IBM Research

Lattice-based cryptography (II)

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PQCrypto Summer School 2017
(June 20, 2017)

IBM Research

Part 2: Lattice algorithms for solving the shortest vector problem

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Lattices

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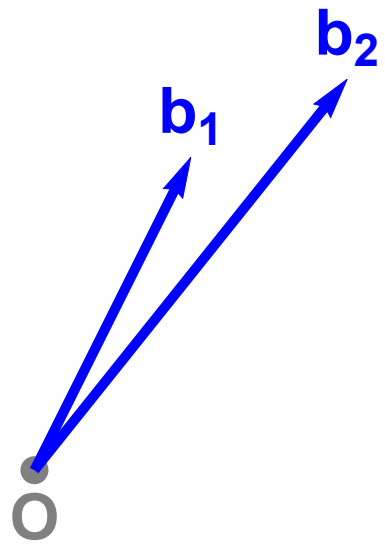

Lattices

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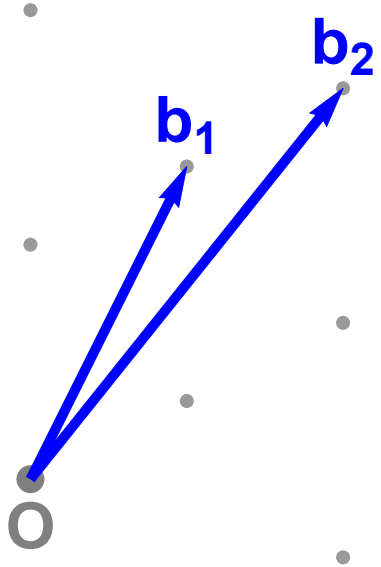
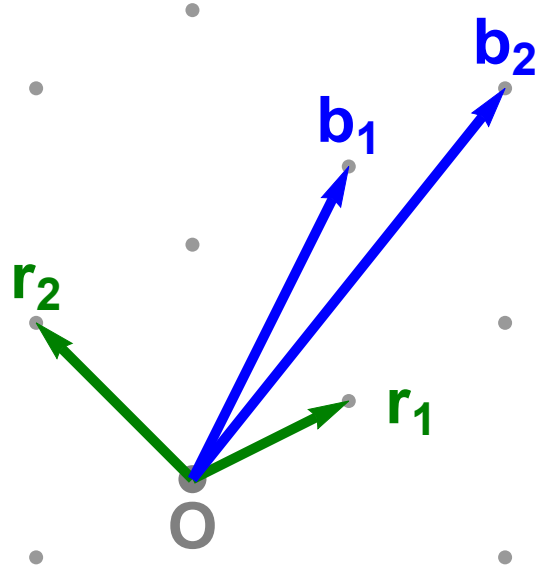


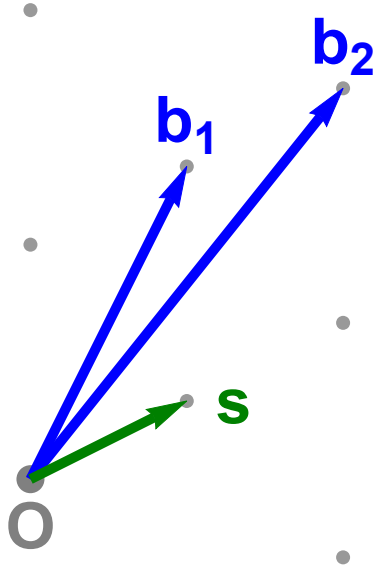
Lattices

Lattice basis reduction



Lattices

Shortest Vector Problem (SVP)



Outline

Enumeration algorithms

- Fincke-Pohst enumeration
- Kannan enumeration
- Pruning the enumeration tree

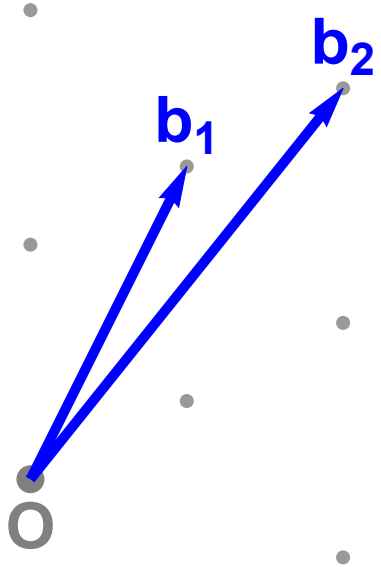
Sieving algorithms

- Nguyen-Vidick sieve
- Multiple levels
- Near neighbor techniques

Practical comparison

Fincke-Pohst enumeration

1. Determine possible coefficients of b_2



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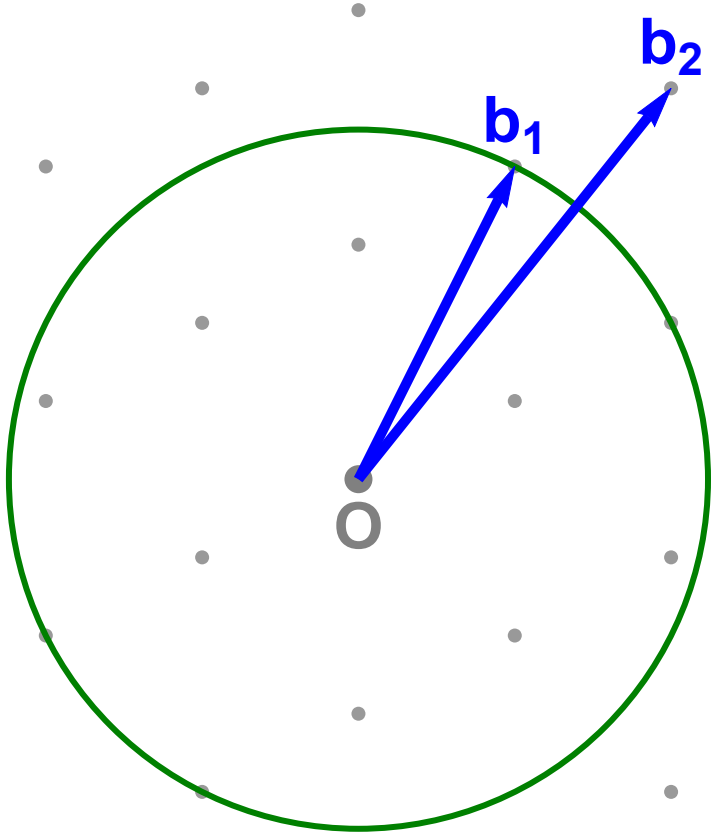

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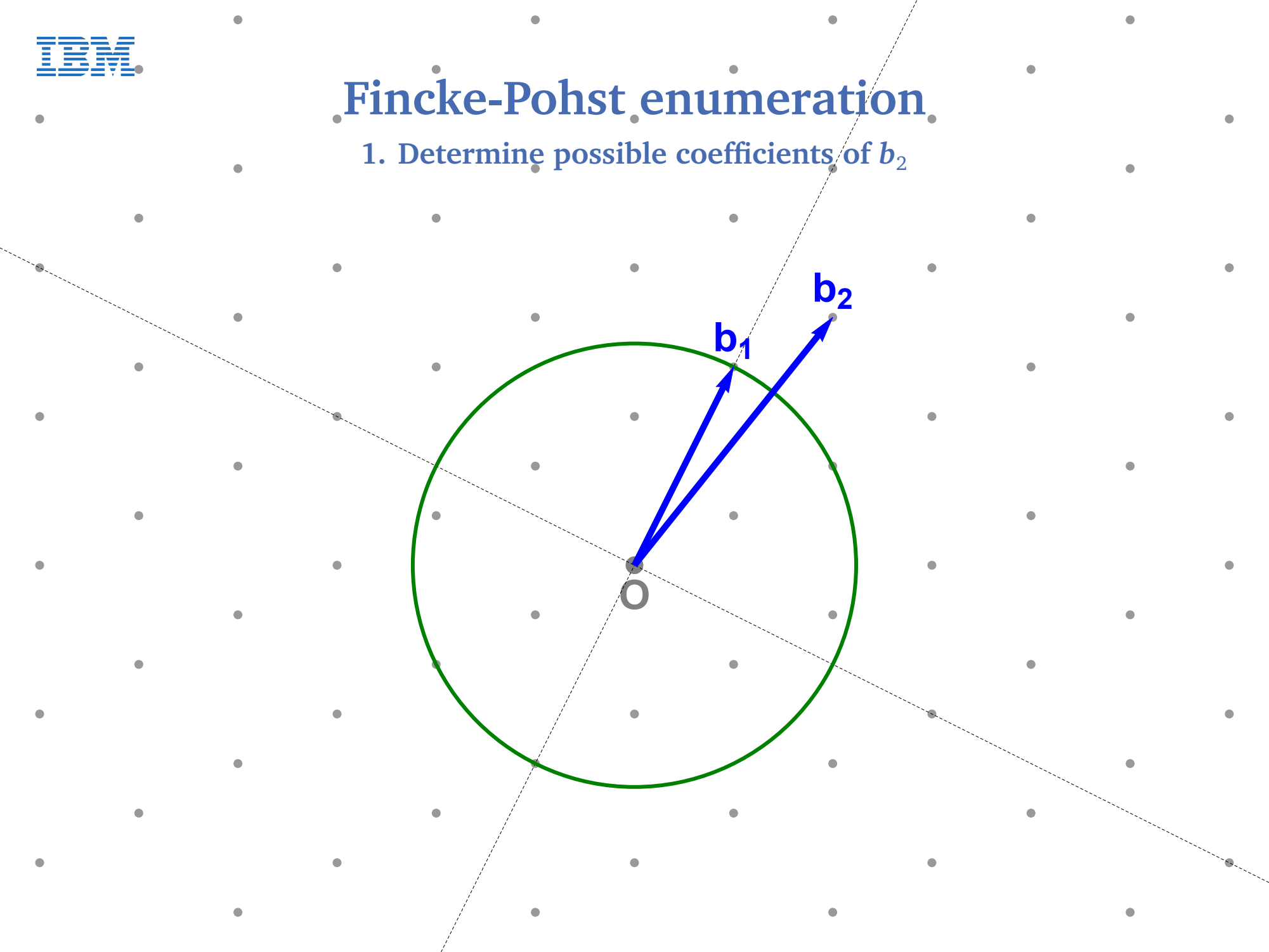

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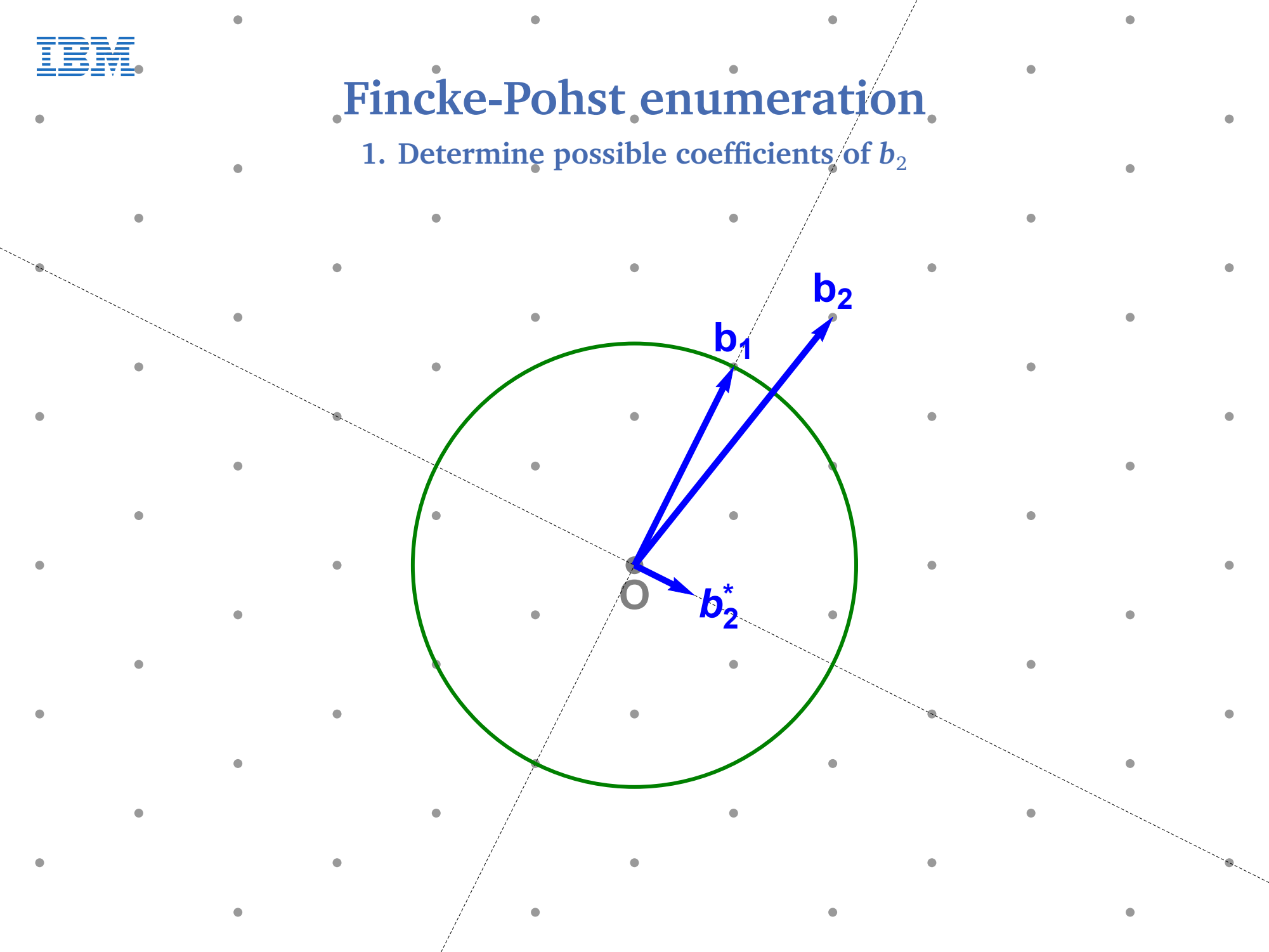

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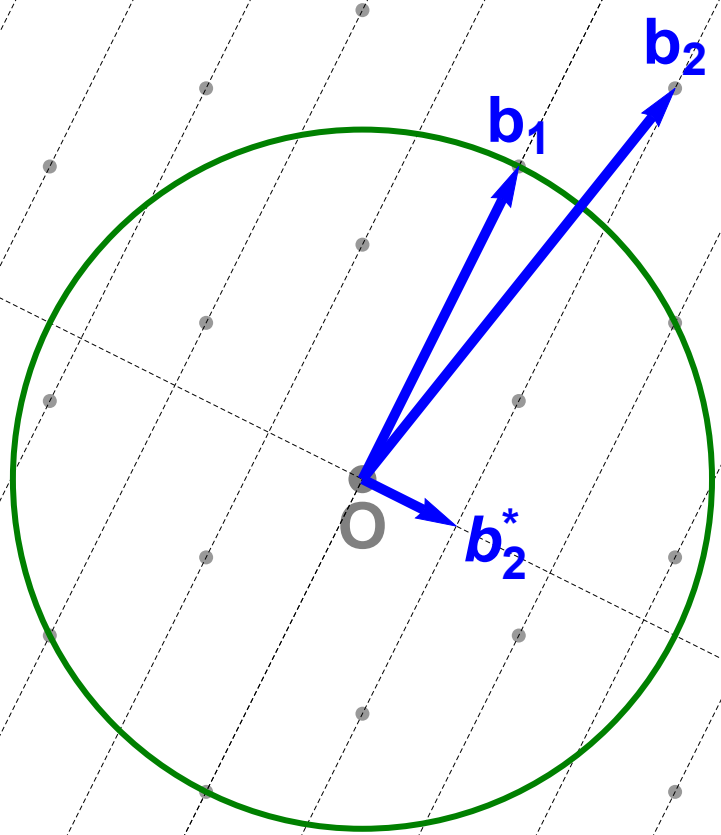
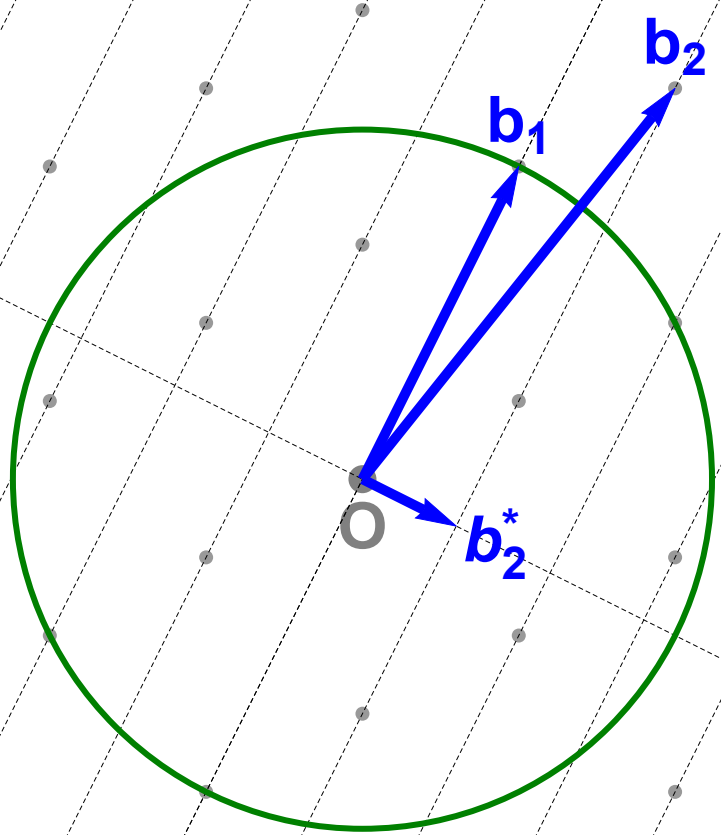


