IBM Research

Hypercube locality-sensitive hashing for approximate near neighbors

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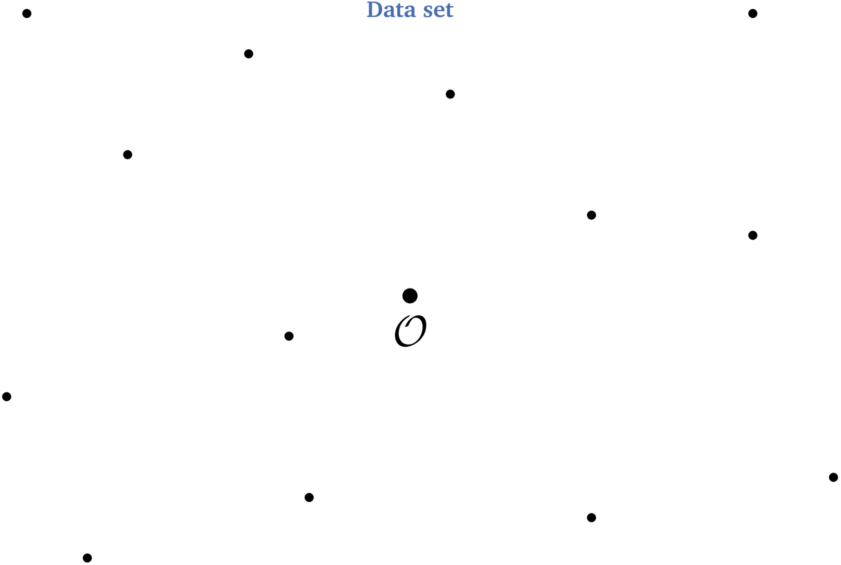
Nearest neighbor searching

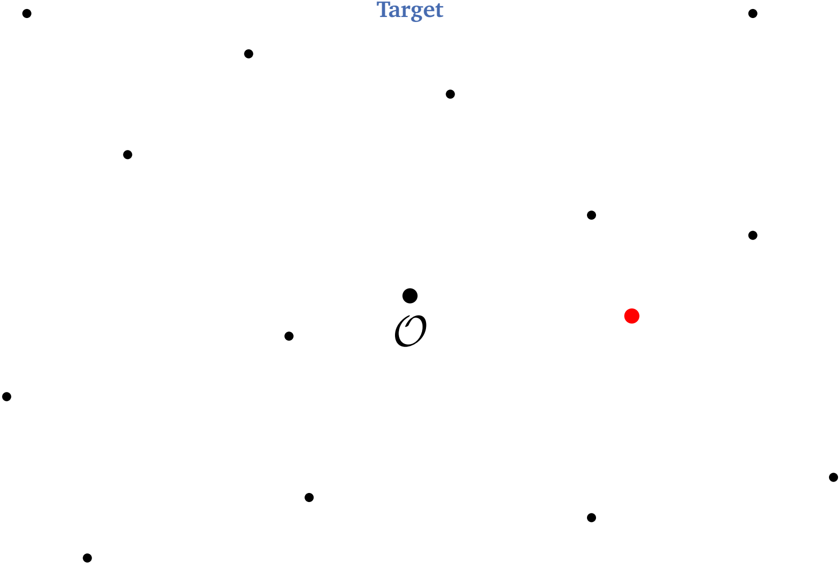
Nearest neighbor searching

Data set



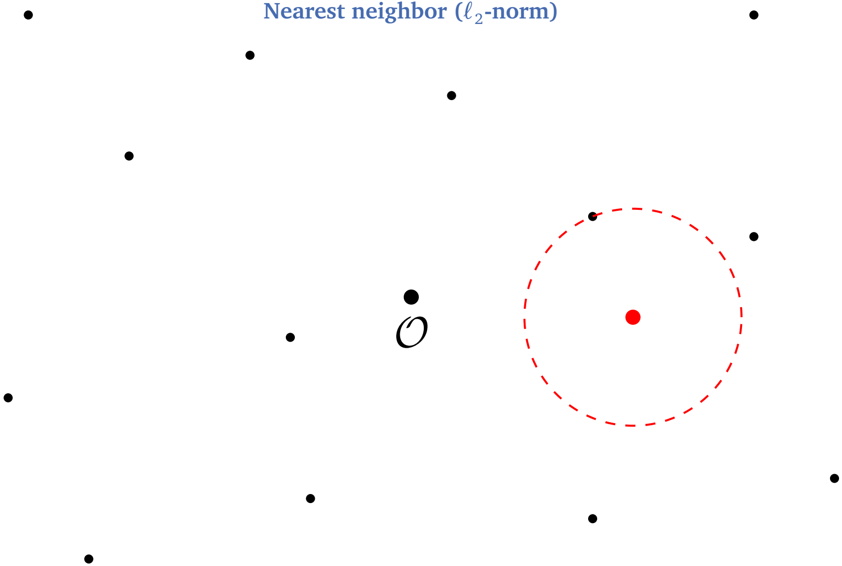
Nearest neighbor searching

Target



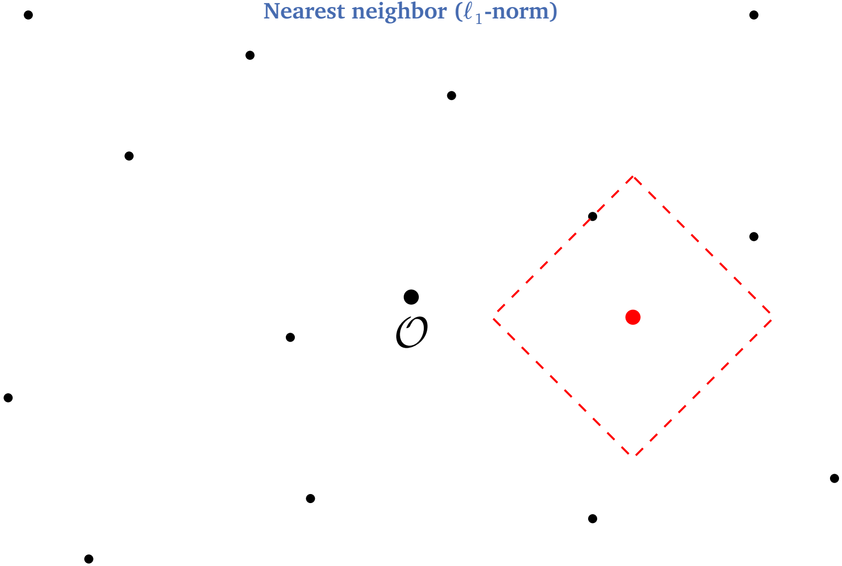
Nearest neighbor searching

Nearest neighbor (ℓ_2-norm)



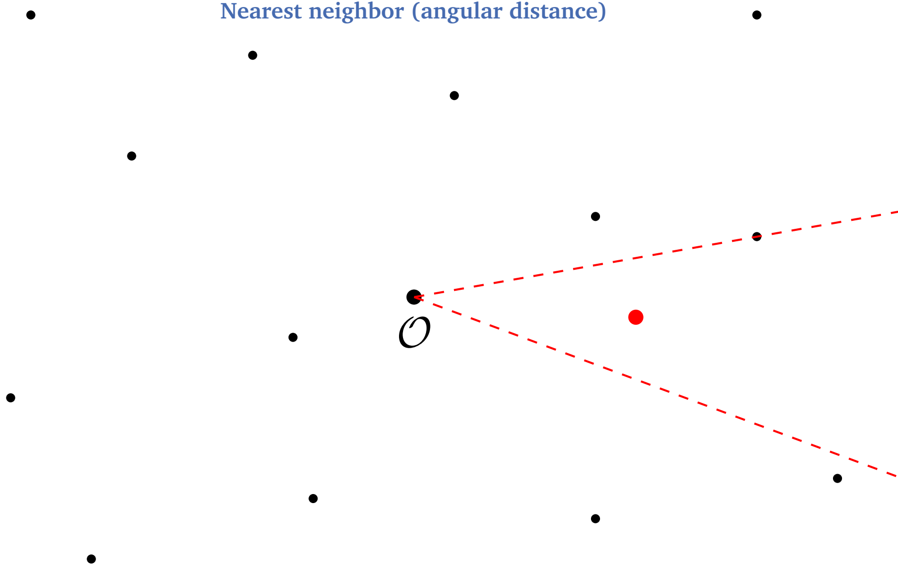
Nearest neighbor searching

Nearest neighbor (ℓ_1-norm)



