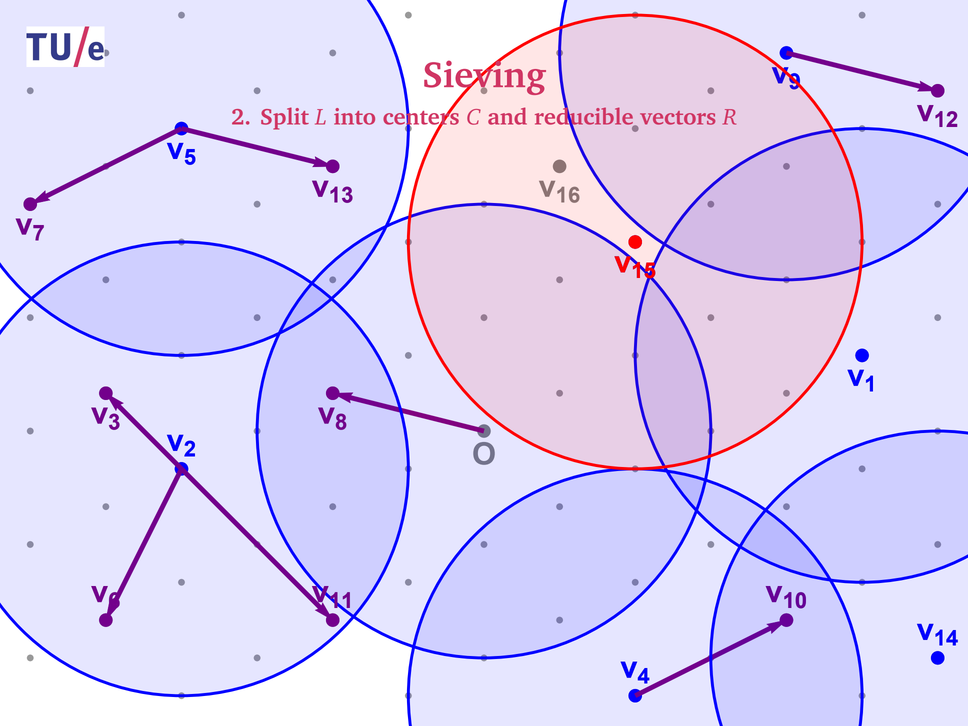

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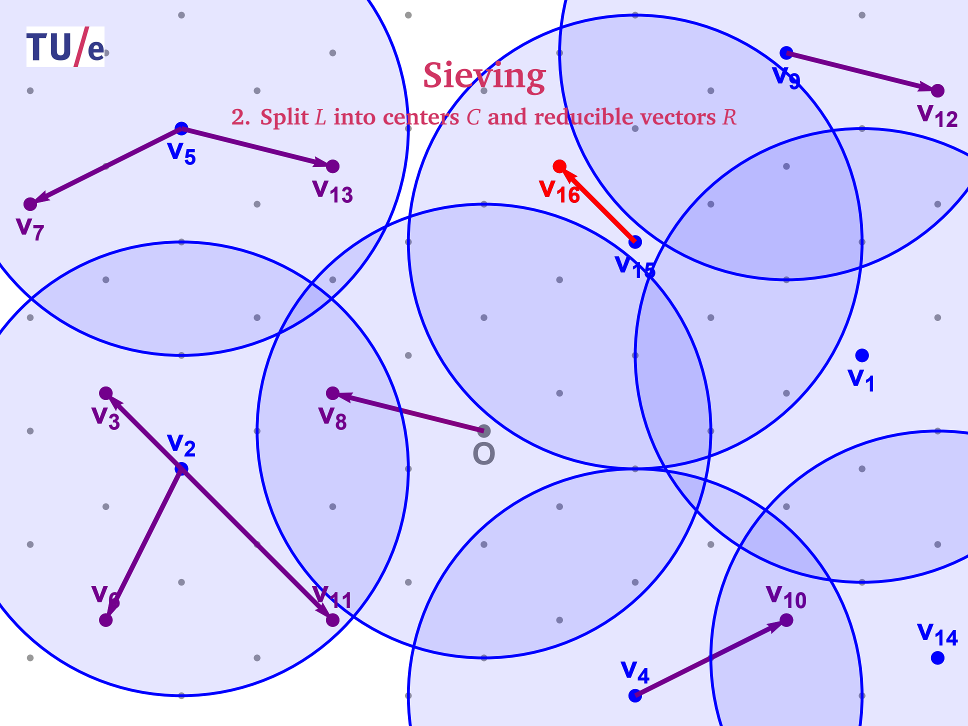

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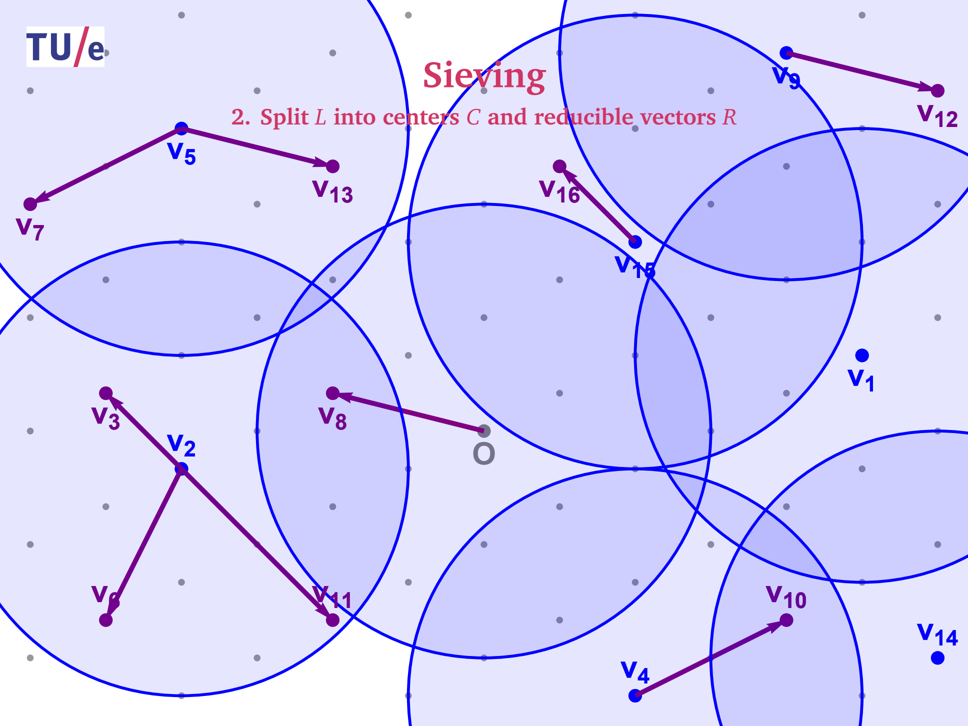
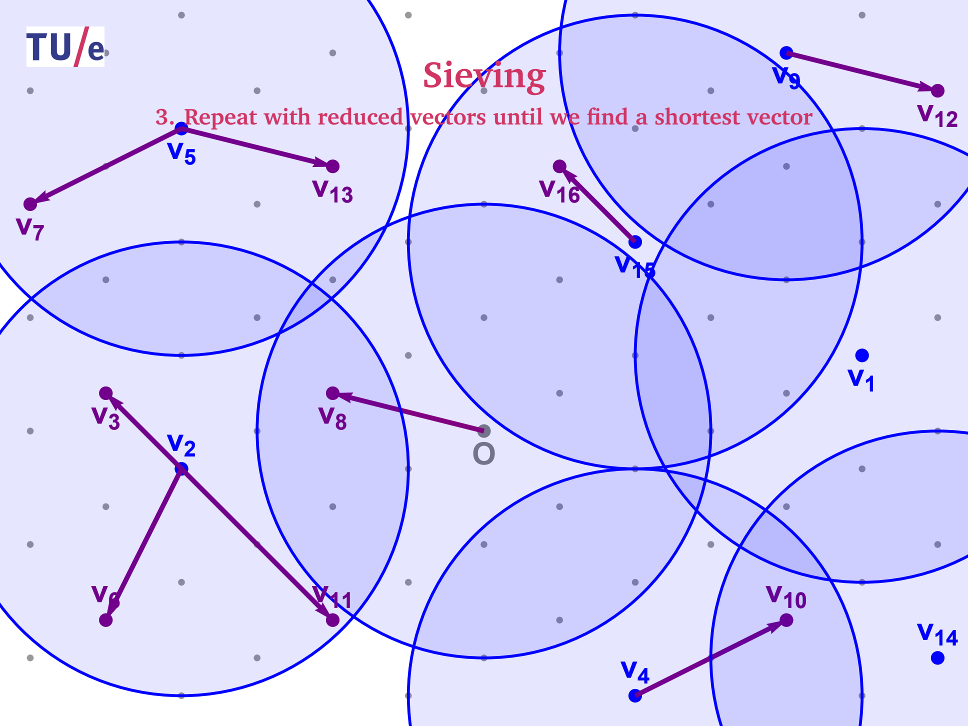


Sieving

3. Repeat with reduced vectors until we find a shortest vector



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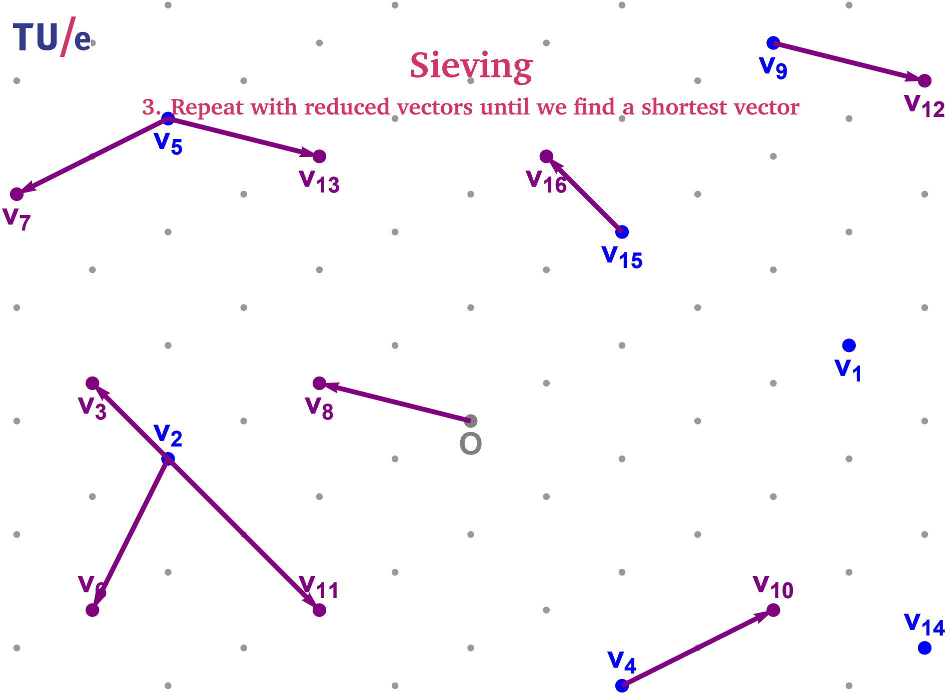

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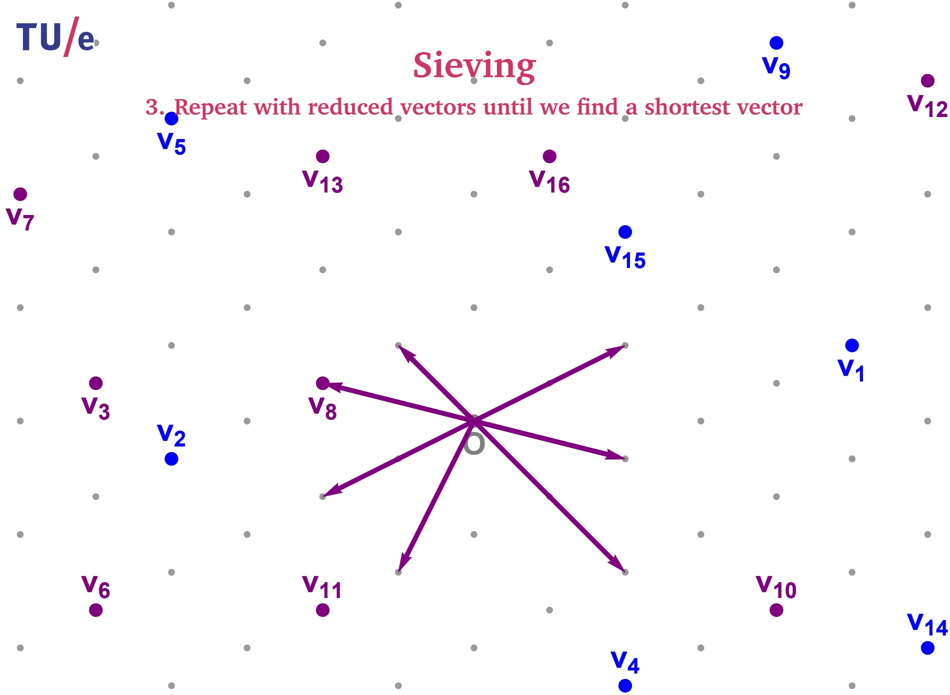

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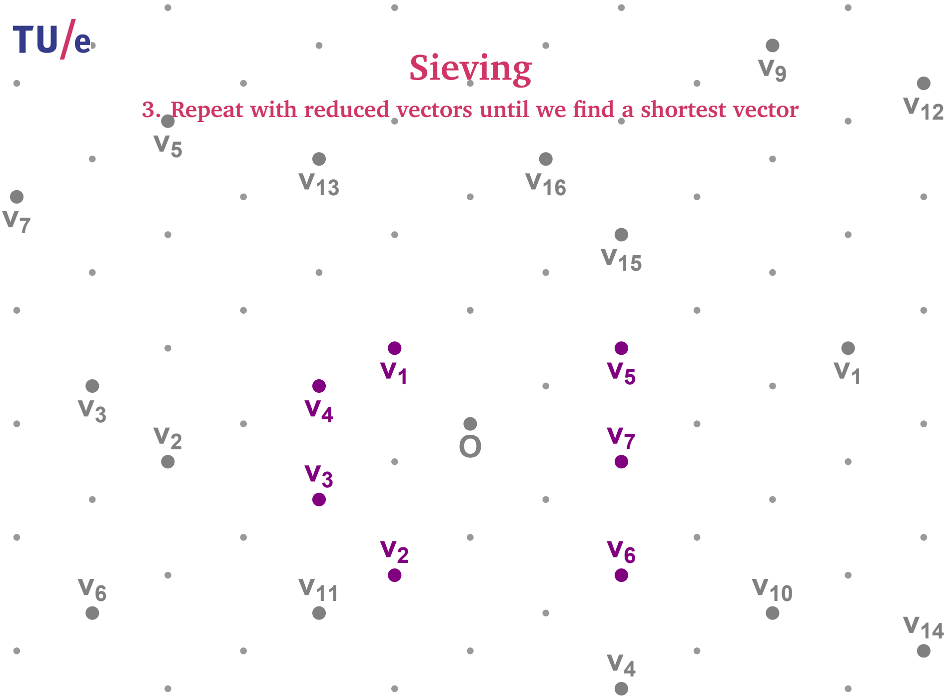


Sieving

Overview

Sieving

Overview

Heuristic (Nguyen–Vidick, J. Math. Crypt. '08)

Sieving solves SVP in time $(4/3)^{n+o(n)}$ and space $(4/3)^{n/2+o(n)}$.