Fincke-Pohst enumeration

2. Find short vectors for each coefficient of b_2



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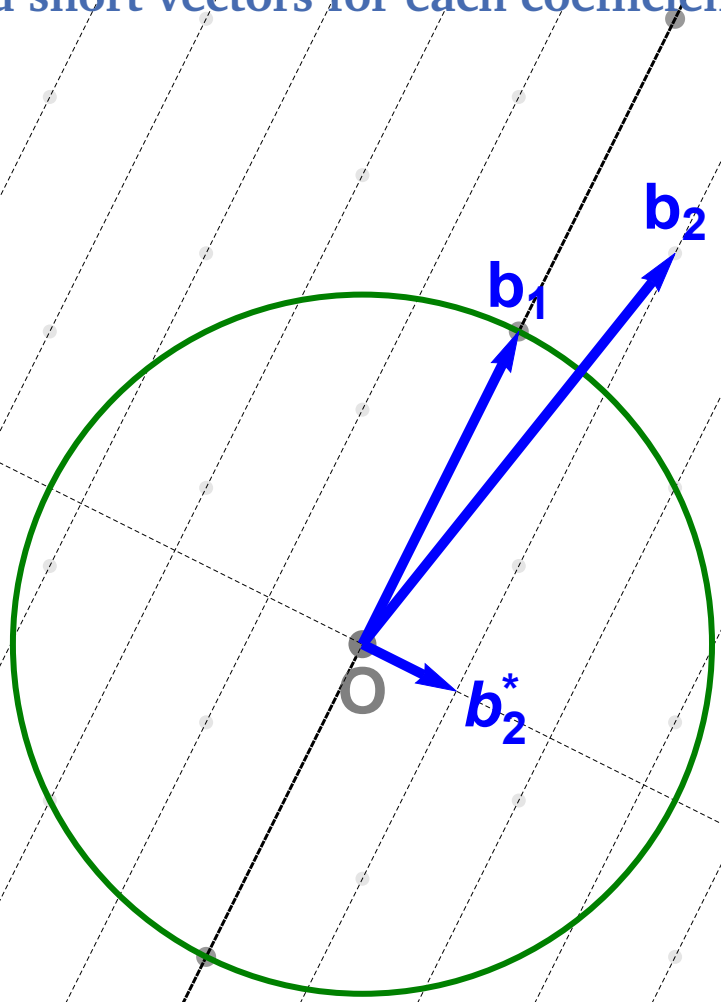

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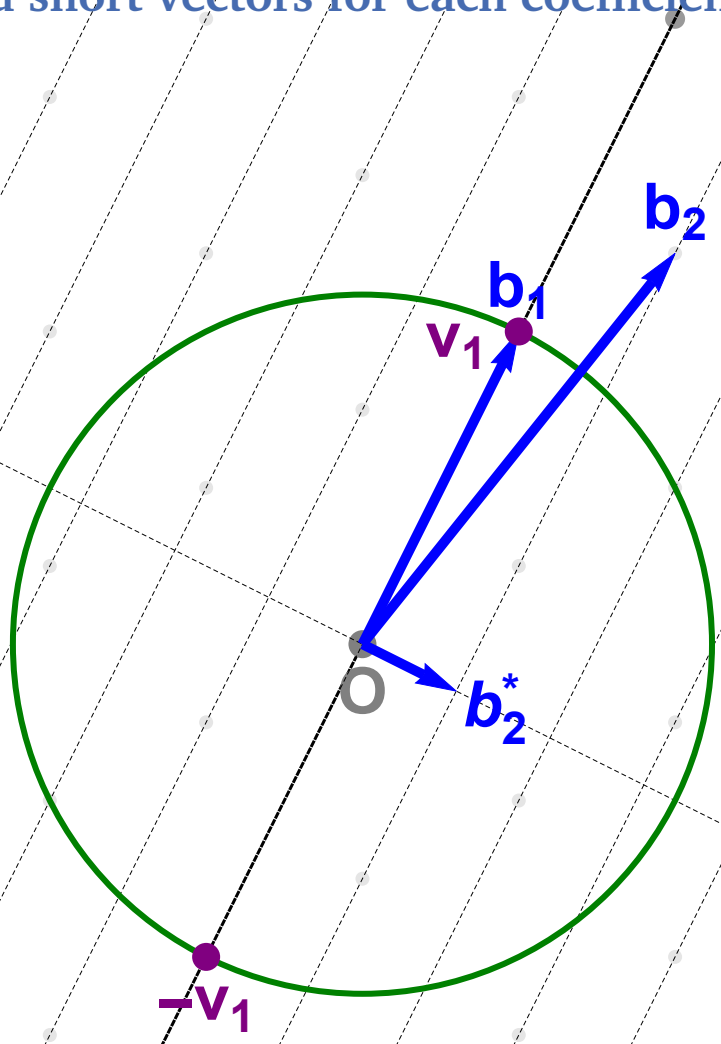

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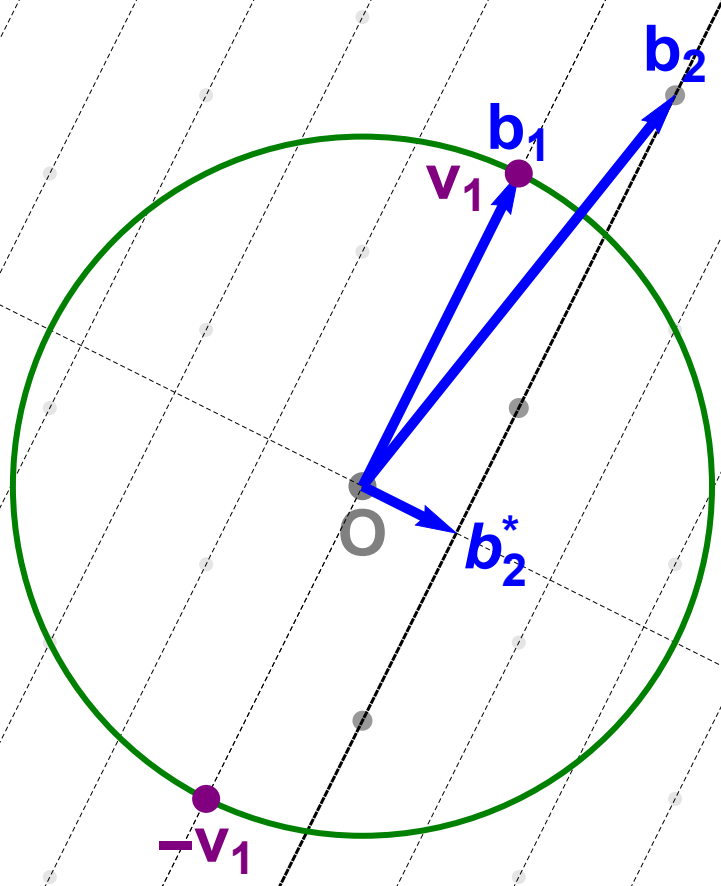

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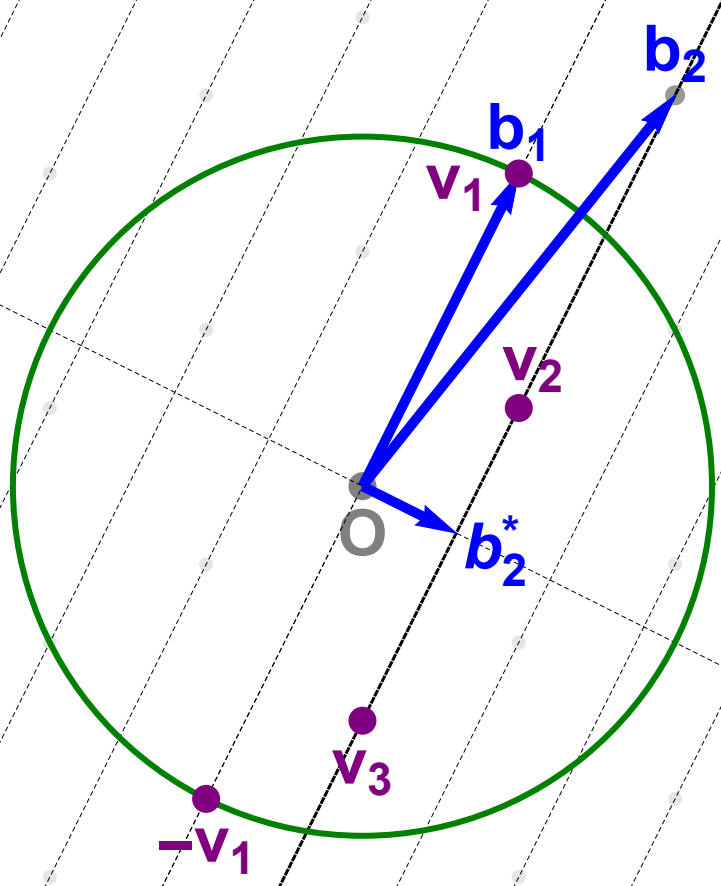

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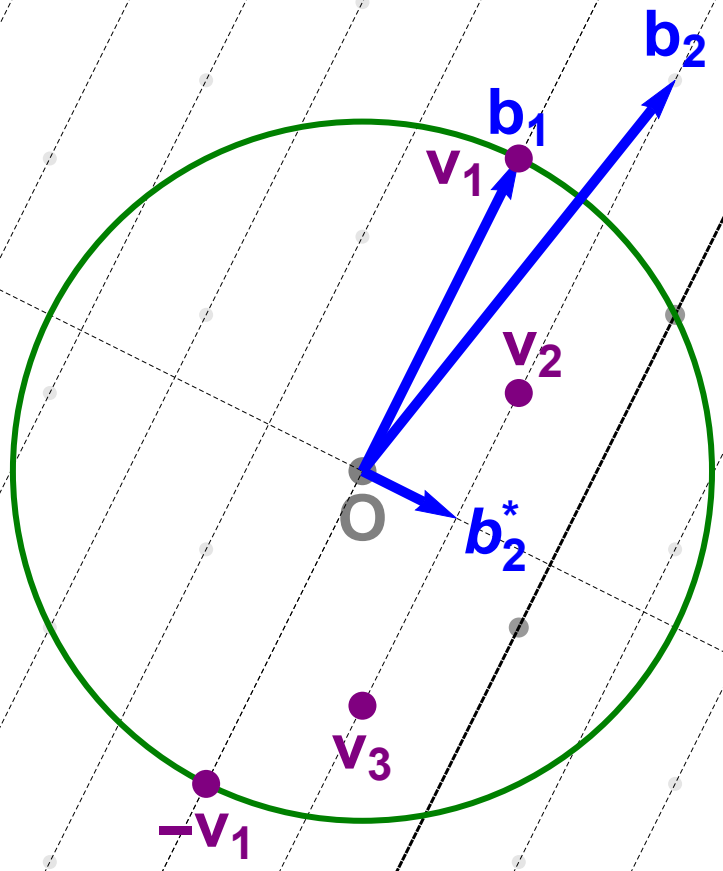

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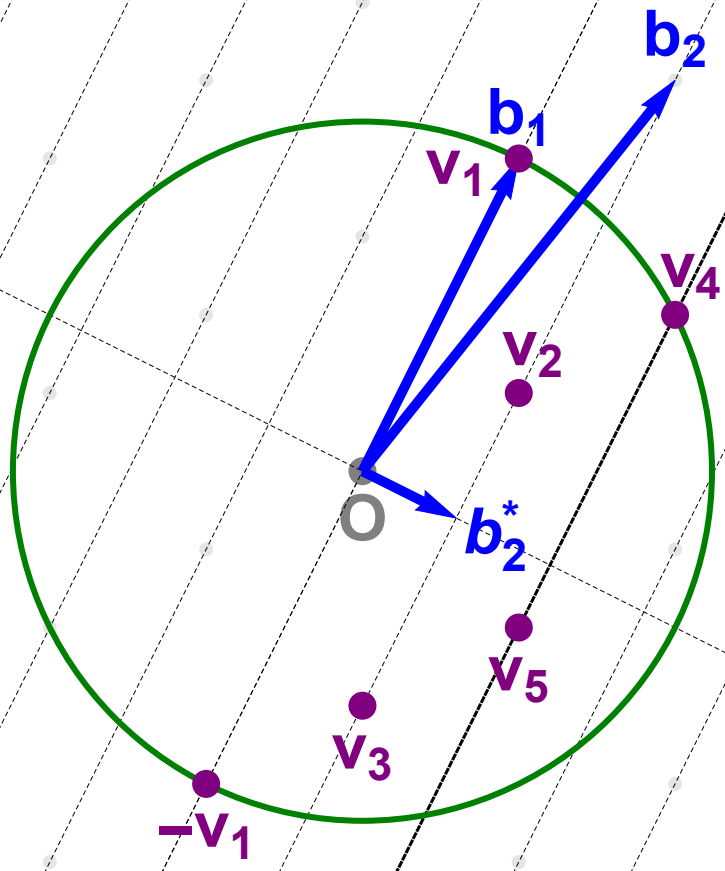

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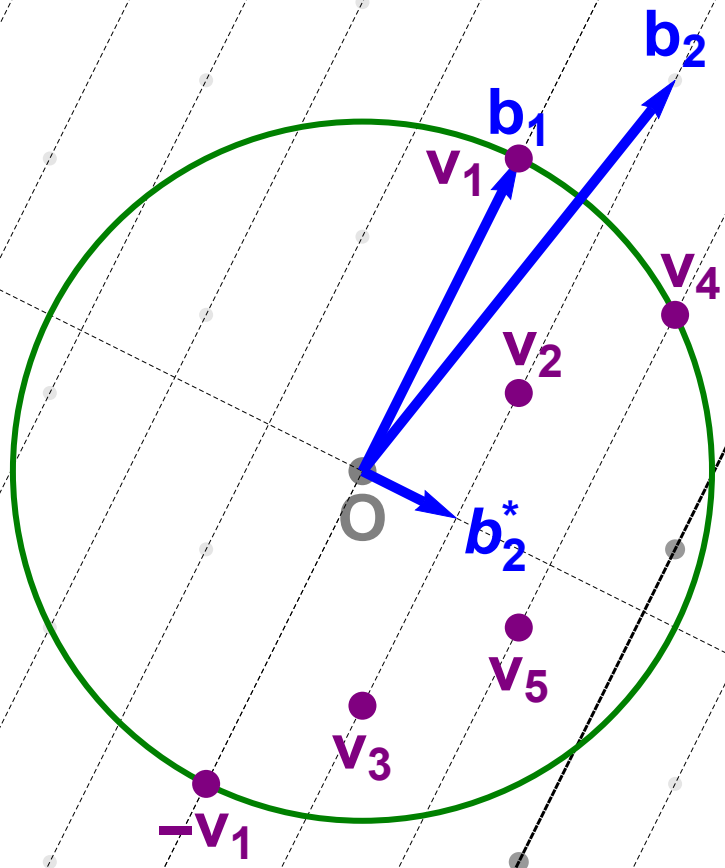

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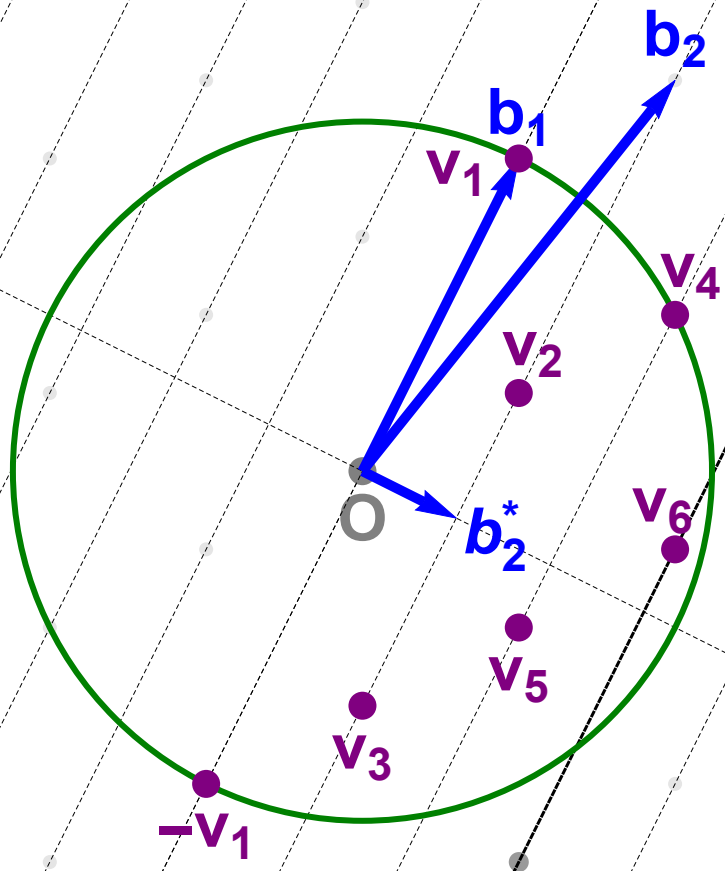

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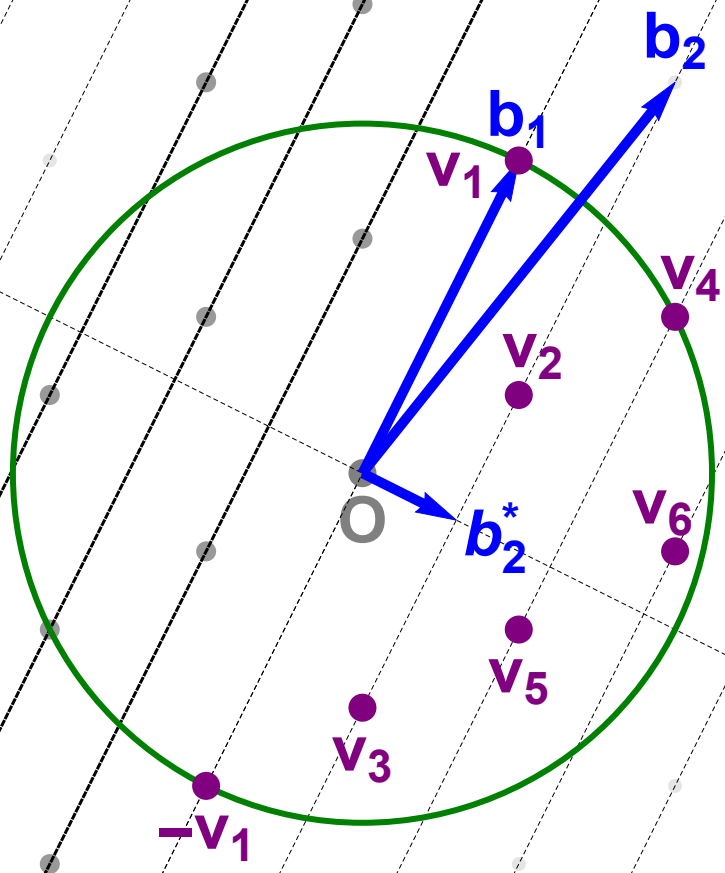

