IBM Research

Sieving for closest lattice vectors (with preprocessing)

Thijs Laarhoven

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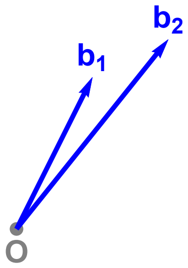
Lattices

Lattices



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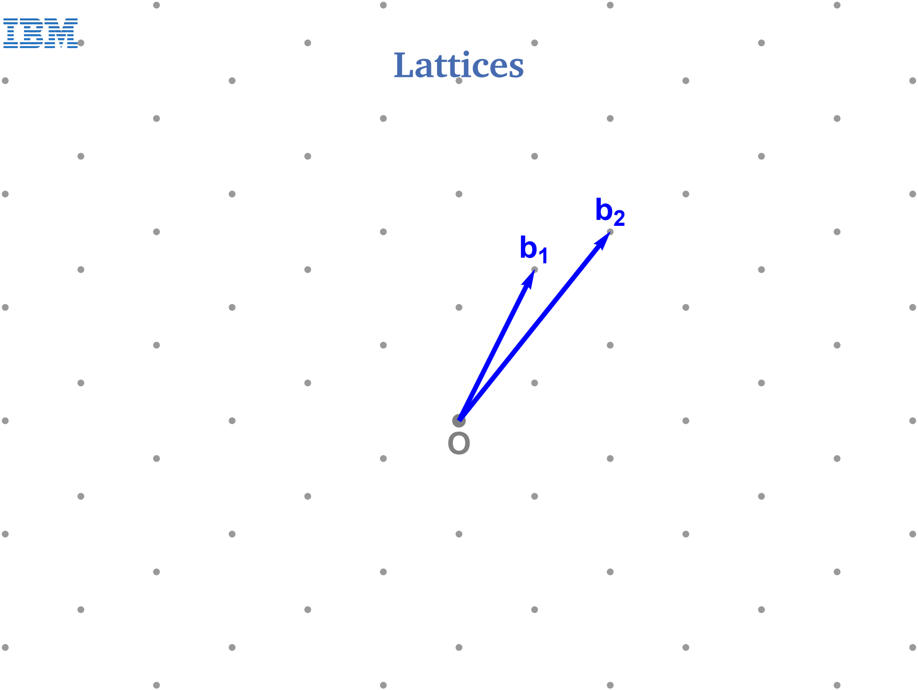
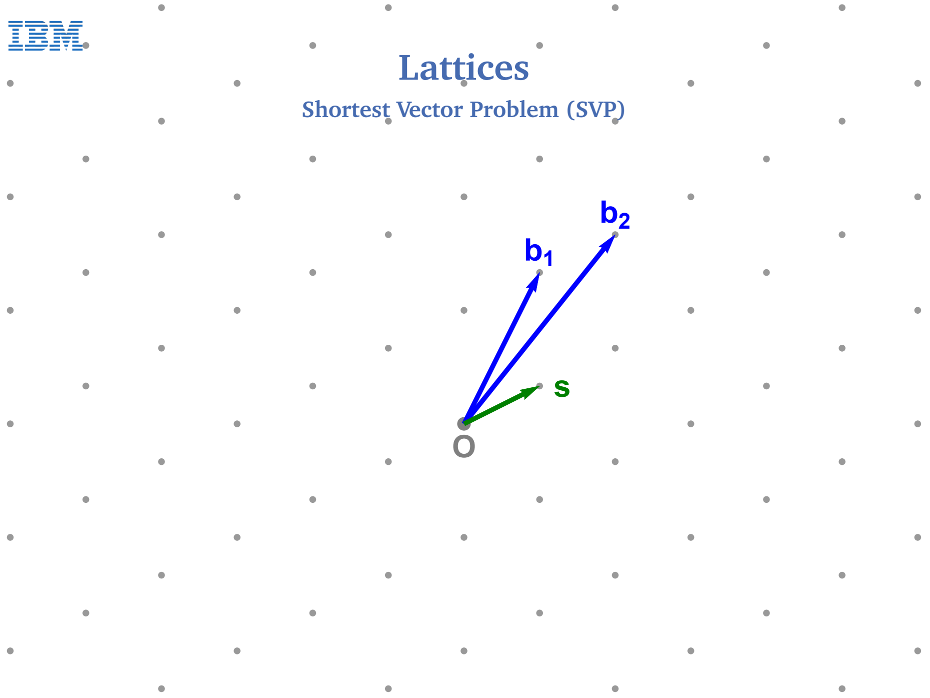


Lattices

Shortest Vector Problem (SVP)



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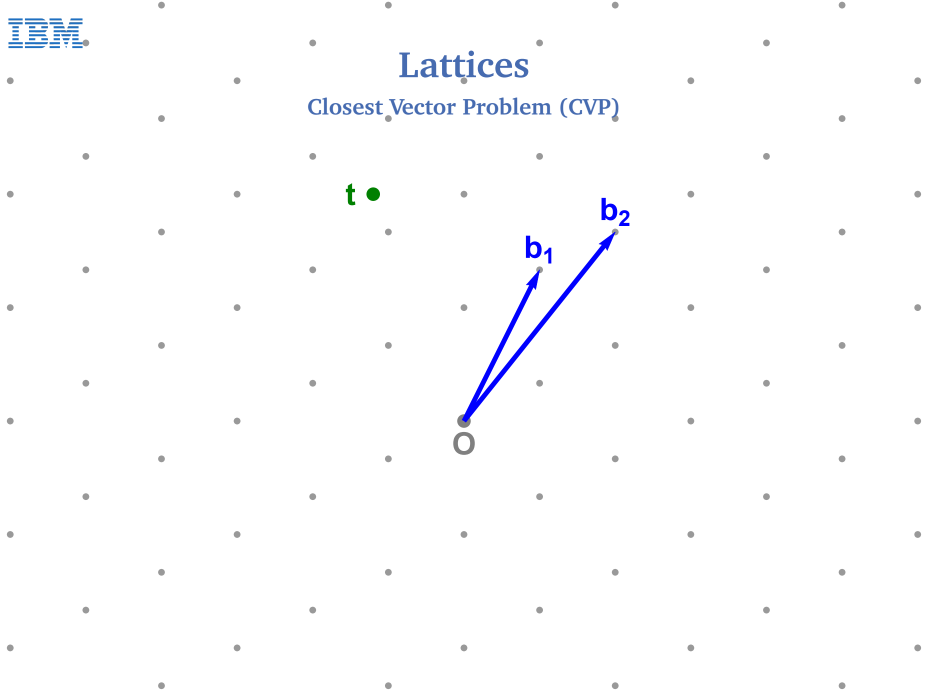

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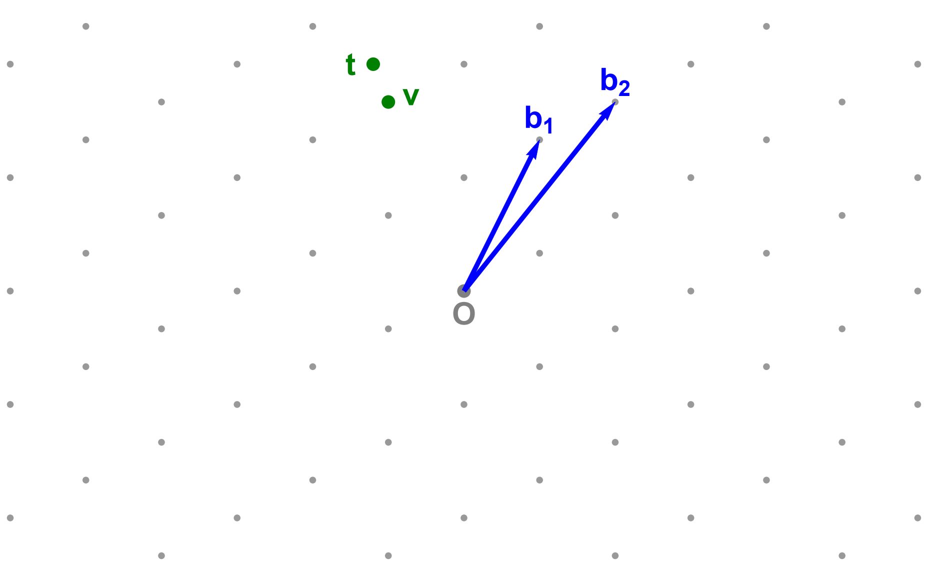

Outline

Sieving for SVP

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Generate random lattice vectors

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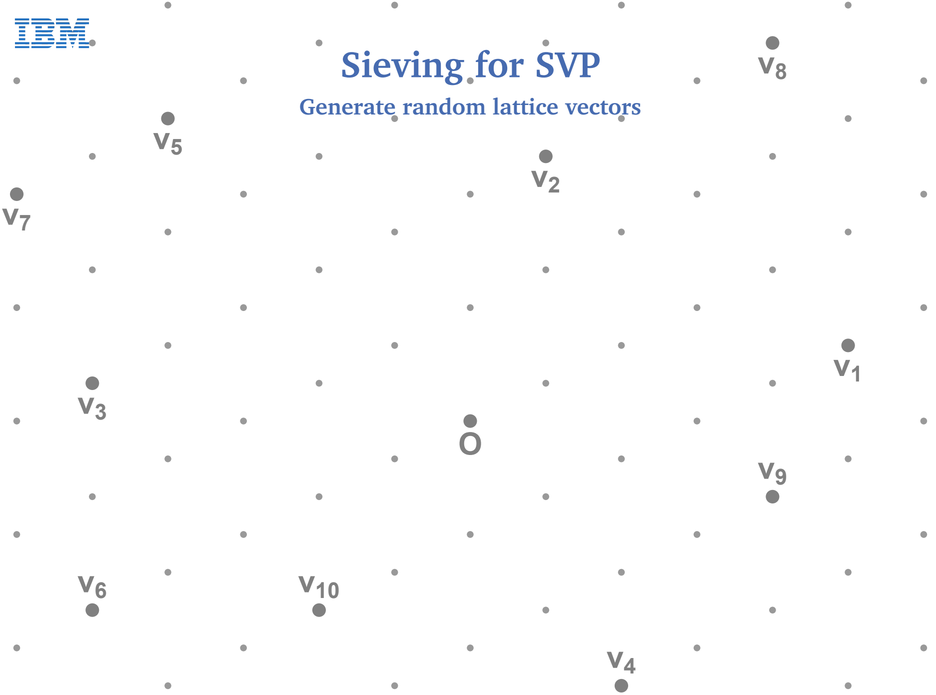
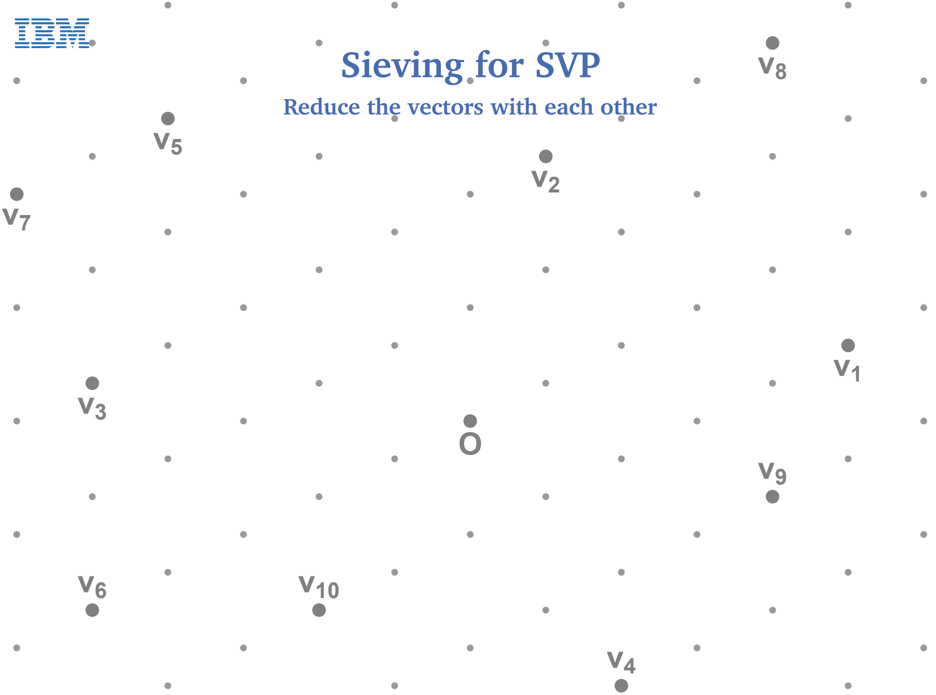


Sieving for SVP

Reduce the vectors with each other



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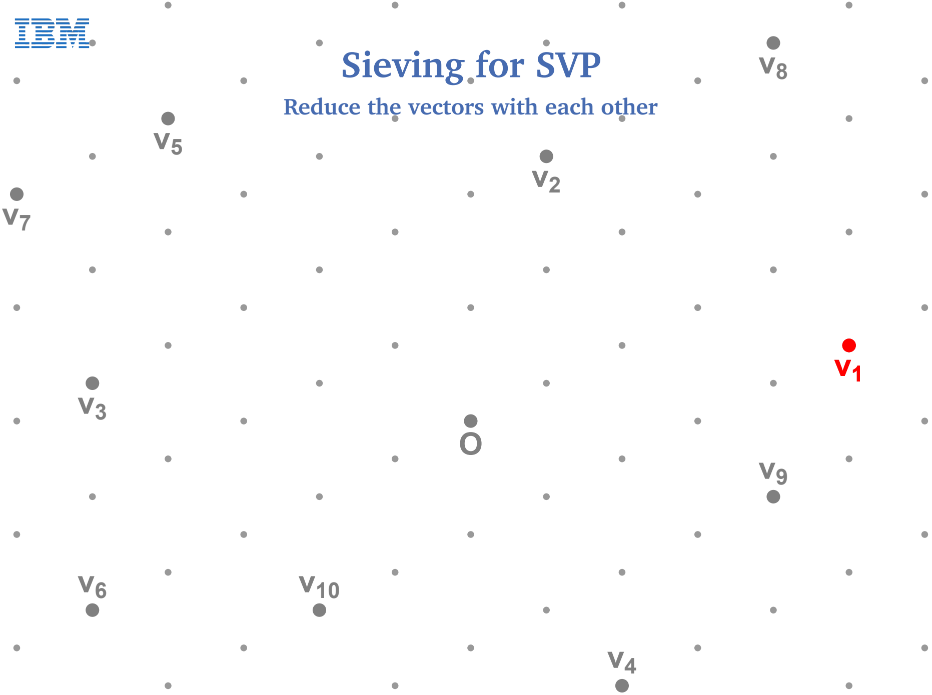

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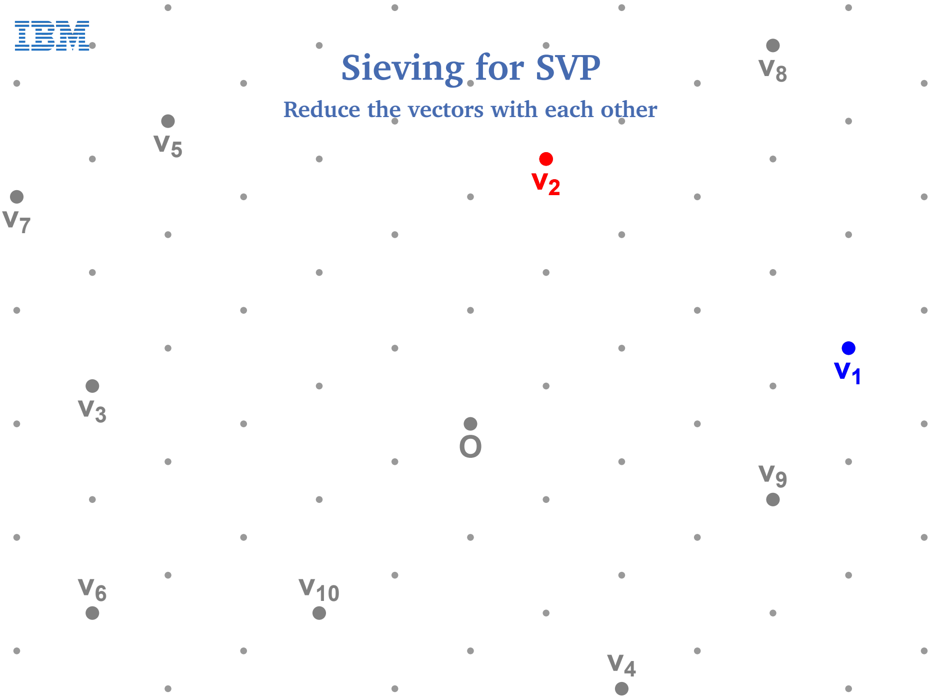