Nearest neighbor searching

Nearest neighbor (angular distance)



Nearest neighbor searching

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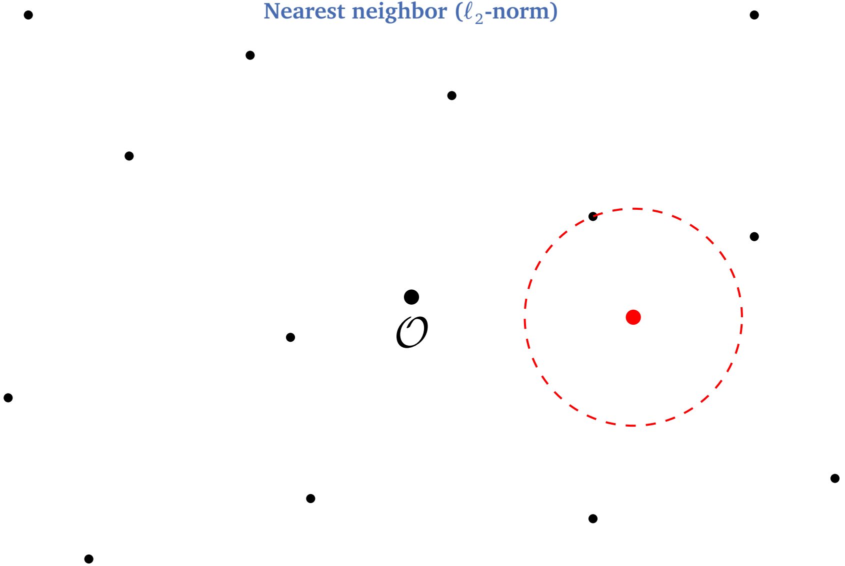
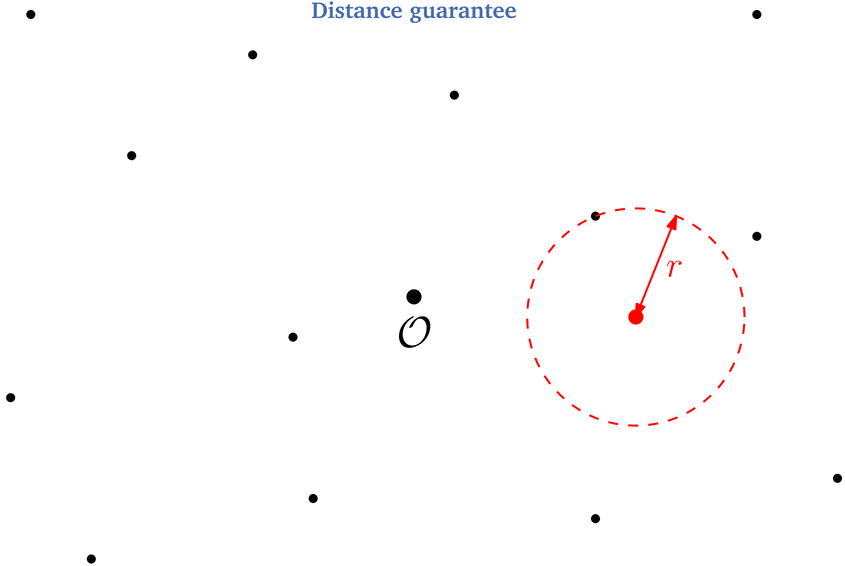