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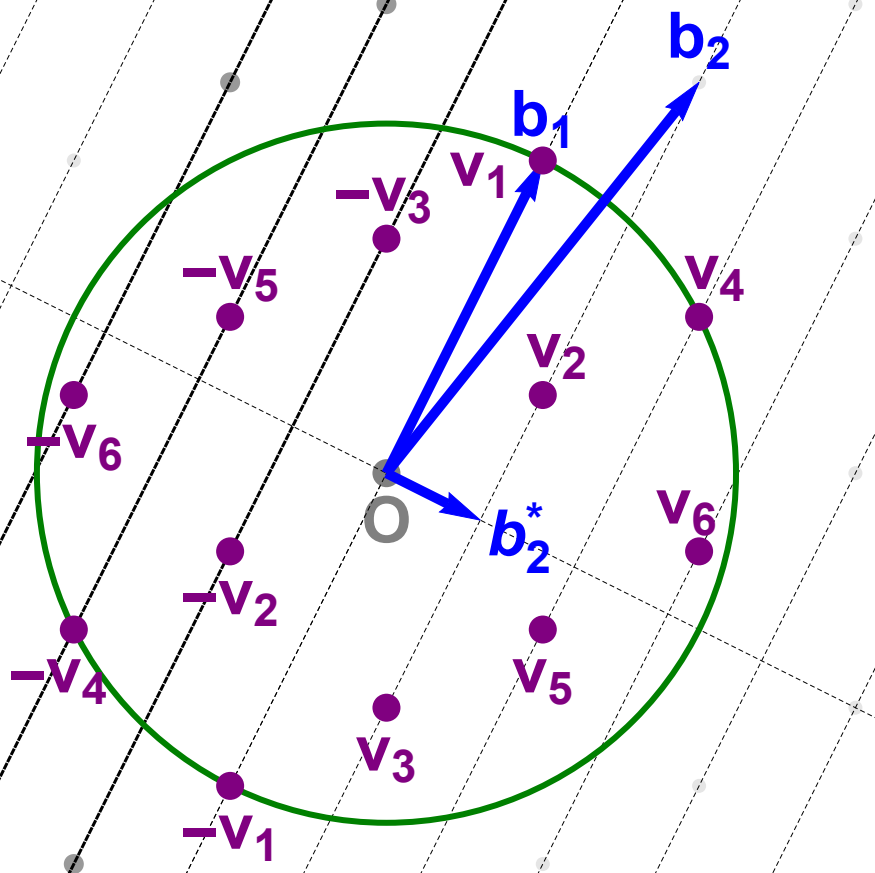

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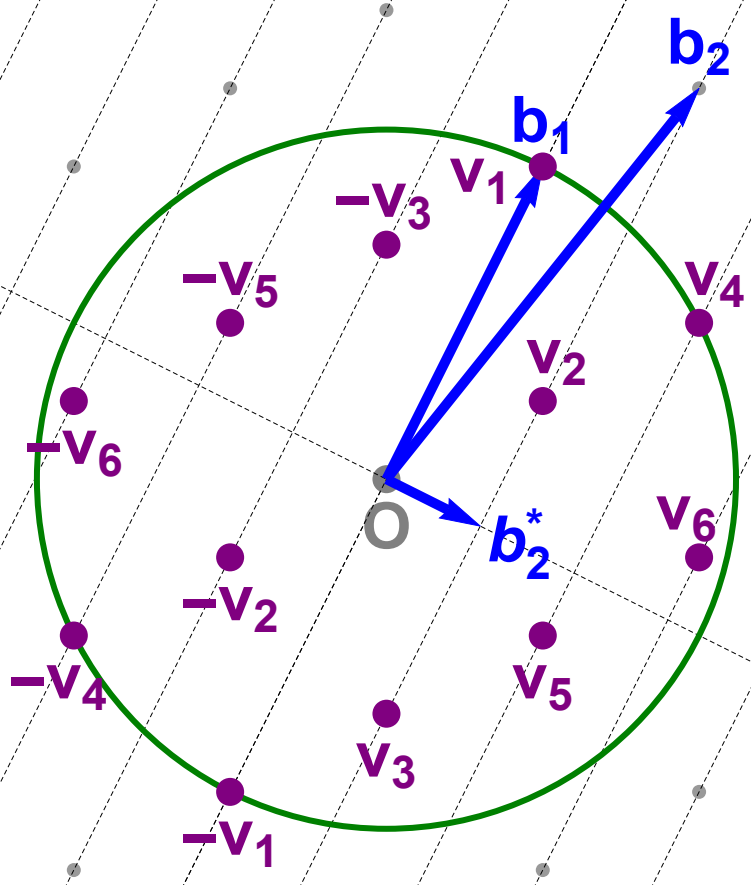
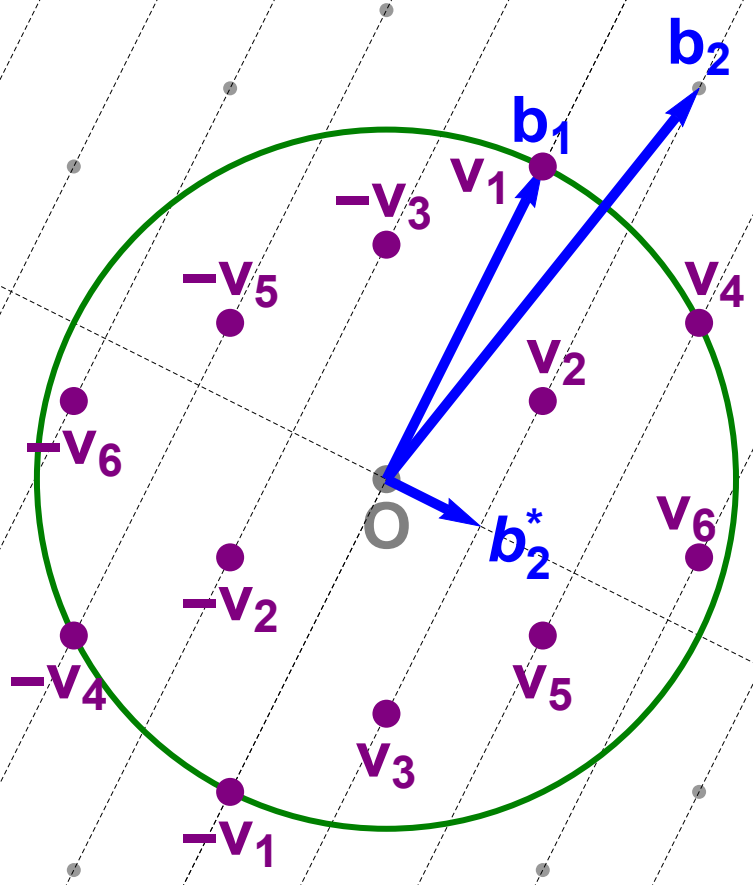


Fincke-Pohst enumeration

3. Find a shortest vector among all found vectors



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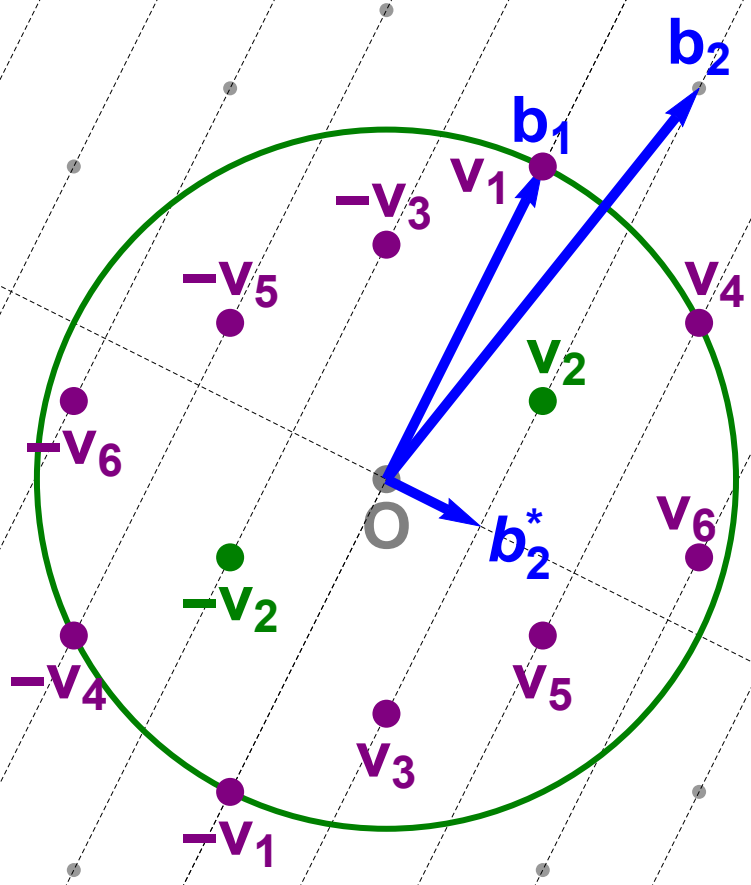

Fincke-Pohst enumeration

Overview

Fincke-Pohst enumeration

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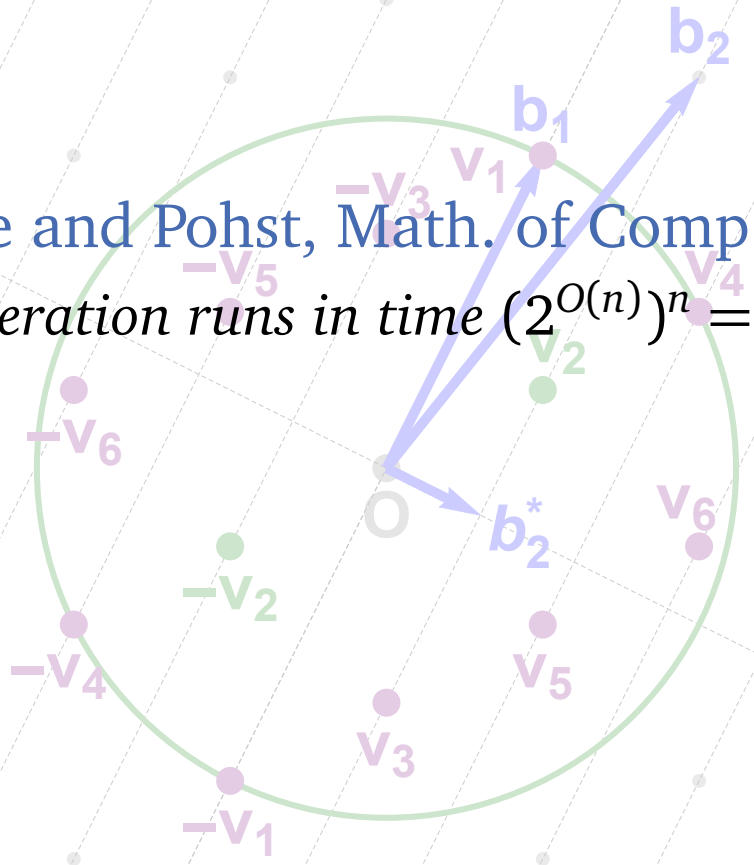

Theorem (Fincke and Pohst, Math. of Comp. '85)

Fincke-Pohst enumeration runs in time $(2^{O(n)})^n = 2^{O(n^2)}$ *and space* poly(n).

Fincke-Pohst enumeration

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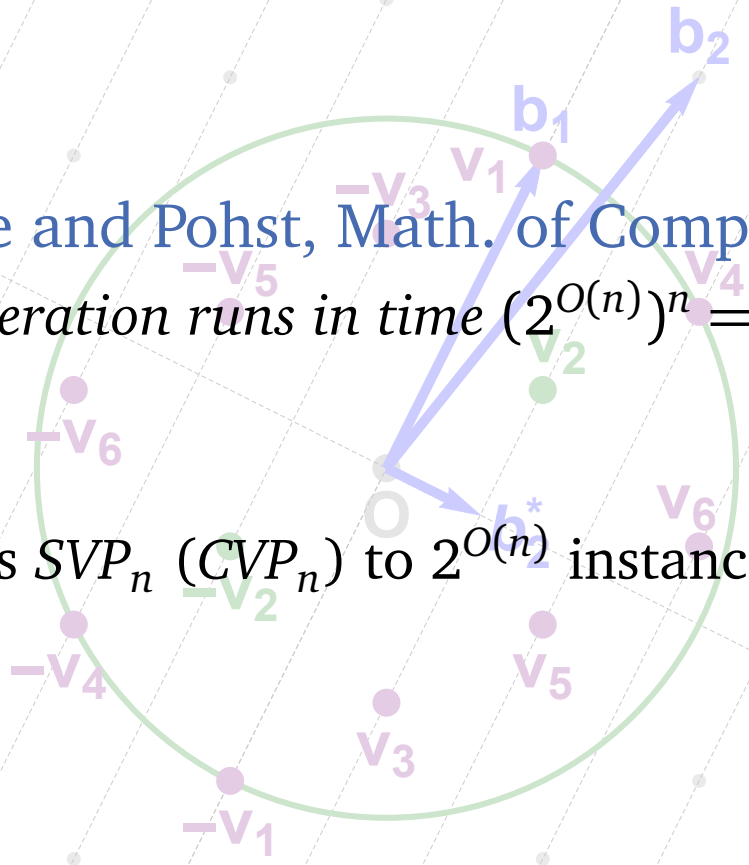


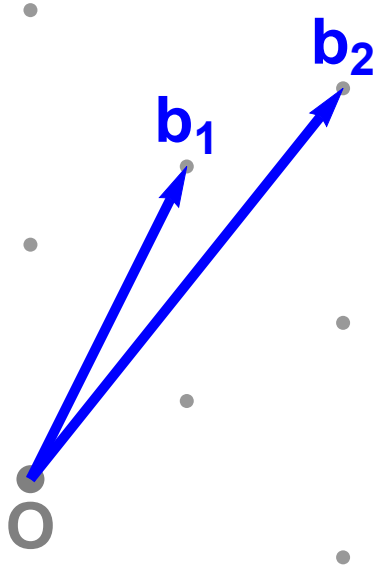
Theorem (Fincke and Pohst, Math. of Comp. '85)

Fincke-Pohst enumeration runs in time $(2^{O(n)})^n = 2^{O(n^2)}$ *and space* poly(n).

Essentially reduces SVP_n (CVP_n) to $2^{O(n)}$ instances of CVP_{n-1}

Kannan enumeration

Better bases



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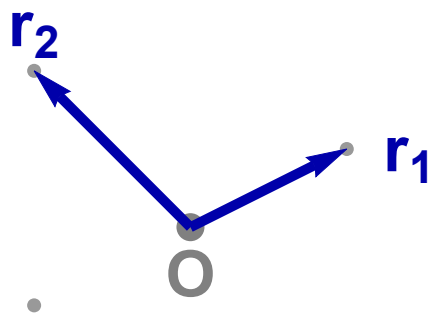

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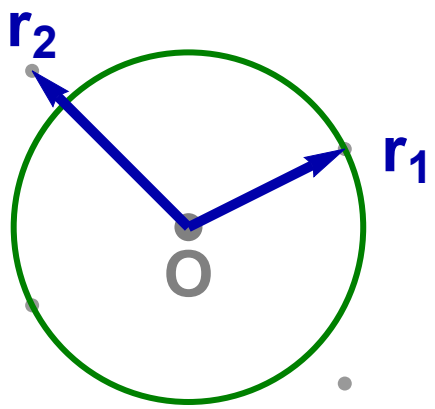

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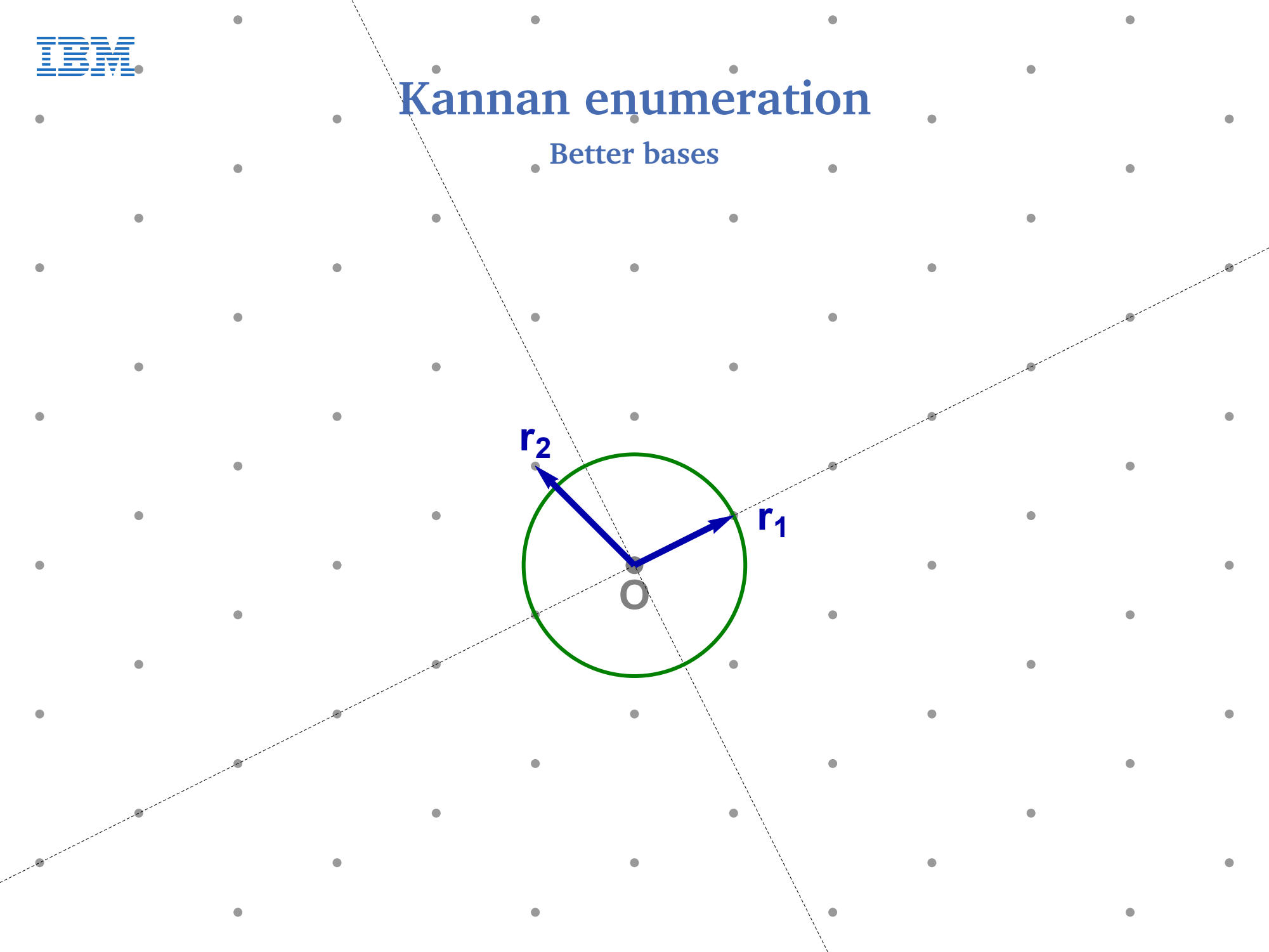