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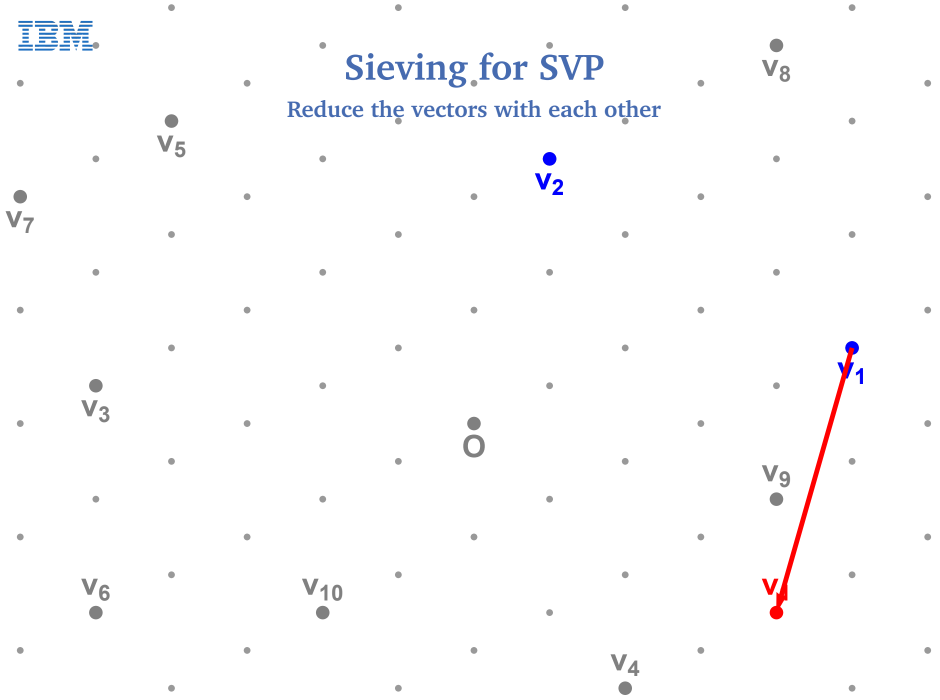

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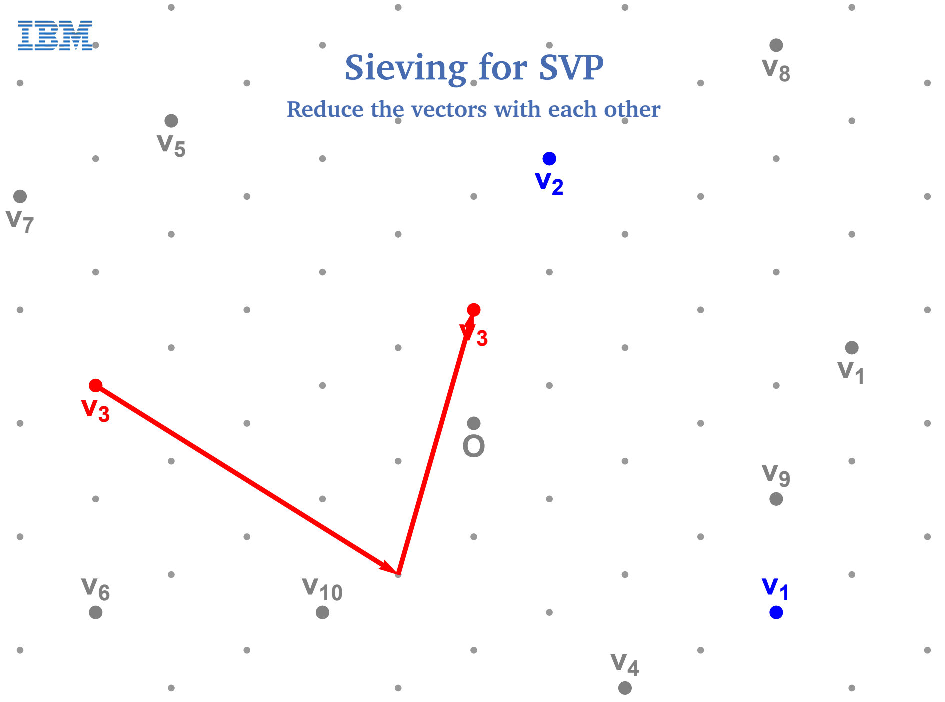

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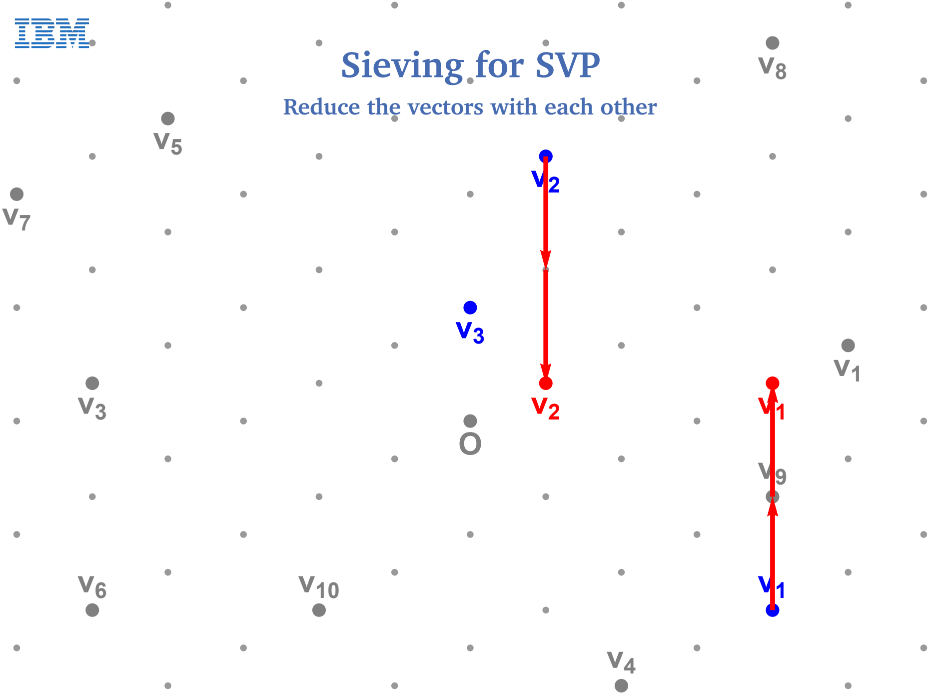

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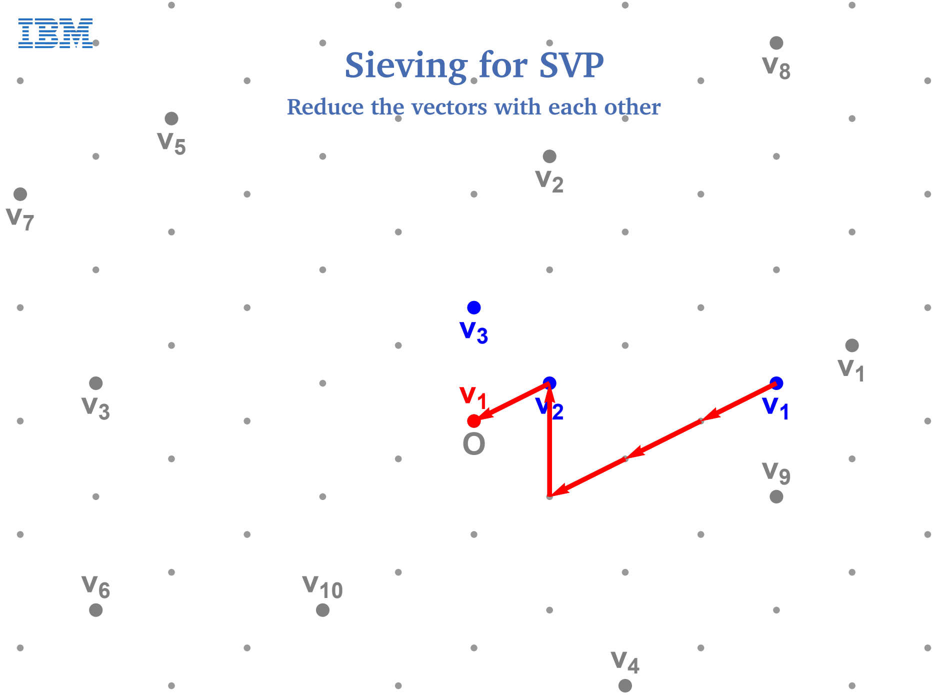

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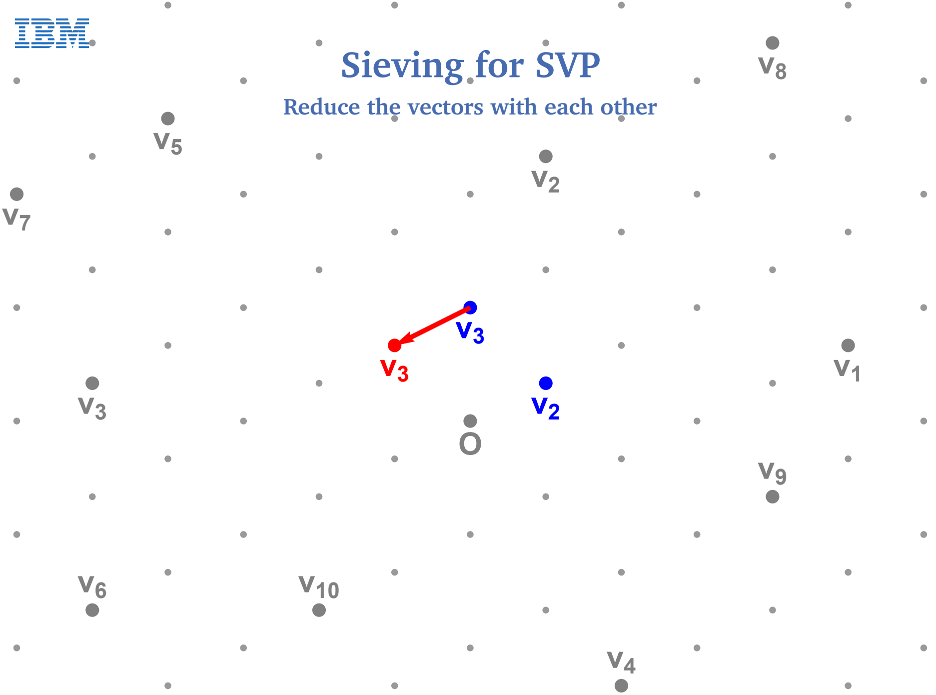

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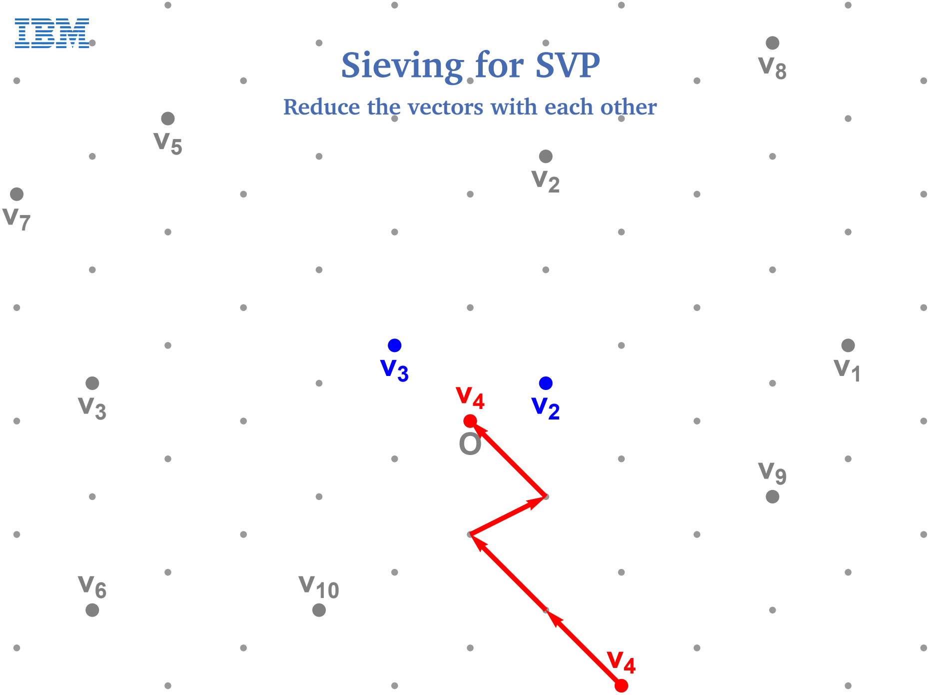

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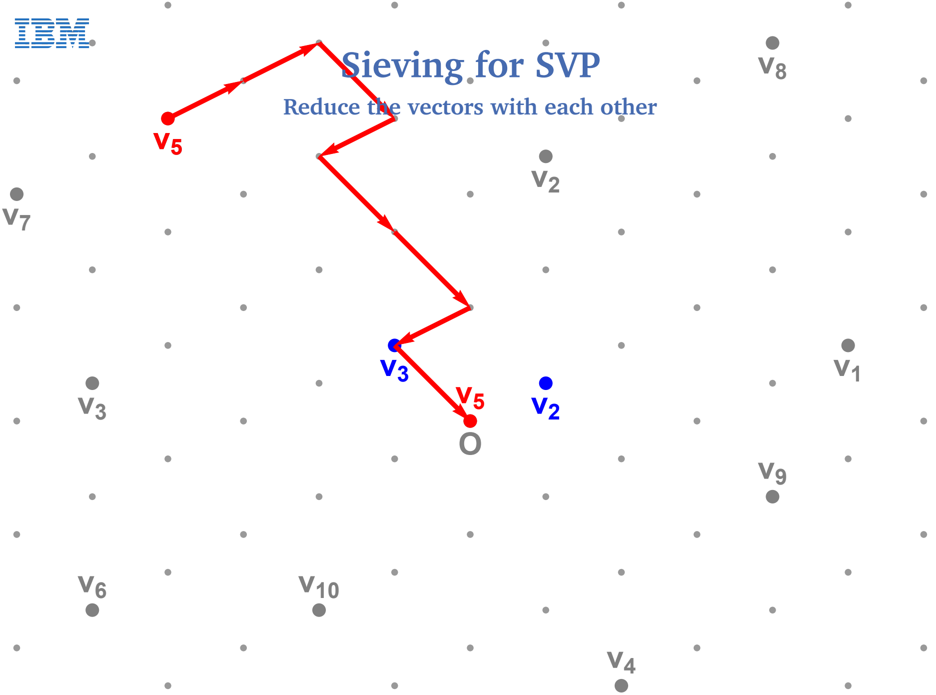

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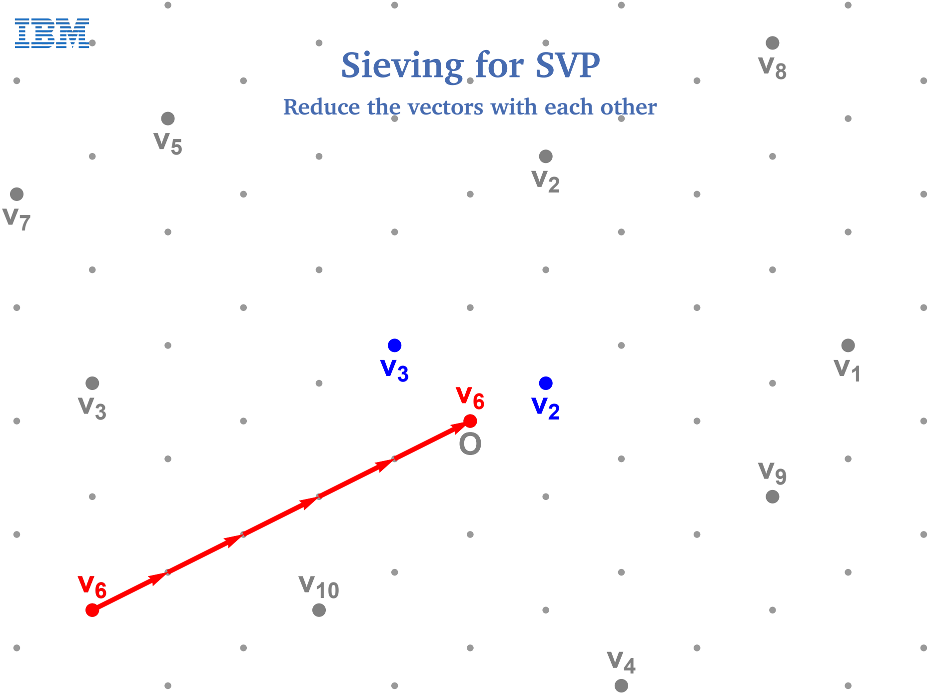