Sieving

Overview

Heuristic (Nguyen–Vidick, J. Math. Crypt. '08)

Sieving solves SVP in time $(4/3)^{n+o(n)}$ and space $(4/3)^{n/2+o(n)}$.

The list size comes from heuristic packing/saturation arguments, the time complexity is quadratic in the list size.

Sieving

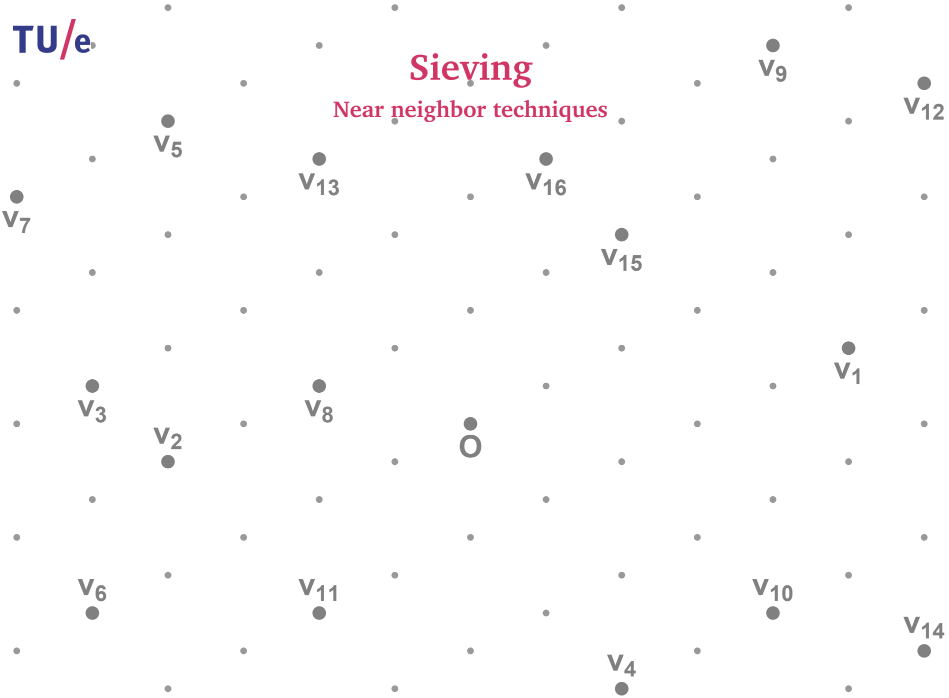
Near neighbor techniques

Sieving

Near neighbor techniques

Sieving

Near neighbor techniques



Sieving

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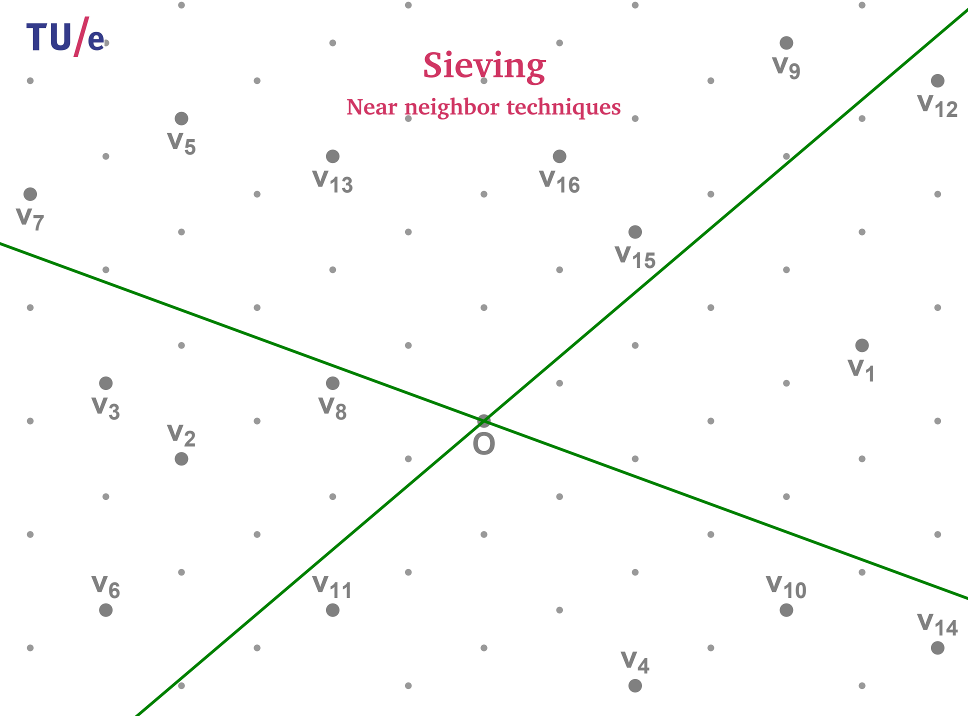

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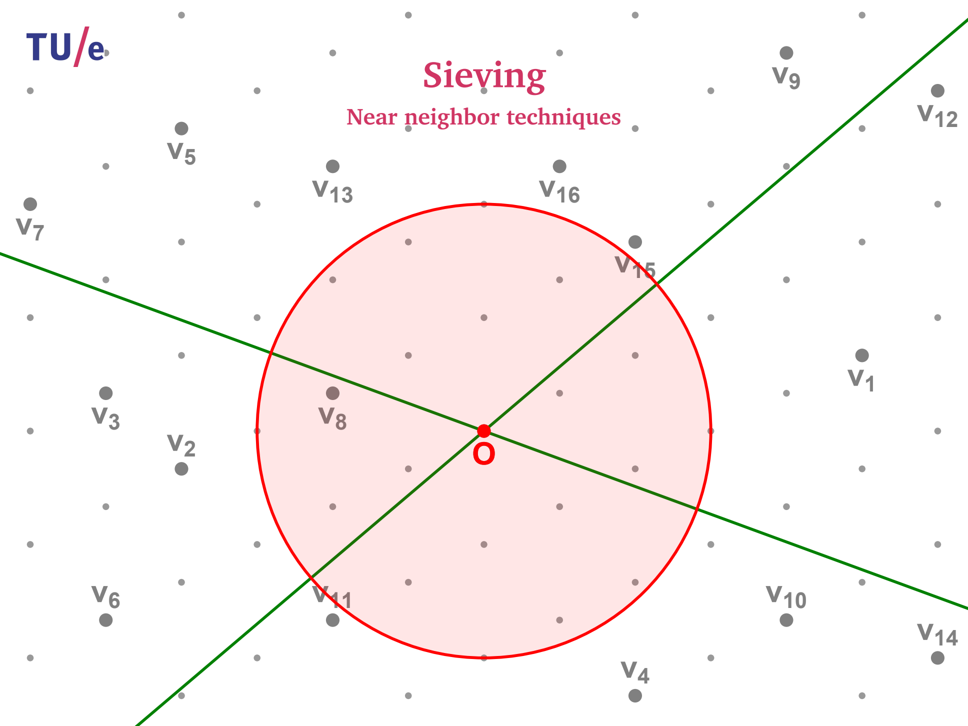
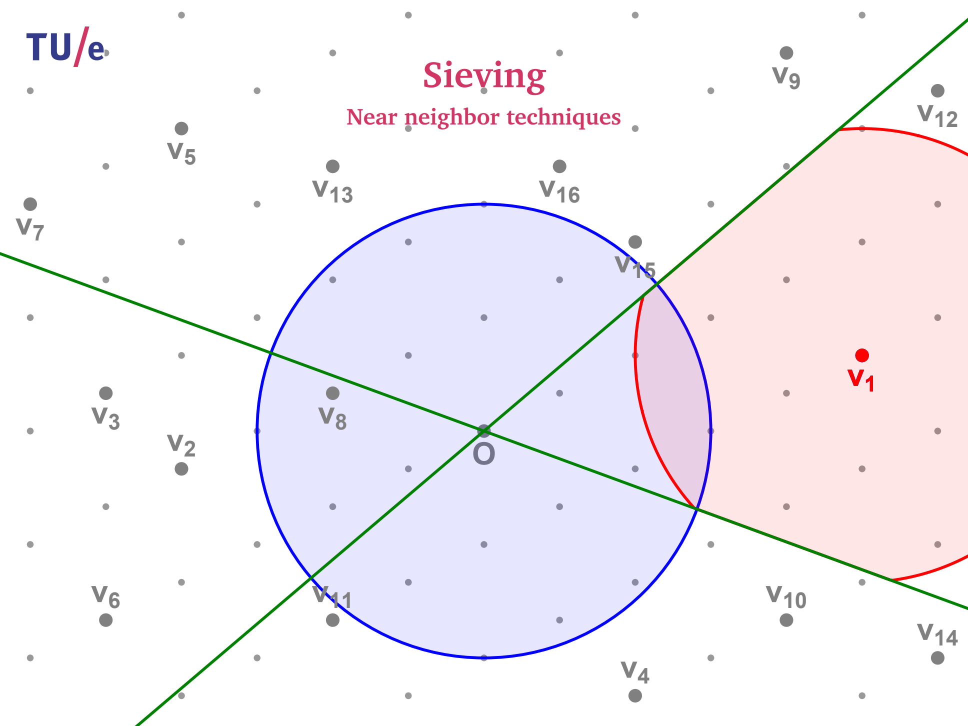


TU/e

Sieving

Near neighbor techniques



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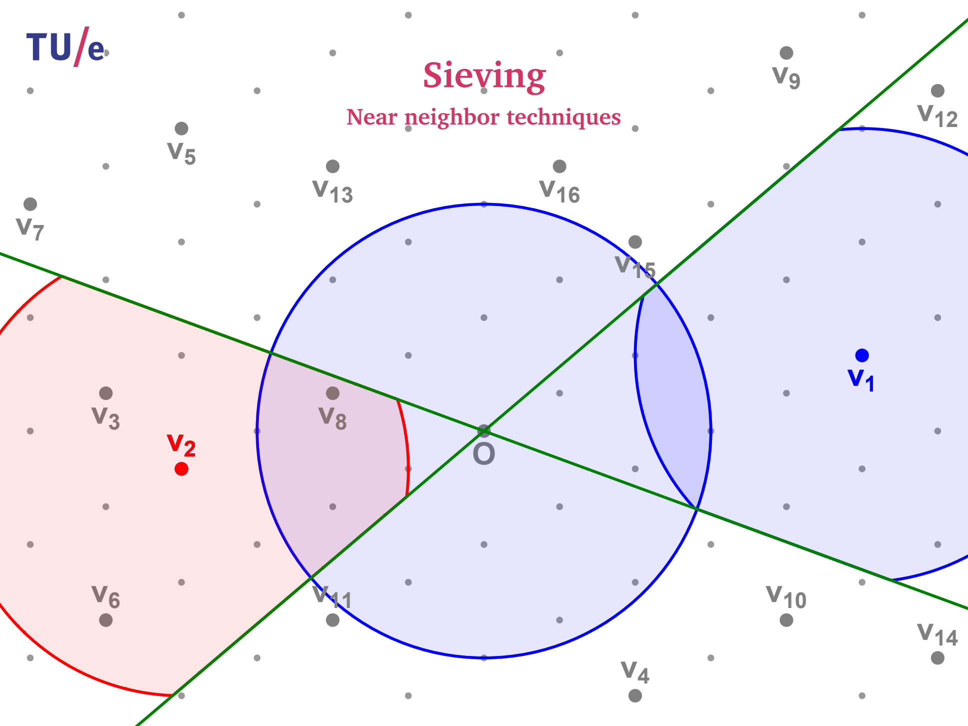