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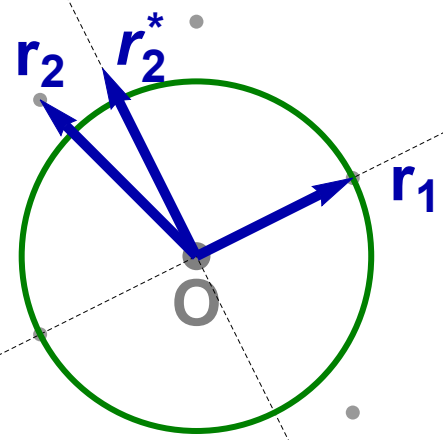

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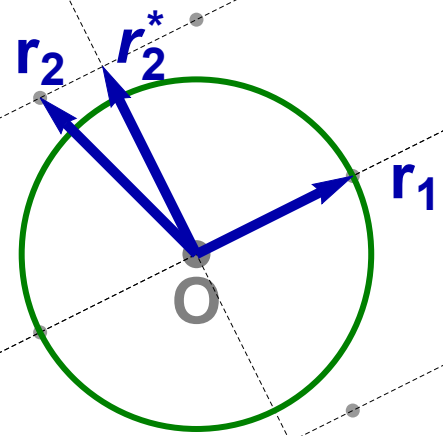

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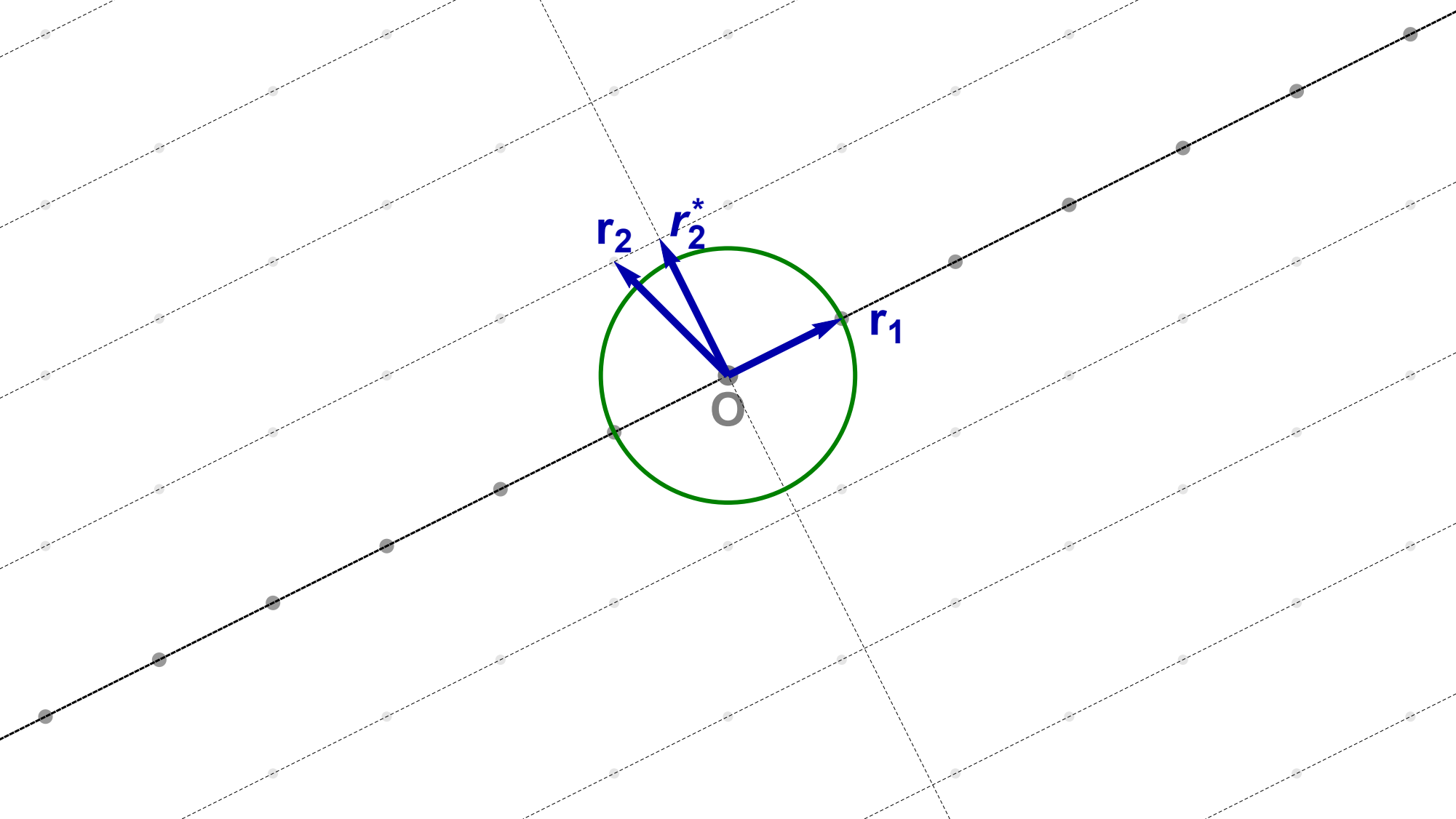

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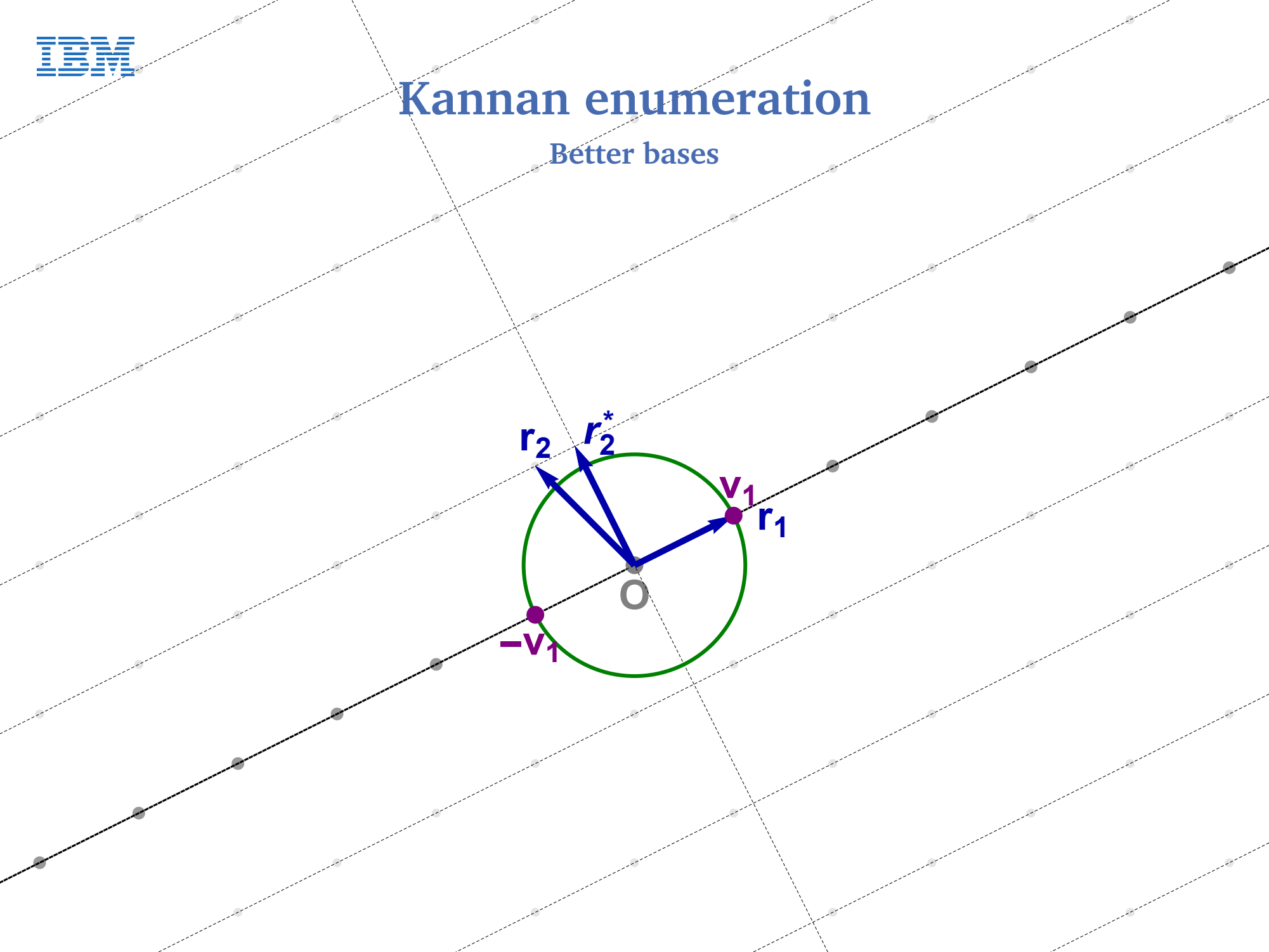

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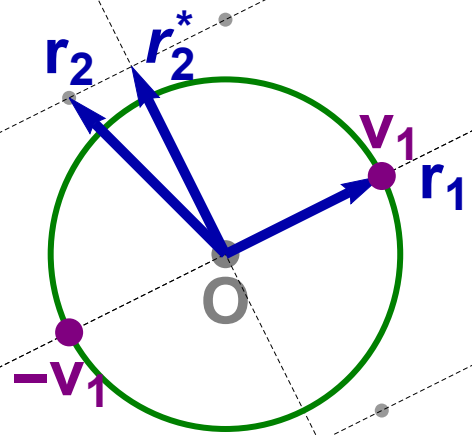

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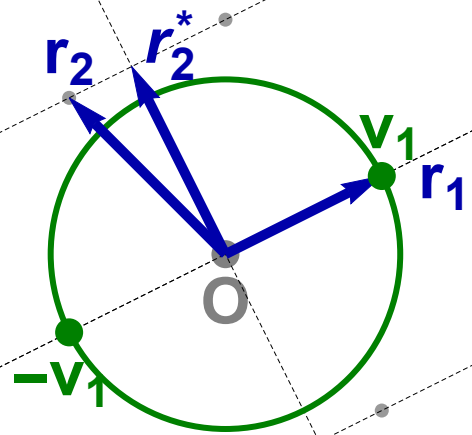

Kannan enumeration

Overview

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Theorem (Kannan, STOC'83)

Kannan enumeration runs in time $2^{O(n \log n)}$ *and space* $\mathrm{poly}(n)$.

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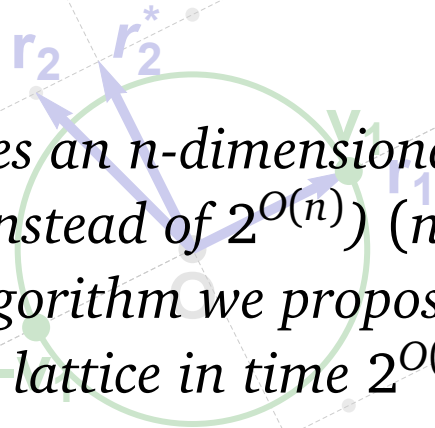
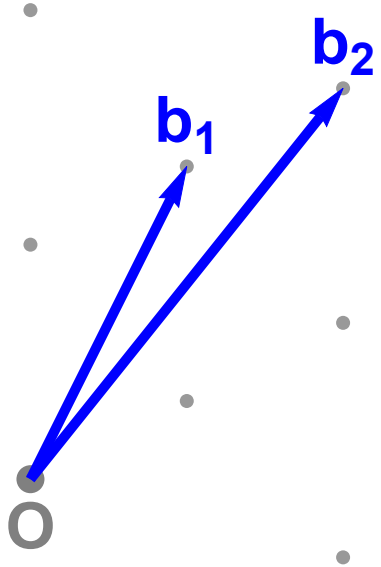


> *"Our algorithm reduces an n-dimensional problem to polynomially many (instead of* $2^{O(n)}$*)* $(n-1)$*-dimensional problems. [...] The algorithm we propose, first finds a more orthogonal basis for a lattice in time* $2^{O(n \log n)}$*."*
>
> – Kannan, *STOC'83*

Pruned enumeration

Reducing the search space



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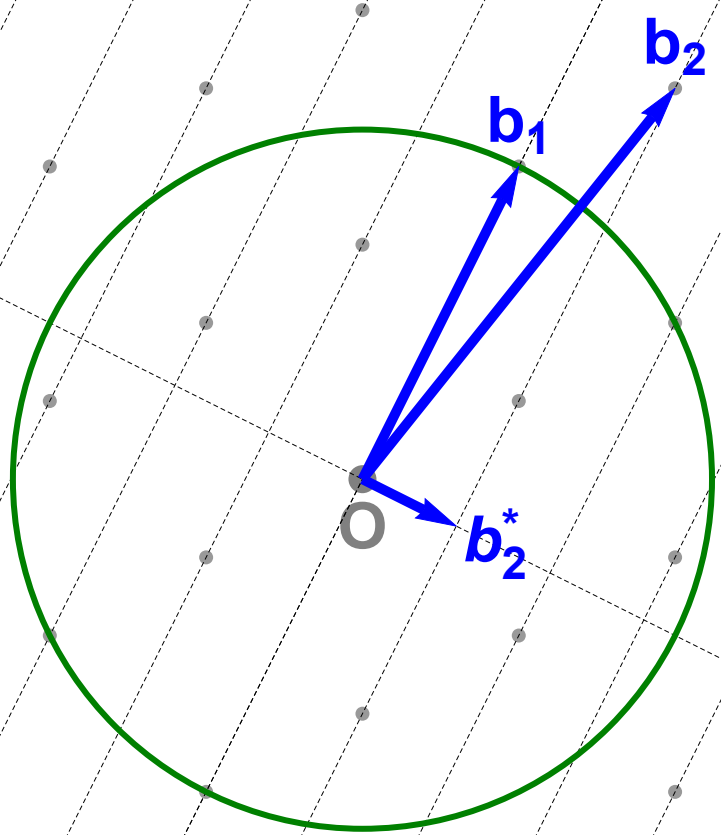
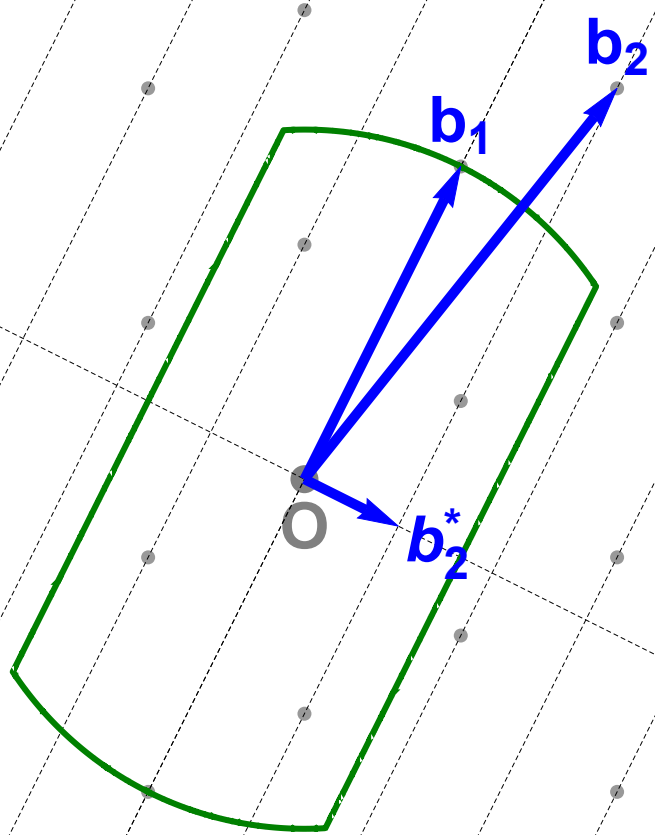


Pruned enumeration

Reducing the search space



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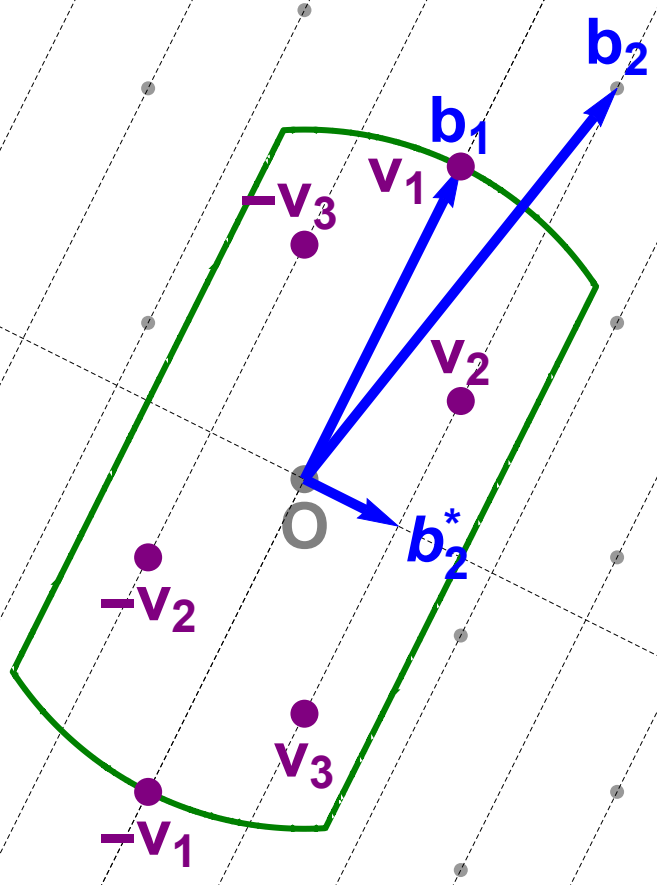