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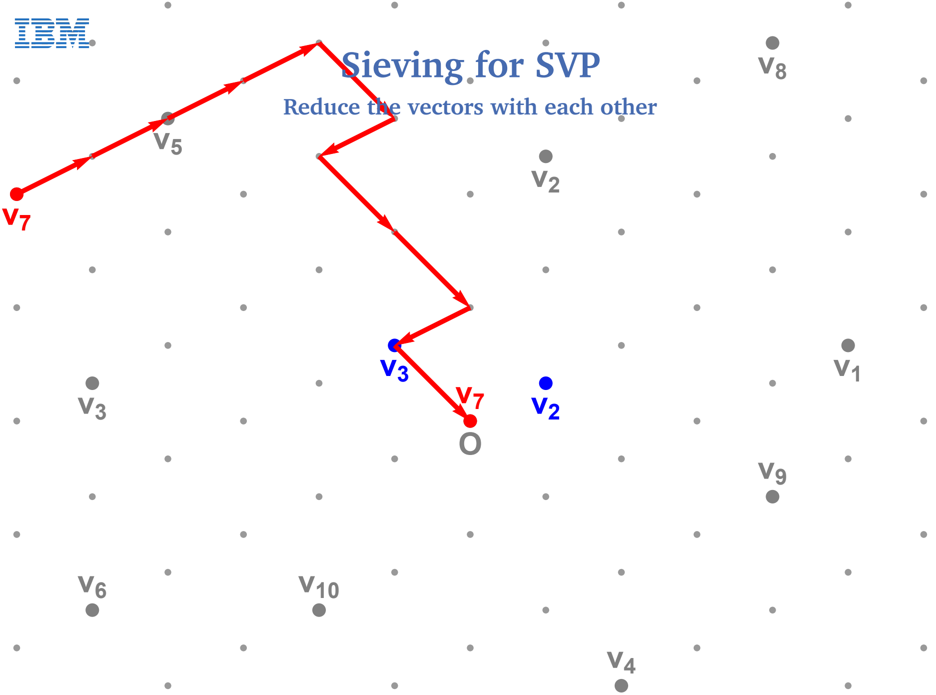

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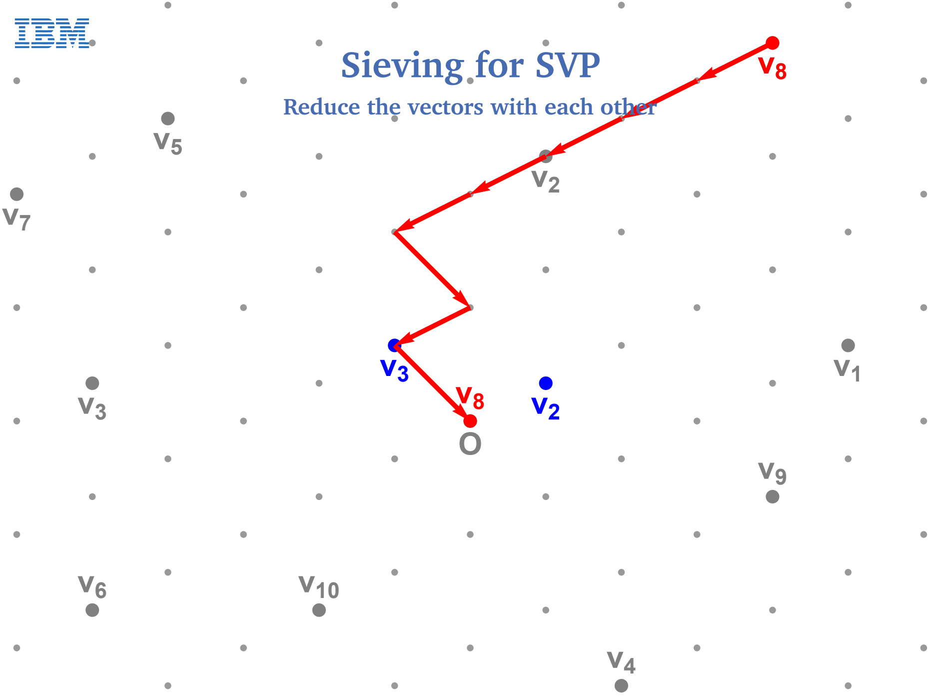

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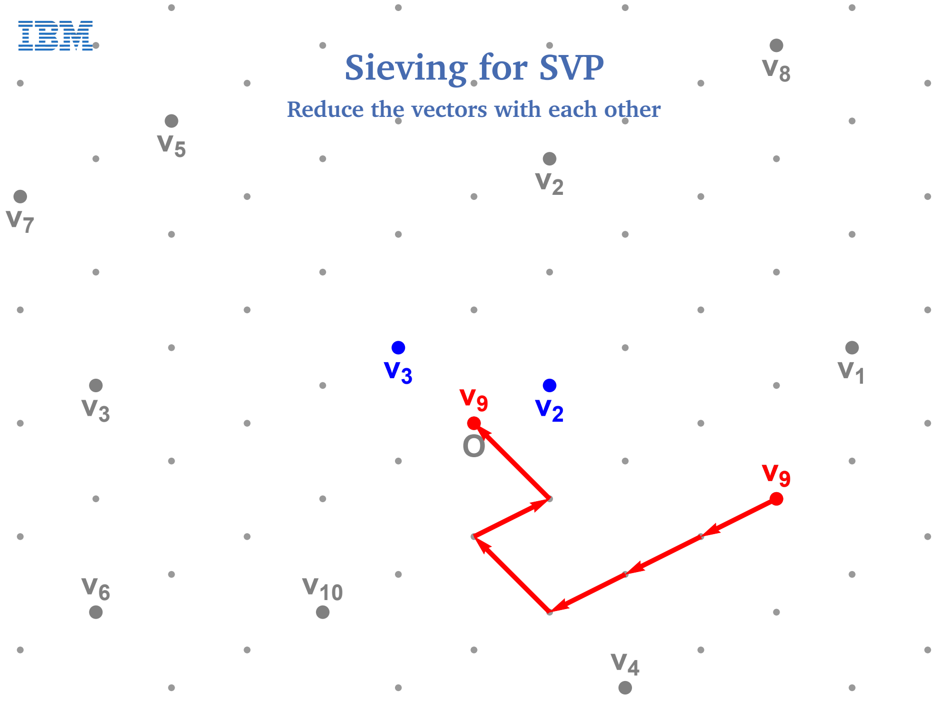

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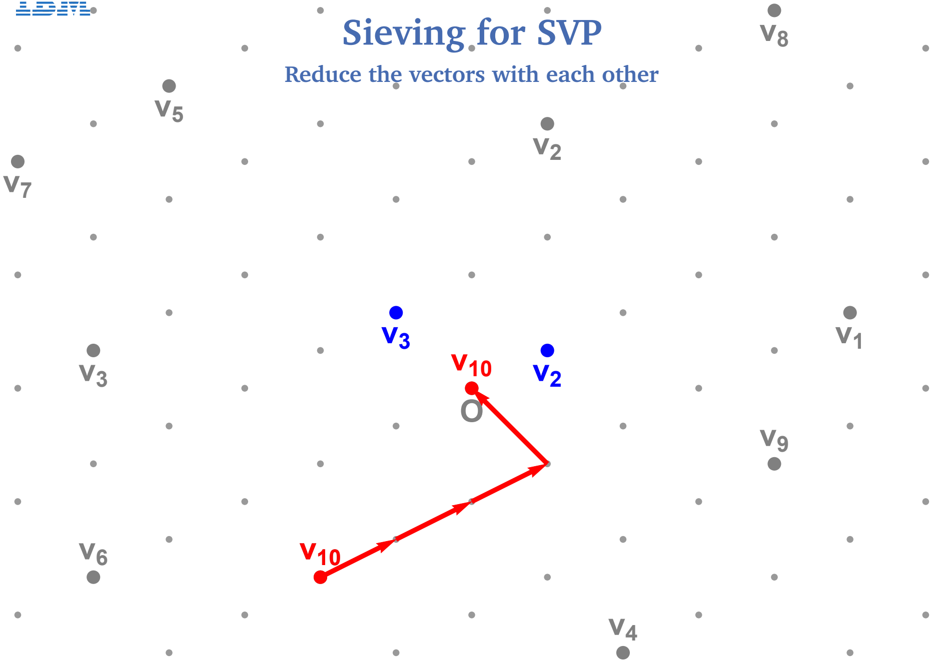

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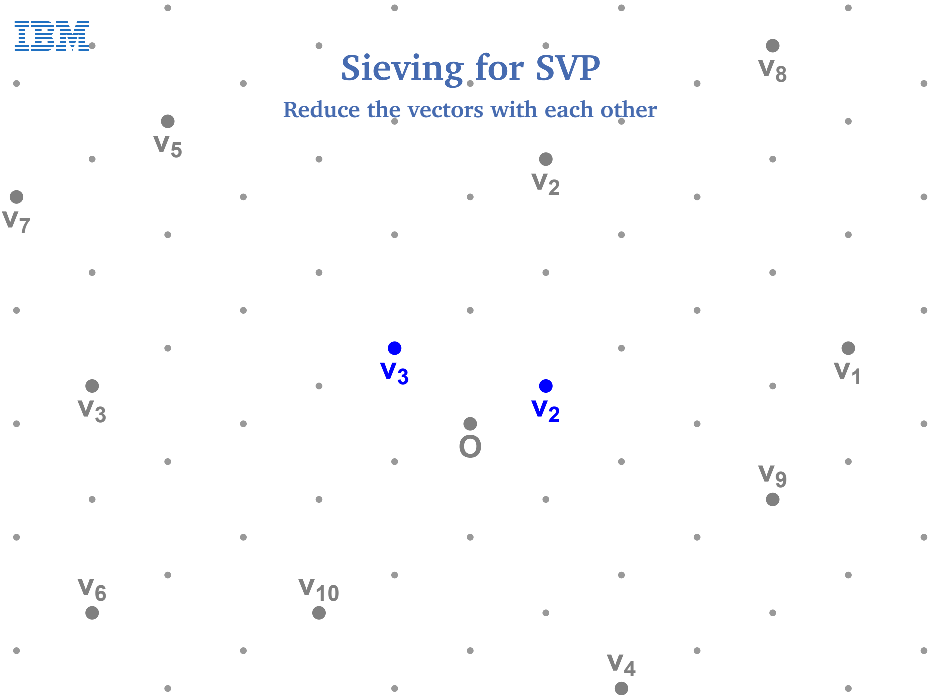

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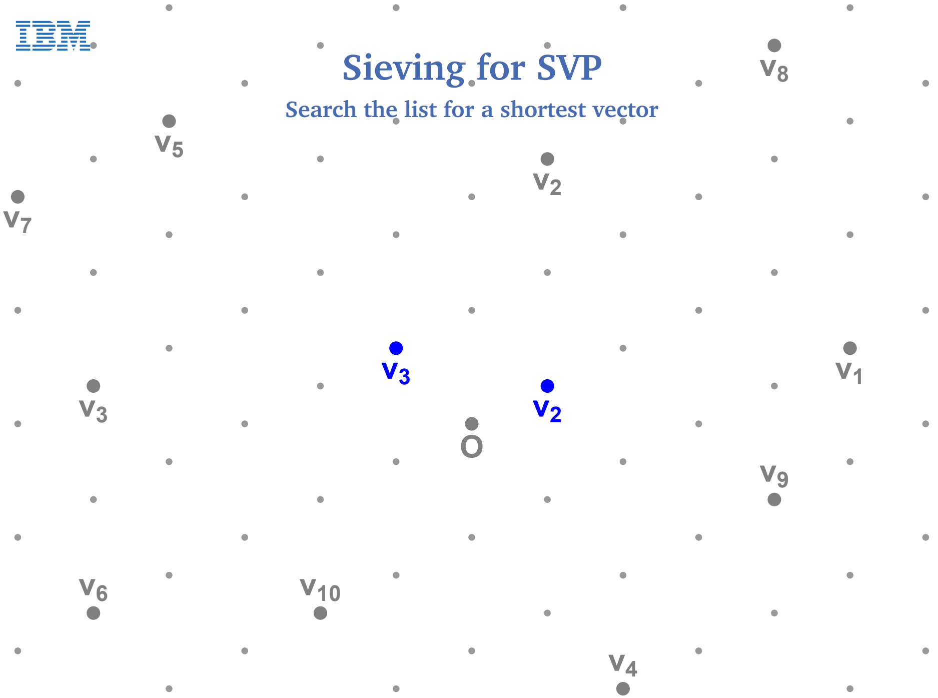

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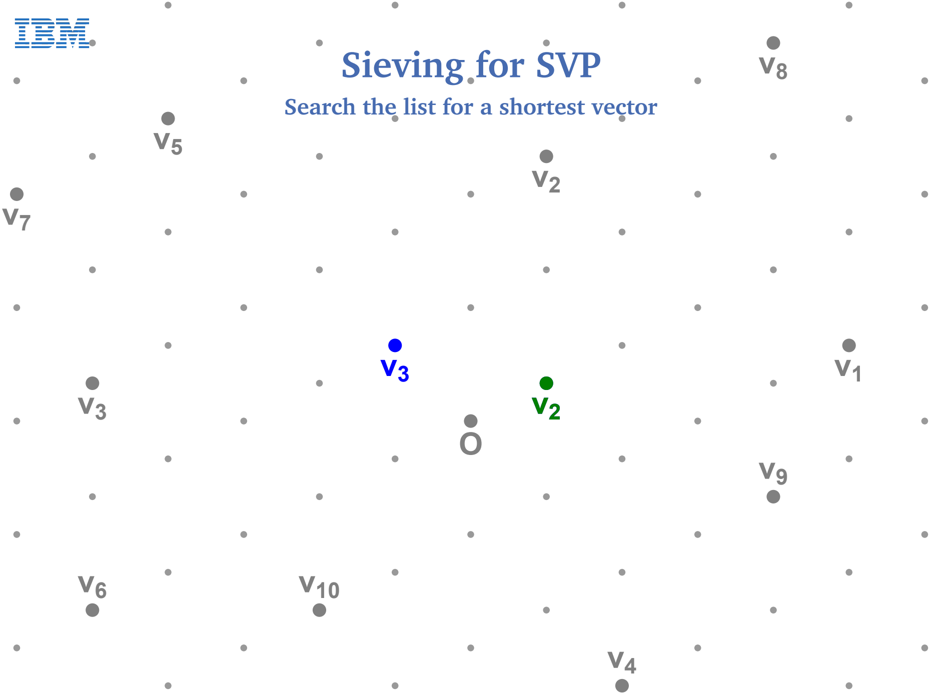
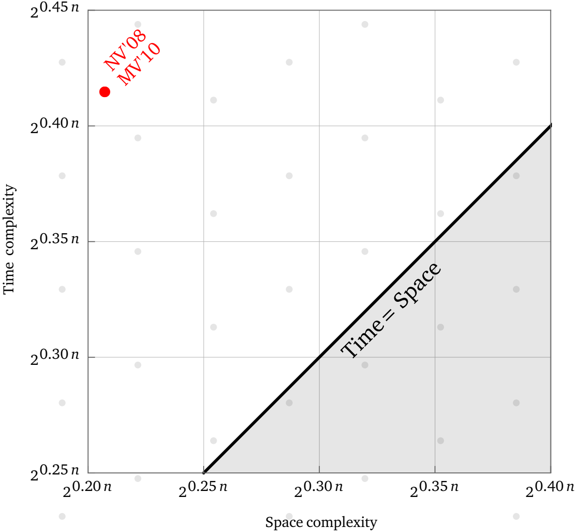


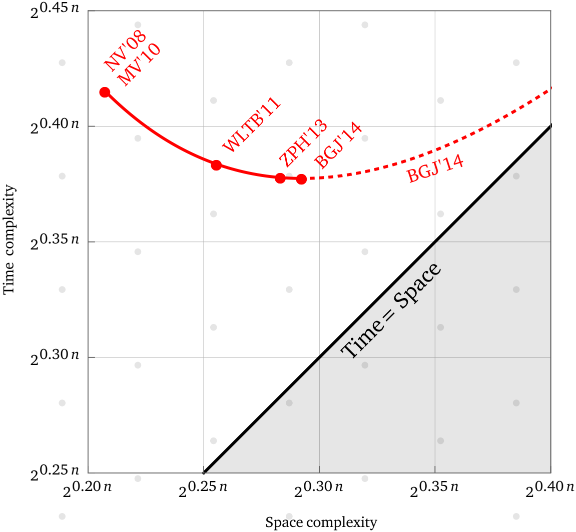
Sieving for SVP

The GaussSieve and Nguyen-Vidick sieve



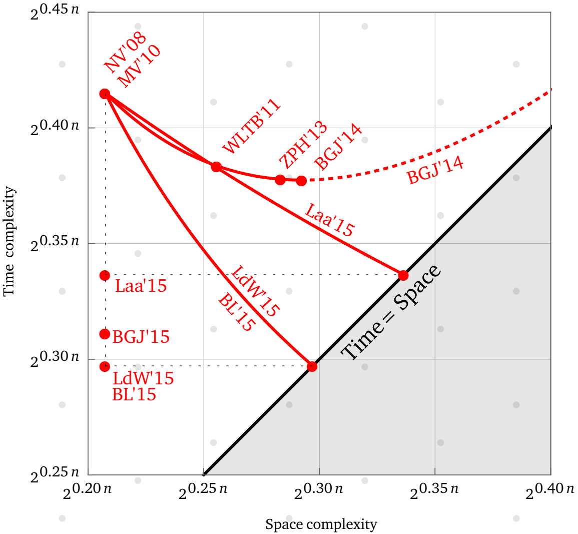
Sieving for SVP

Leveled sieving approaches



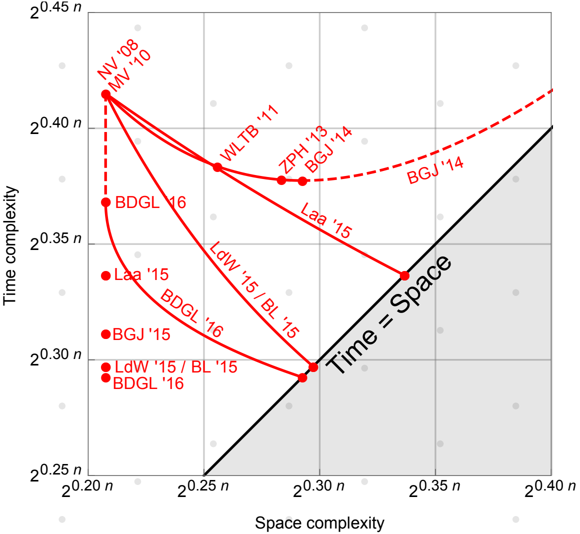
Sieving for SVP

Locality-Sensitive Hashing (LSH)