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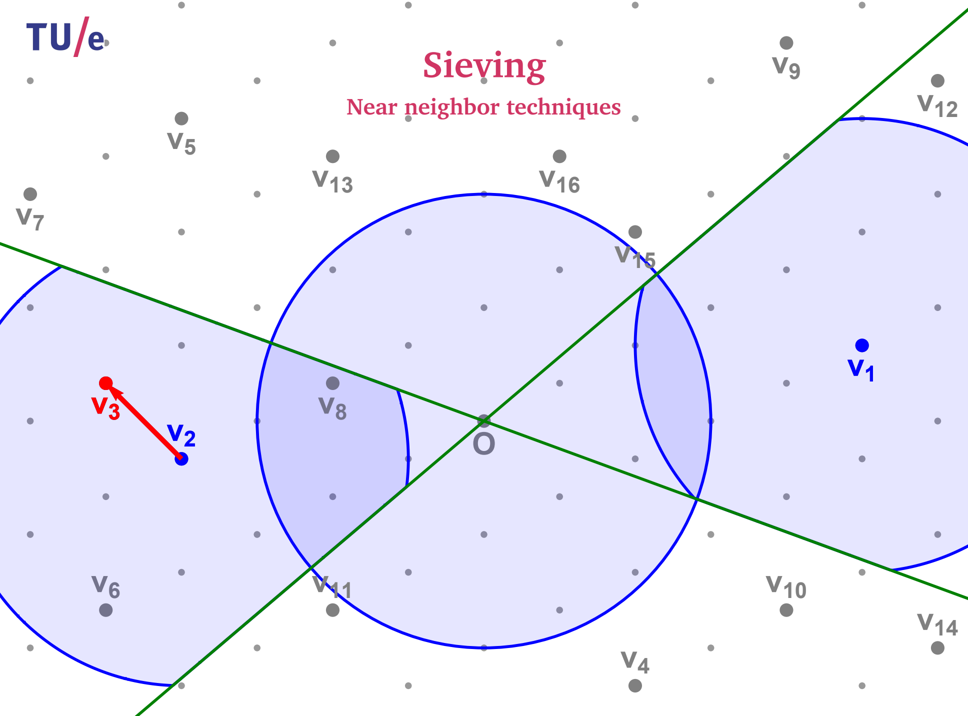

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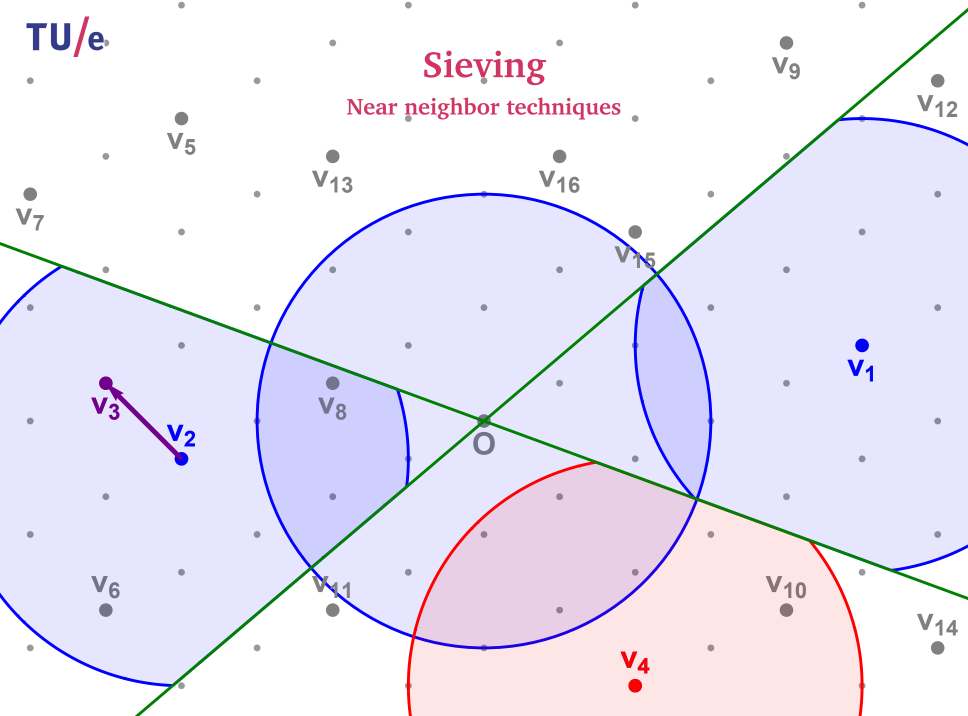

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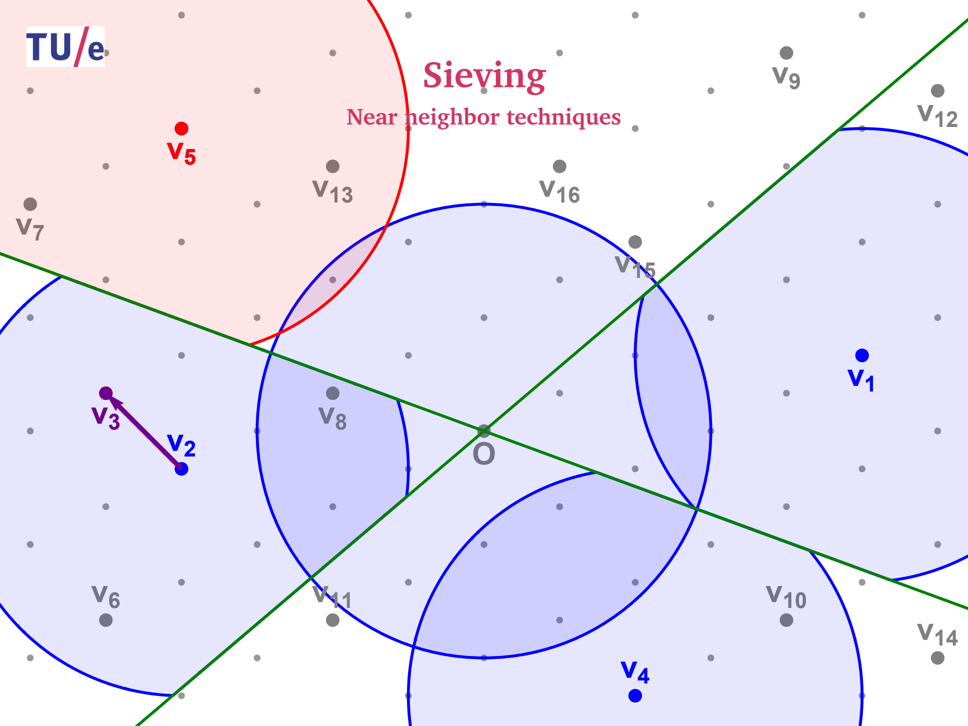

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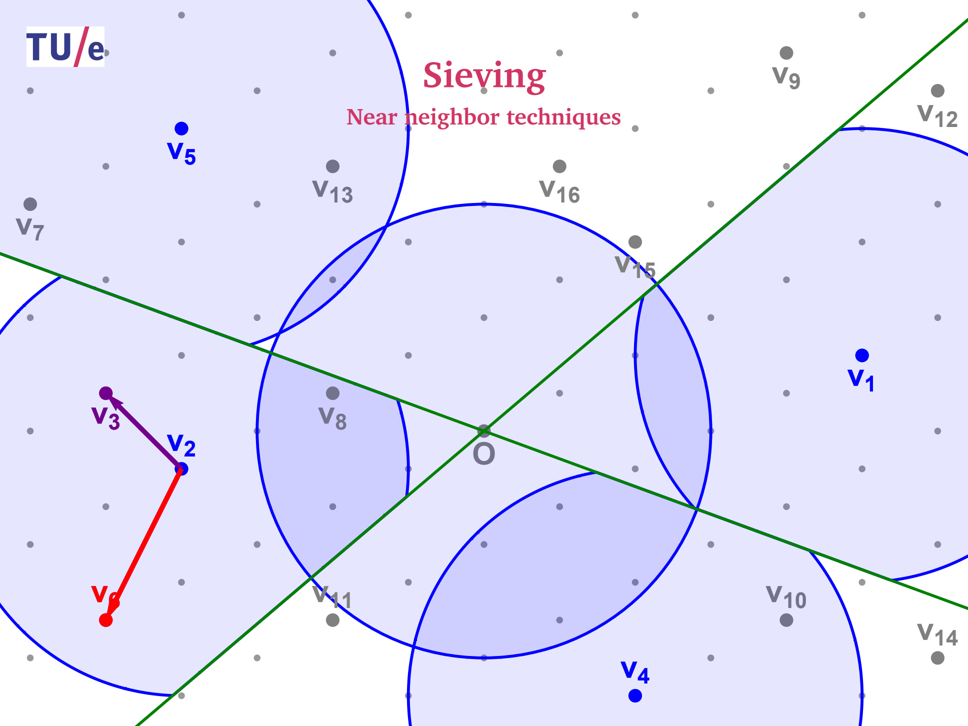

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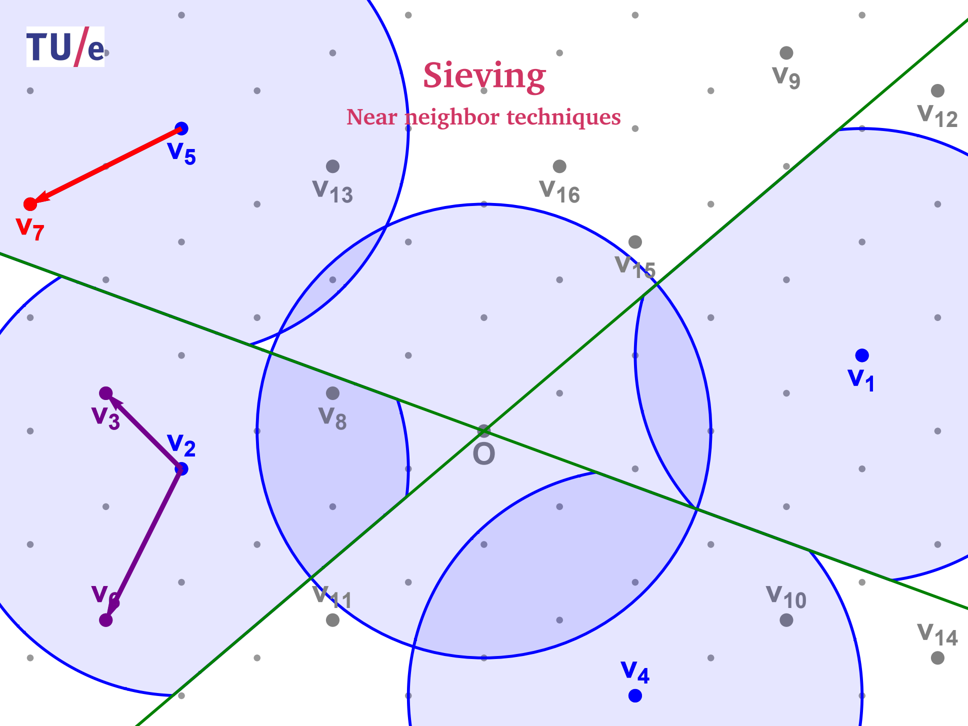

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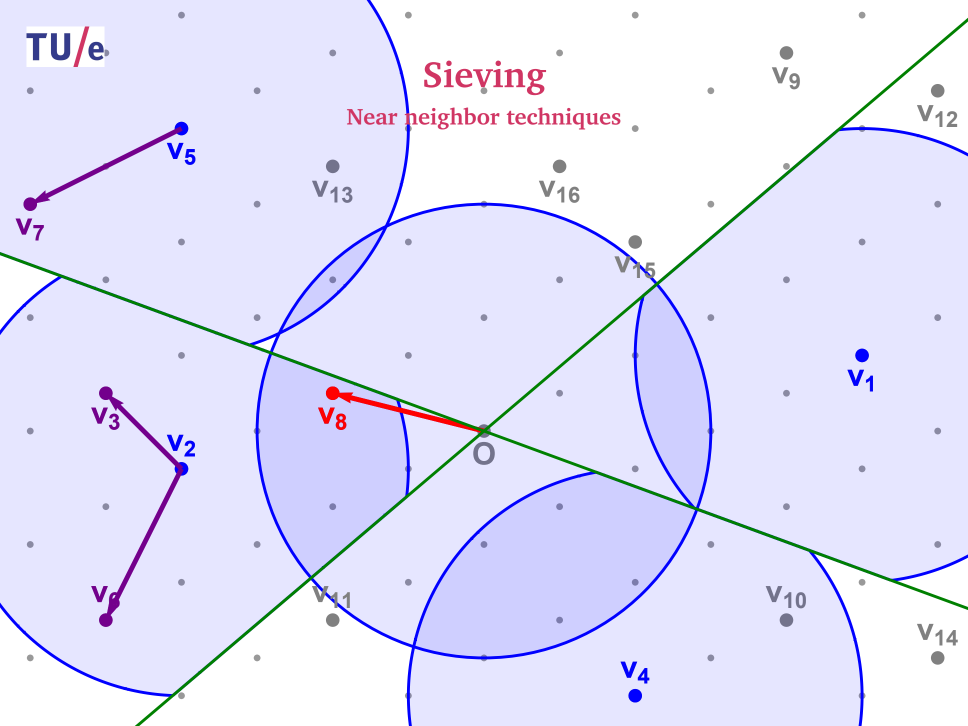