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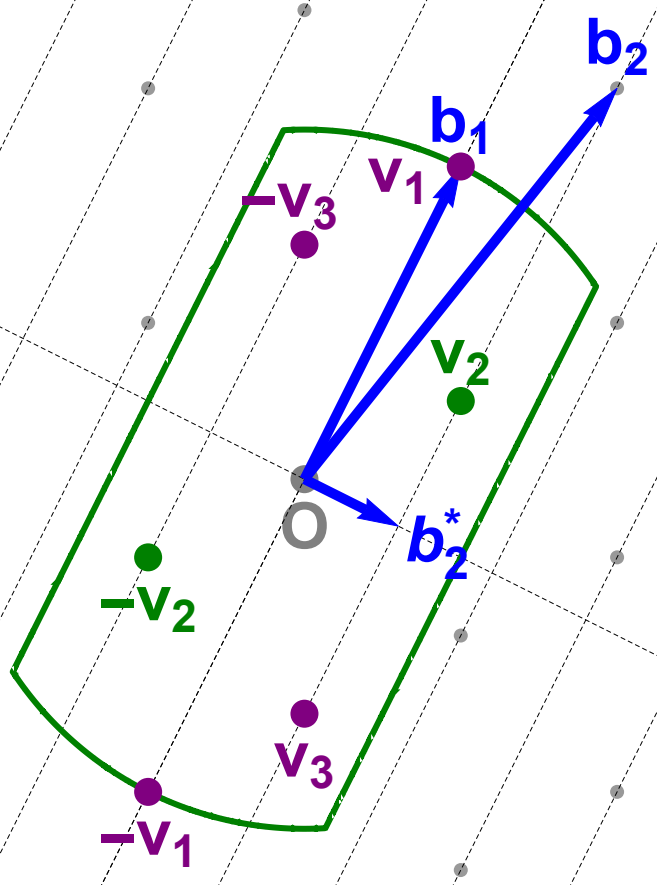

Pruned enumeration

Overview

Pruned enumeration

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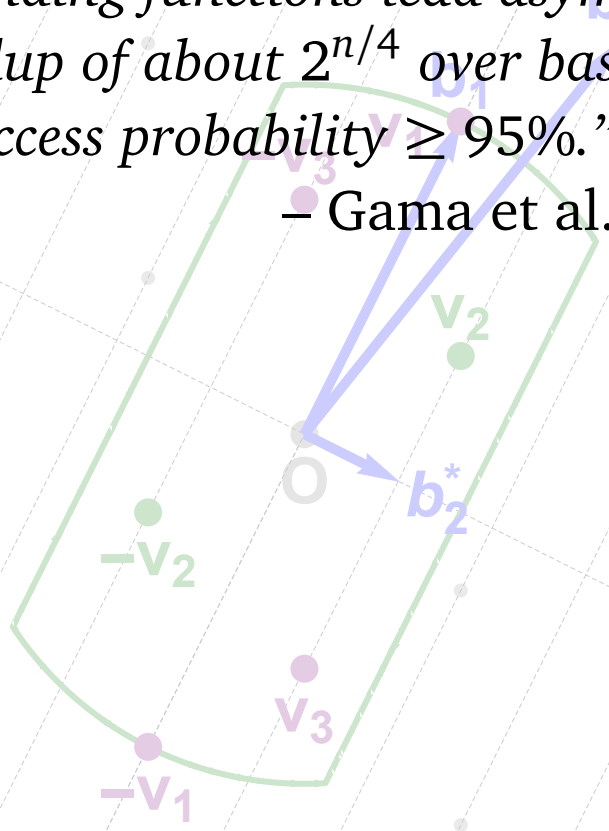
"Well-chosen bounding functions lead asymptotically to an exponential speedup of about $2^{n/4}$ over basic enumeration, maintaining a success probability $\geq 95\%$."

– Gama et al., *EUROCRYPT'10*



Pruned enumeration

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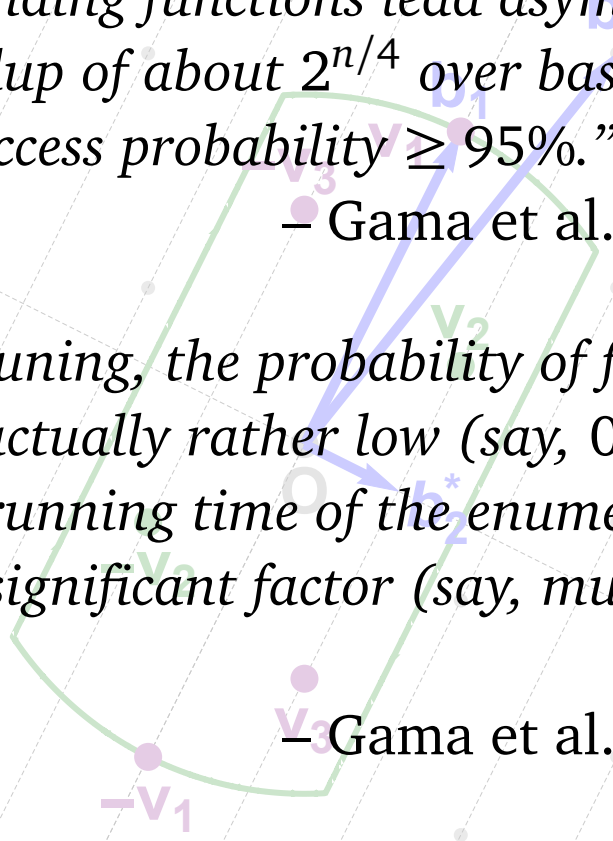
"With extreme pruning, the probability of finding the desired vector is actually rather low (say, 0.1%), but surprisingly, the running time of the enumeration is reduced by a much more significant factor (say, much more than 1000)."

– Gama et al., *EUROCRYPT'10*



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Nguyen-Vidick sieve

1. Sample a list L of random lattice vectors

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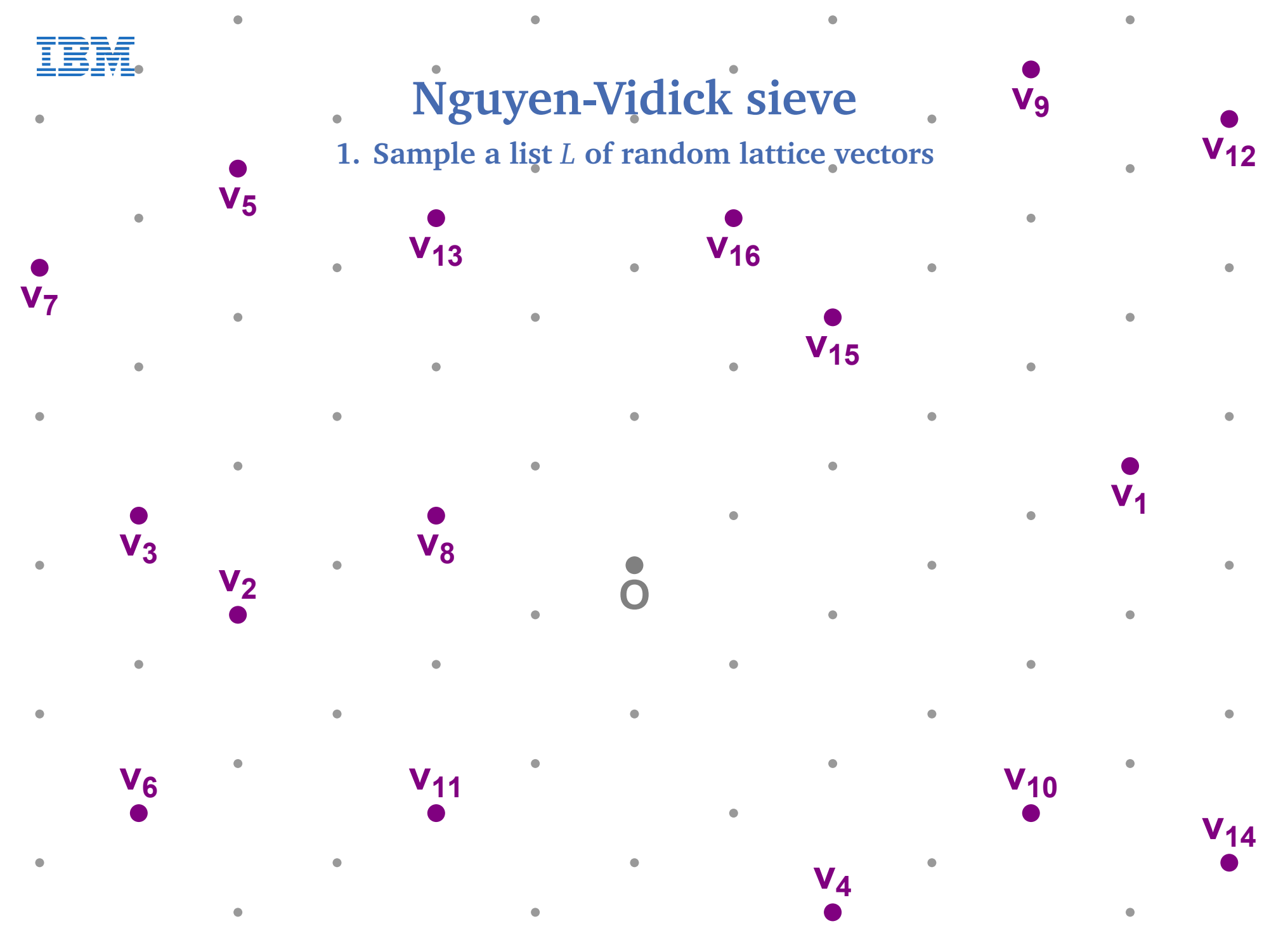

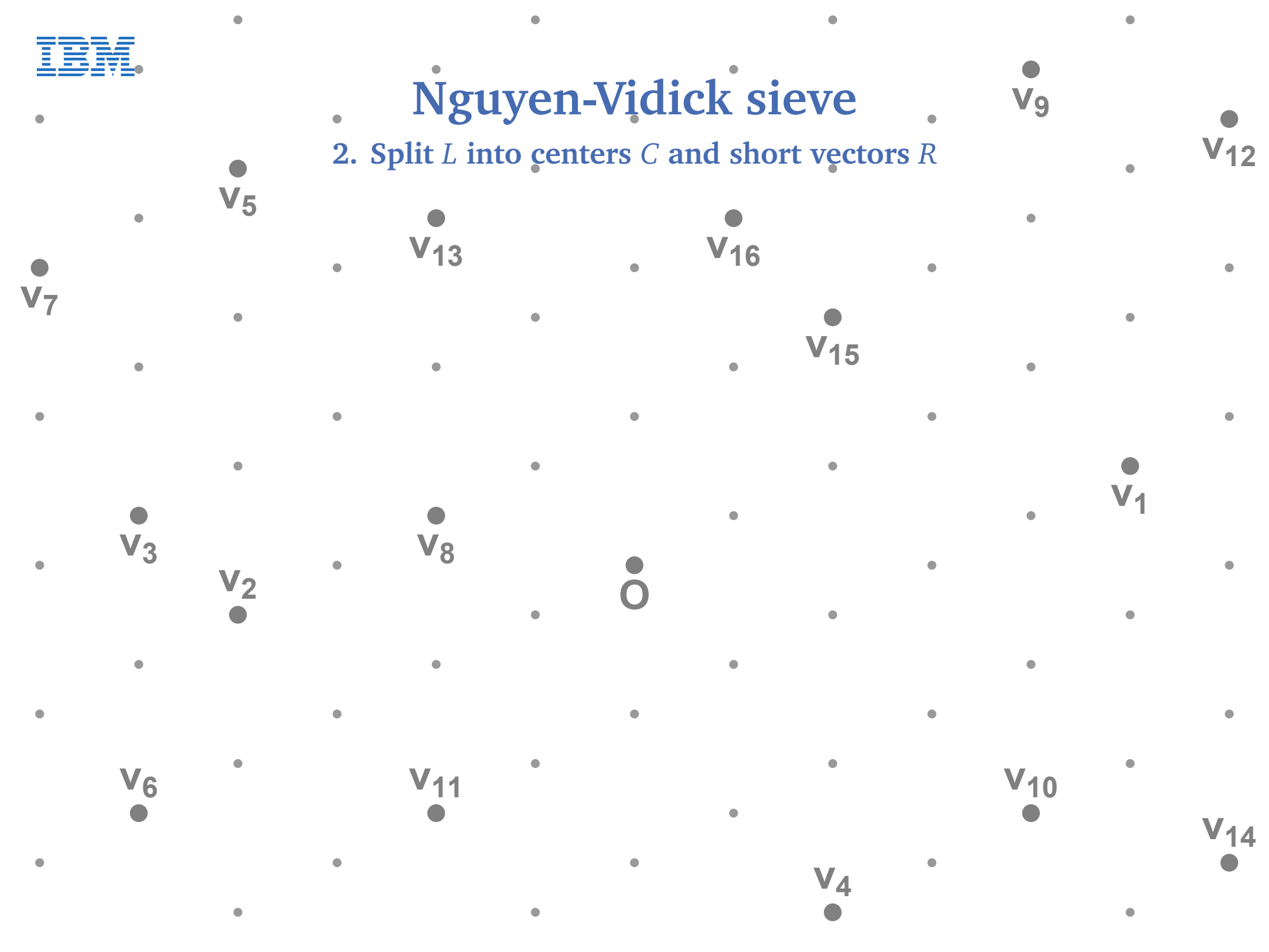
IBM
Nguyen-Vidick sieve
2. Split L into centers C and short vectors R
v_9
v_{12}
v_5
v_{13}
v_{16}
v_7
v_{15}
v_1
v_3
v_8
v_2
O
v_6
v_{11}
v_{10}
v_{14}
v_4

Nguyen-Vidick sieve

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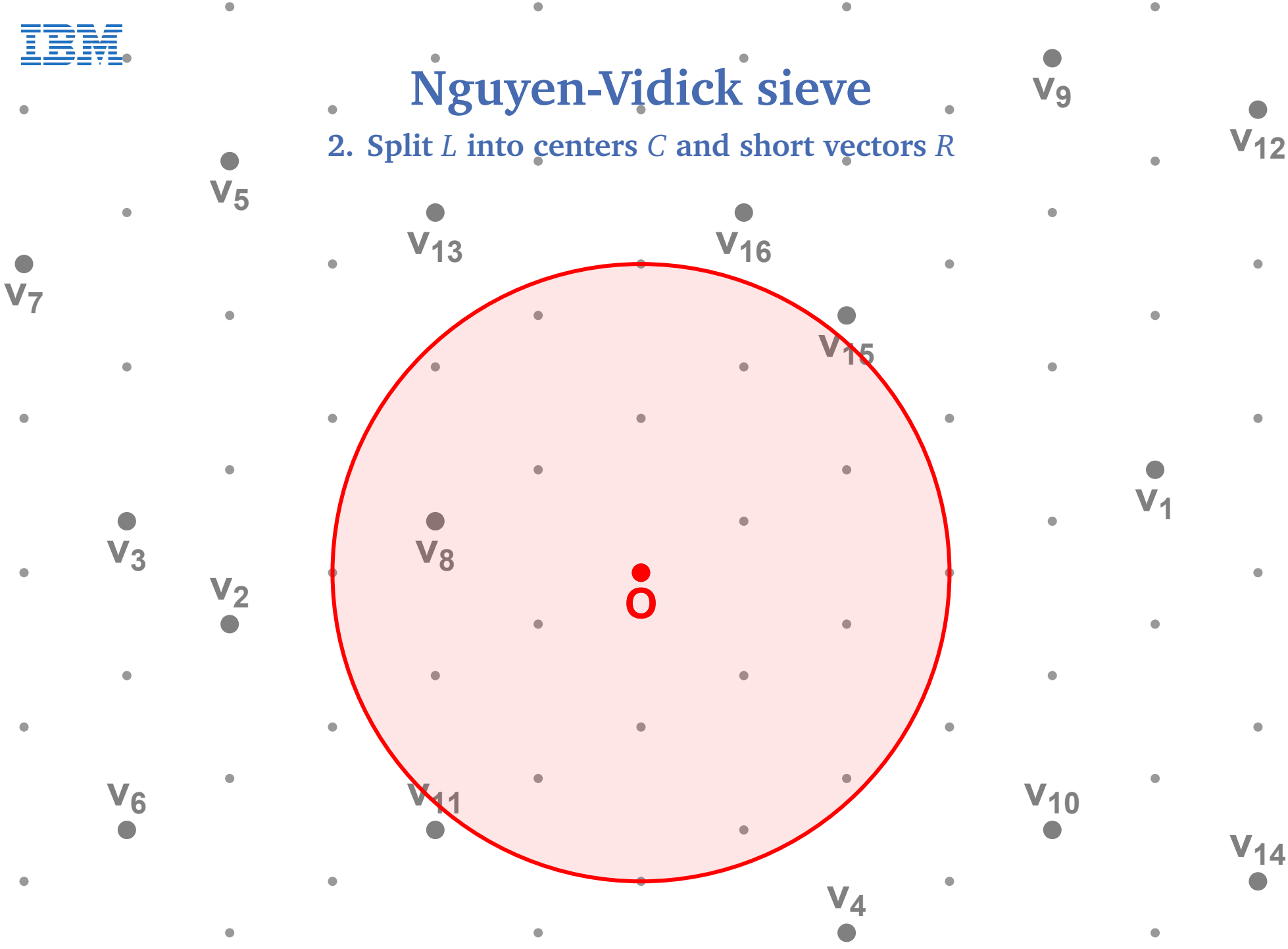