Sieving for SVP

Locality-Sensitive Filters (LSF)



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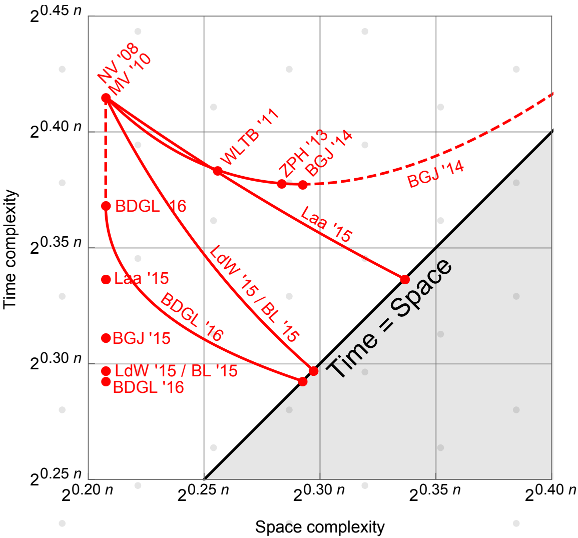

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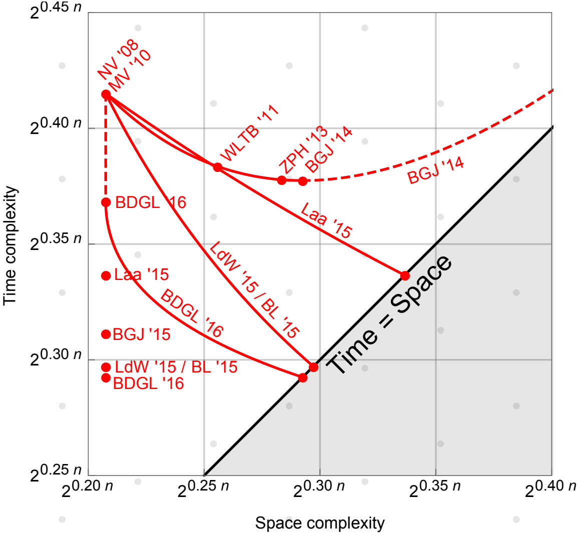

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Reduce the target vectors with the list

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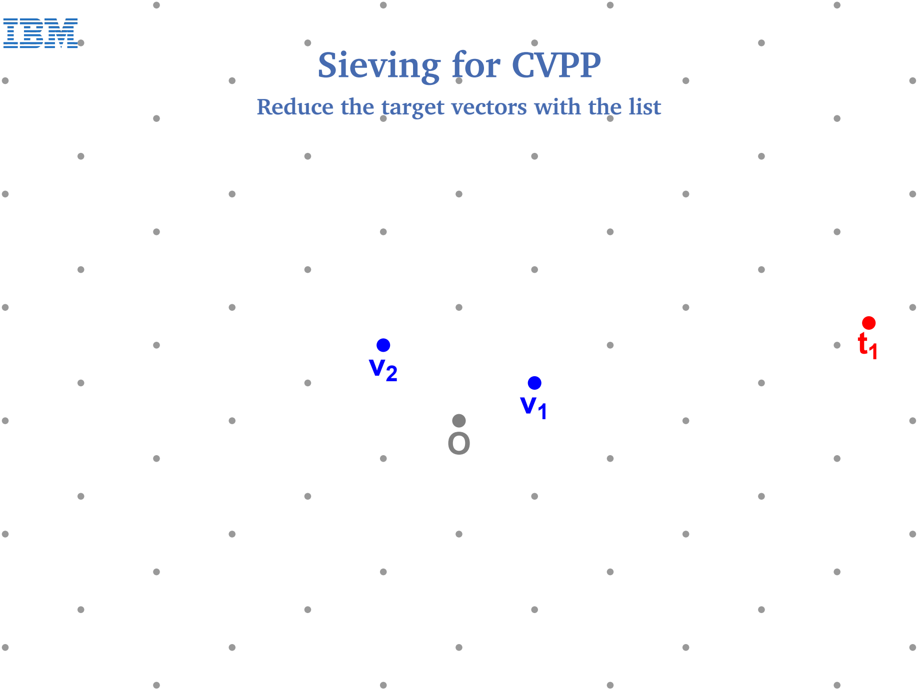

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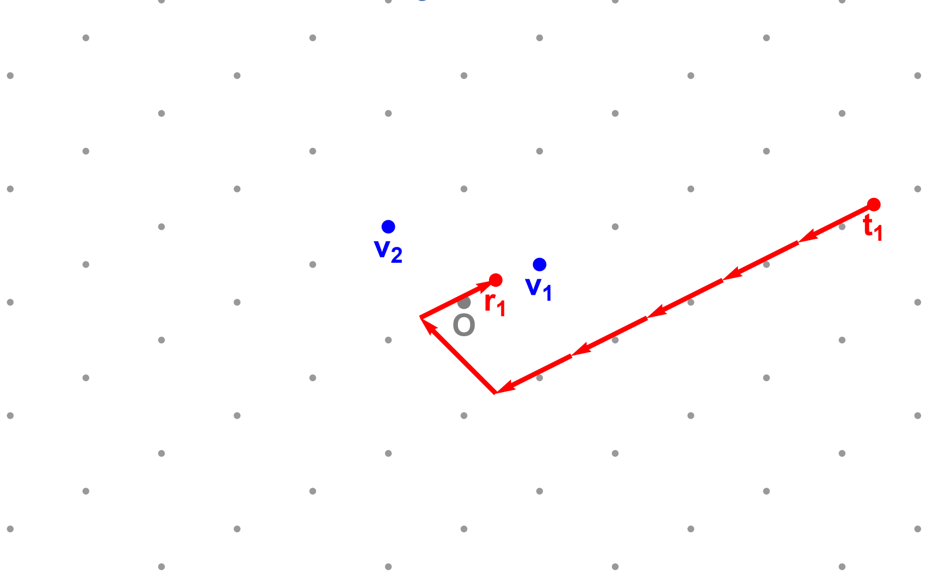

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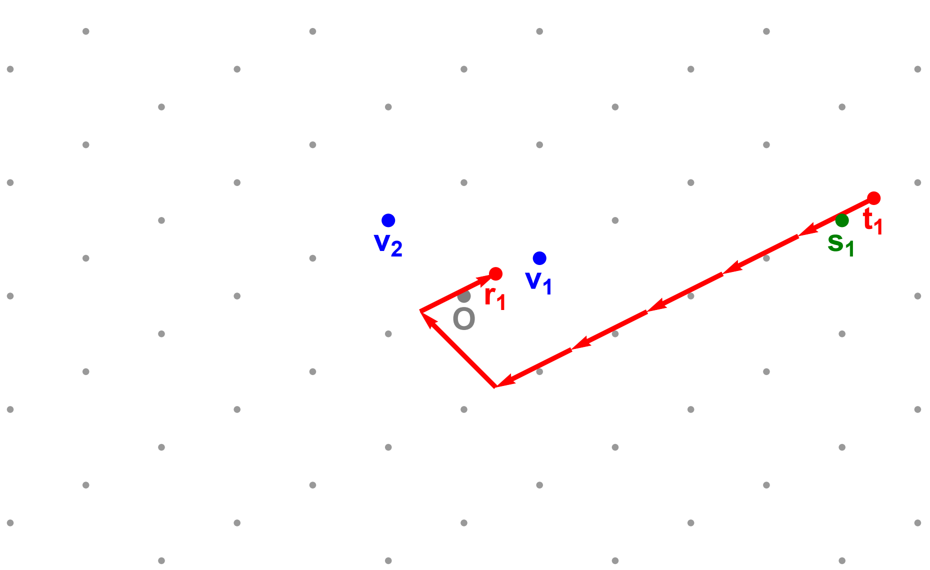

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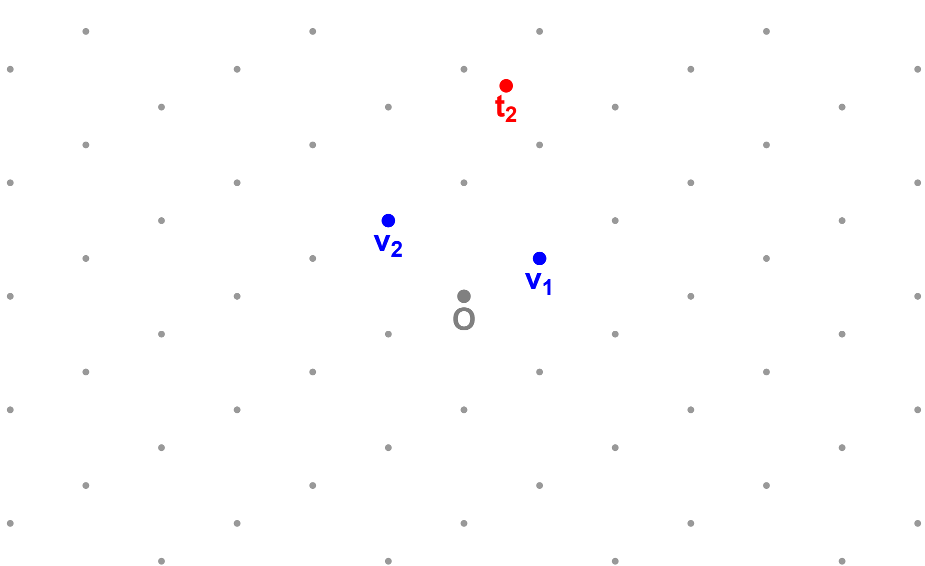

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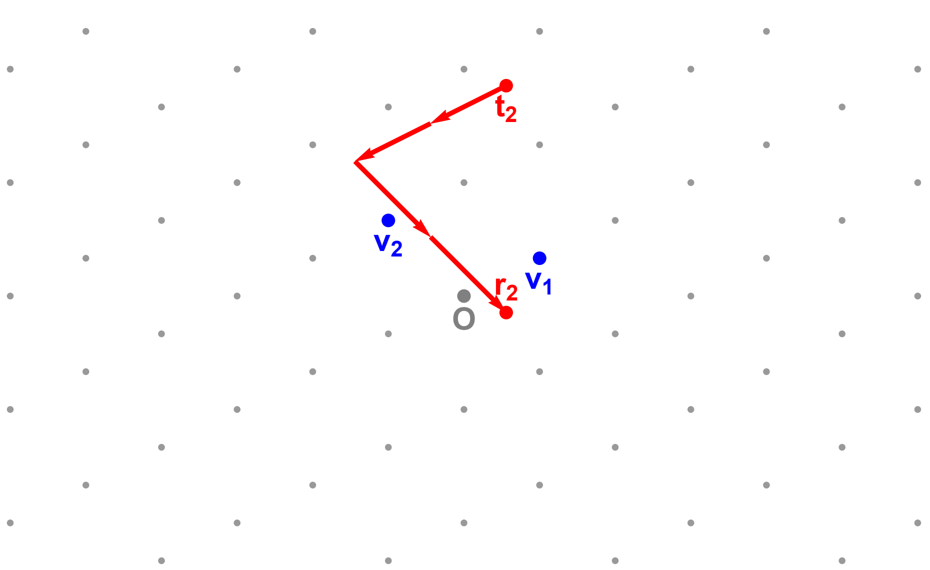

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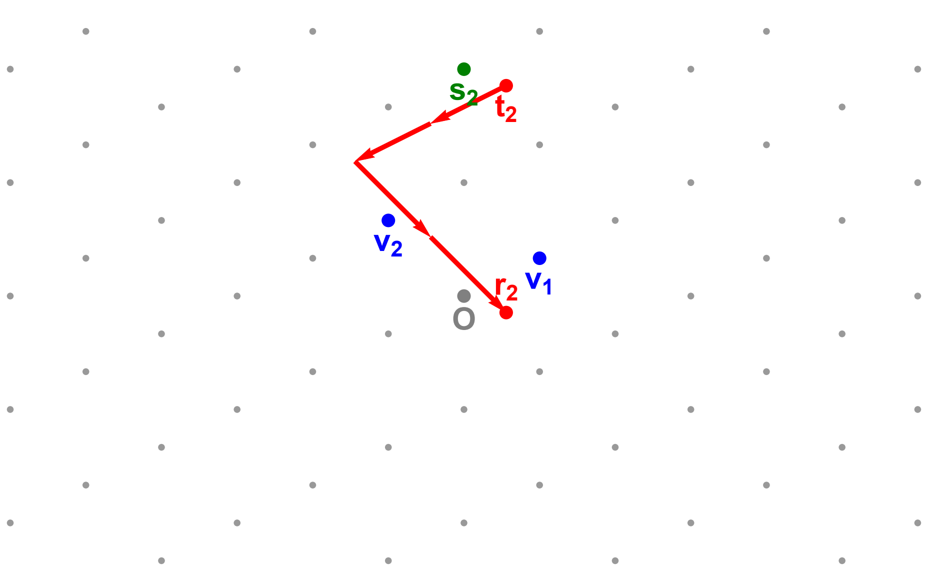

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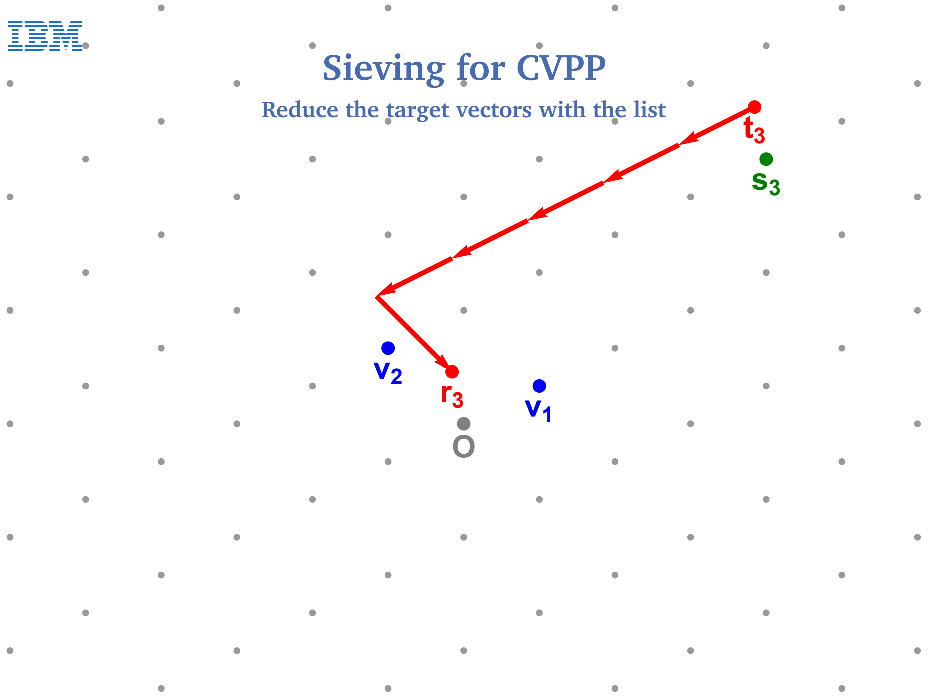