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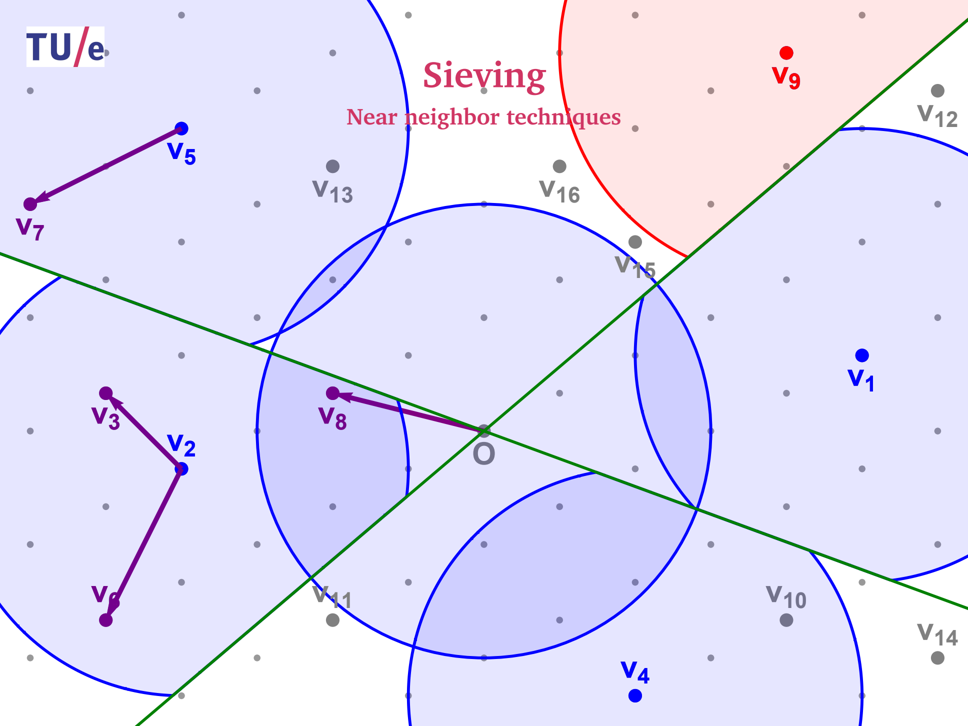

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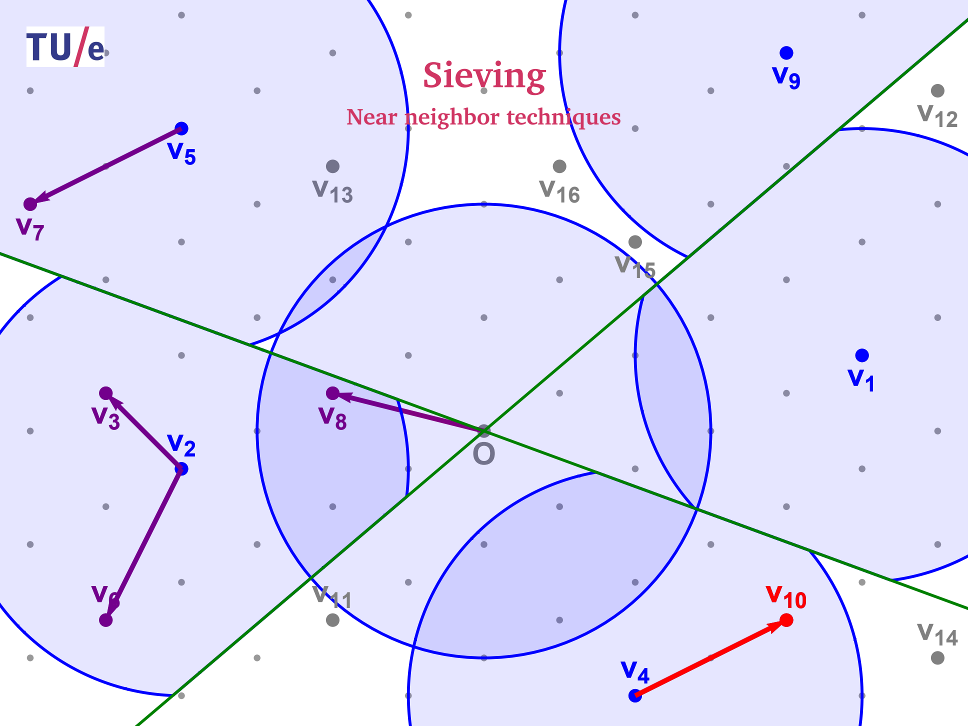

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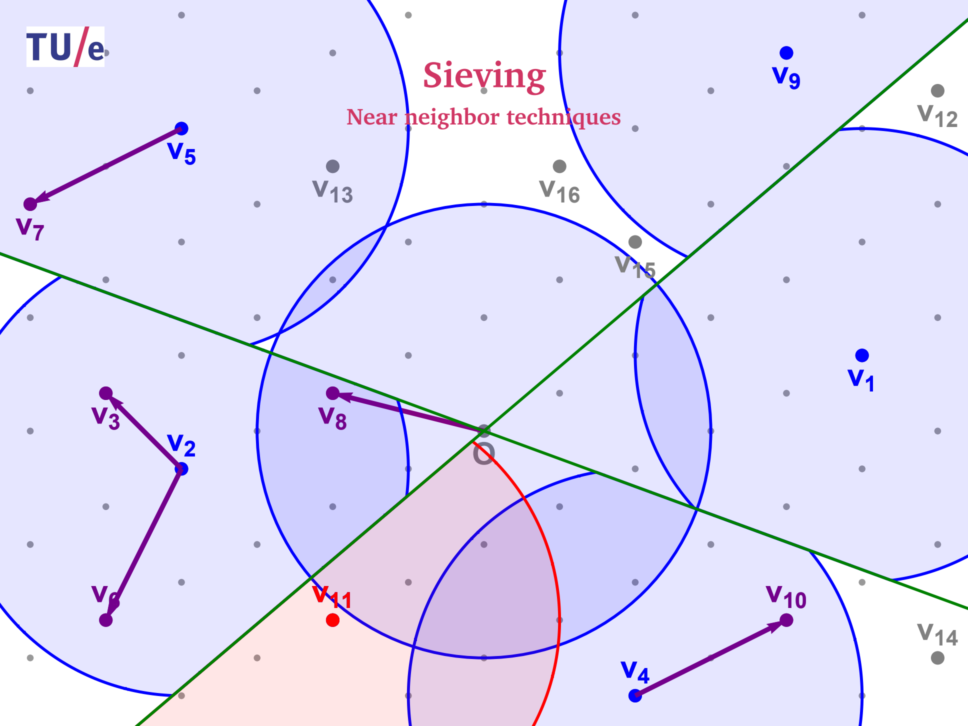

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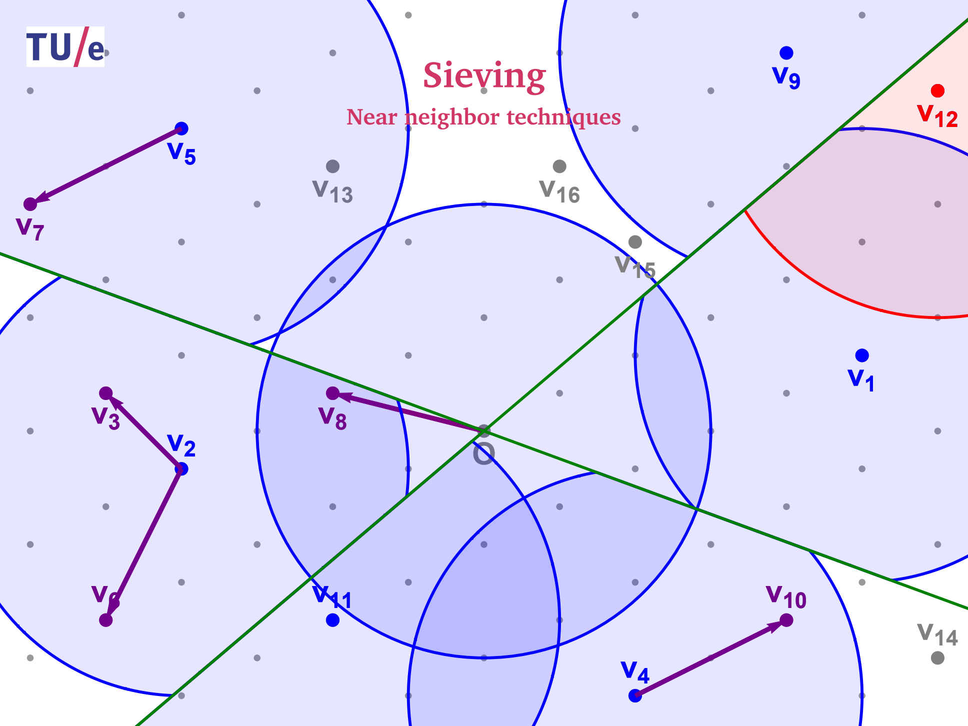

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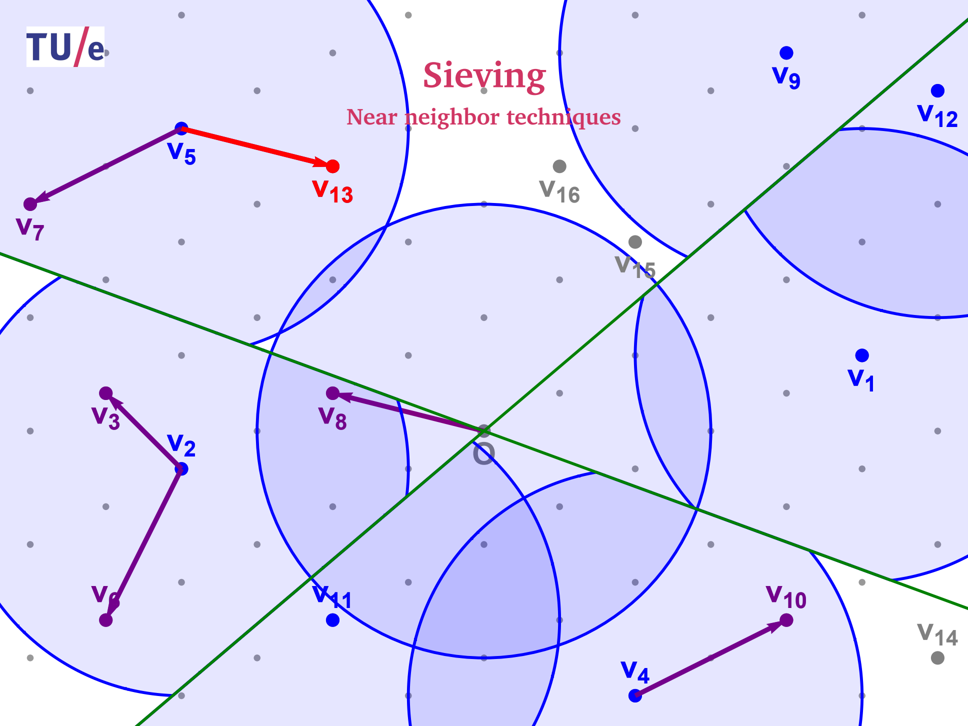

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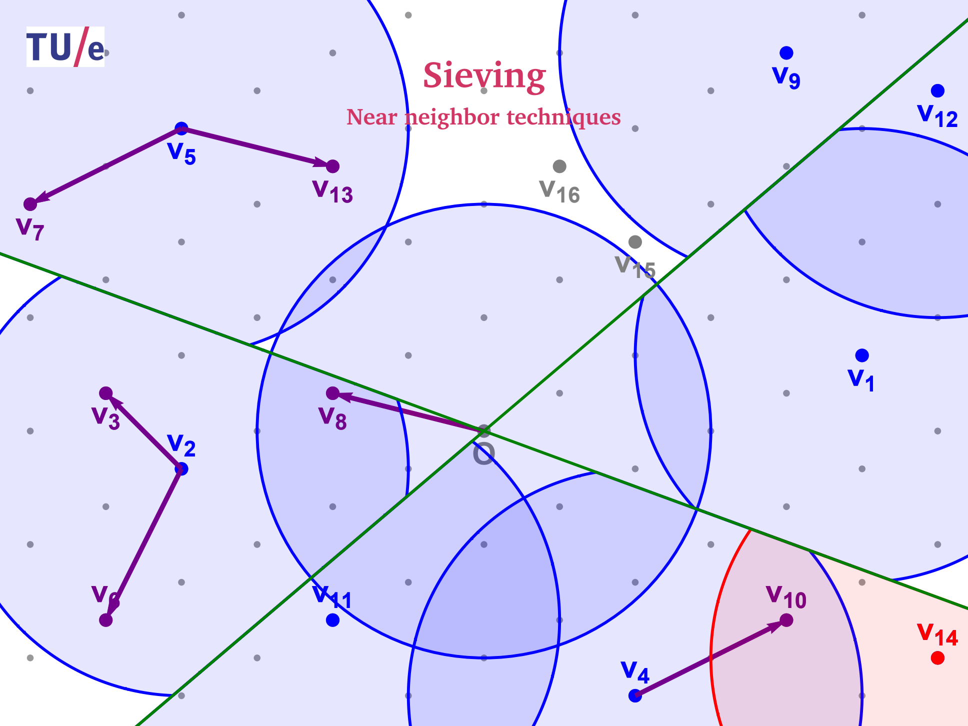

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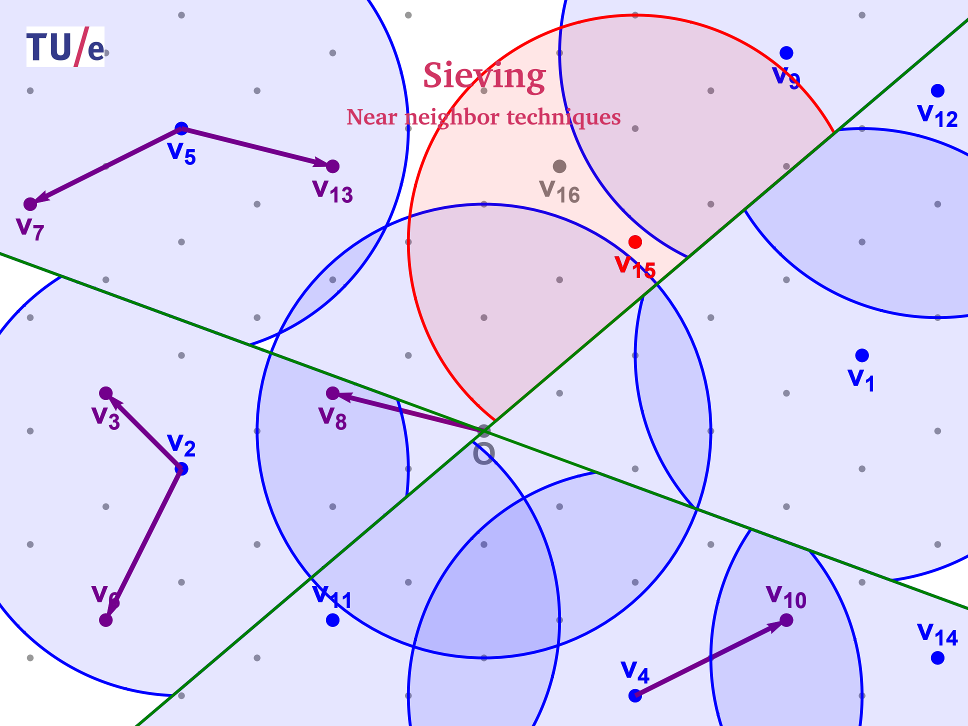