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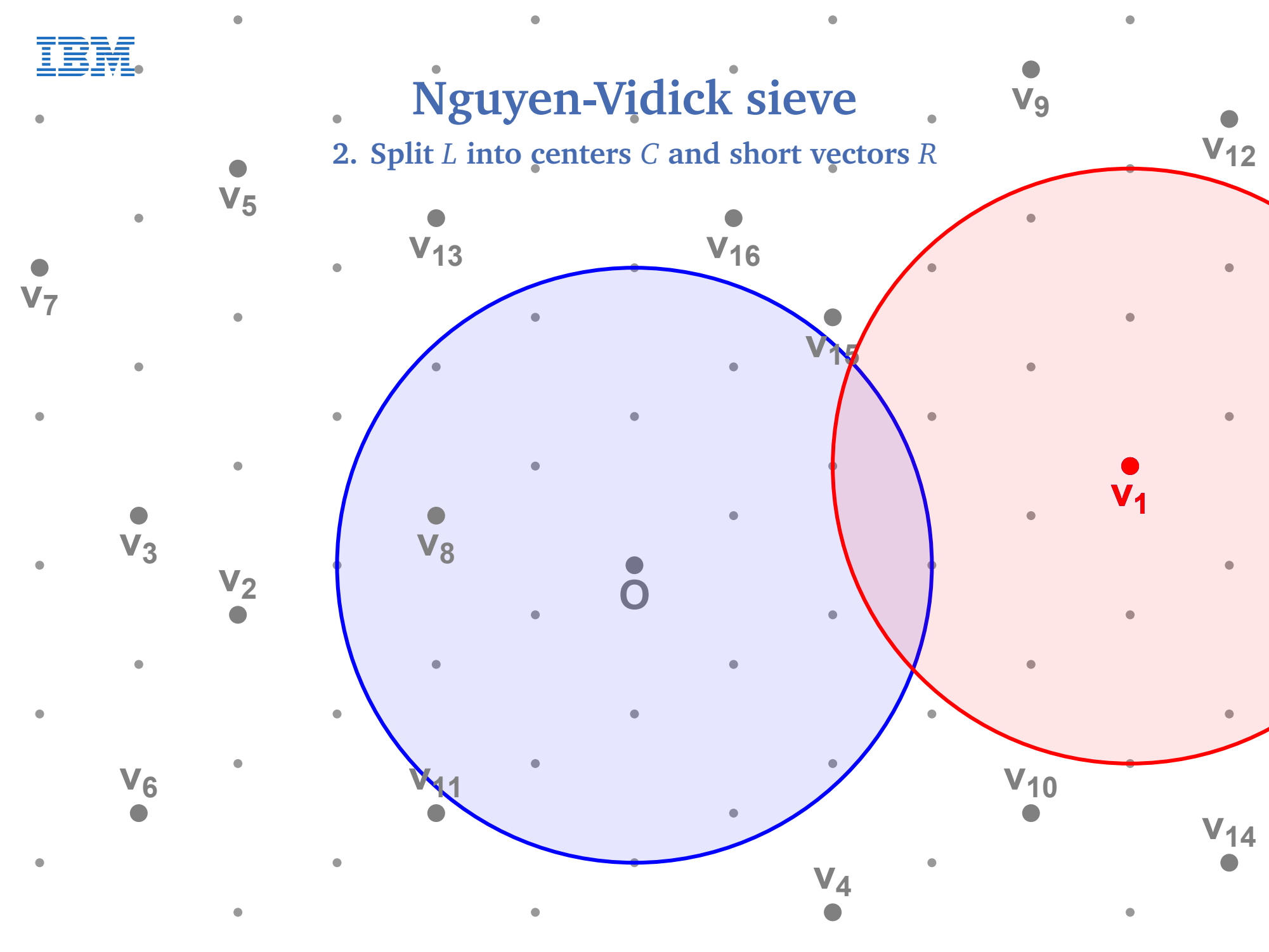

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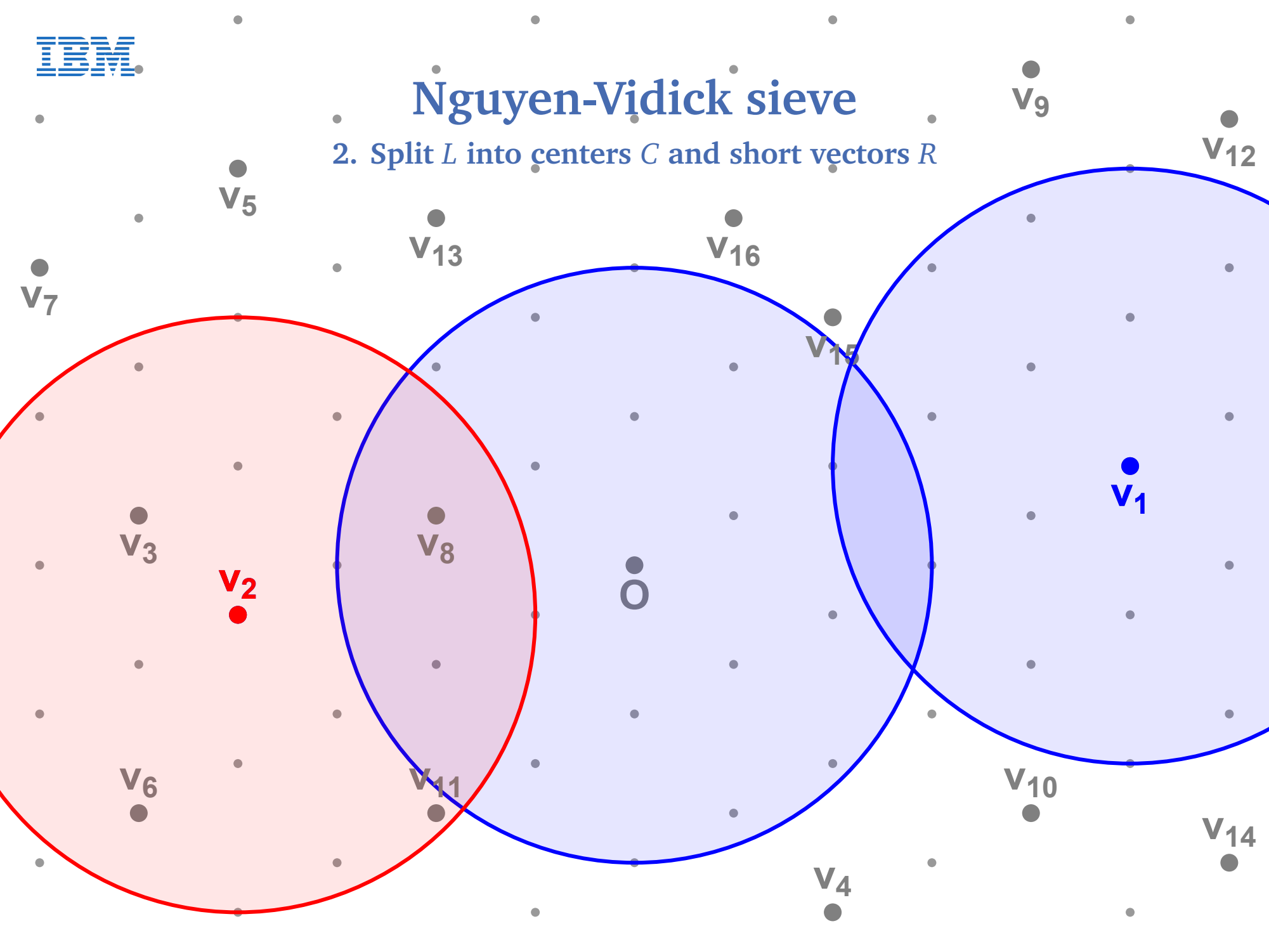

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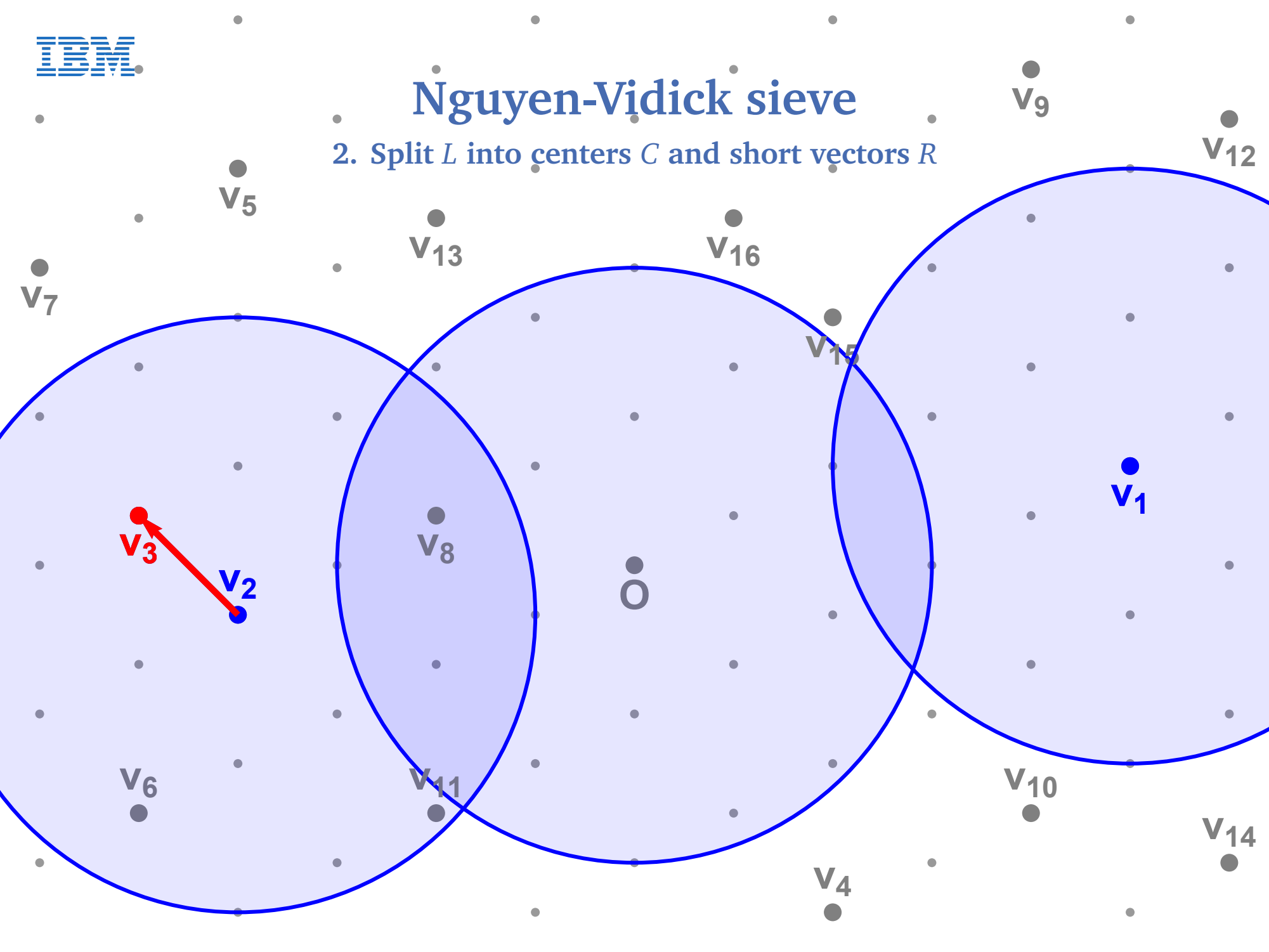

Nguyen-Vidick sieve

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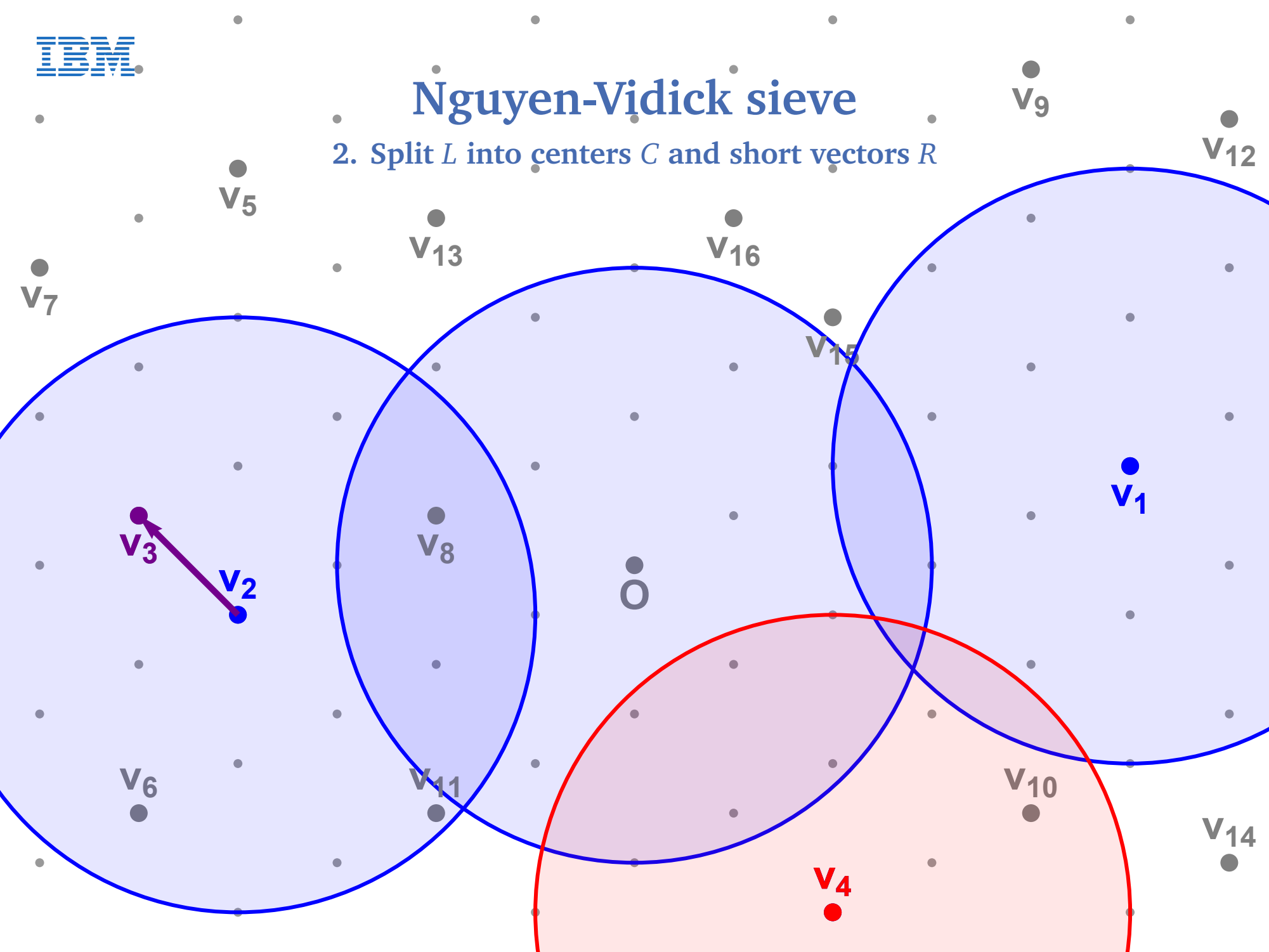

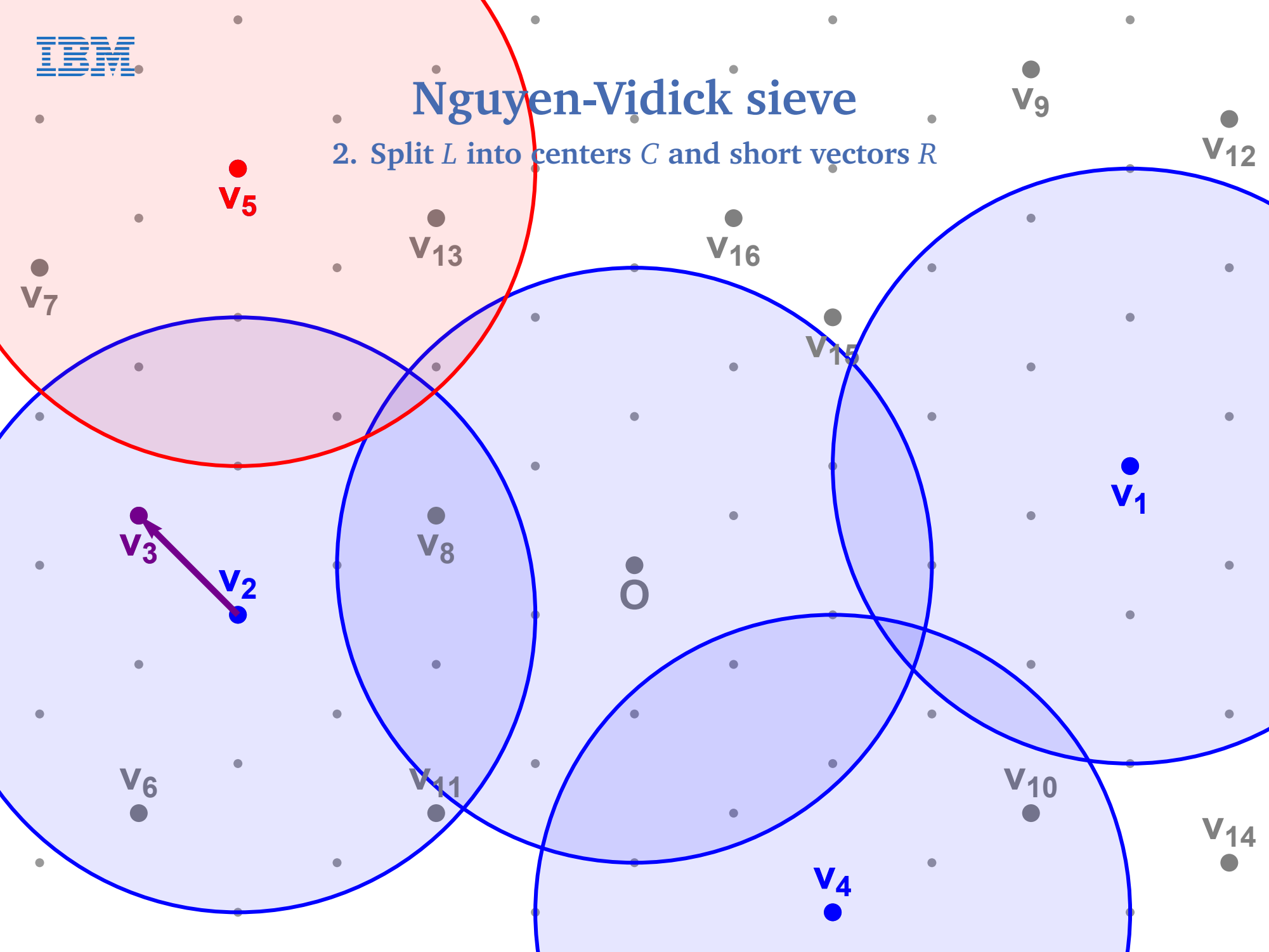
IBM
Nguyen-Vidick sieve
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v_5
v_{13}
v_7
v_{16}
v_9
v_{12}
v_{15}
v_1
v_3
v_2
v_8
O
v_6
v_{11}
v_{10}
v_{14}
v_4