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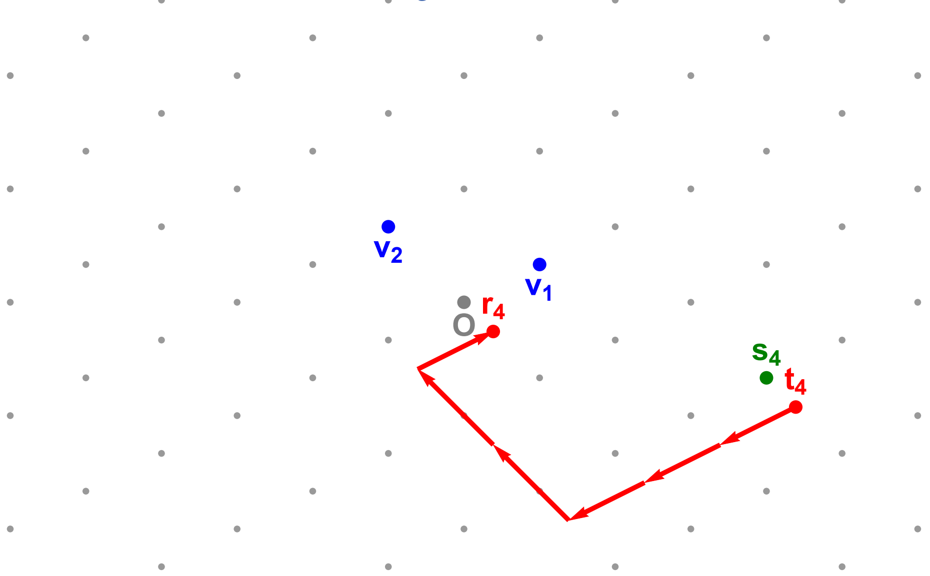

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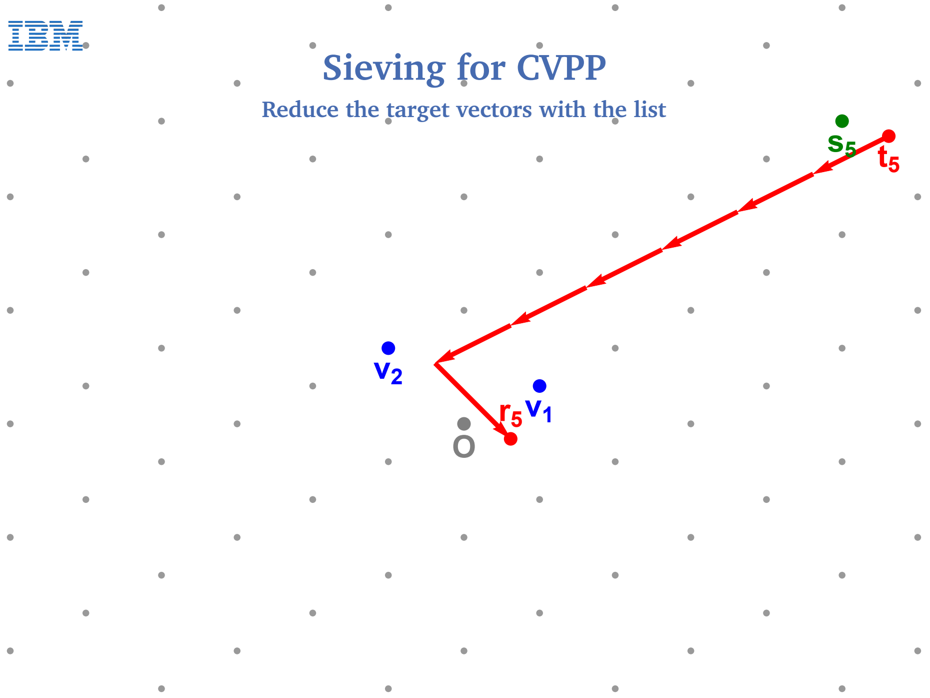

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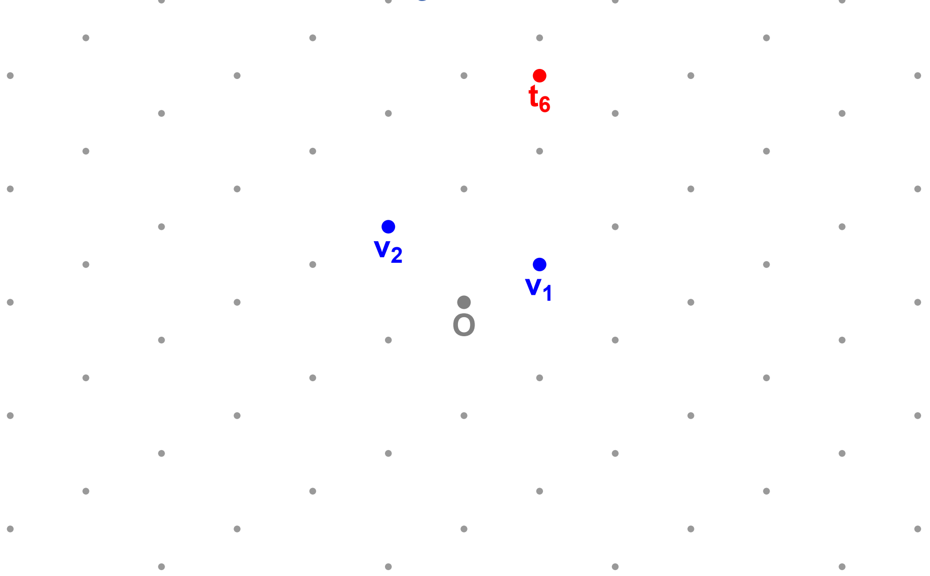

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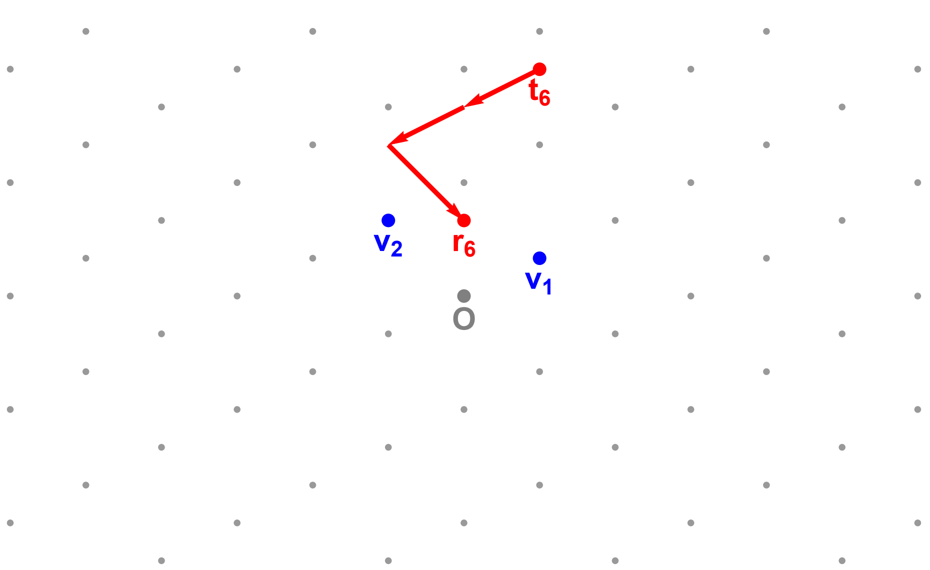

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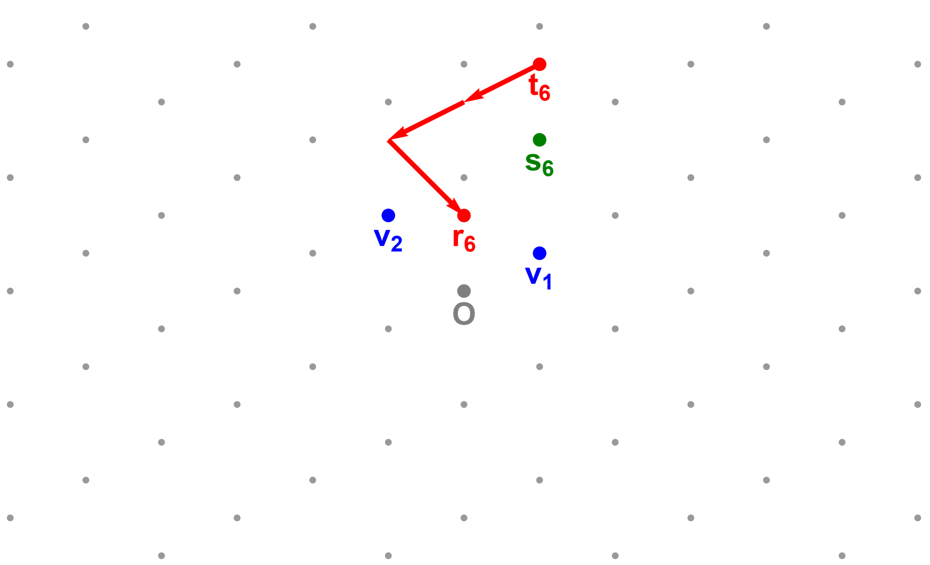

Sieving for CVPP

Relation with the Voronoi cell

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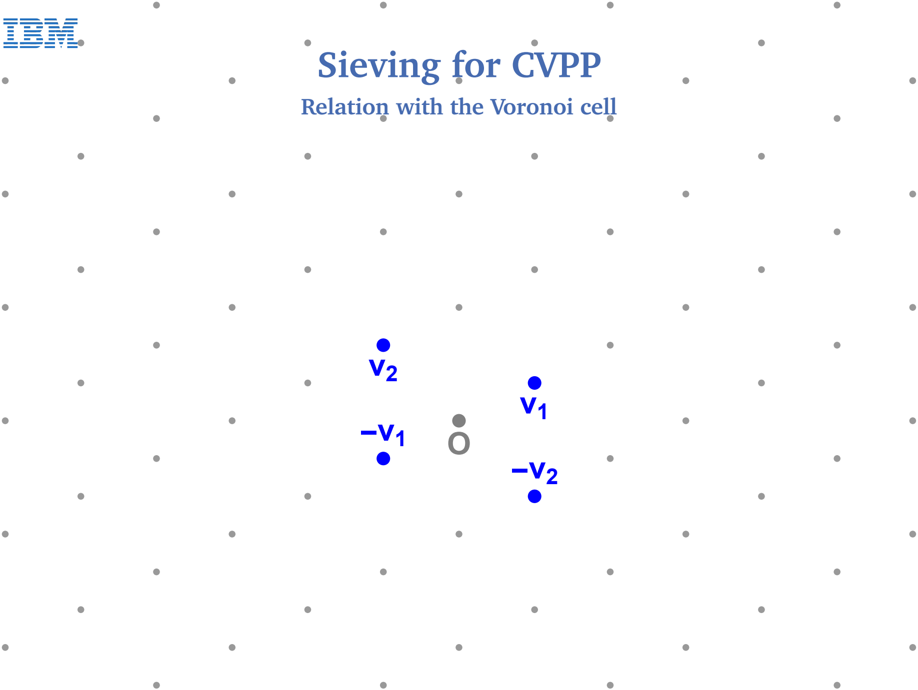

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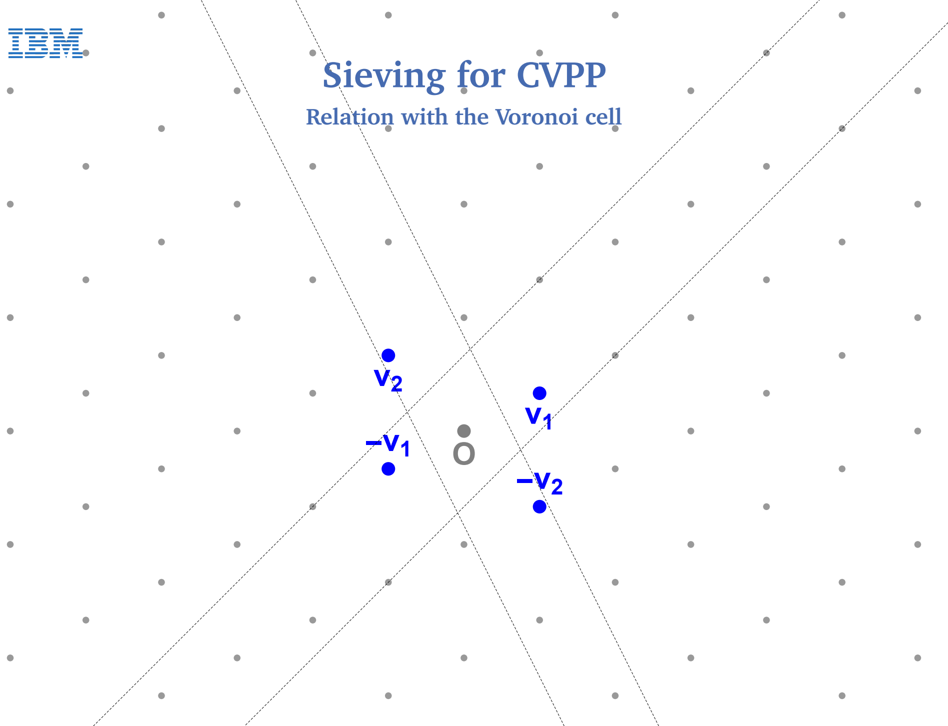

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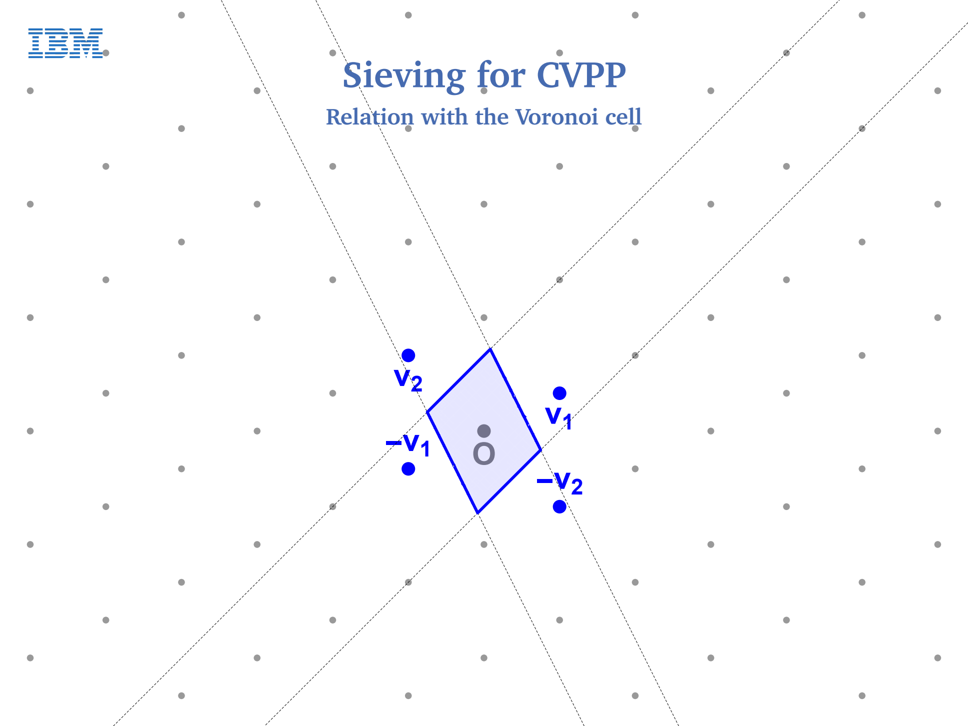

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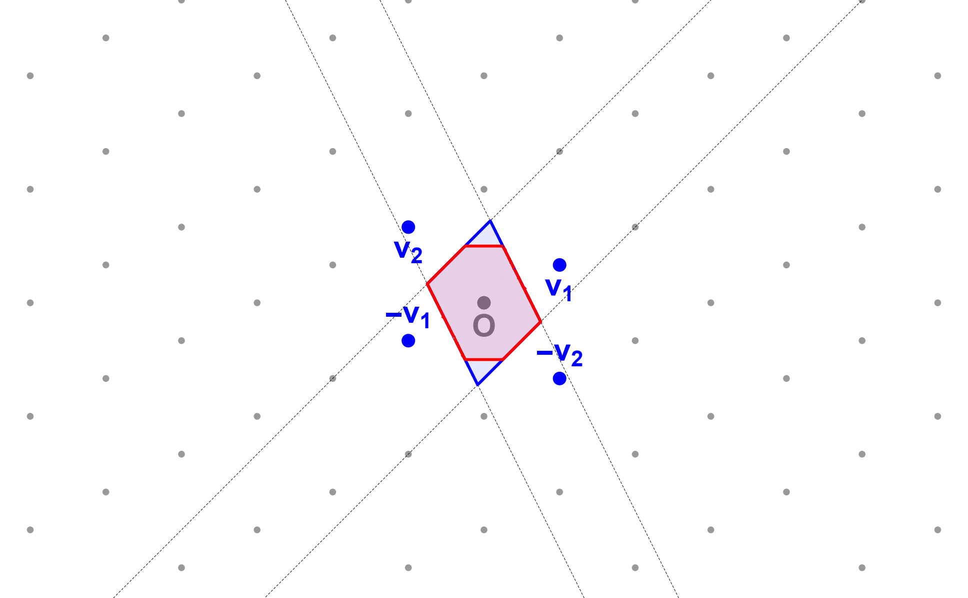