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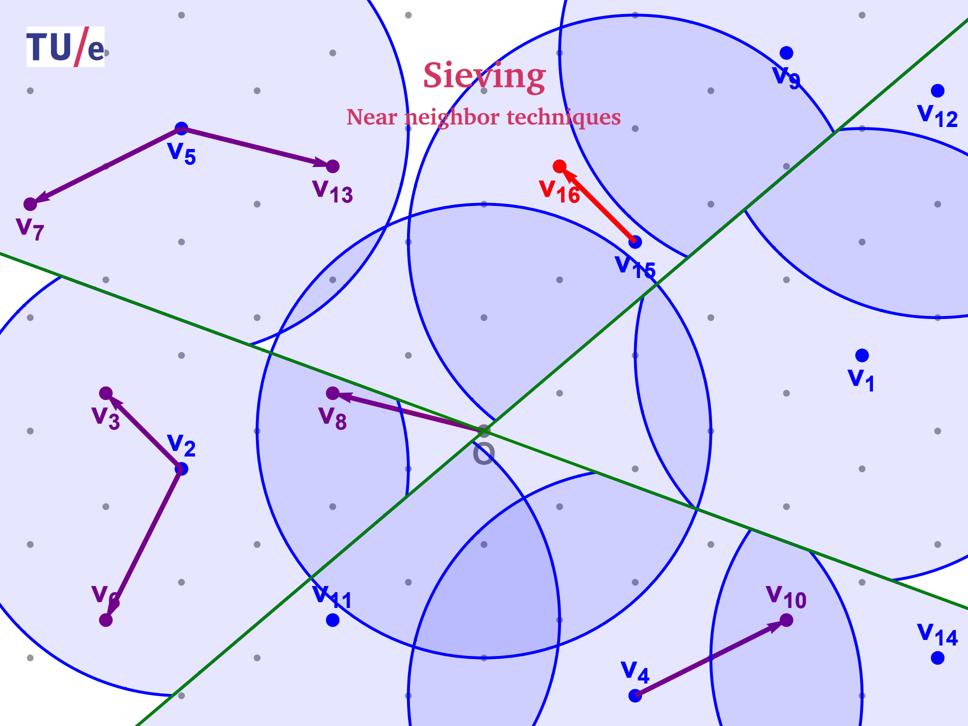

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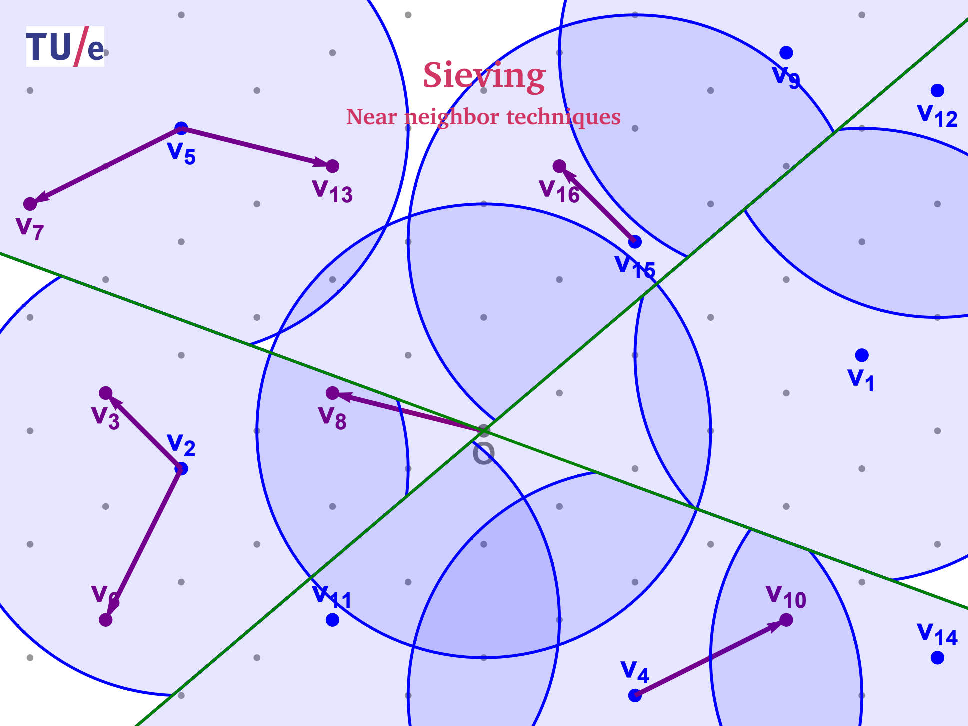

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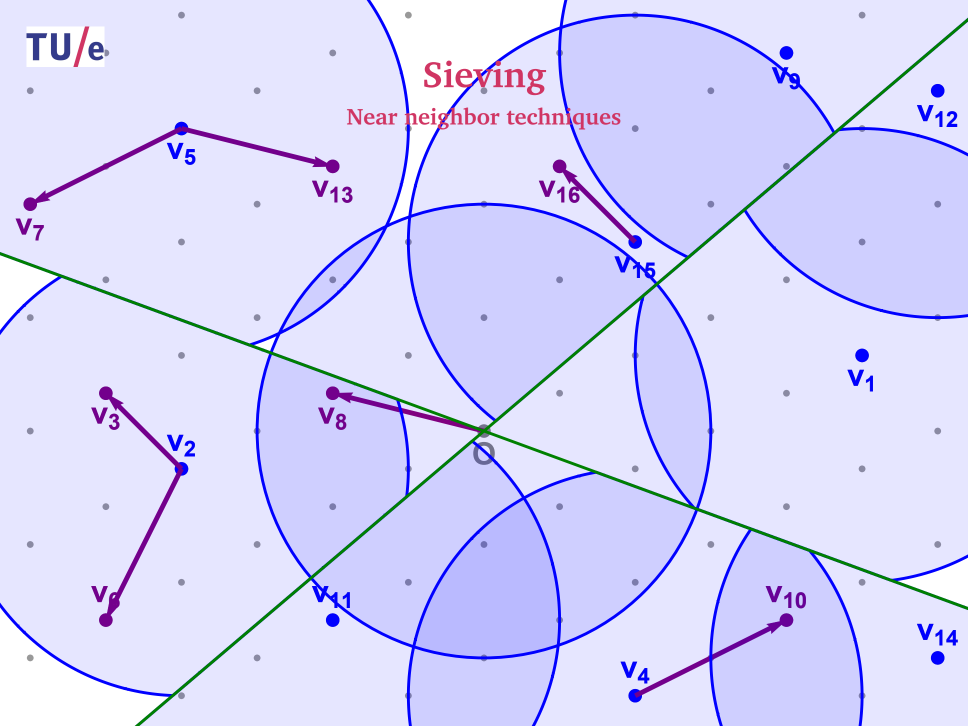

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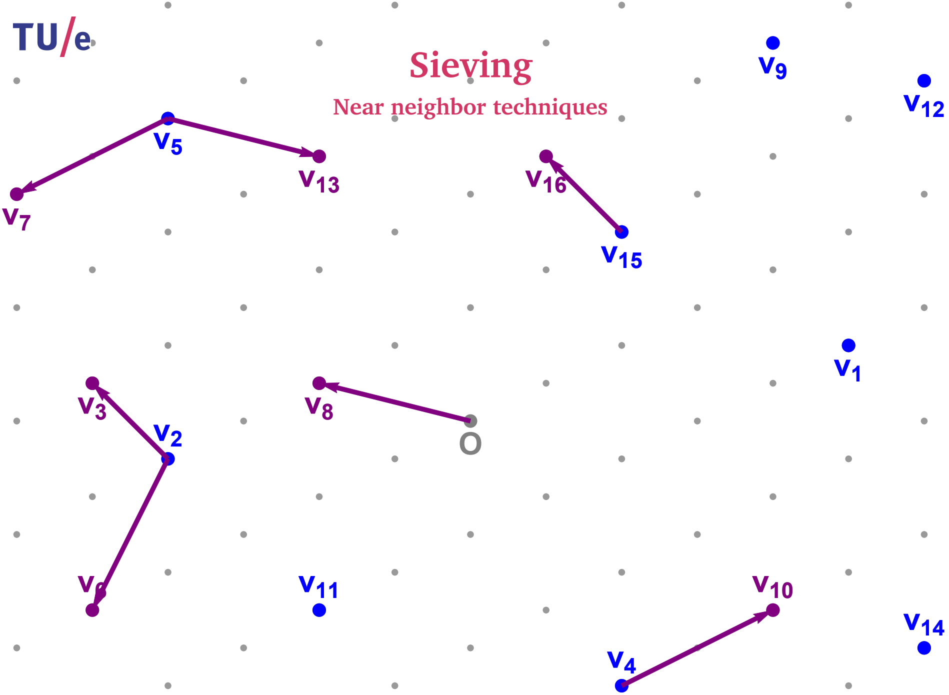

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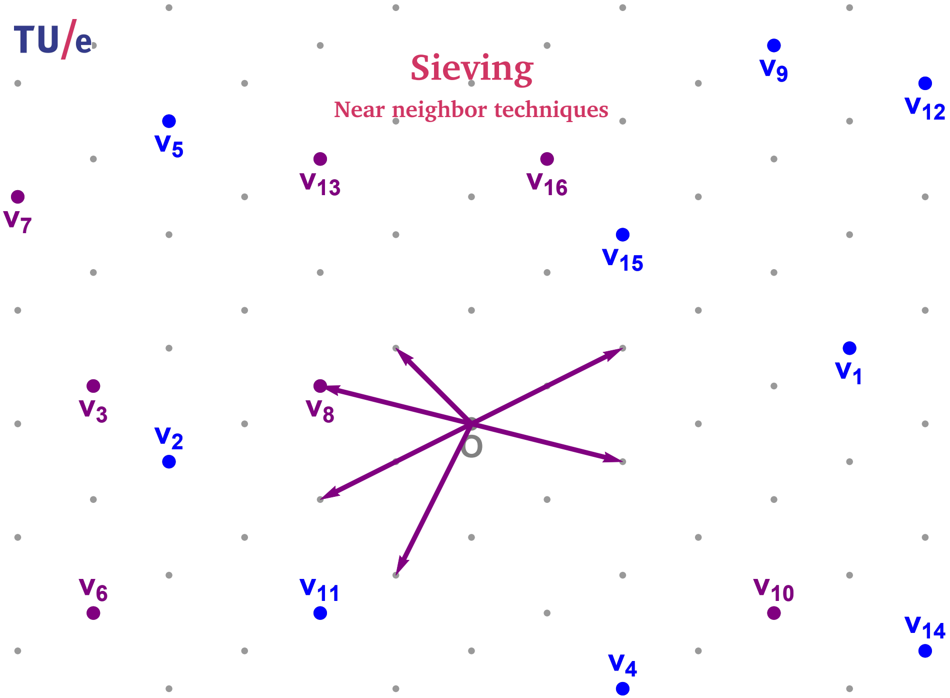