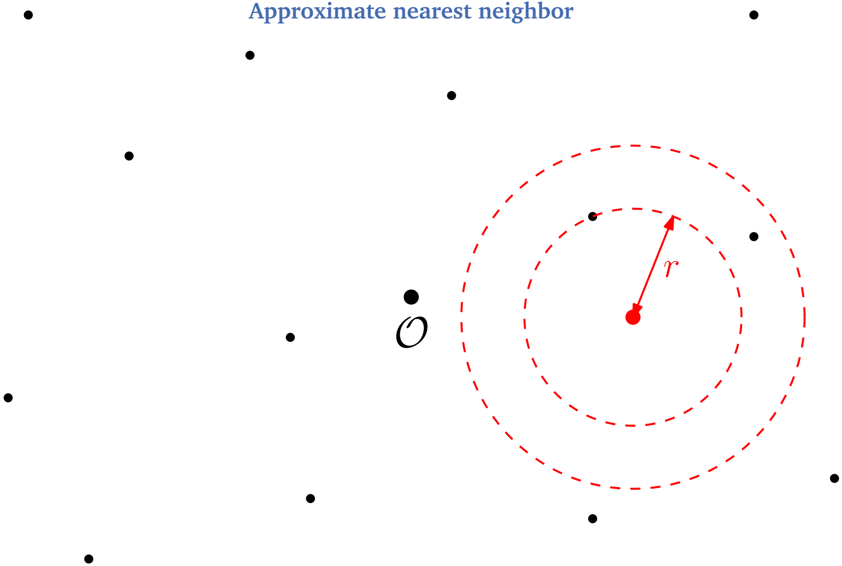
Nearest neighbor searching

Distance guarantee



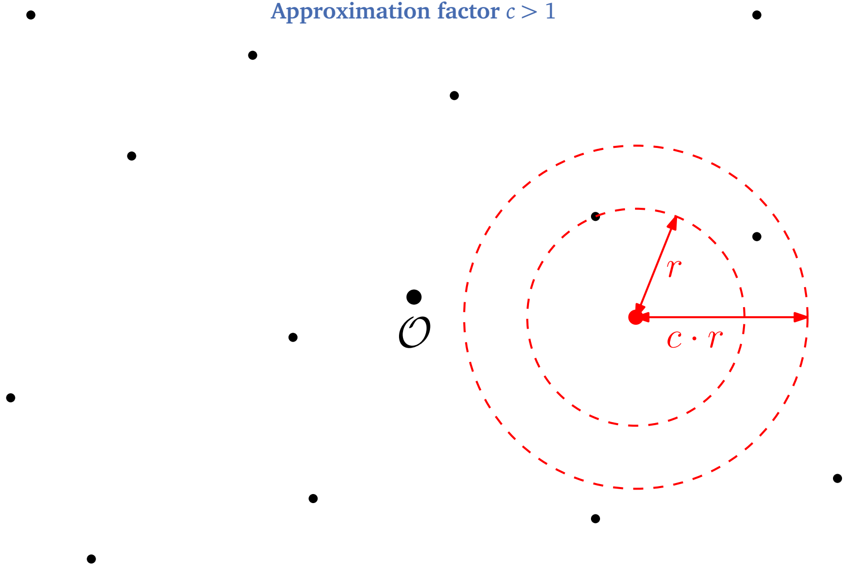
Nearest neighbor searching

Approximate nearest neighbor



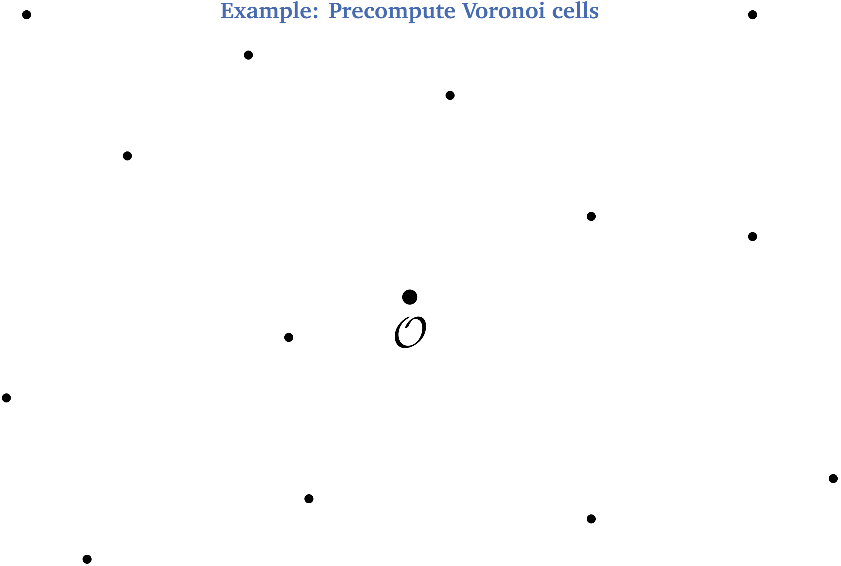
Nearest neighbor searching

Approximation factor $c > 1$



Nearest neighbor searching

Example: Precompute Voronoi cells

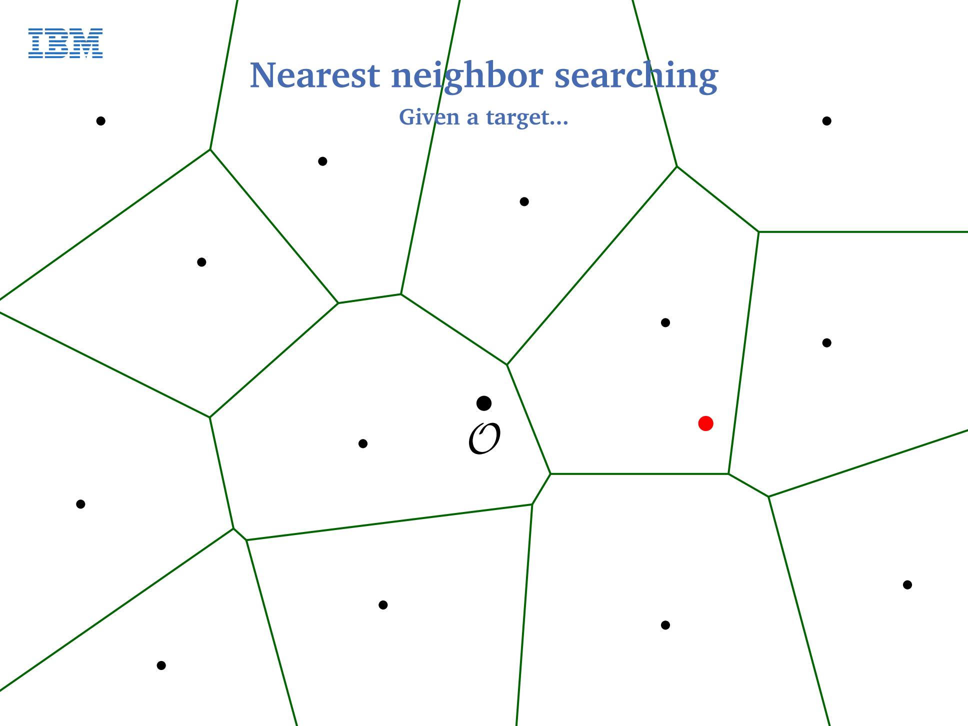


Nearest neighbor searching

Example: Precompute Voronoi cells

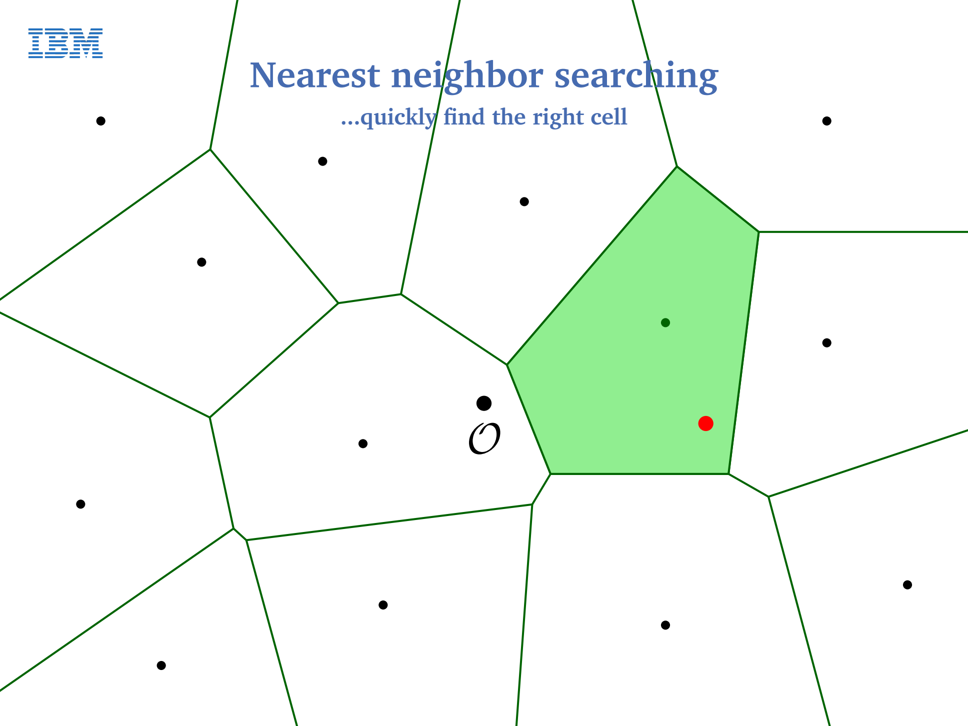
Nearest neighbor searching

Given a target…



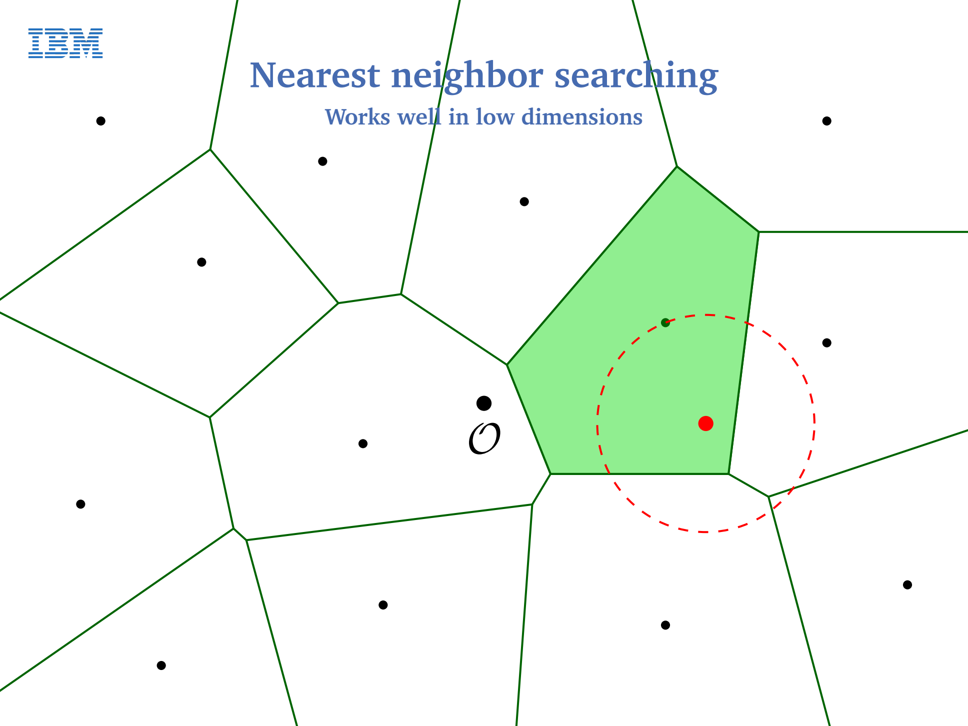
Nearest neighbor searching

...quickly find the right cell



Nearest neighbor searching

Works well in low dimensions



Nearest neighbor searching

Problem setting

- High dimensions d

Nearest neighbor searching

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- Large data set of size $n = 2^{\Omega(d/\log d)}$
  - Smaller n? $\implies$ Use JLT to reduce d