Sieving for CVPP

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- Blue region: **Gauss cell** $\mathscr{G}$

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  - Defined by $2^{0.21n+o(n)}$ short lattice vectors
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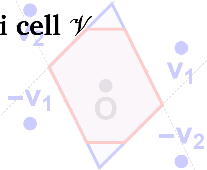

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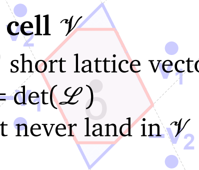

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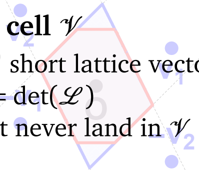

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- Problems:
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  - Probability only over randomness of targets

Sieving for CVPP

Solving the problems

- Idea 1: **Larger lists, weaker reductions**

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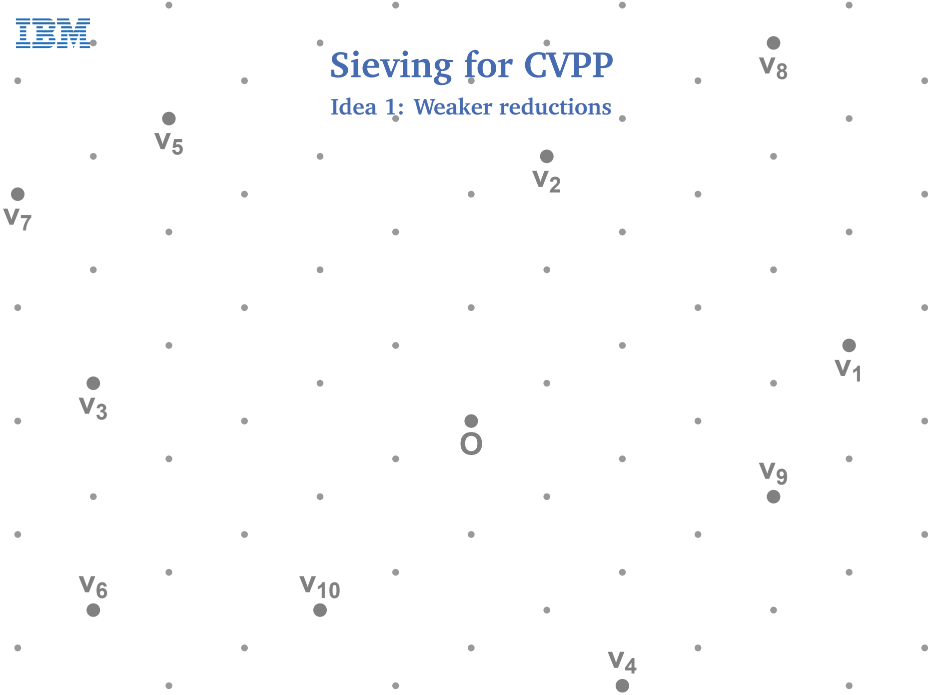
Solving the problems

- Idea 1: **Larger lists, weaker reductions**
  - Problem: Exponentially small success probability
  - To guarantee $\mathrm{Vol}(\mathscr{G}) \approx \mathrm{Vol}(\mathscr{V})$, need $2^{n/2+o(n)}$ vectors
  - Preprocessing: reduce $\boldsymbol{v}_1$ with $\boldsymbol{v}_2$ iff $\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| \ll \|\boldsymbol{v}_1\|$

Sieving for CVPP

Idea 1: Weaker reductions



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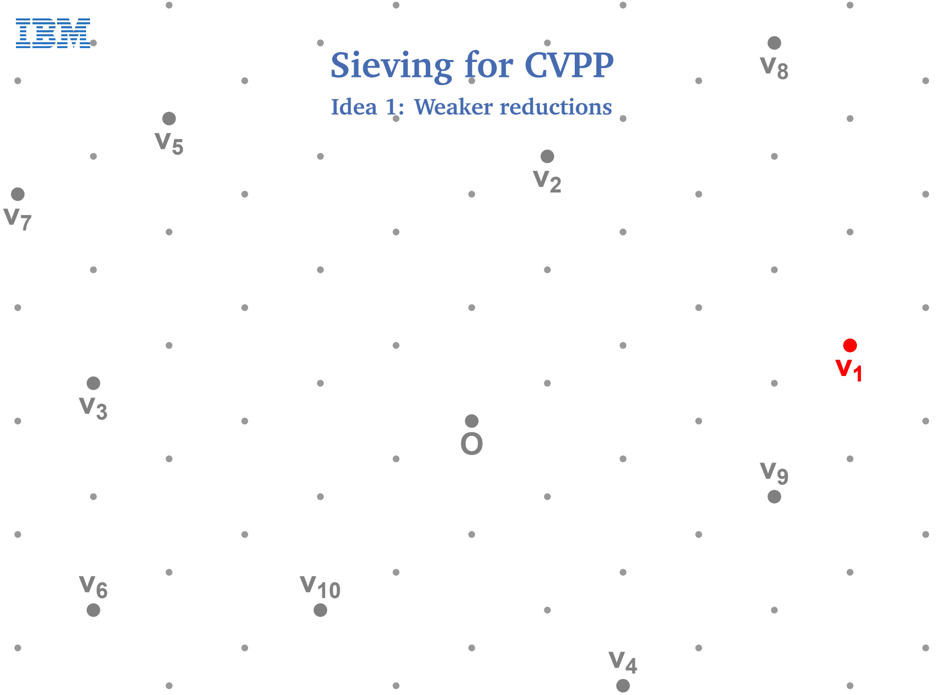

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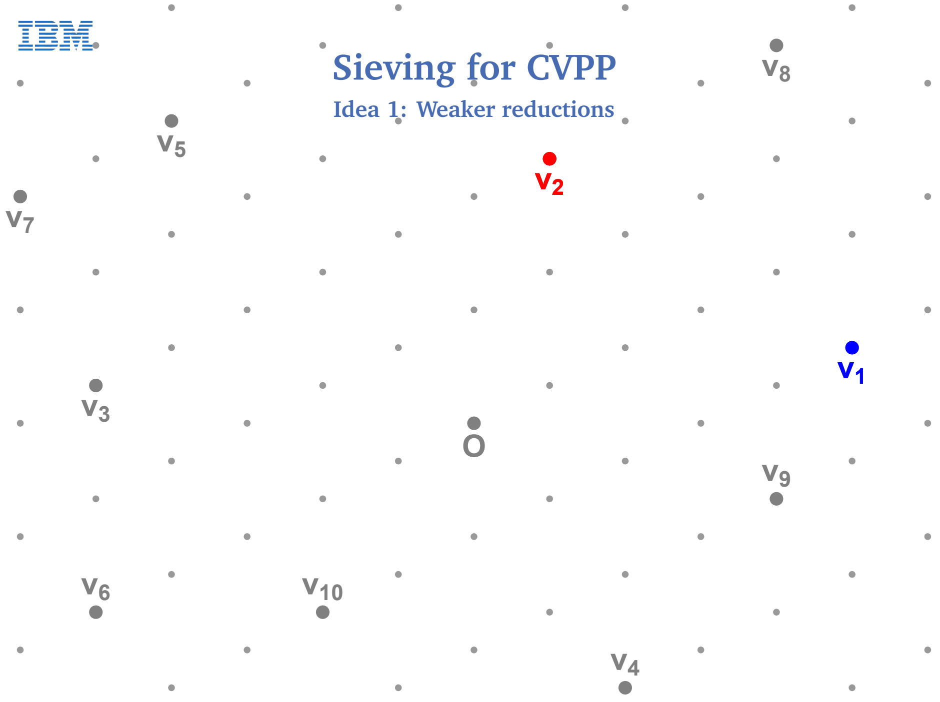

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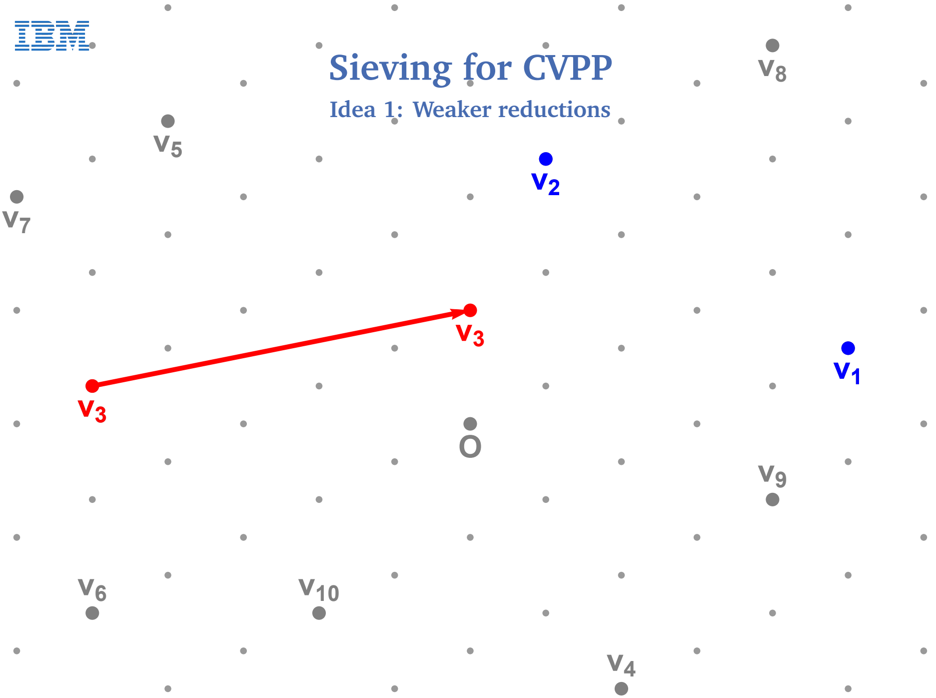

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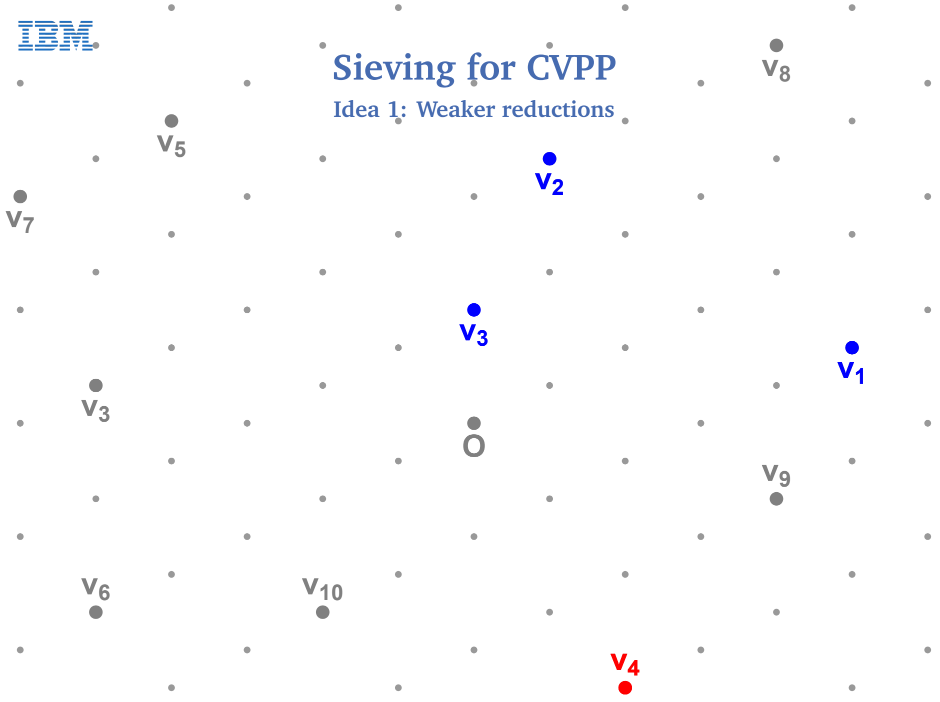

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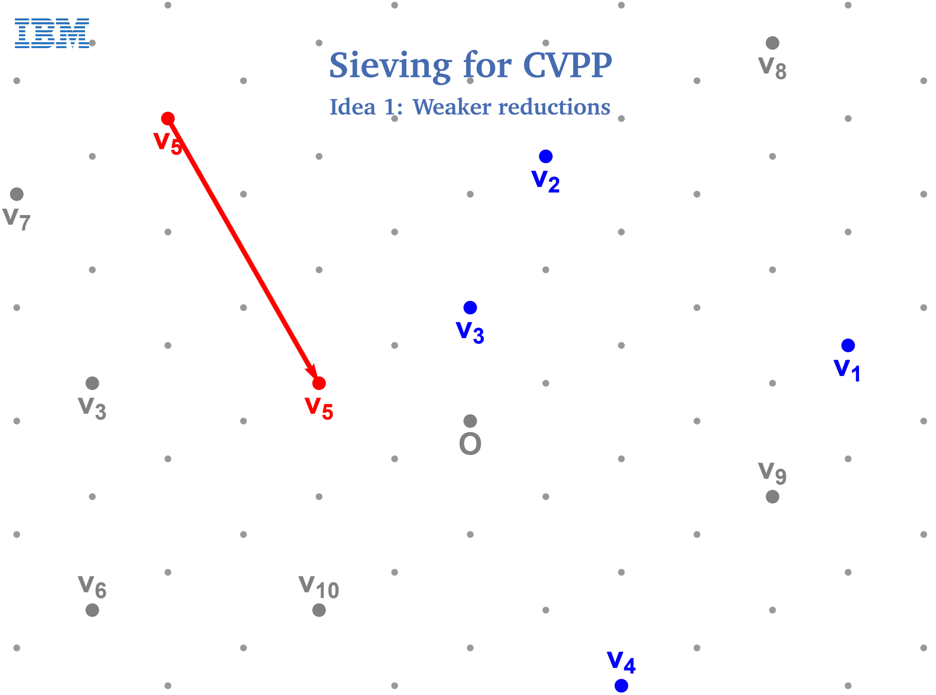

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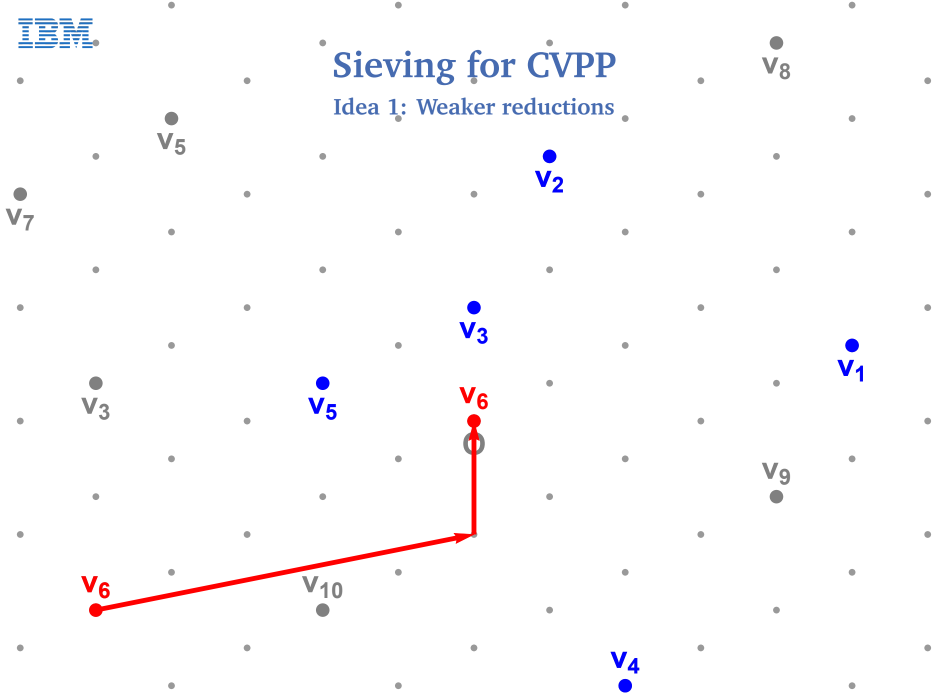