Nguyen-Vidick sieve

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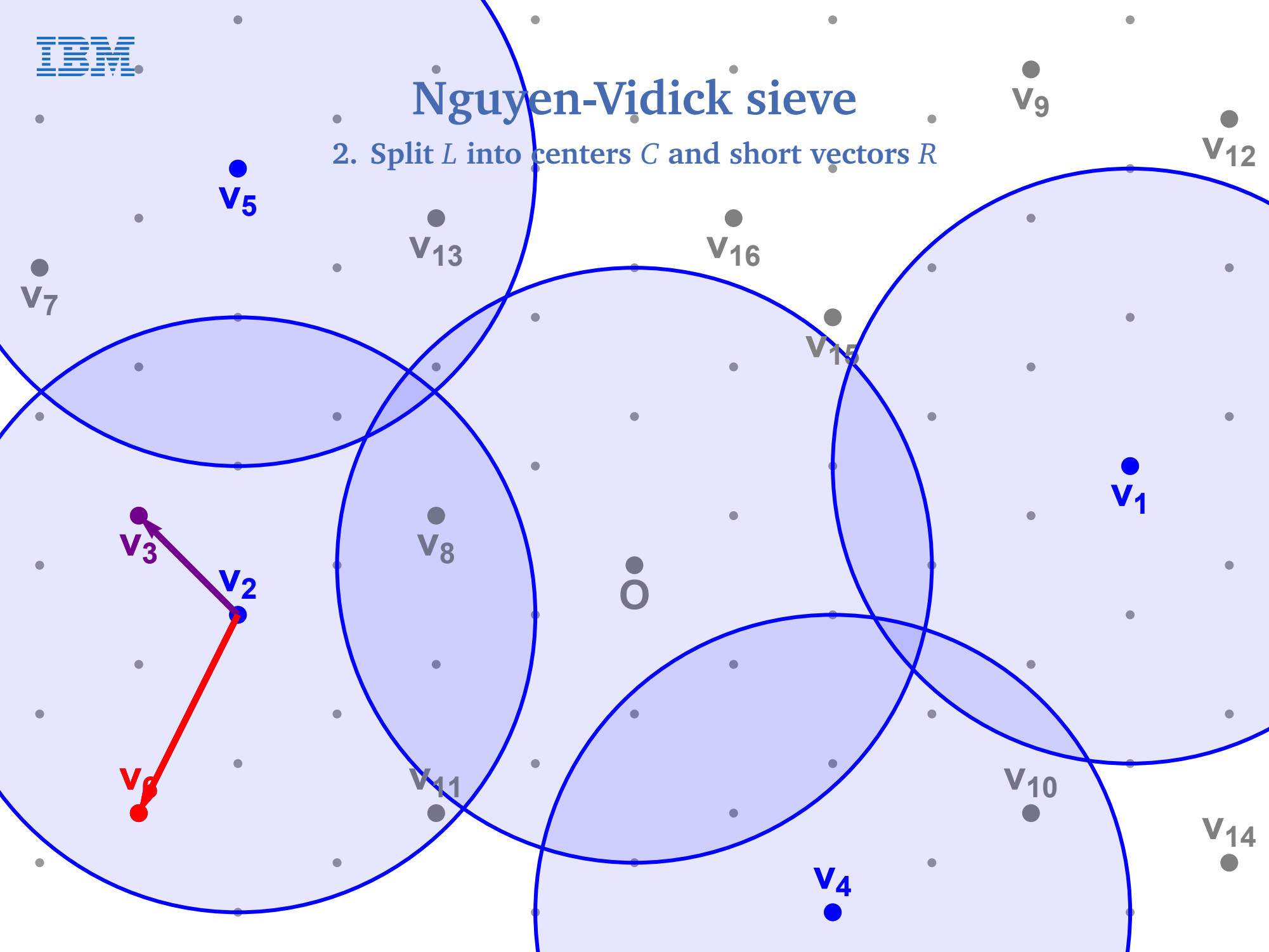

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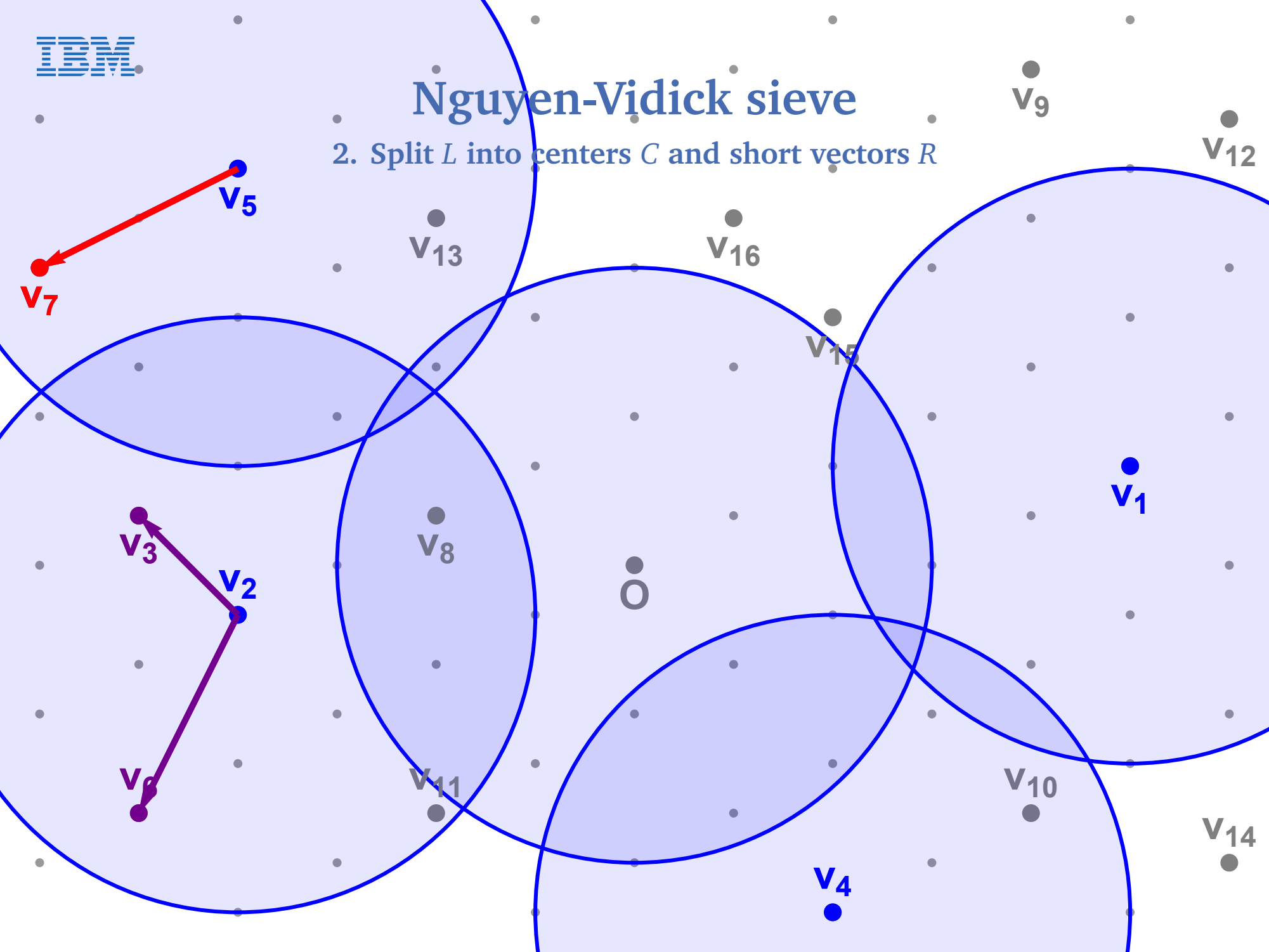

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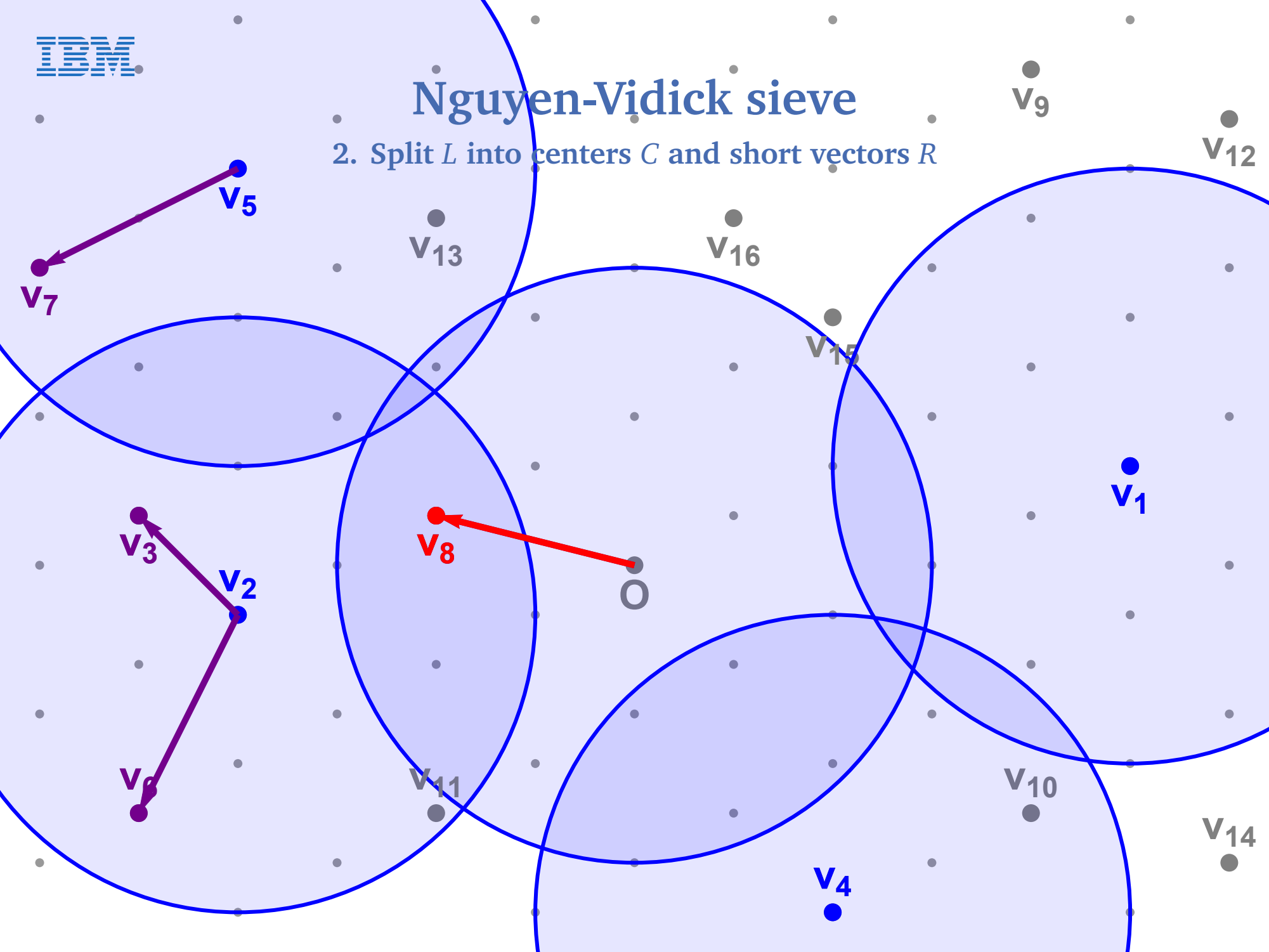

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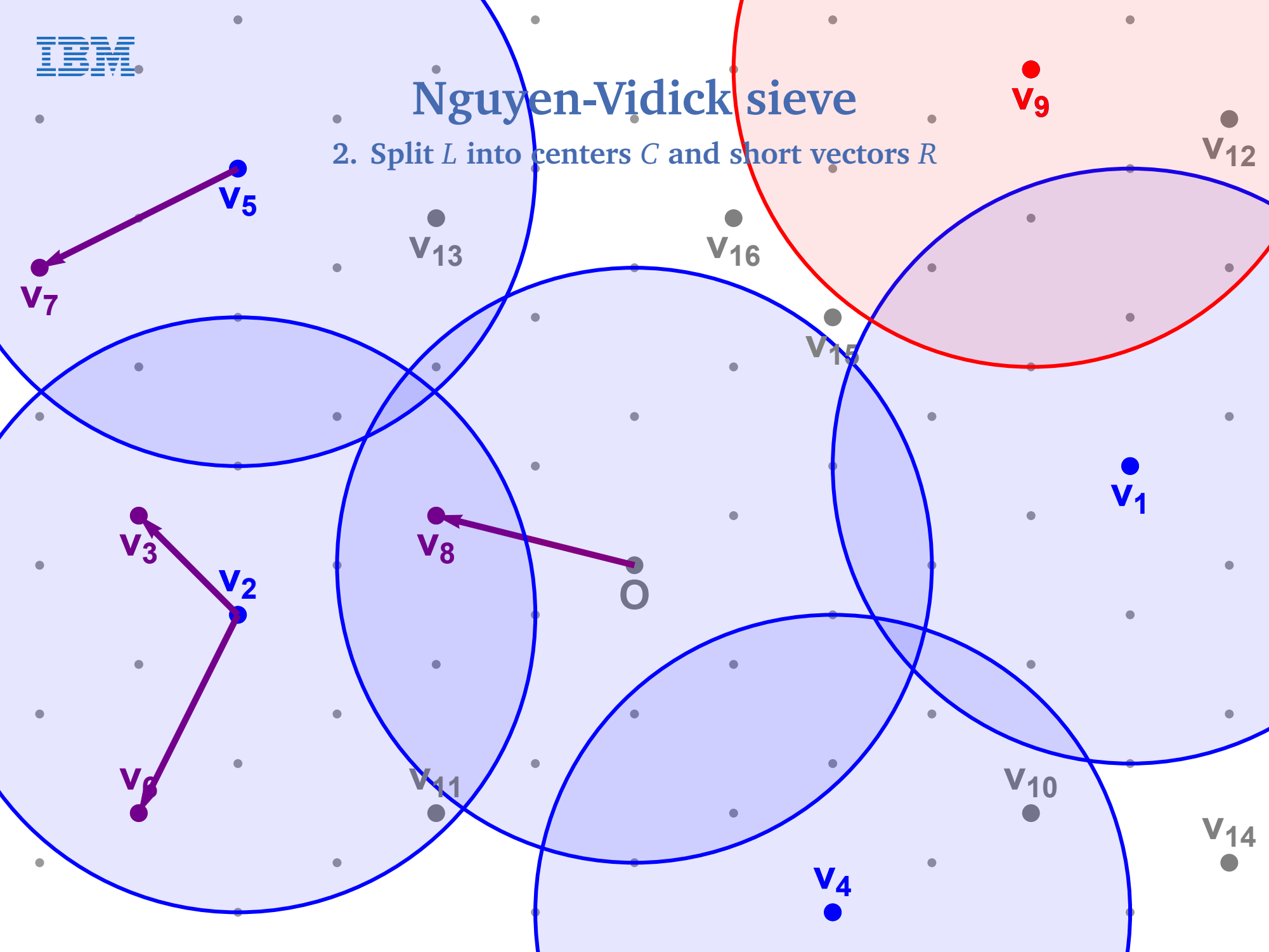

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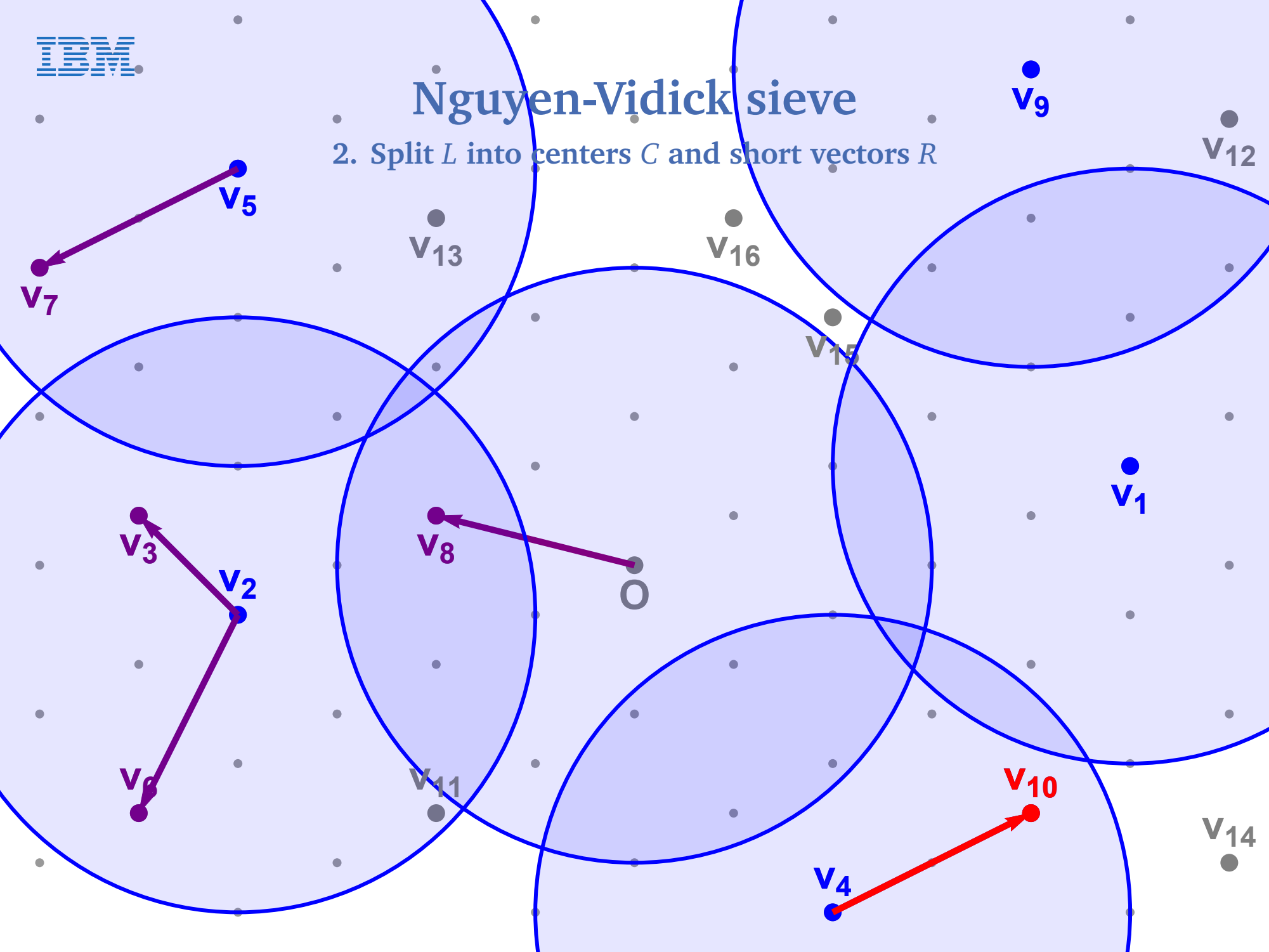