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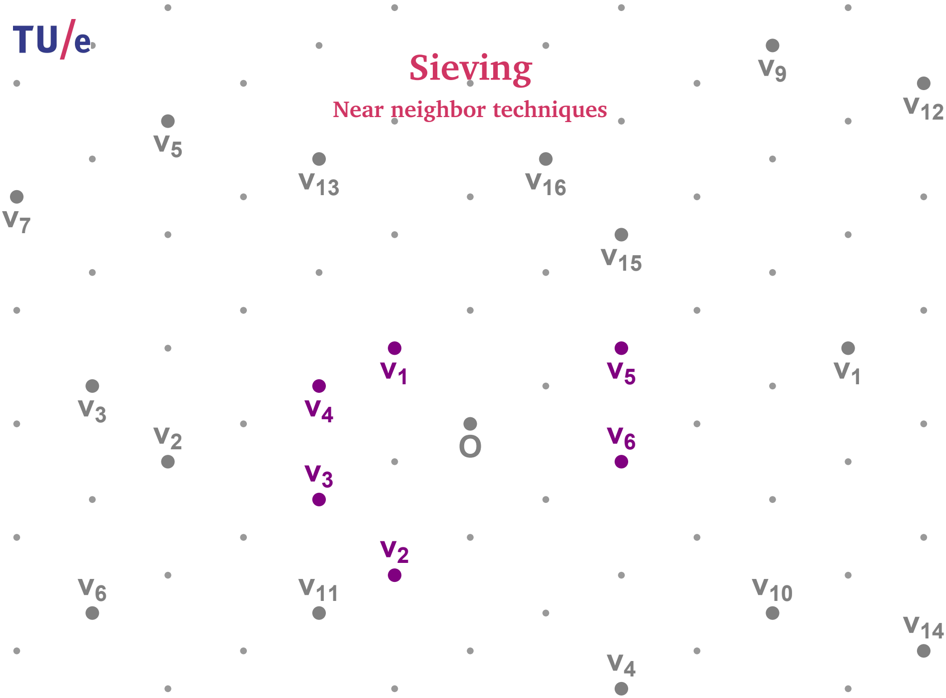
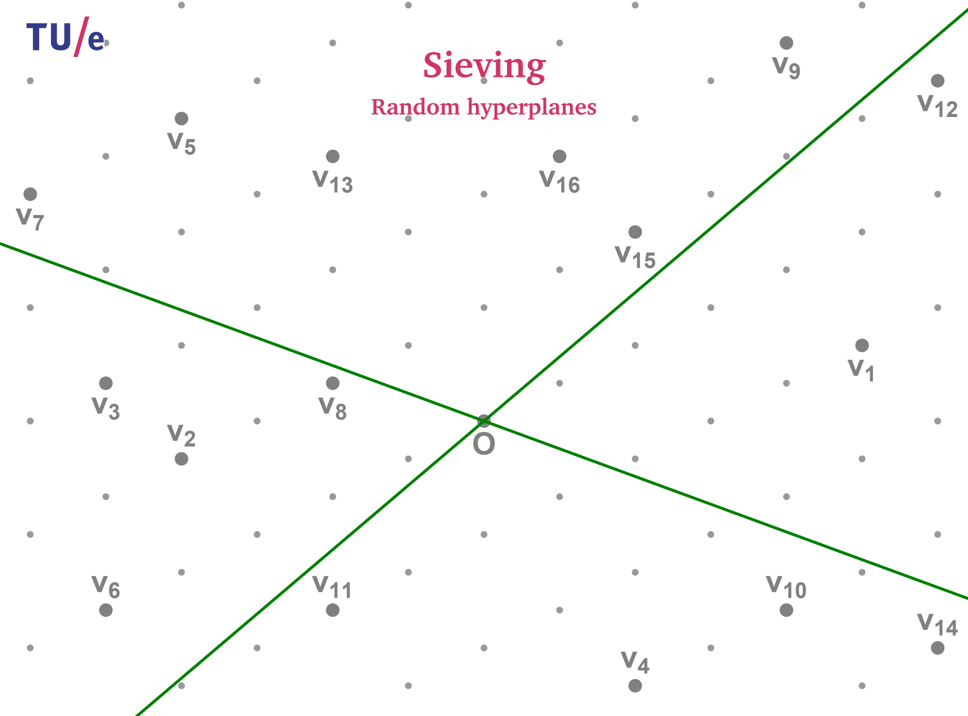


Sieving

Random hyperplanes

Sieving

Random hyperplanes



Sieving

Random hypercones

TU/e

Sieving

Random hypercones

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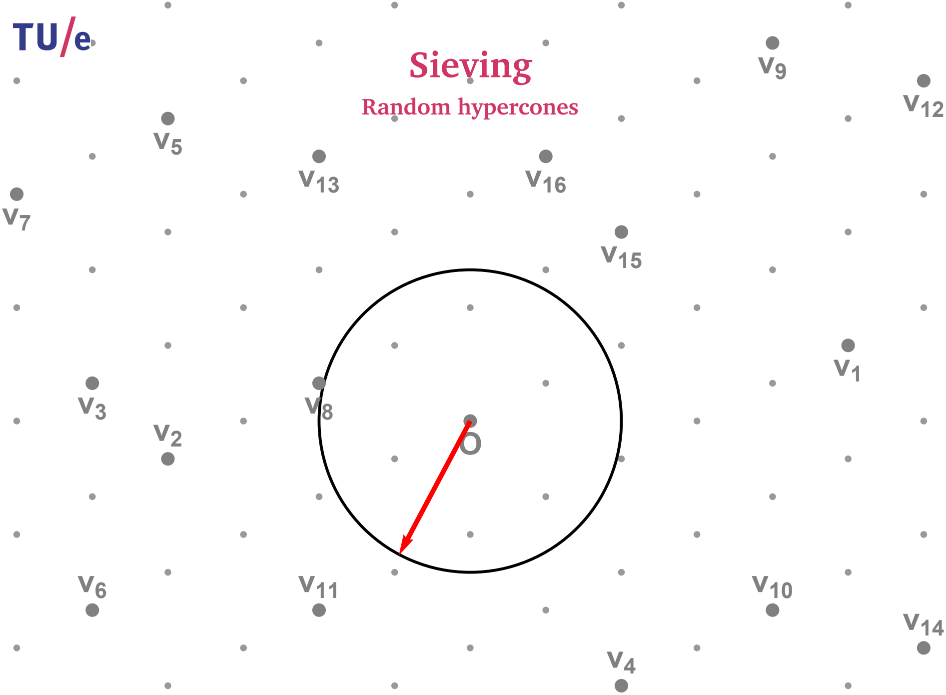

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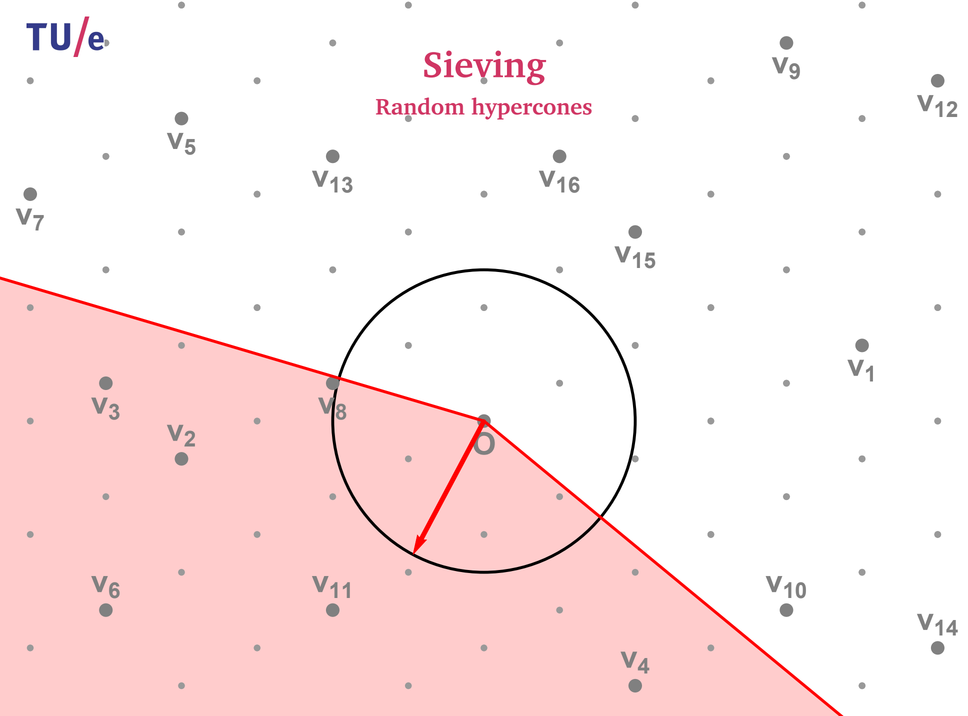

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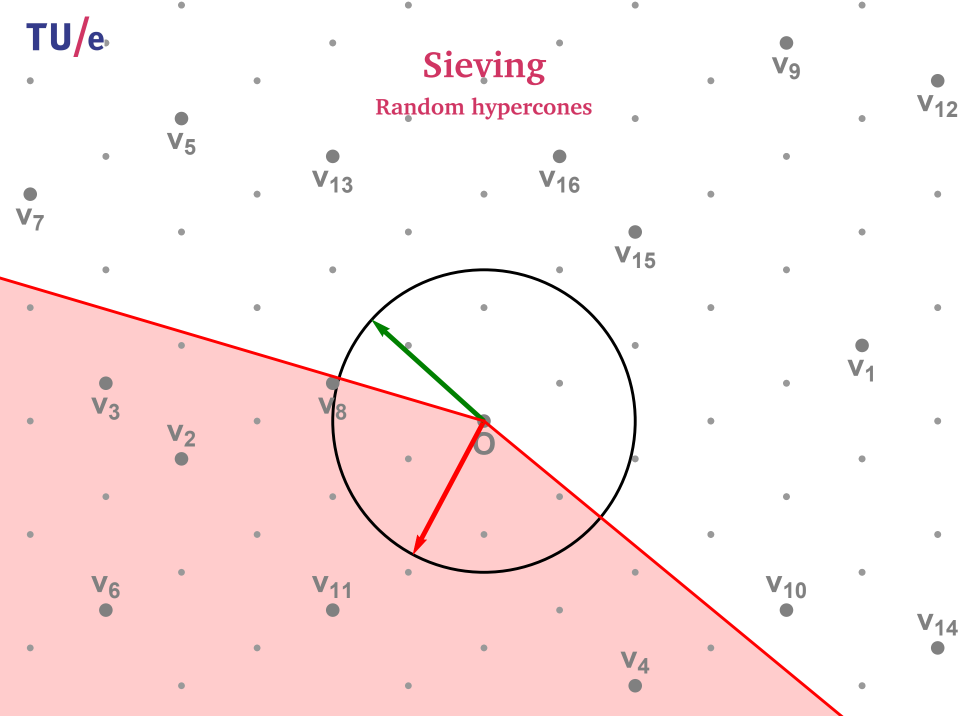

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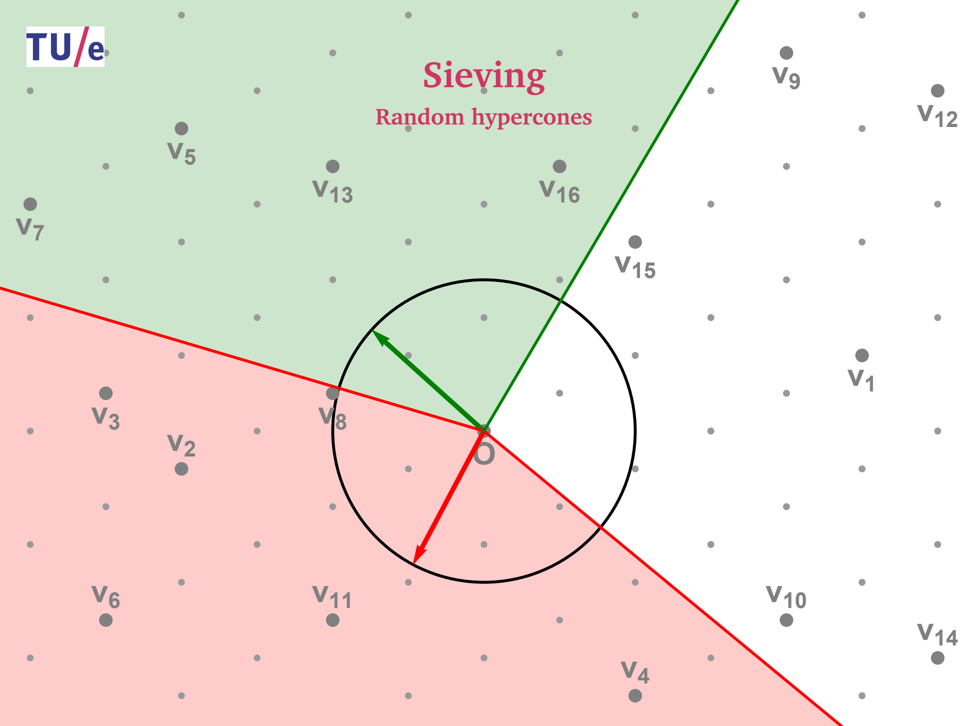

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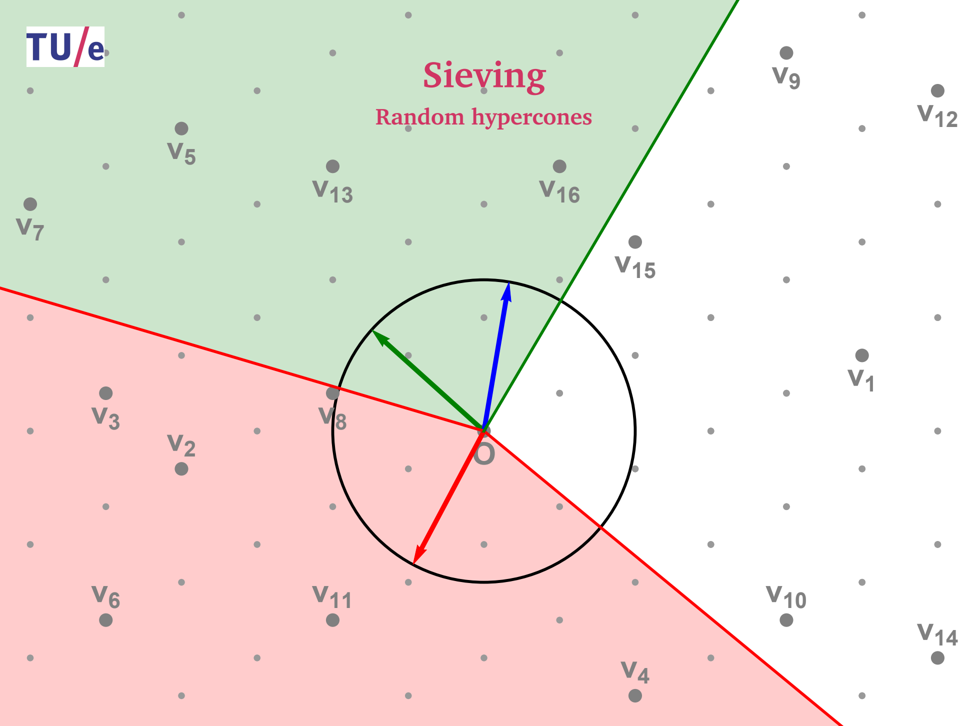