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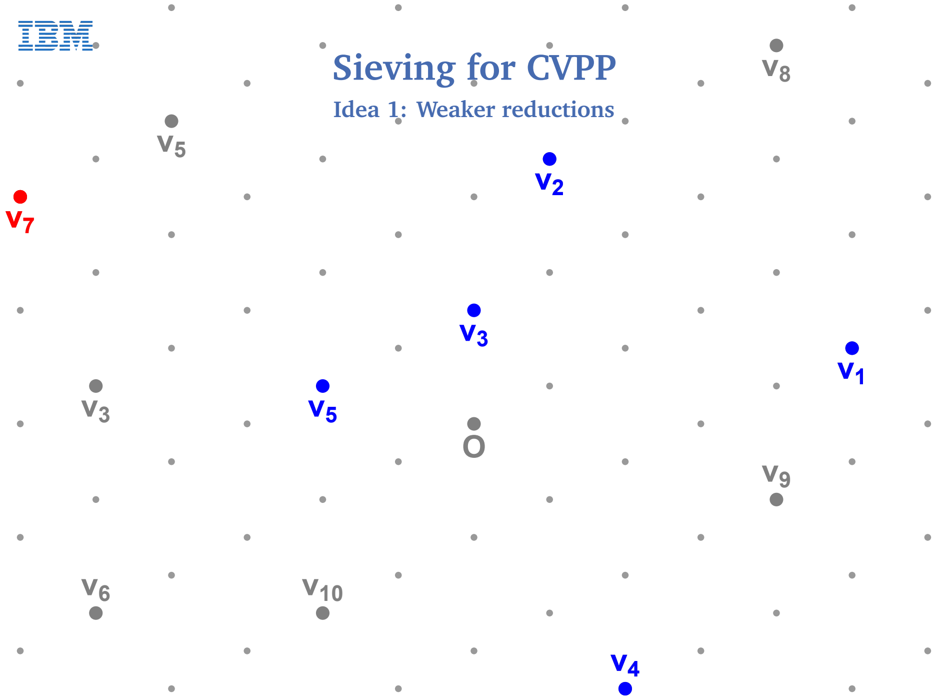

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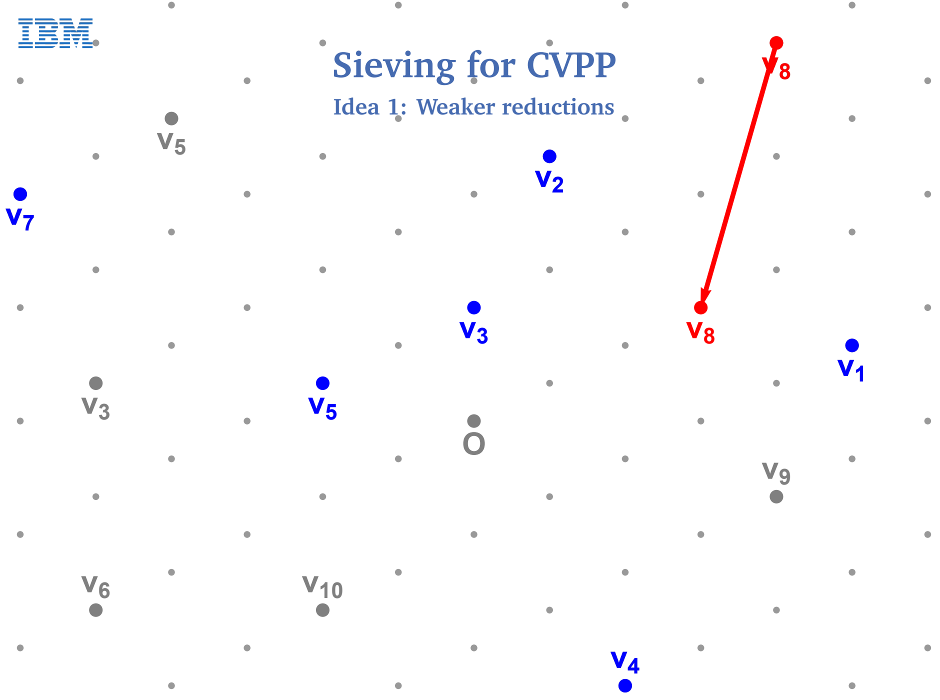

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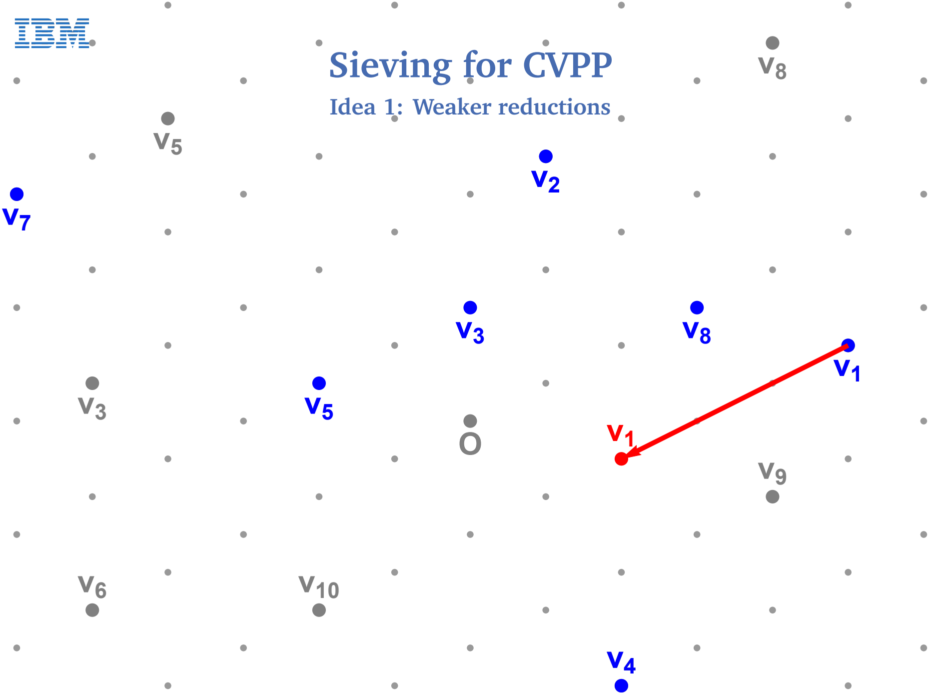

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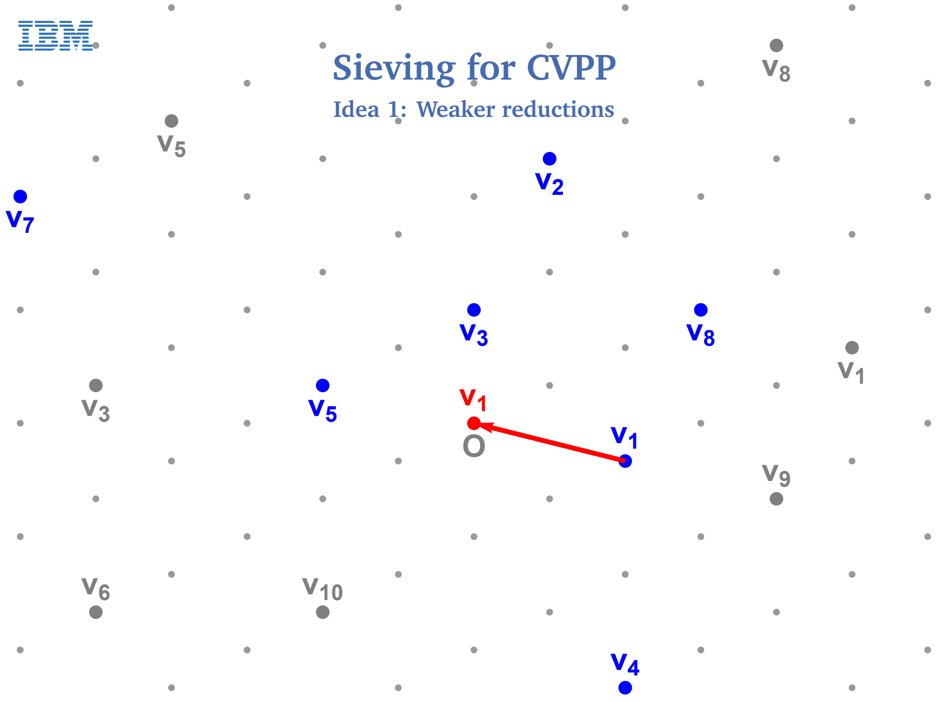

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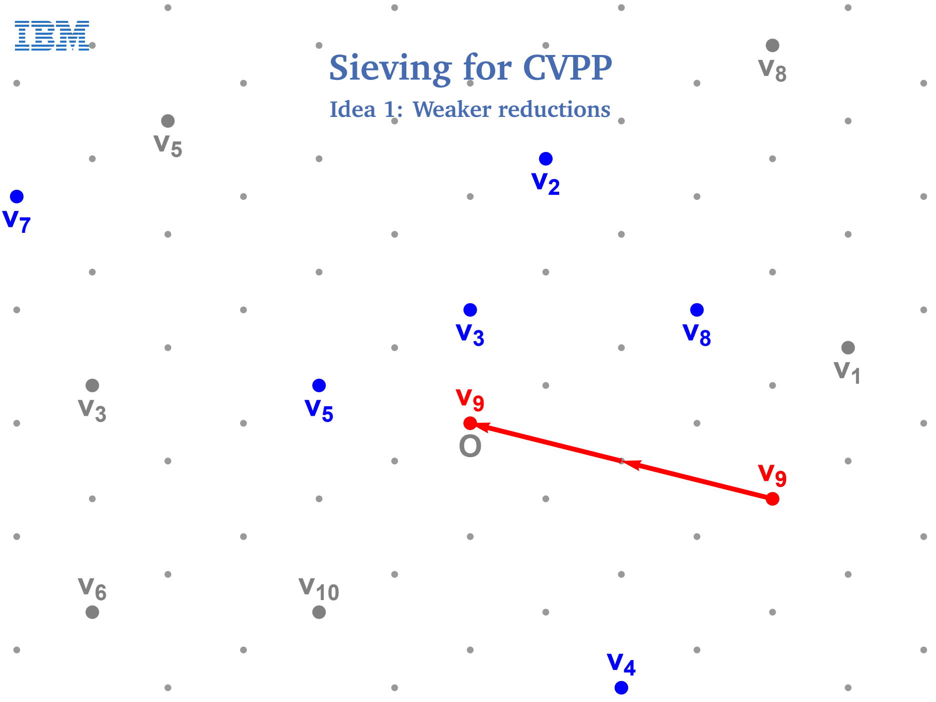

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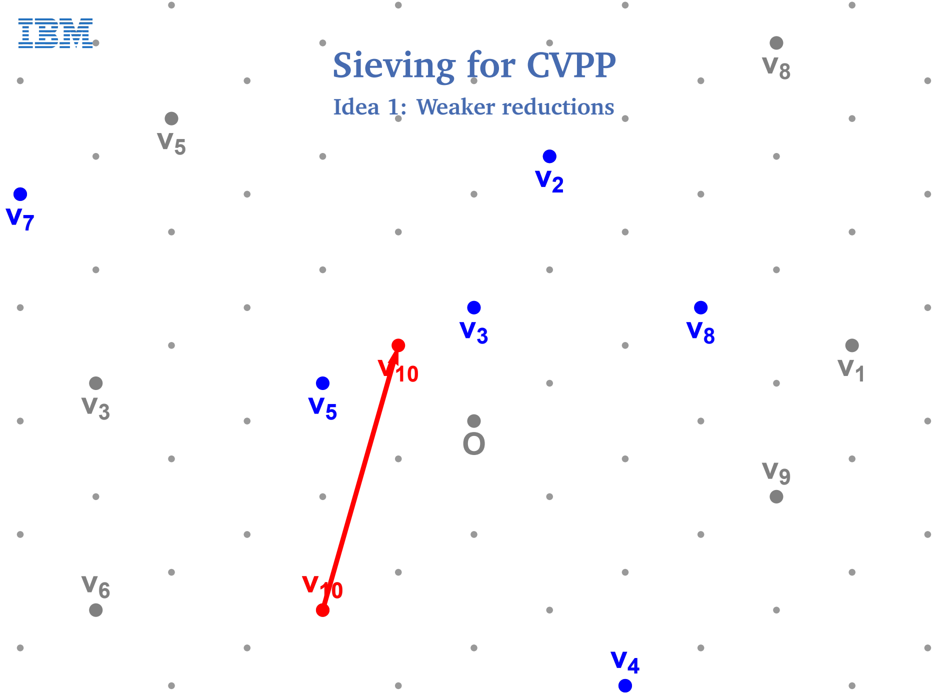

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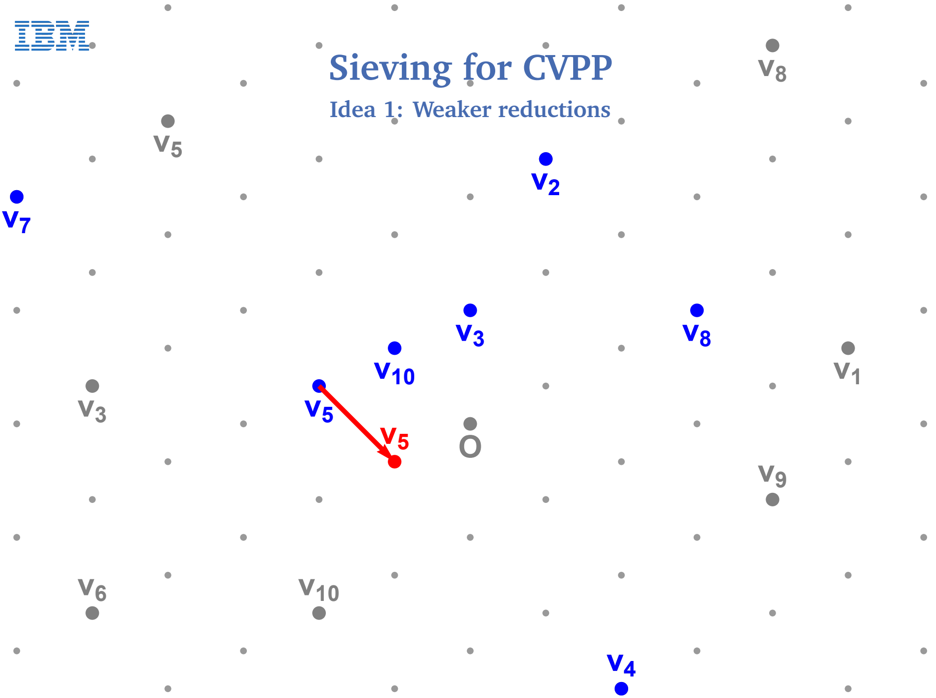

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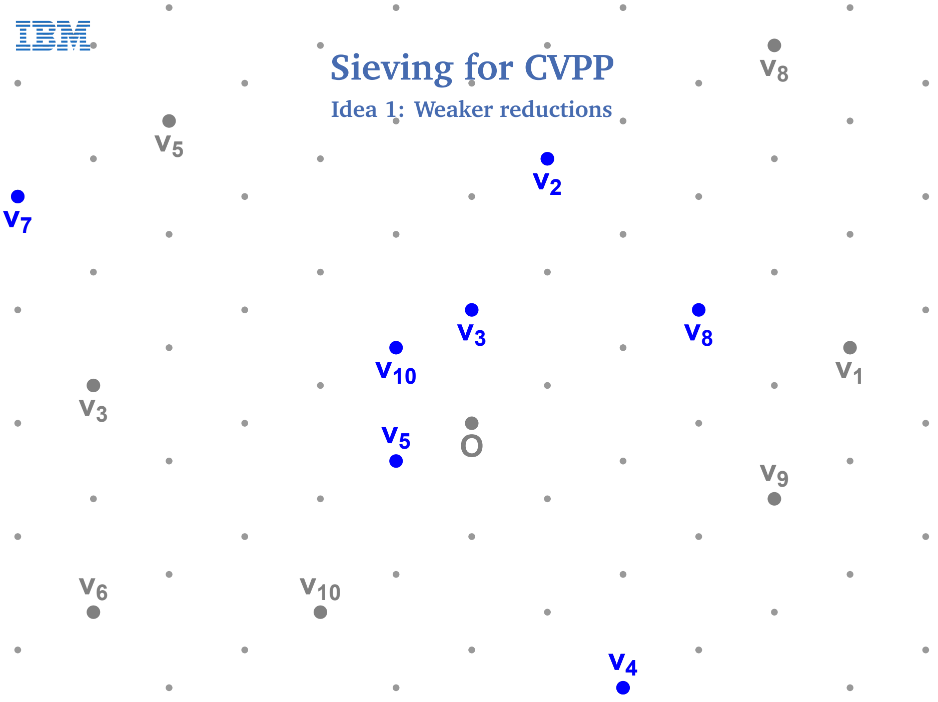

Sieving for CVPP

Solving the problems

- Idea 1: **Larger lists, weaker reductions**
  - Problem: Exponentially small success probability
  - To guarantee $\mathrm{Vol}(\mathscr{G}) \approx \mathrm{Vol}(\mathscr{V})$, need $2^{n/2+o(n)}$ vectors
  - Preprocessing: reduce $\boldsymbol{v}_1$ with $\boldsymbol{v}_2$ iff $\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| \ll \|\boldsymbol{v}_1\|$