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- Assumption: Data set lies on the sphere
  - Equivalent to angular distance/cosine similarity in all of $\mathbb{R}^d$
  - Reduction from Eucl. NNS in $\mathbb{R}^d$ to Eucl. NNS on the sphere [AR'15]

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Nearest neighbor searching

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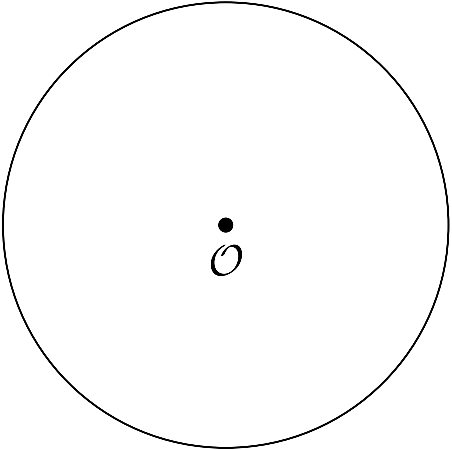
- Goal: Query time $O(n^\rho)$ with $\rho < 1$

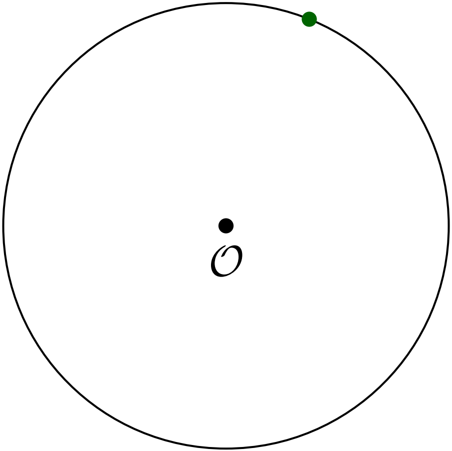
Hyperplane LSH

[Charikar, STOC'02]