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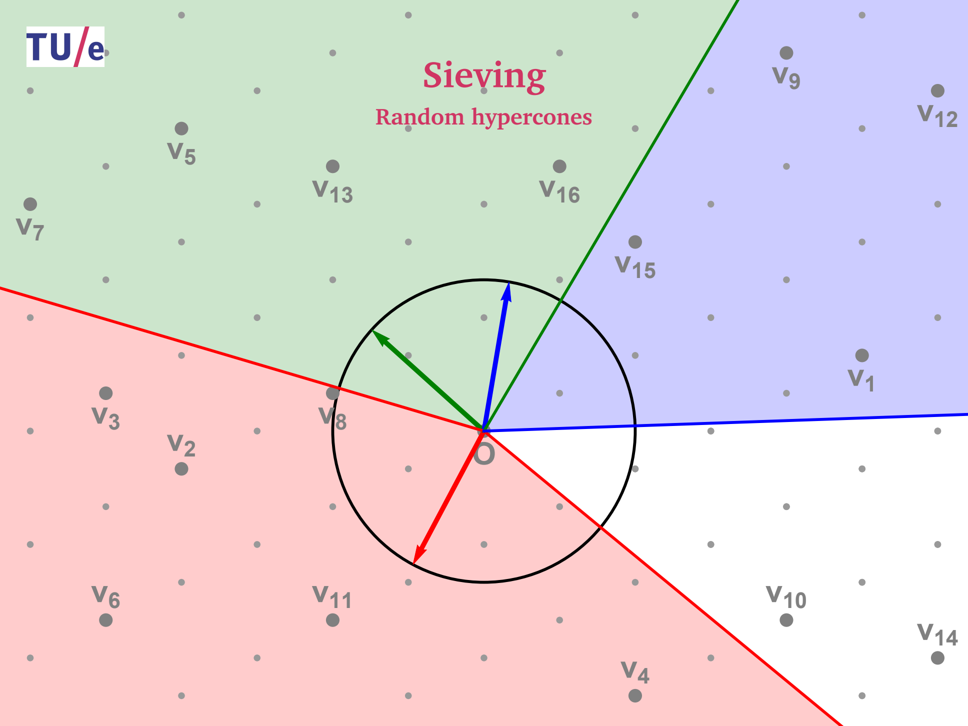

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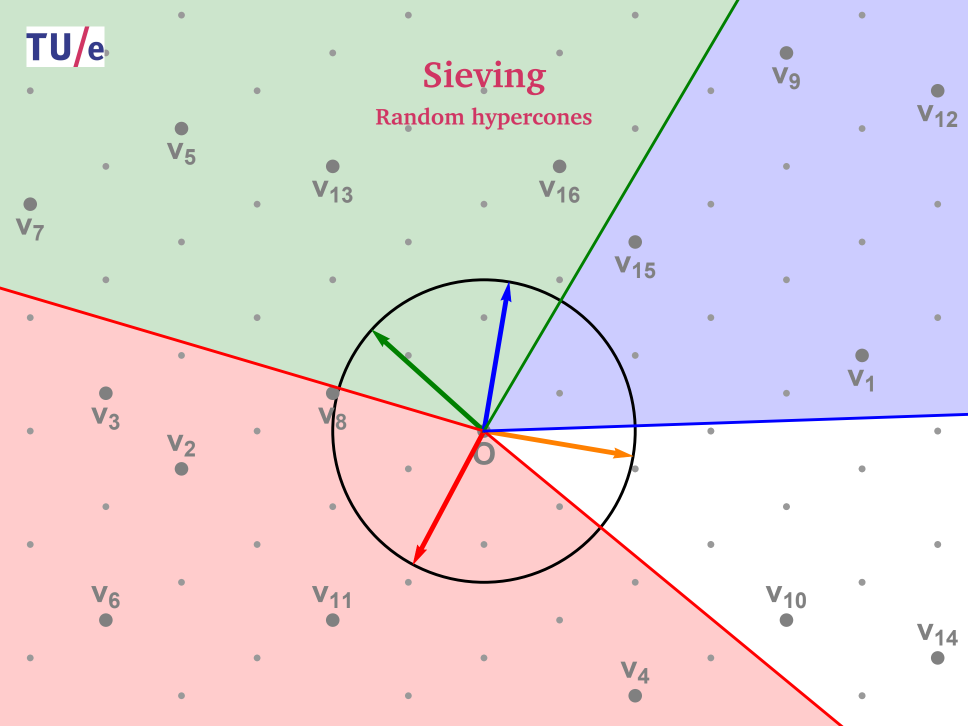

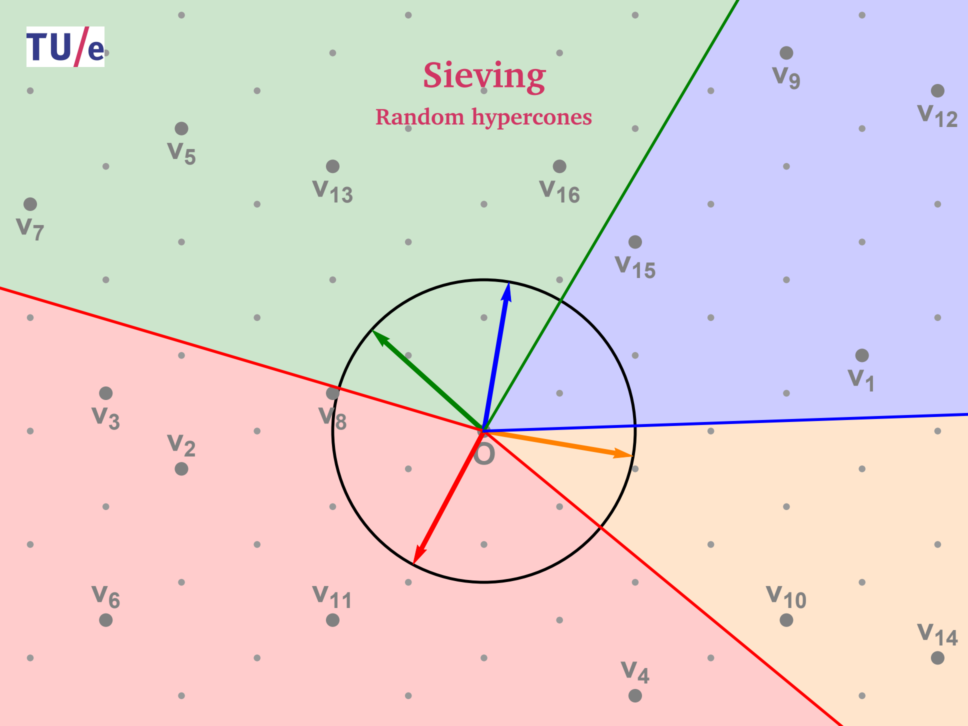
Sieving
Random hypercones
v9
v12
v5
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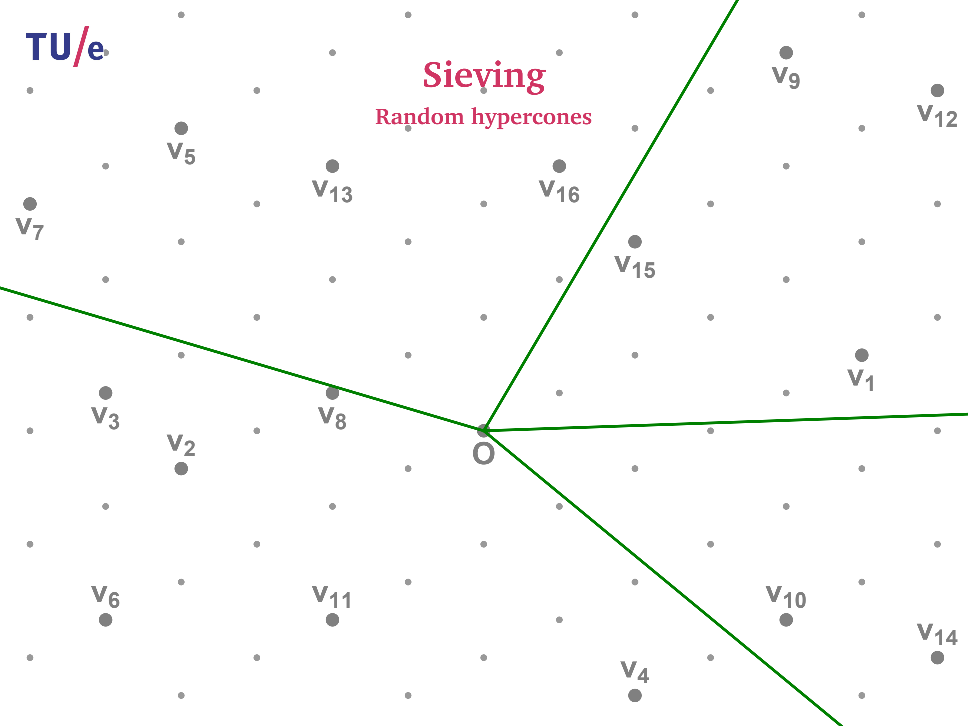
v4

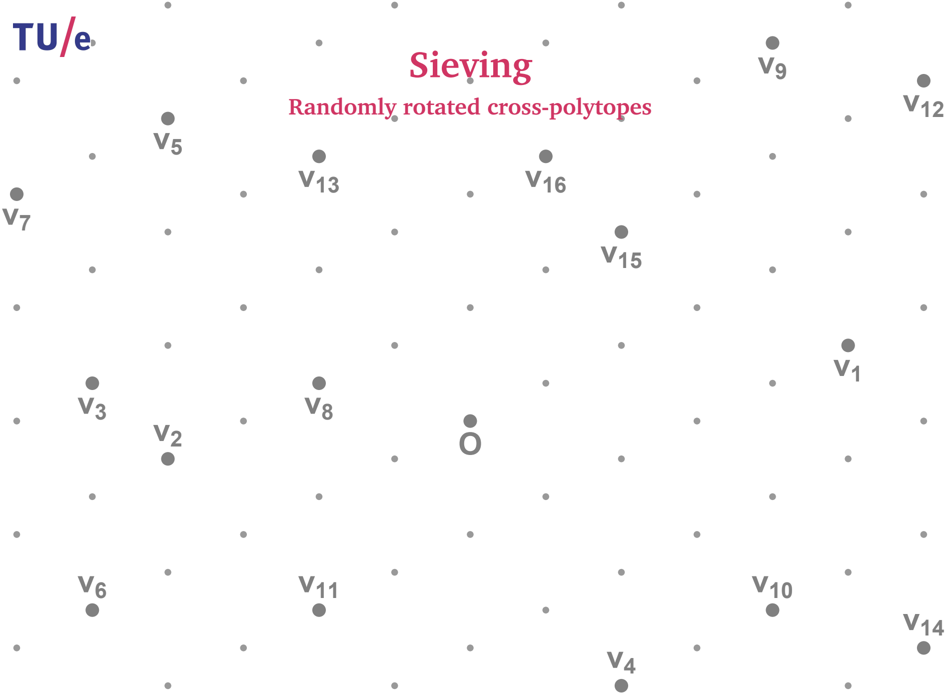
Sieving

Random hypercones



Sieving

Randomly rotated cross-polytopes



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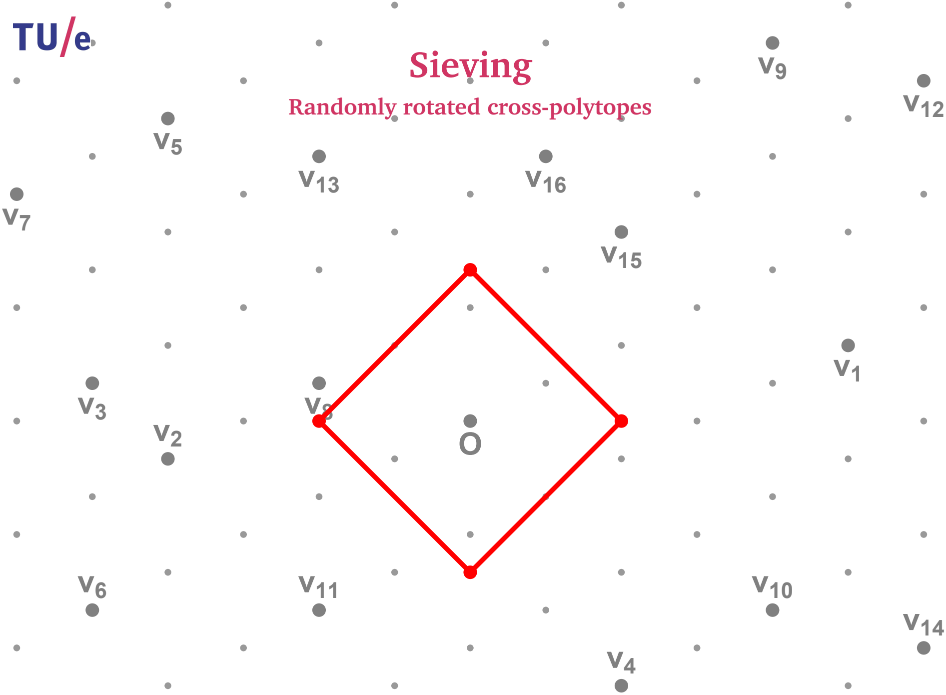