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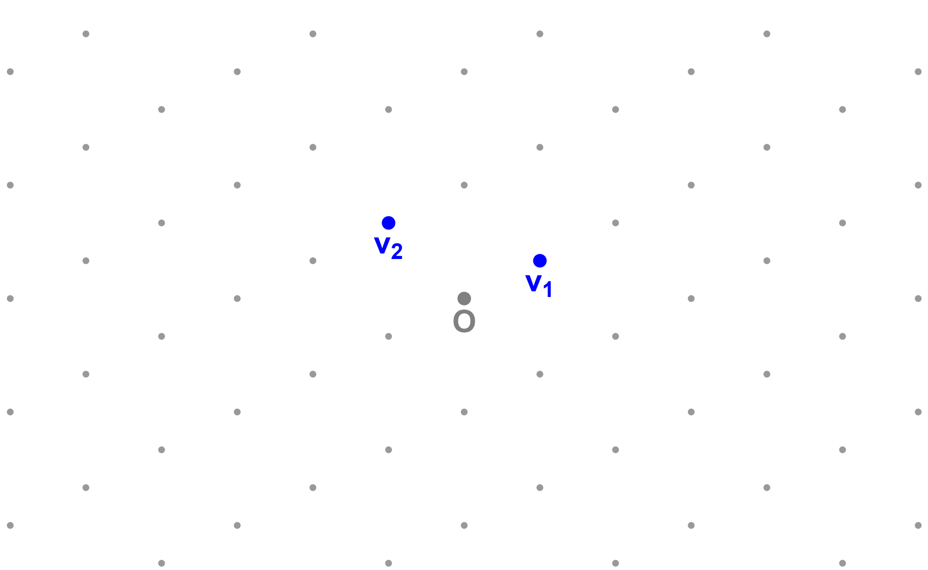
  - Randomize target $\boldsymbol{t}$ before reducing ($\boldsymbol{t}' \in_R \boldsymbol{t} + \mathscr{L}$)
  - Randomness now over algorithm, independently of target

Sieving for CVPP

Idea 2: Rerandomize the target



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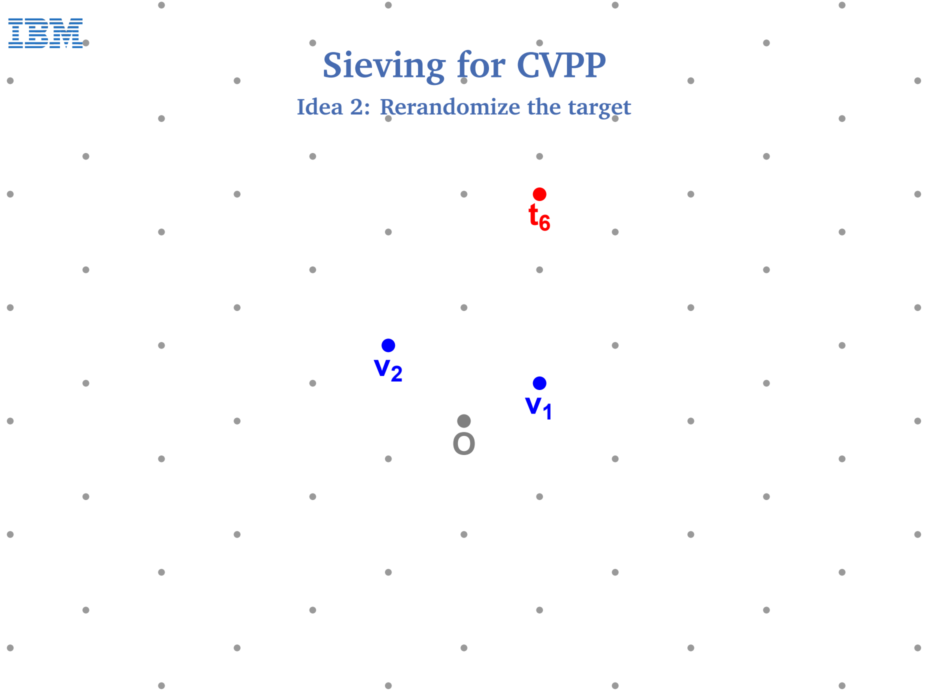

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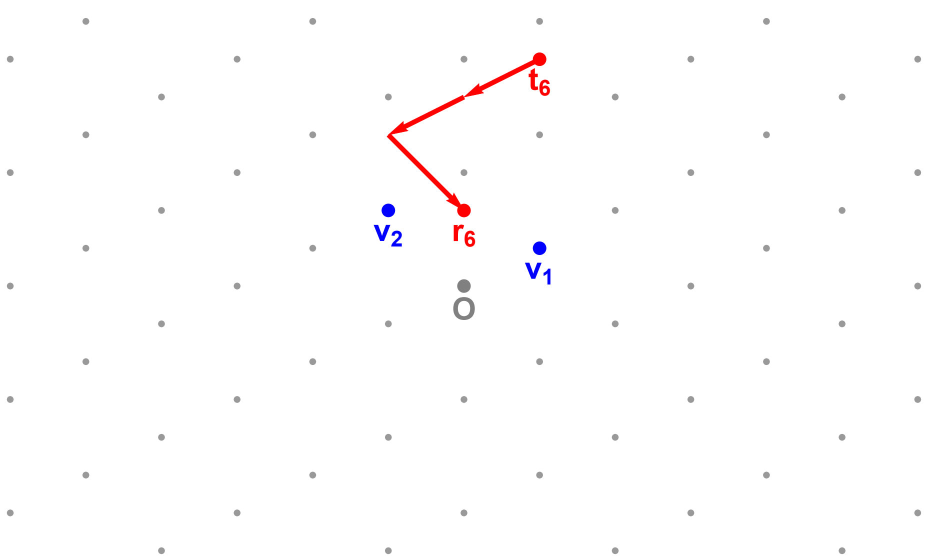

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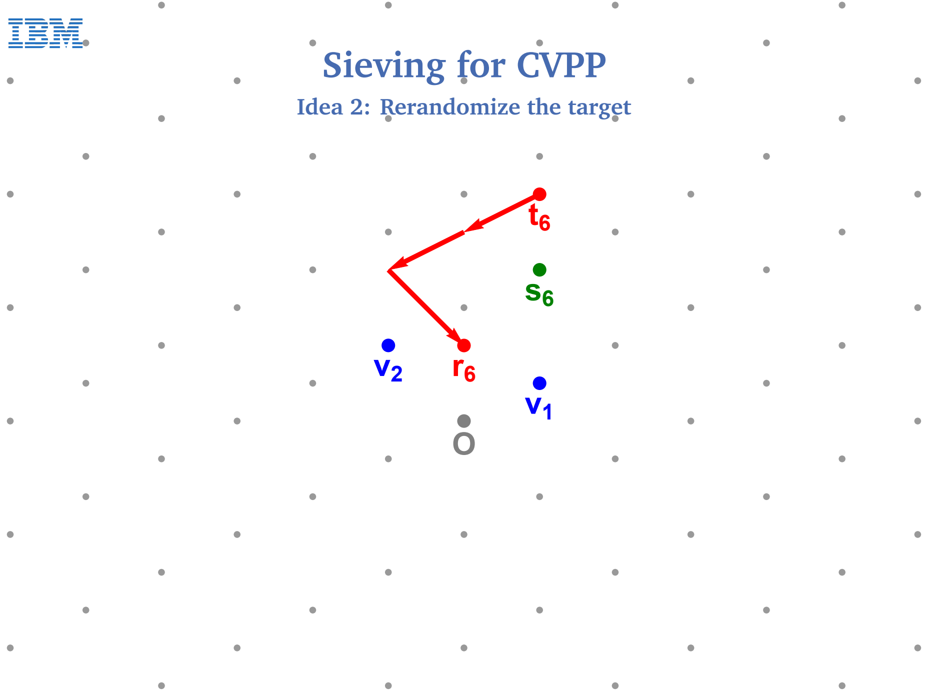

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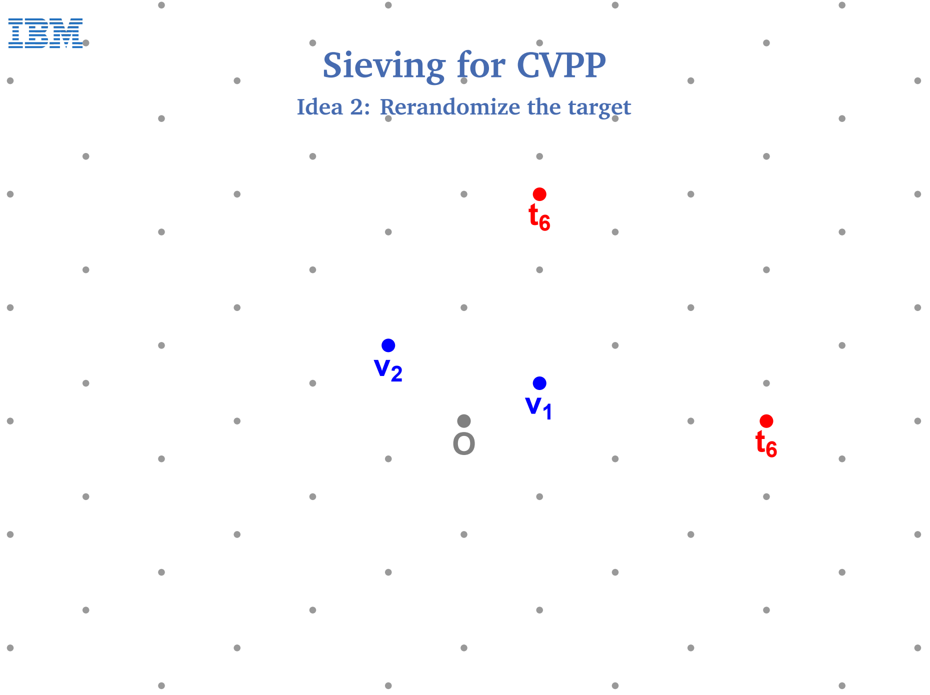

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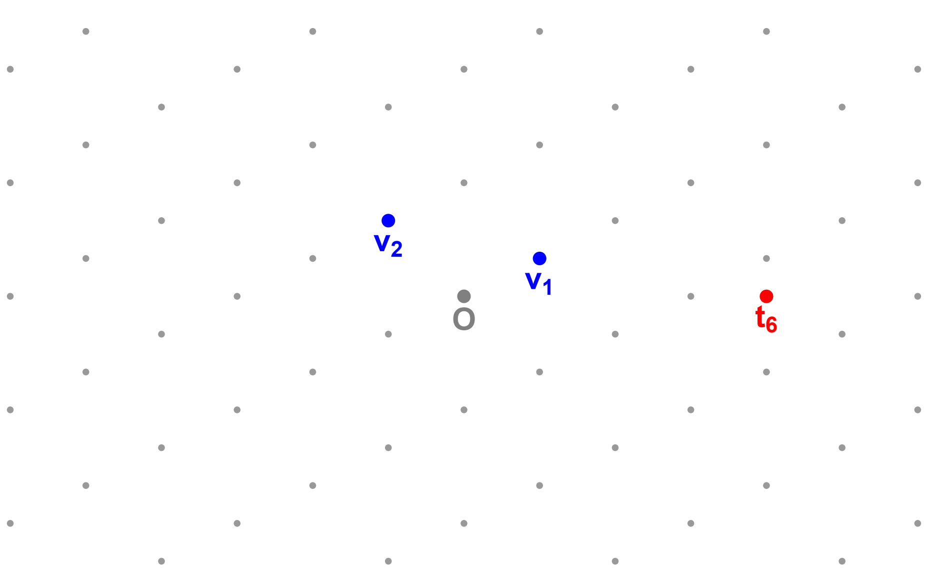

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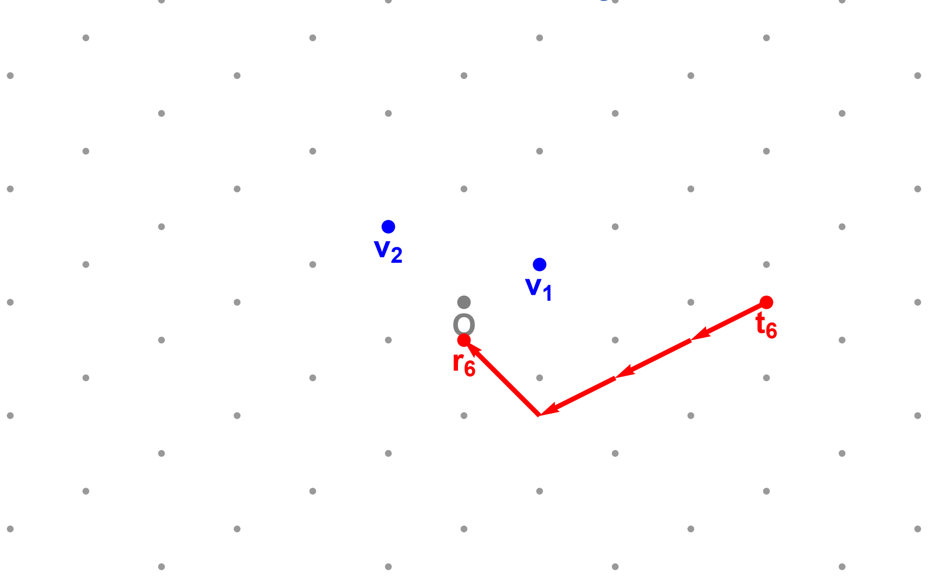

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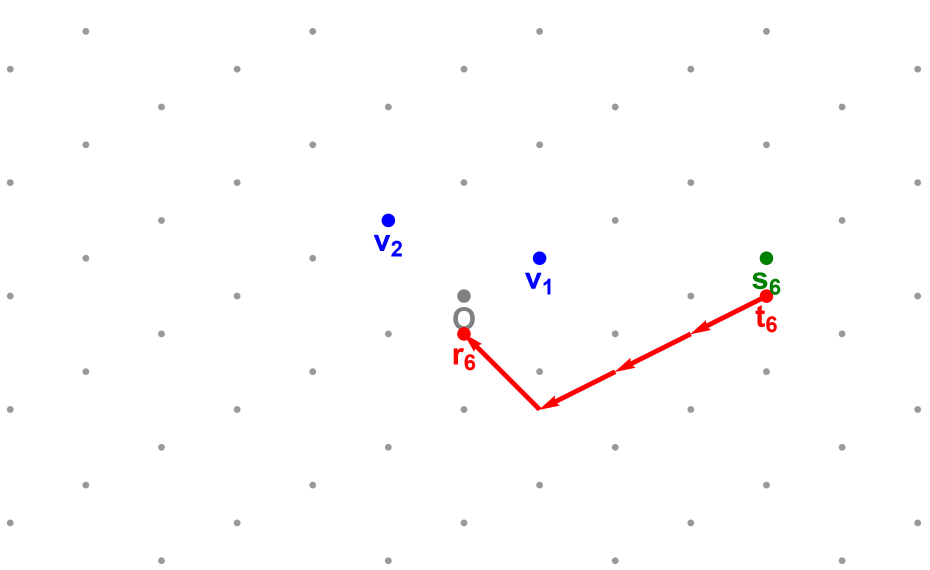

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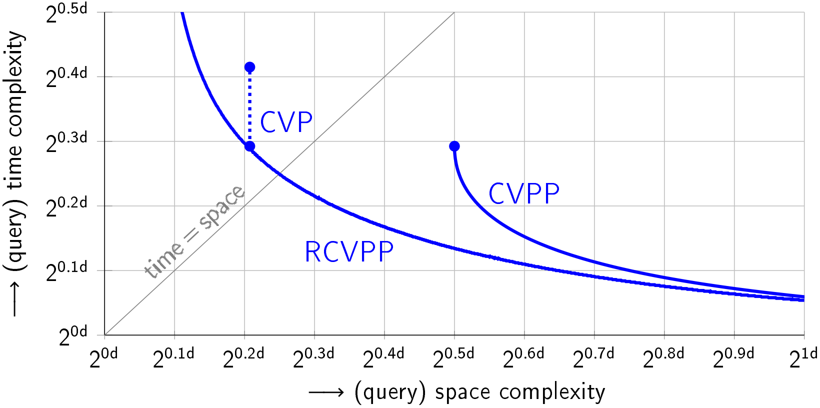
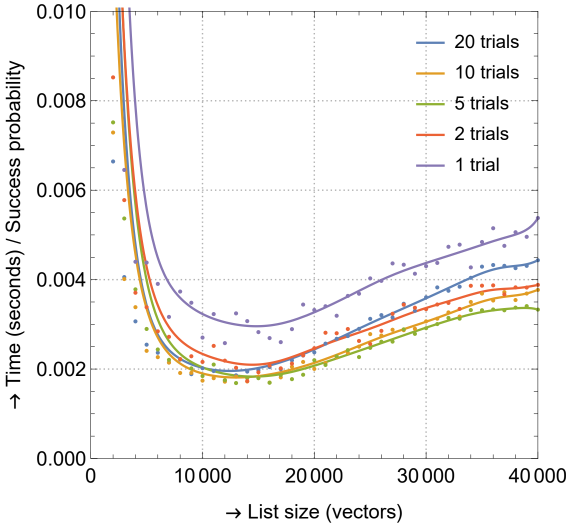
Trade-offs



Sieving for CVPP

Preliminary experiments ($n = 50$, HashSieve)



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Questions?