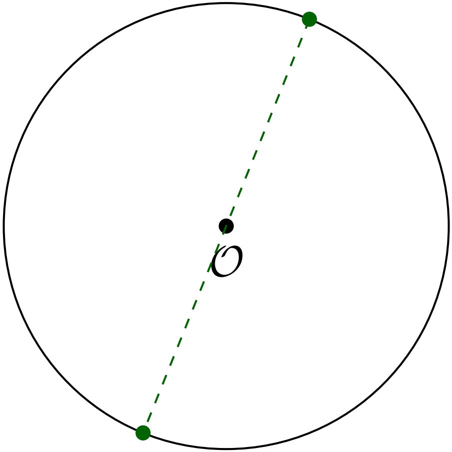
Hyperplane LSH

Random point



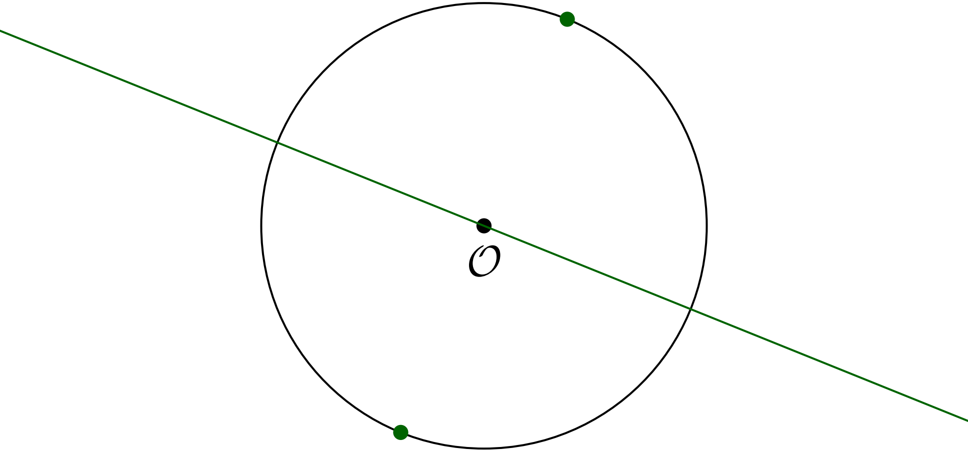
Hyperplane LSH

Opposite point



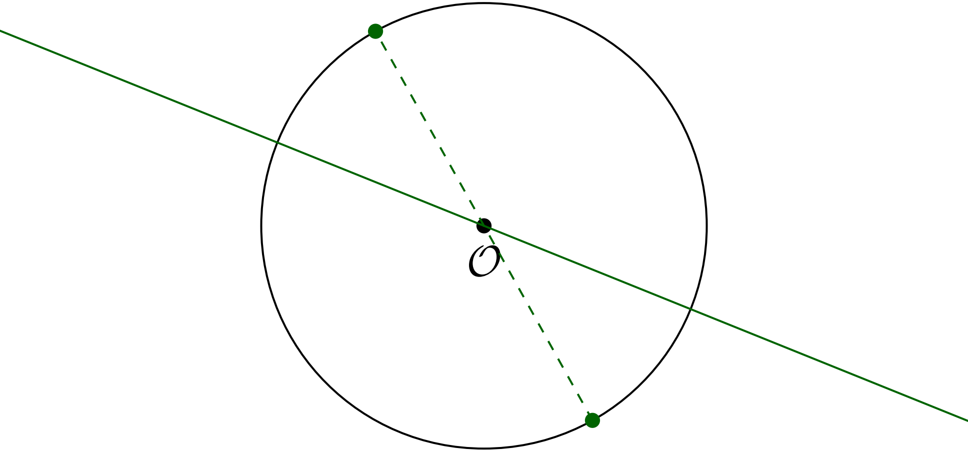
Hyperplane LSH

Two Voronoi cells



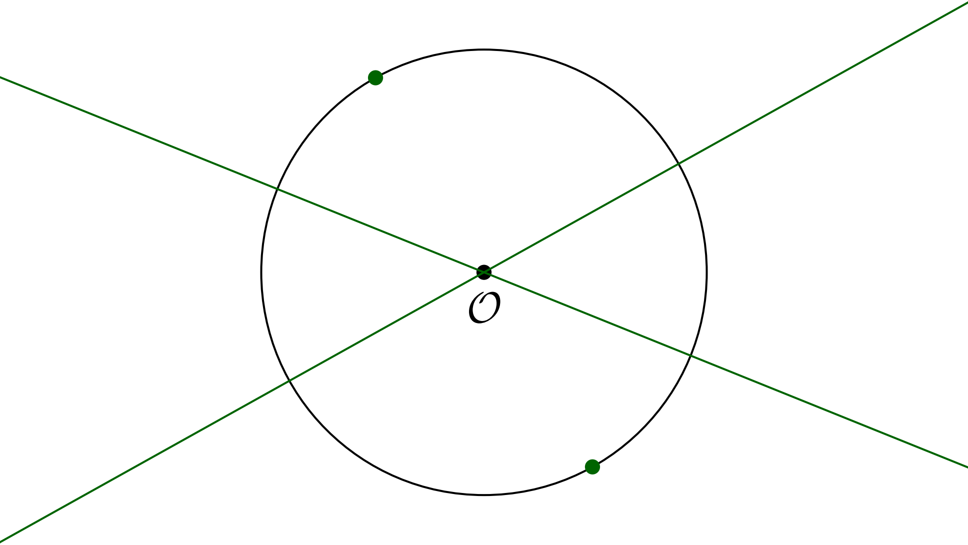
Hyperplane LSH

Another pair of points



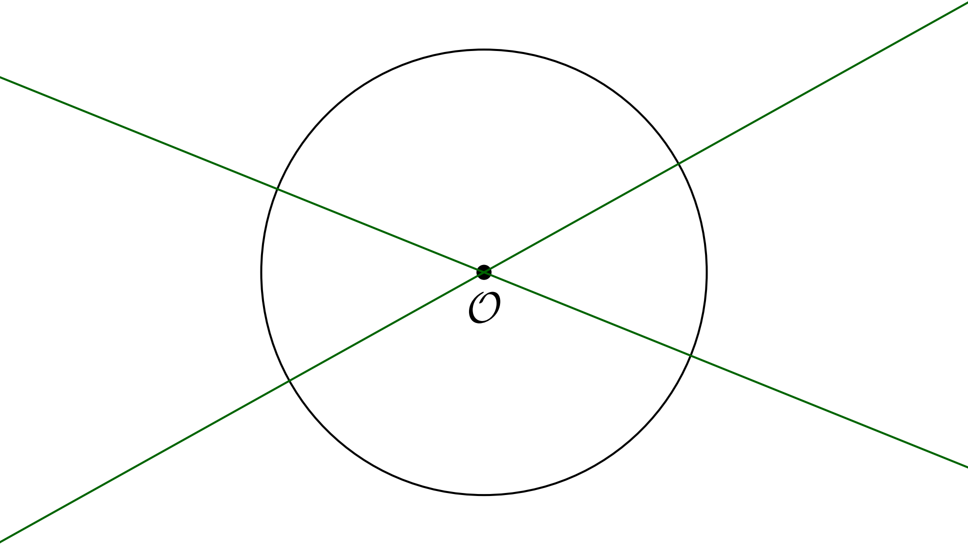
Hyperplane LSH

Another hyperplane



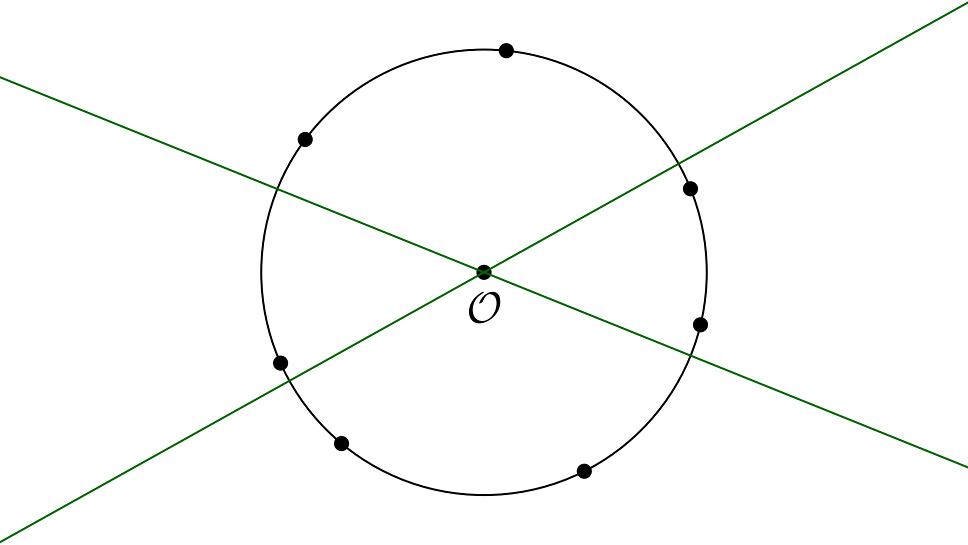
Hyperplane LSH

Defines partition