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Sieving

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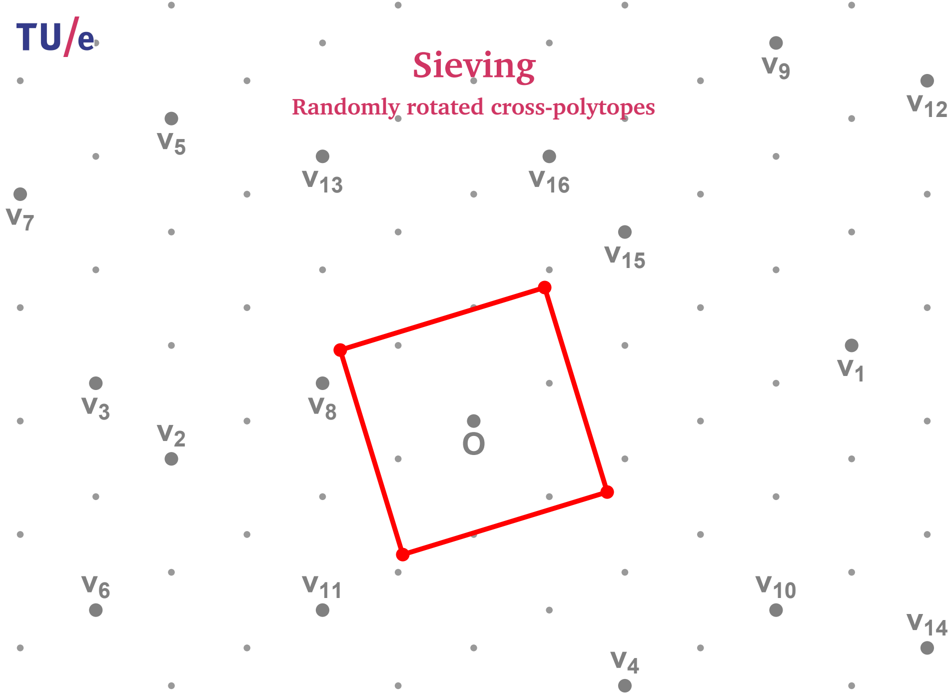

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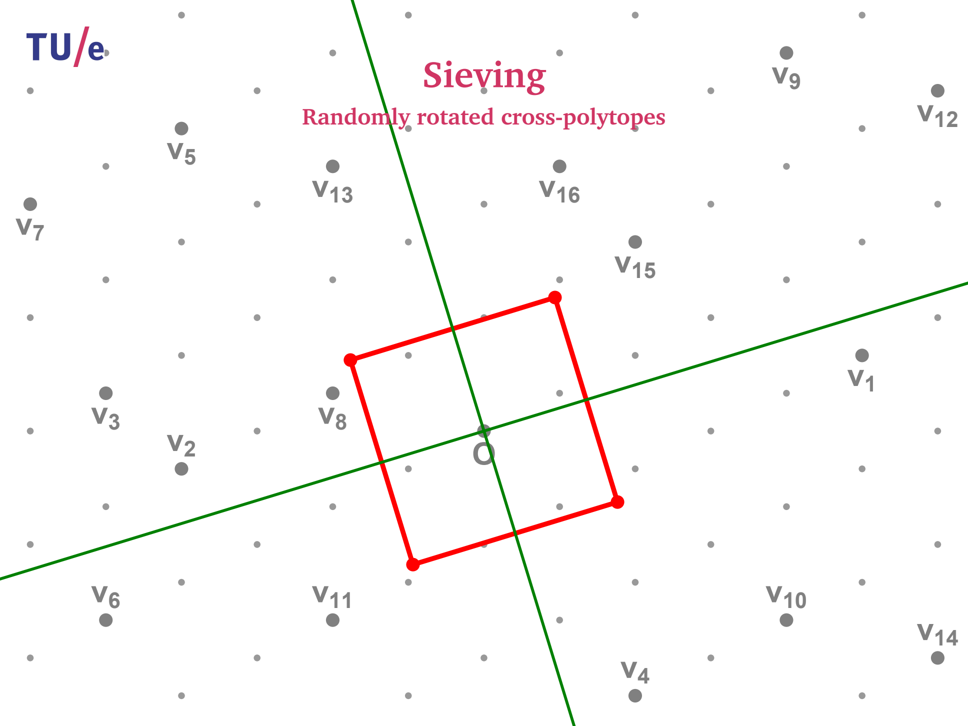

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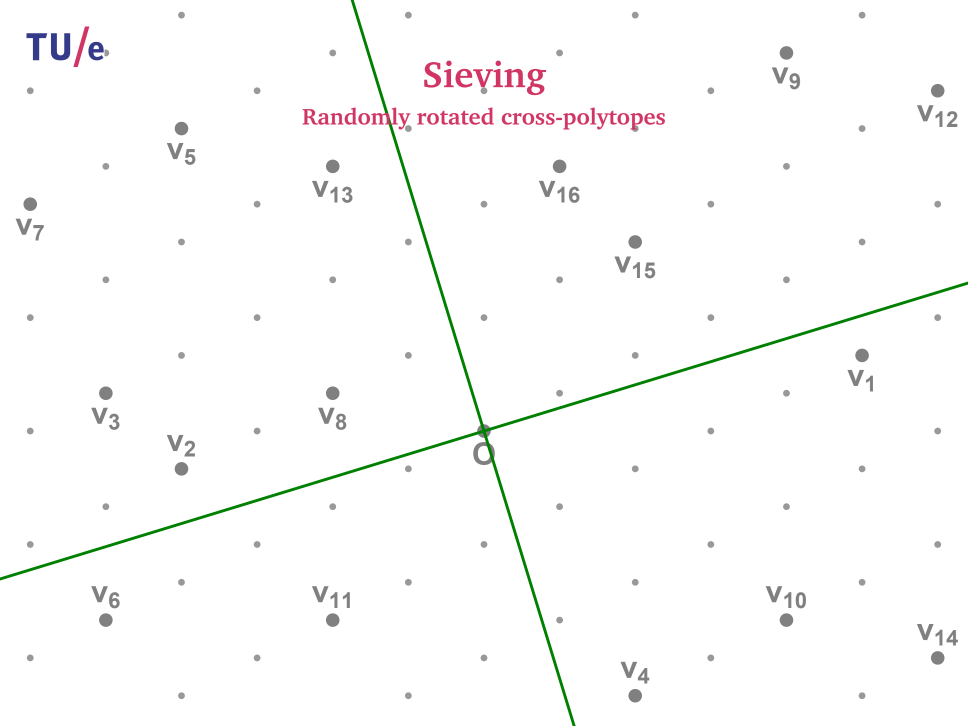


Outline

SVP algorithms

- Enumeration
- Sieving

SVP hardness

- Theory
- Practice
- NIST submissions

Conclusion

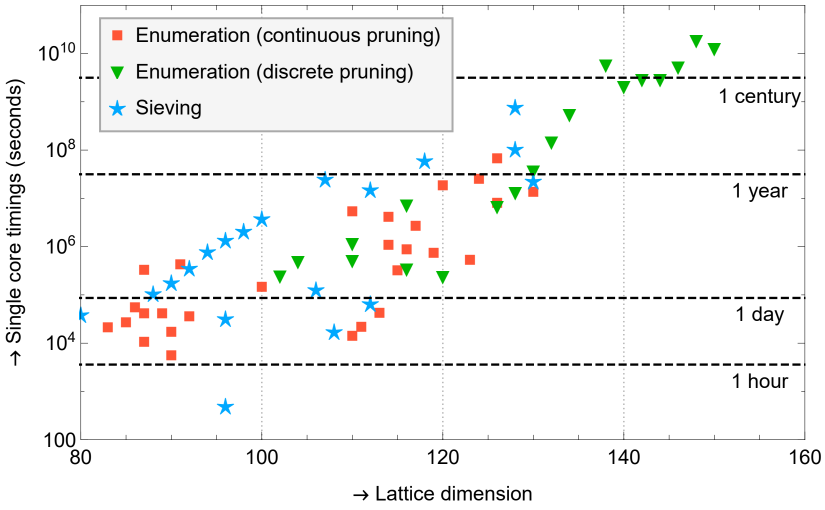
SVP hardness

Theory

	Algorithm	$\log_2$(Time)	$\log_2$(Space)
Proven SVP	Enumeration [Poh81, Kan83, ..., MW15, AN17]	$O(n \log n)$	$O(\log n)$
	AKS-sieve [AKS01, NV08, MV10, HPS11]	$3.398n$	$1.985n$
	ListSieve [MV10, MDB14]	$3.199n$	$1.327n$
	Birthday sieves [PS09, HPS11]	$2.465n$	$1.233n$
	Enumeration/DGS hybrid [CCL17]	$2.048n$	$0.500n$
	Voronoi cell algorithm [AEVZ02, MV10b]	$2.000n$	$1.000n$
	Quantum sieve [LMP13, LMP15]	$1.799n$	$1.286n$
	Quantum enum/DGS [CCL17]	$1.256n$	**0.500n**
	Discrete Gaussian sampling [ADRS15, ADS15, AS18]	**1.000n**	$1.000n$
Sieving	The Nguyen–Vidick sieve [NV08]	$0.415n$	$0.208n$
	GaussSieve [MV10, ..., IKMT14, BNvdP16, YKYC17]	$0.415n$	$0.208n$
	Triple sieve [BLS16, HK17]	$0.396n$	$0.189n$
	Leveled sieving [WLTB11, ZPH13]	$0.3778n$	$0.283n$
	Overlattice sieve [BGJ14]	$0.3774n$	$0.293n$
	Quantum sieve [LMP13]	$0.312n$	$0.208n$
Sieving + NNS	Triple sieve with NNS [HK17, HKL18]	$0.359n$	**0.189n**
	Hyperplane LSH [Cha02, Laa15, ..., LM18, Duc18]	$0.337n$	$0.337n$
	Graph-based NNS [EPY99, DCL11, MPLK14, Laa18]	$0.327n$	$0.282n$
	Hypercube LSH [TT07, Laa17]	$0.322n$	$0.322n$
	May–Ozerov NNS [MO15, BGJ15]	$0.311n$	$0.311n$
	Spherical LSH [AINR14, LdW15]	$0.298n$	$0.298n$
	Cross-polytope LSH [TT07, AILRS15, BL16, KW17]	$0.298n$	$0.298n$
	Spherical LSF [BDGL16, MLB17, ALRW17, Chr17]	**0.293n**	$0.293n$
	Quantum NNS sieve [LMP15, Laa16]	**0.265n**	$0.265n$

SVP hardness

Practice



SVP hardness

NIST submissions

Title	S	E	O	Submitters
CRYSTALS–Dilithium	•			**Lyubashevsky**, Ducas, Kiltz, Lepoint, Schwabe, Seiler, Stehlé
CRYSTALS–Kyber	•			**Schwabe**, Avanzi, Bos, Ducas, Kiltz, Lepoint, Lyubashevsky, Schanck, ...
Ding Key Exchange	•			**Ding**, Takagi, Gao, Wang
DRS			•	**Plantard**, Sipasseuth, Dumondelle, Susilo
(R.)EMBLEM	•			**Seo**, Park, Lee, Kim, Lee
FALCON	•			**Prest**, Fouque, Hoffstein, Kirchner, Lyubashevsky, Pornin, Ricosset, ...
FrodoKEM	•			**Naehrig**, Alkim, Bos, Ducas, Easterbrook, LaMacchia, Longa, Mironov, ...
Giophantus	•			**Akiyama**, Goto, Okumura, Takagi, Nuida, Hanaoka, Shimizu, Ikematsu
HILA5	•			**Saarinen**
KCL	•			**Zhao**, Jin, Gong, Sui
KINDI	•			**El Bansarkhani**
LAC	•			**Lu**, Liu, Jia, Xue, He, Zhang
LIMA	•			**Smart**, Albrecht, Lindell, Orsini, Osheter, Paterson, Peer
Lizard	•			**Cheon**, Park, Lee, Kim, Song, Hong, Kim, Kim, Hong, Yun, Kim, Park, ...
LOTUS		•		**Phong**, Hayashi, Aono, Moriai
NewHope	•			**Pöppelmann**, Alkim, Avanzi, Bos, Ducas, De La Piedra, Schwabe, Stebila
NTRUEncrypt	◦	◦		**Zhang**, Chen, Hoffstein, Whyte
NTRU-HRSS-KEM	•			**Schanck**, Hülsing, Rijneveld, Schwabe
NTRU Prime		•		**Bernstein**, Chuengsatiansup, Lange, Van Vredendaal
Odd Manhattan			•	**Plantard**
pqNTRUSign	◦	◦		**Zhang**, Chen, Hoffstein, Whyte
qTESLA	•			**Bindel**, Akleylek, Alkim, Barreto, Buchmann, Eaton, Gutoski, Krämer, ...
Round2	•			**Garcia-Morchon**, Zhang, Bhattacharya, Rietman, Tolhuizen, Torre-Arce
SABER	•			**D'Anvers**, Karmakar, Roy, Vercauteren
Three Bears	•			**Hamburg**
Titanium	•			**Steinfeld**, Sakzad, Zhao
Totals:	**21**	**3**	**2**	**Total: 26 proposals with SVP hardness estimates**

*Not included in the overview: Compact LWE, Mersenne, Ramstake, ...

SVP hardness

NIST submissions

Most common hardness estimates:

- Ignore space complexity, ignore $o(n)$ in time complexity
- Classical sieving: $2^{0.292n}$ time
- Quantum sieving: $2^{0.265n}$ time
- "Paranoid bound": $2^{0.208n}$ time
  - Note: $2^{0.208n}$ is not a "lower bound" (Frodo, ...)
- Complexity of BKZ(β) $\geq$ Complexity of SVP(β)

Conclusion

Summary

- Lattice-based crypto relies on hardness of finding short vectors
- State-of-the-art basis reduction: BKZ with fast SVP subroutine
- Enumeration for SVP:
  - Brute-force all combinations of basis vectors
  - Memory-efficient
  - Fast pruning heuristics
- Sieving for SVP:
  - Consider pairwise combinations of many lattice vectors
  - Large memory requirement
  - Practical near neighbor speed-ups
- Theory: Sieving superior in high dimensions
- Practice: Enumeration superior in low dimensions
- Hardness estimates: Commonly based on sieving

Conclusion

Open problems

Challenges in provable SVP algorithms

- Sieving: Control distribution of lattice vectors
- DGS: Combine vectors more efficiently
- Obtain proven bounds for “random” lattices

Challenges in heuristic SVP algorithms

- Enumeration: Find best pruning methods
- Sieving: Find best near neighbor techniques
- Effective time–memory trade-offs

Challenges in SVP hardness estimation

- Analyze sieving in a more realistic memory model
- Lower bounds (SVP, sieving, enumeration, ...)
- Consensus on attack model/memory costs

Questions?