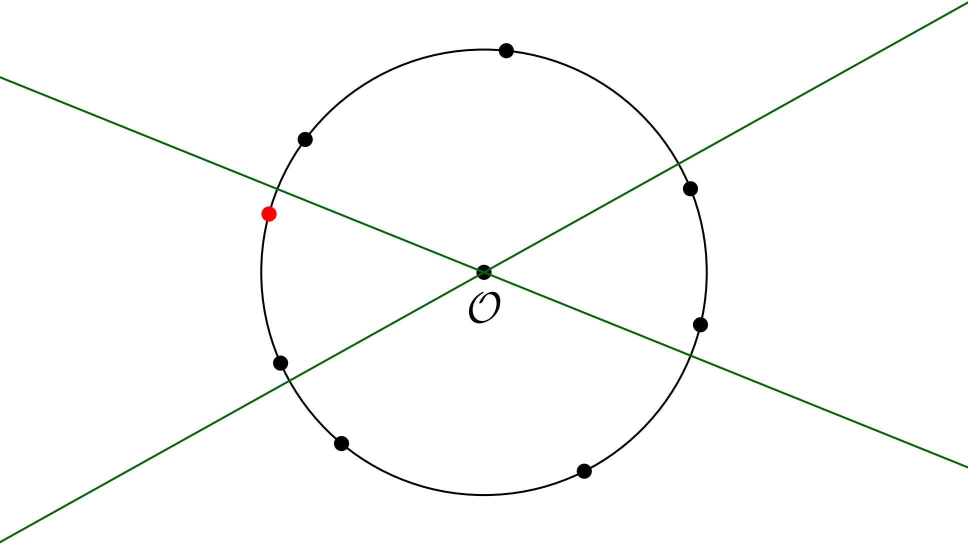
Hyperplane LSH

Preprocessing



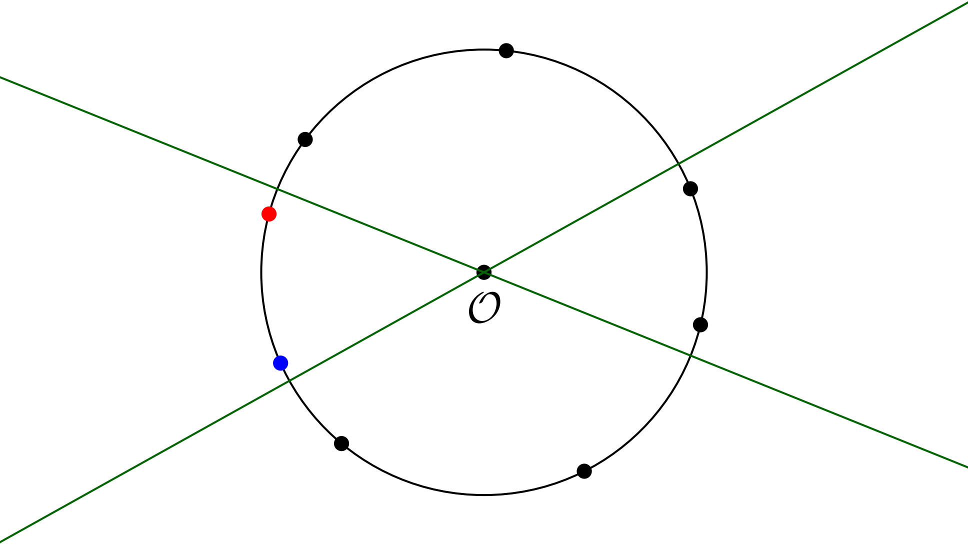
Hyperplane LSH

Query



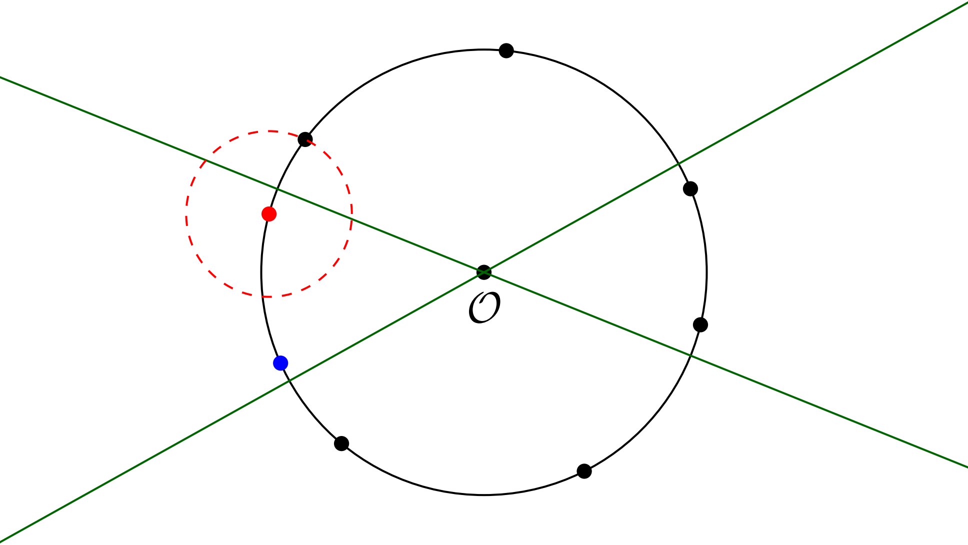
Hyperplane LSH

Collisions



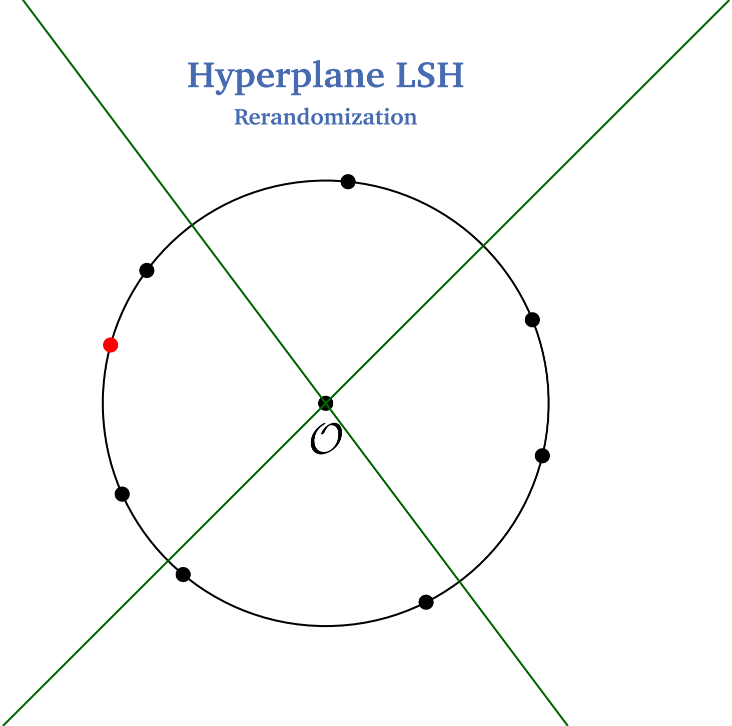
Hyperplane LSH

Failure



Hyperplane LSH

Rerandomization



Hyperplane LSH

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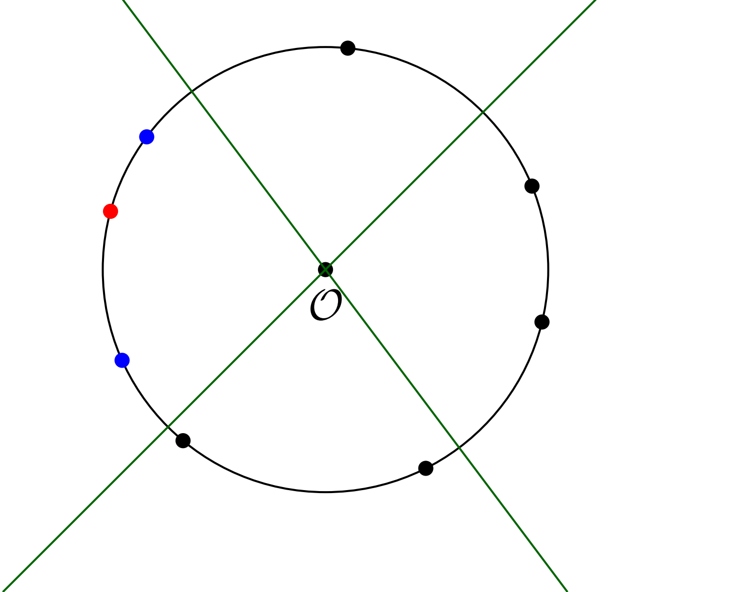
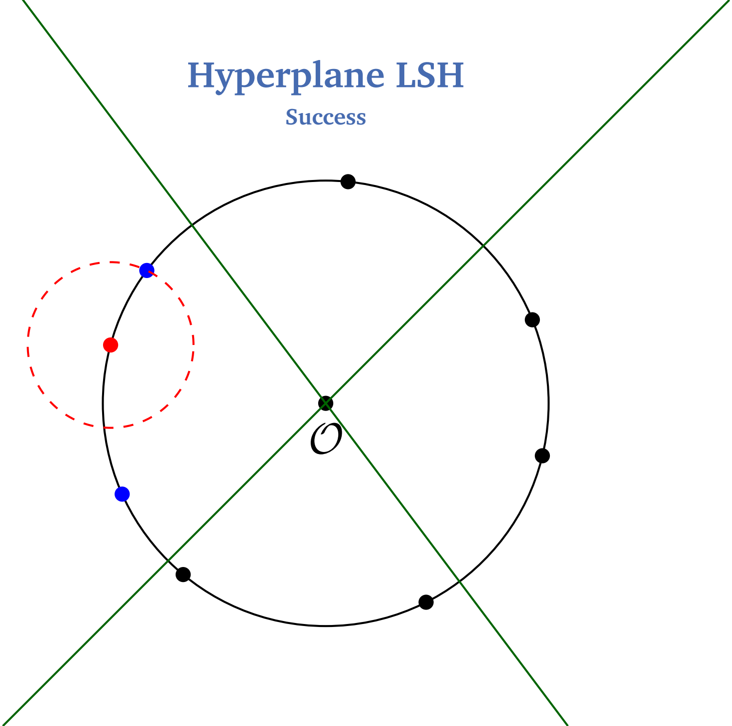


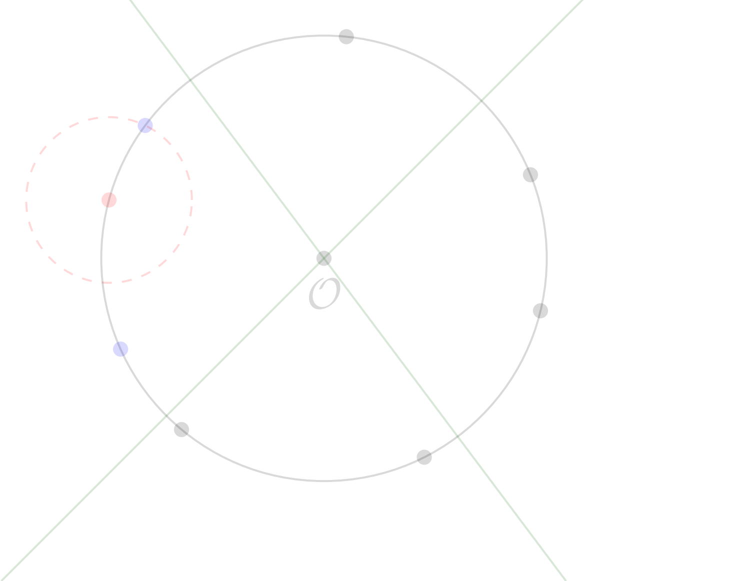
Hyperplane LSH

Success



Hyperplane LSH

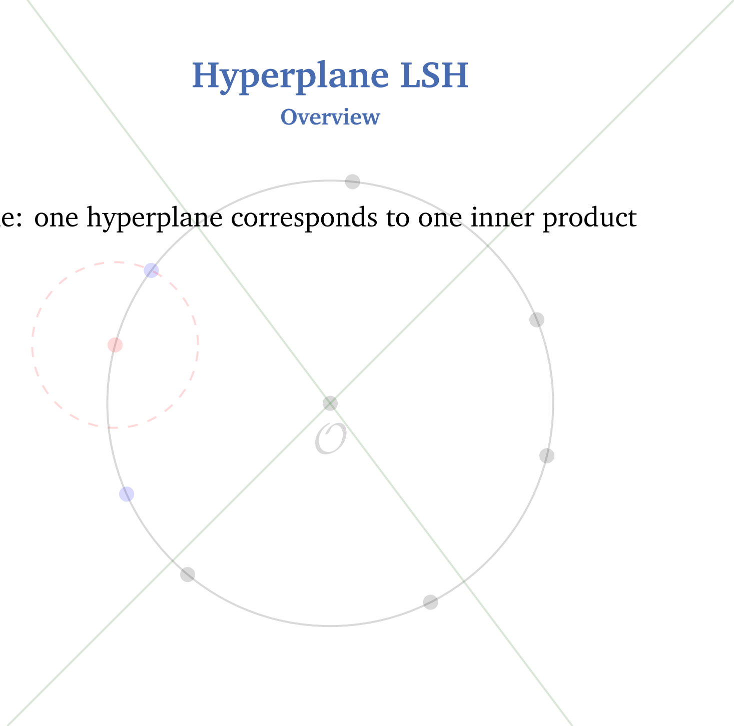
Overview



Hyperplane LSH

Overview

- Simple: one hyperplane corresponds to one inner product



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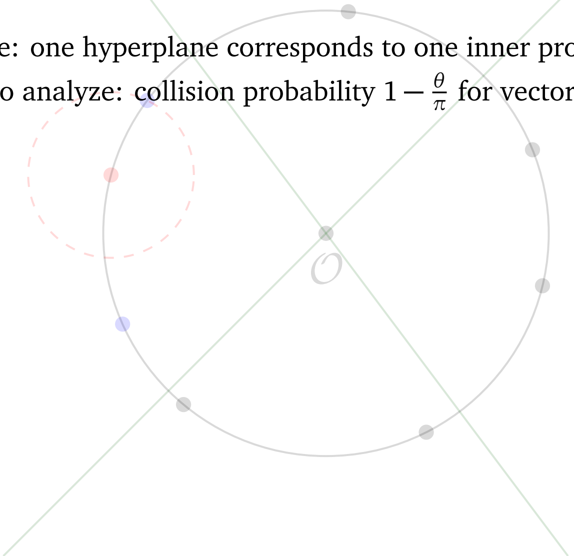

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  - Sparse hyperplane vectors [Ach'01, LHC'06]
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Hyperplane LSH

Asymptotically "optimal"

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Hyperplane LSH

Topic of this paper

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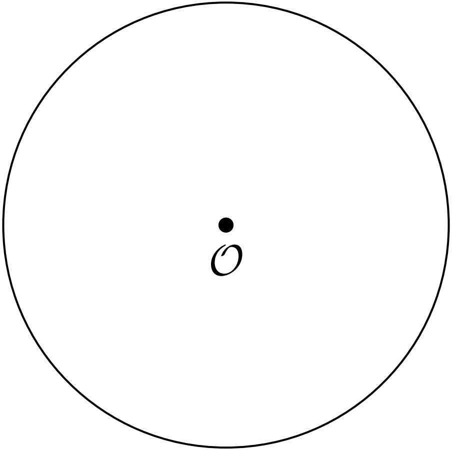
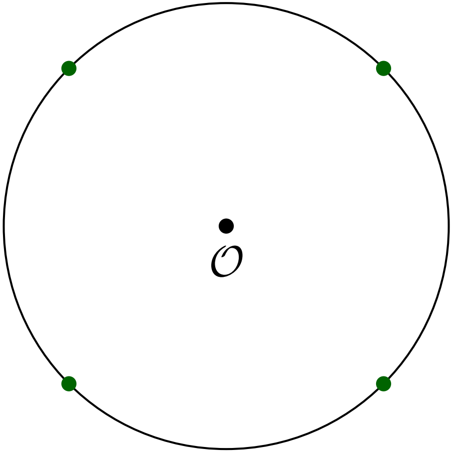
Hypercube LSH

[Terasawa–Tanaka, WADS'07]



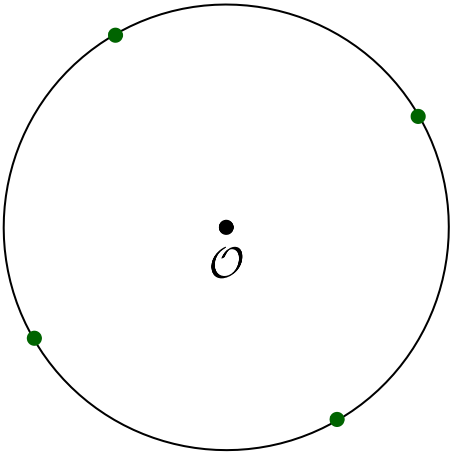
Hypercube LSH

Vertices of hypercube



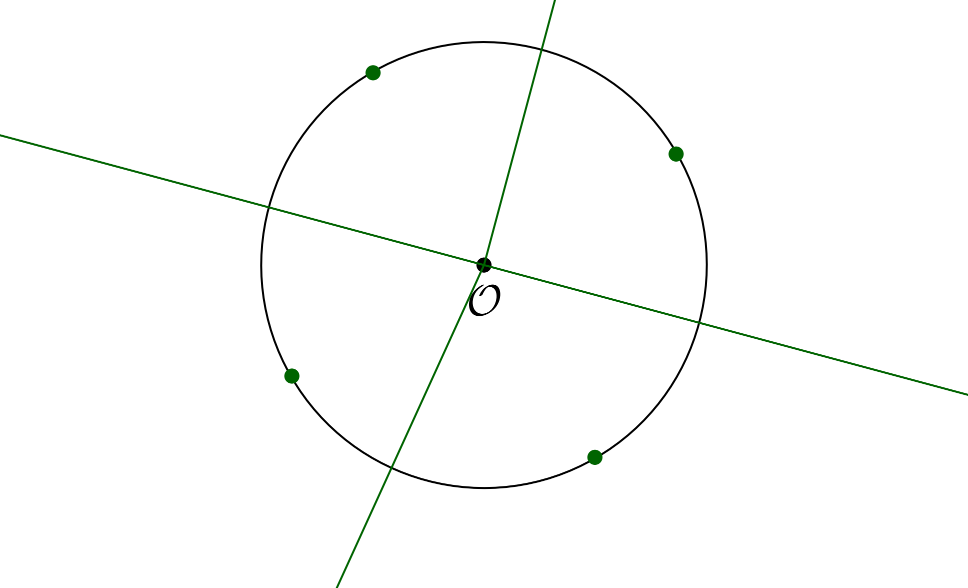
Hypercube LSH

Random rotation



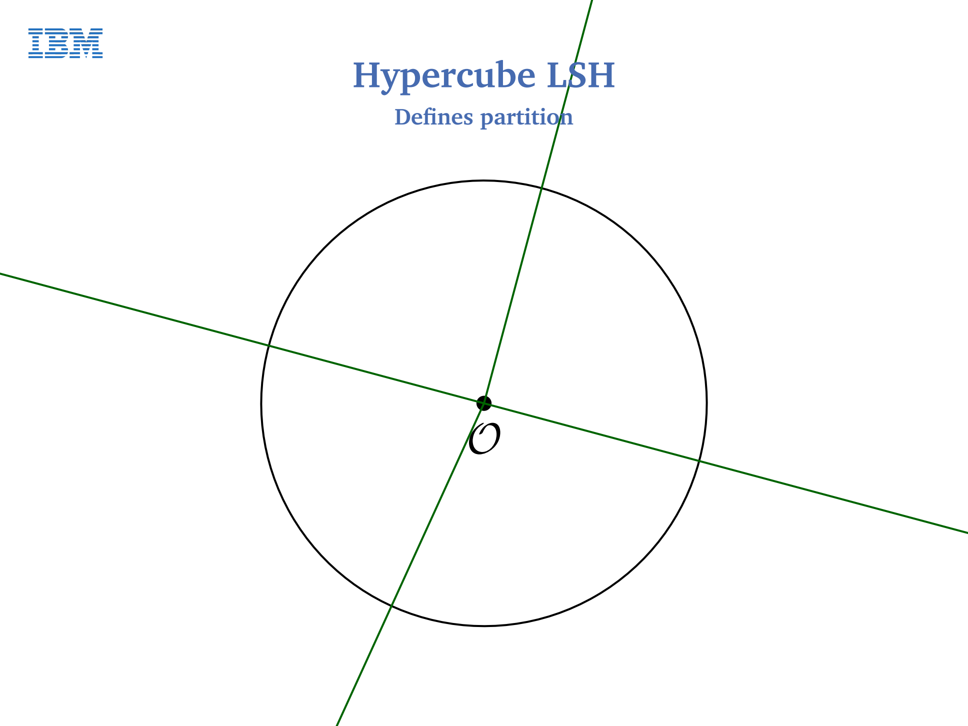
Hypercube LSH

Voronoi regions



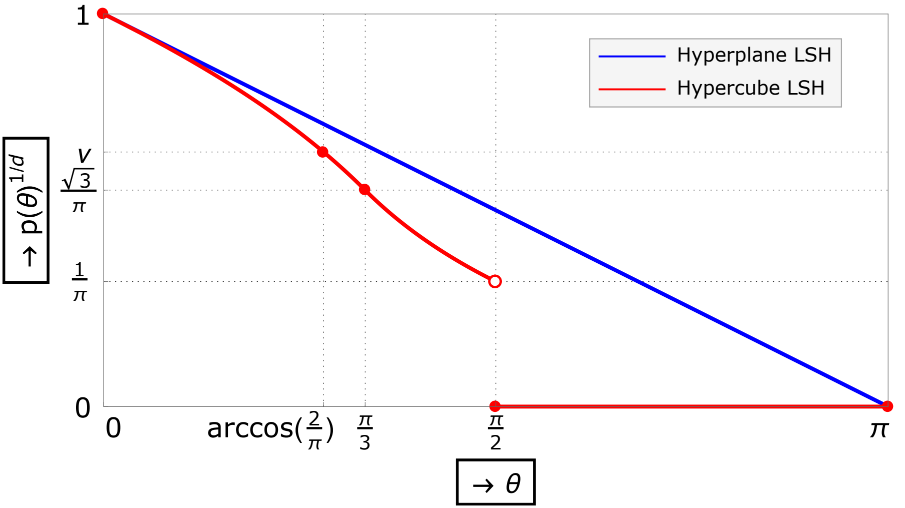
Hypercube LSH

Defines partition



Hypercube LSH

Collision probabilities



Hypercube LSH

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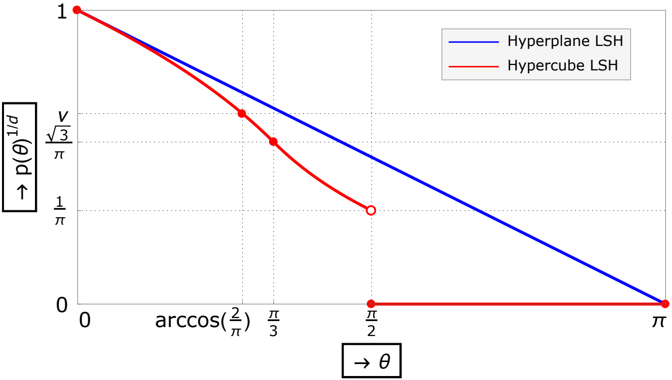
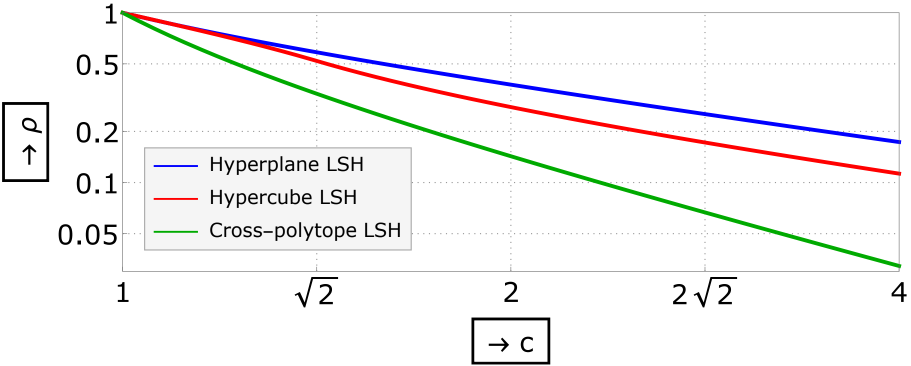


- Two vectors at angle $(\frac{\pi}{2})^-$ lie in the same orthant with probability $(\frac{1}{\pi})^d$
- Two vectors at angle $\frac{\pi}{3}$ lie in the same orthant with probability $(\frac{\sqrt{3}}{\pi})^d$

Hypercube LSH

Asymptotic performance (random data)



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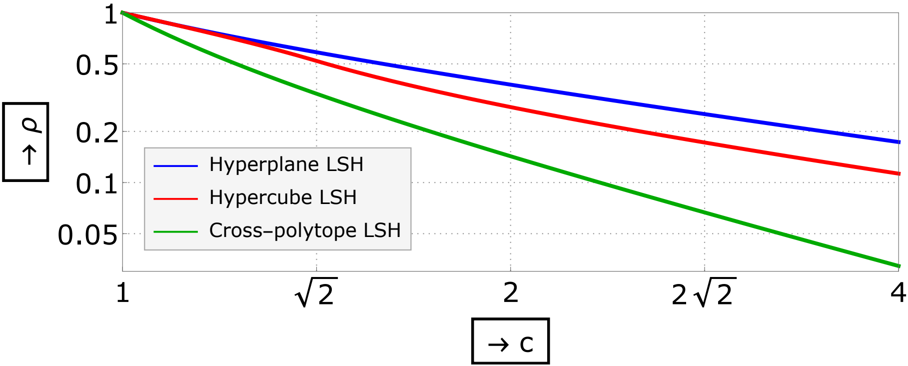

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Hypercube LSH

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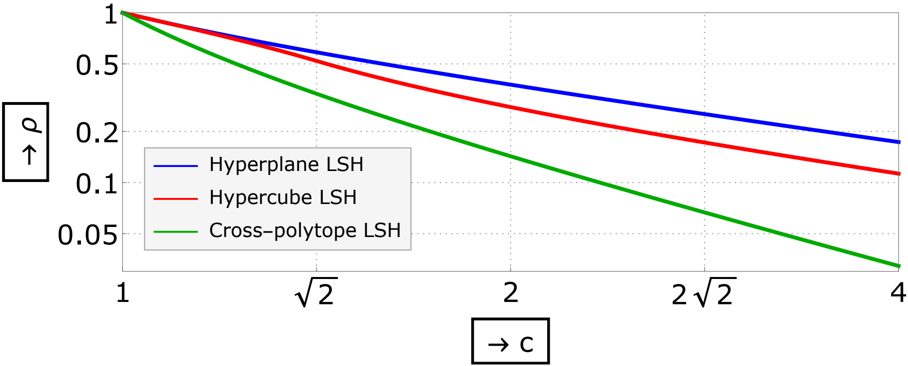

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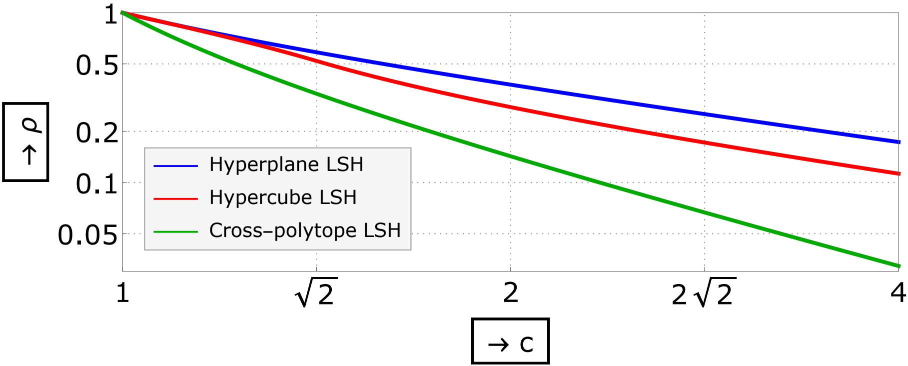

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- Cross-polytope LSH: $\rho = \frac{1}{2c^2-1} + o(\frac{1}{c^2})$

Conclusions

Positive results

- Exact asymptotics for full-dimensional hypercube LSH
- Exact asymptotics for partial hypercube LSH when $d' \leq O(d/\log d)$
- Asymptotically superior to hyperplane LSH
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Thank you for your attention!