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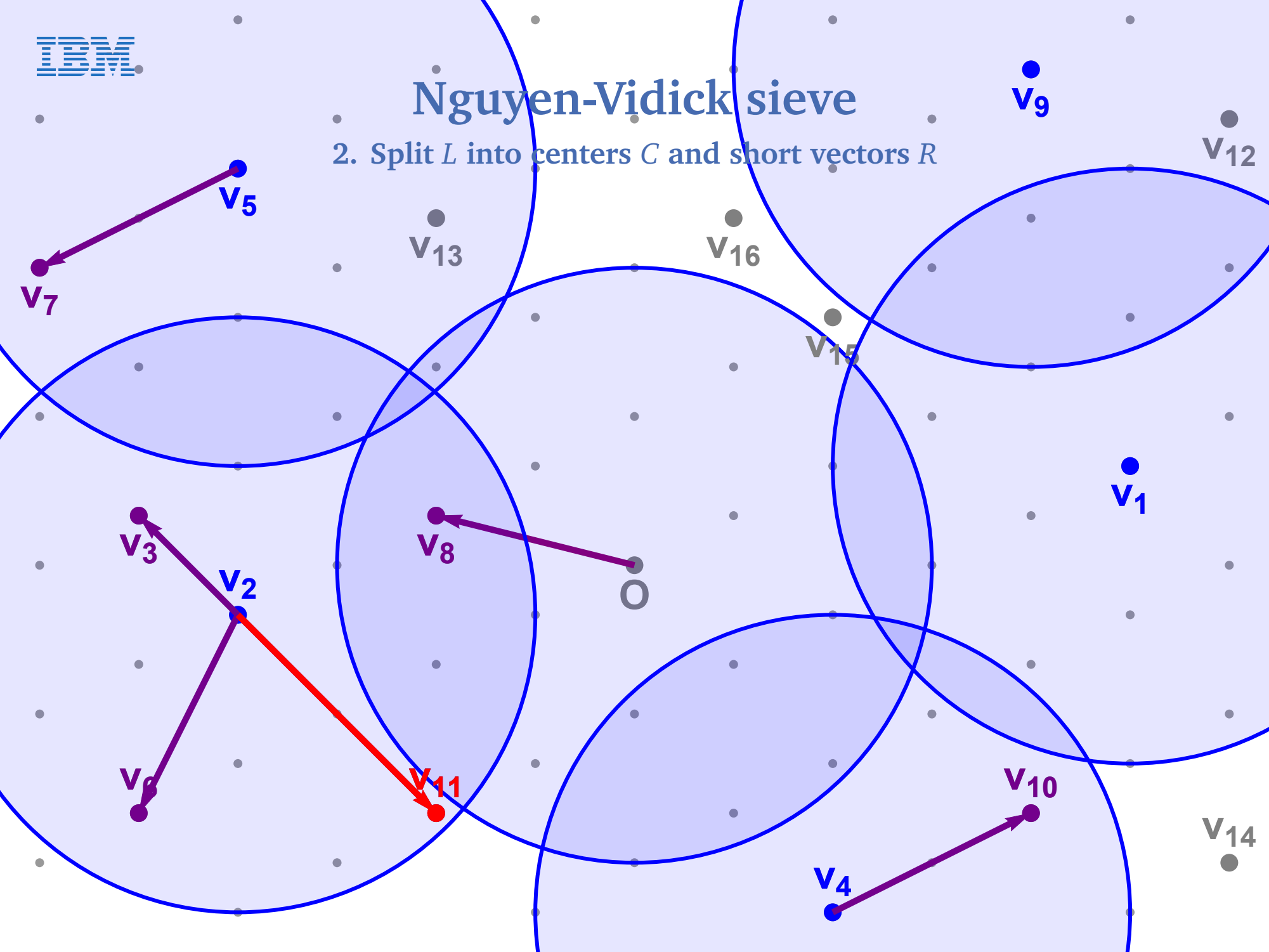

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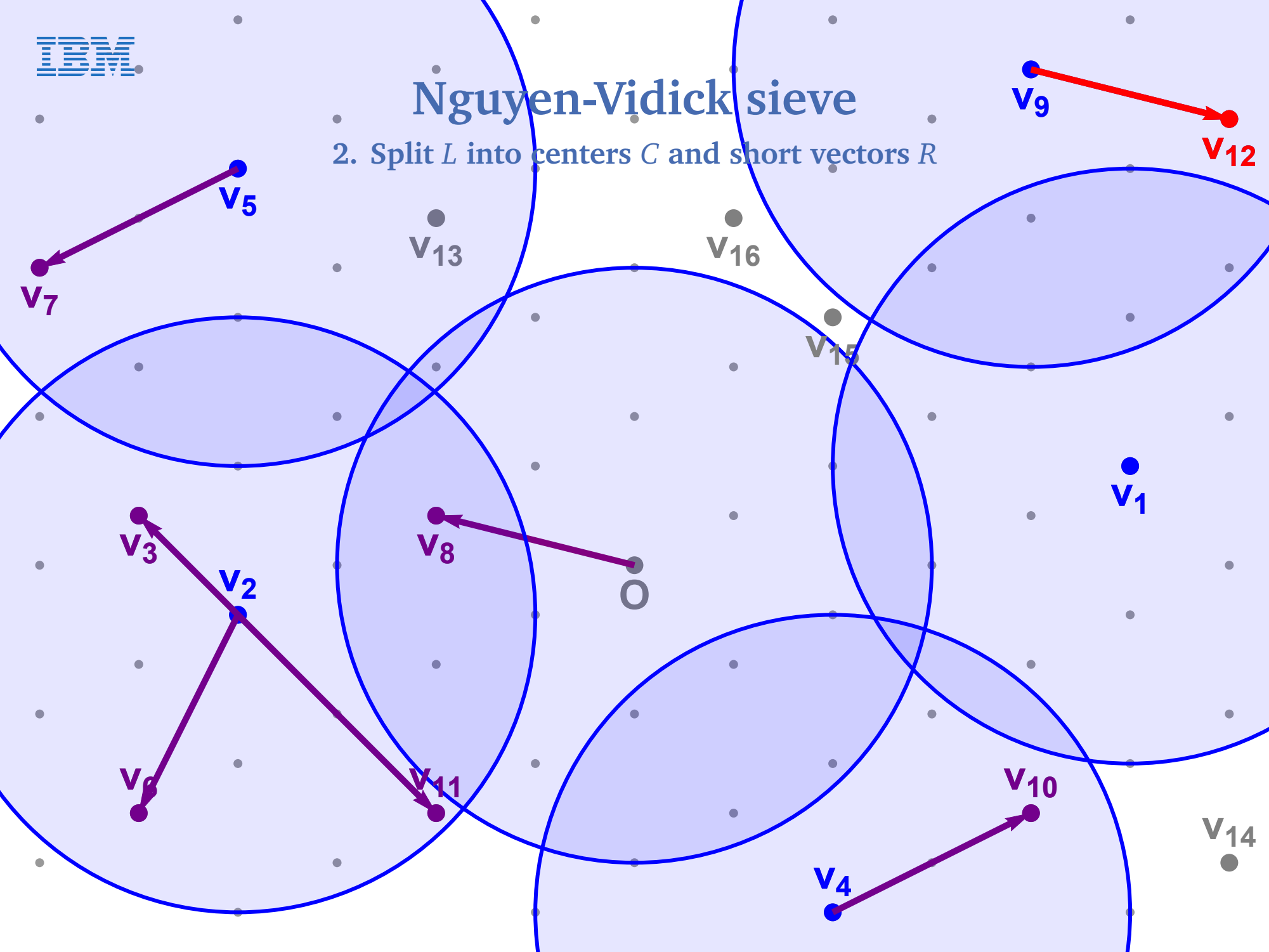

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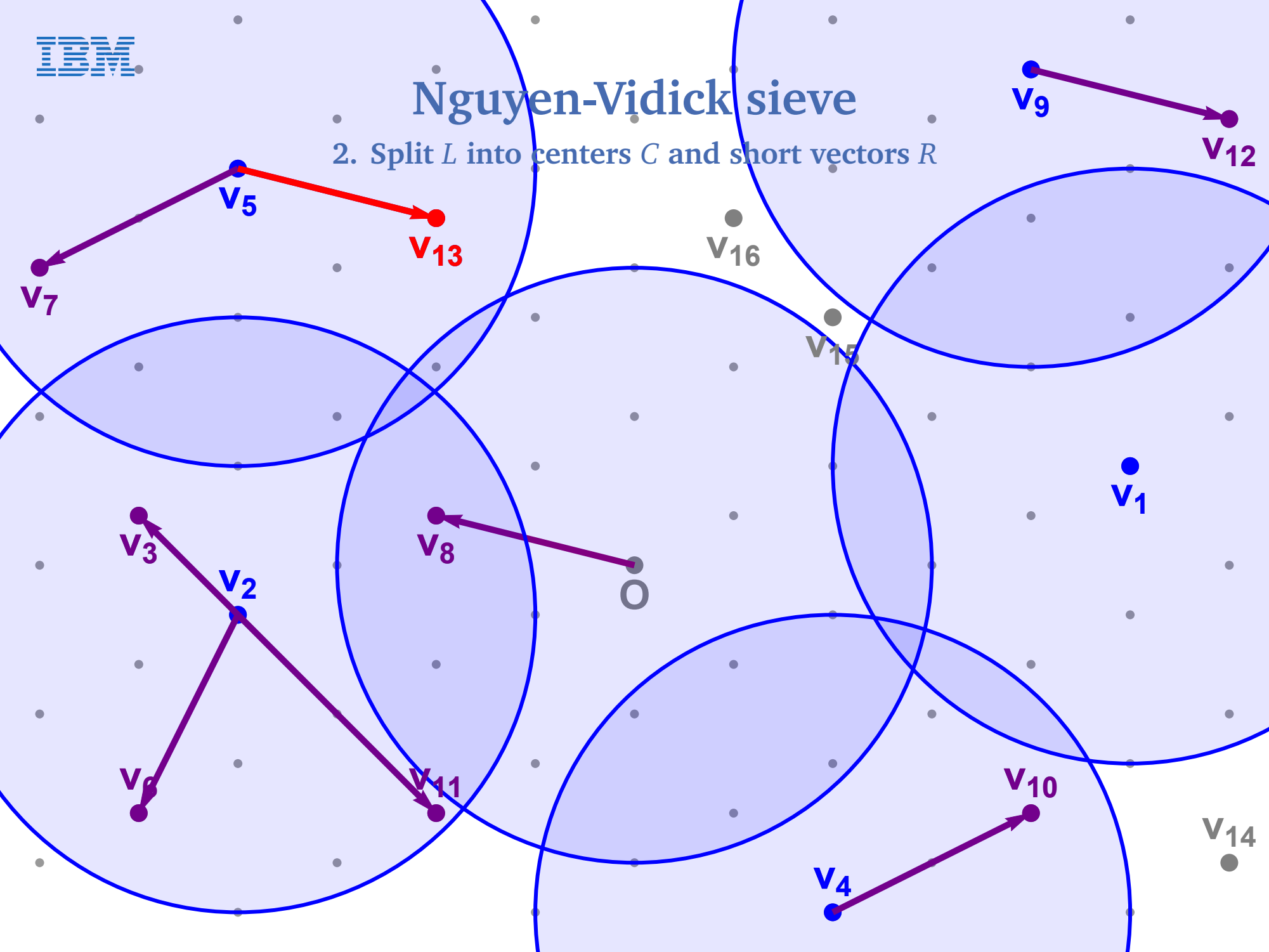

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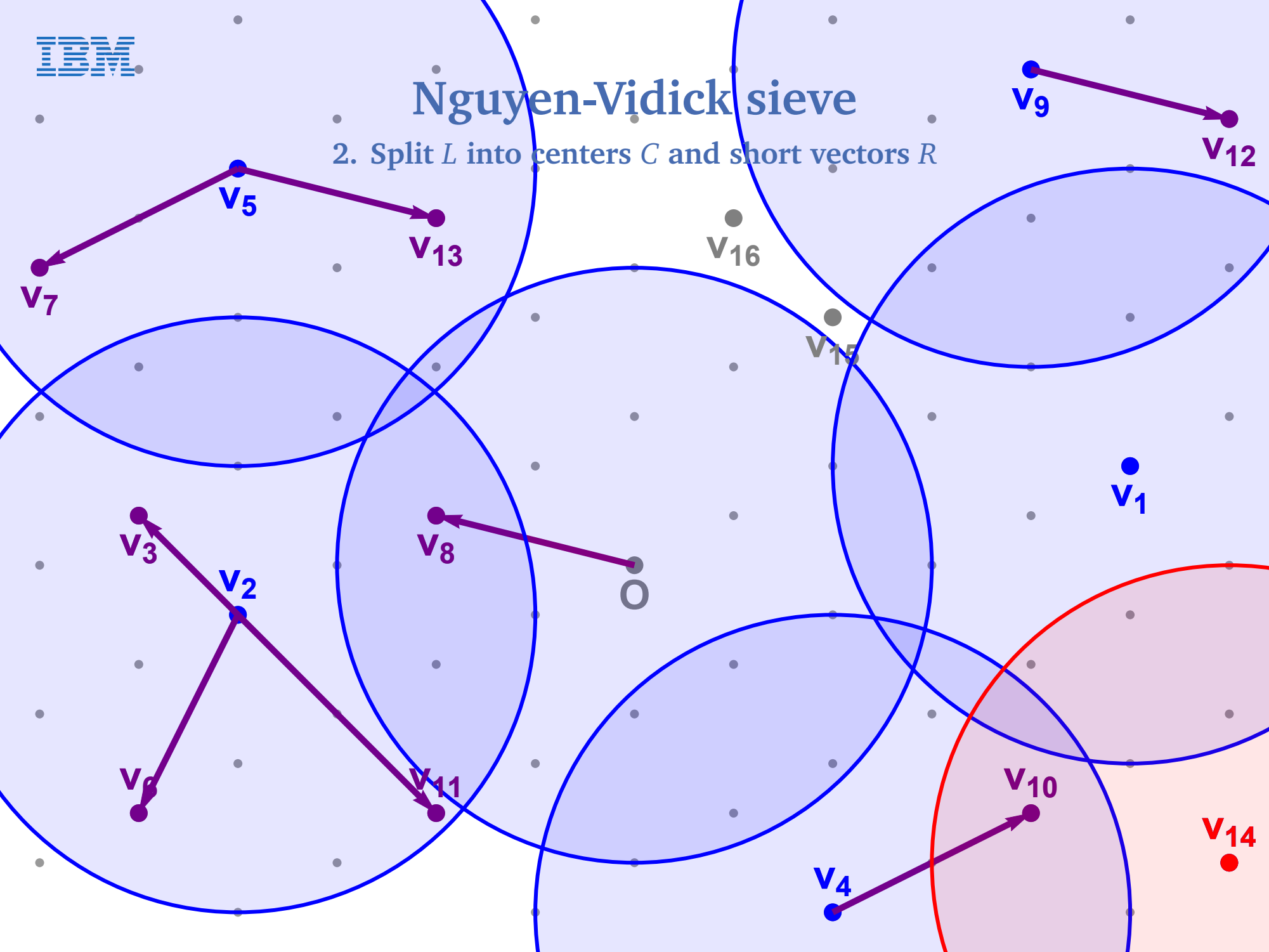

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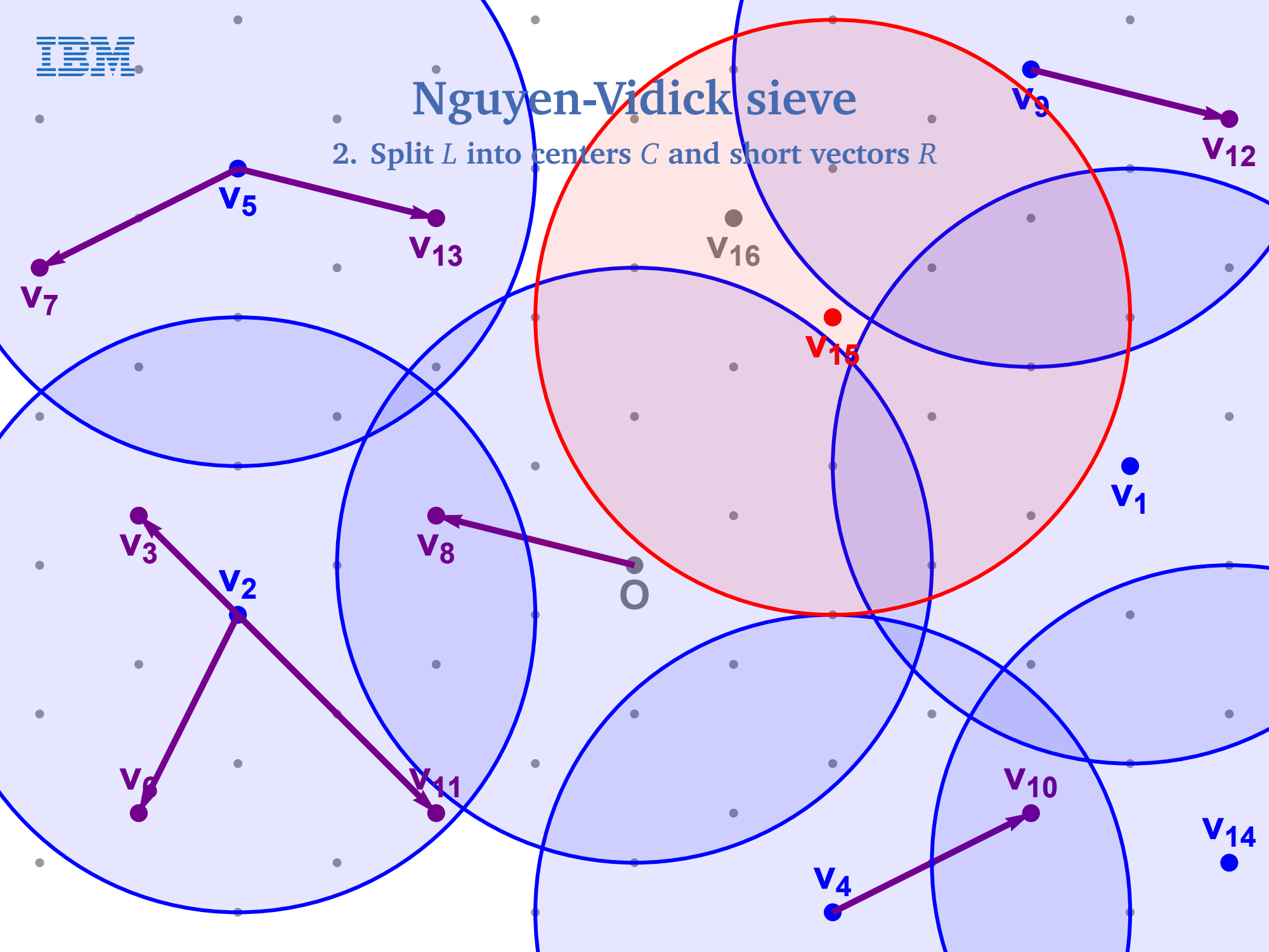

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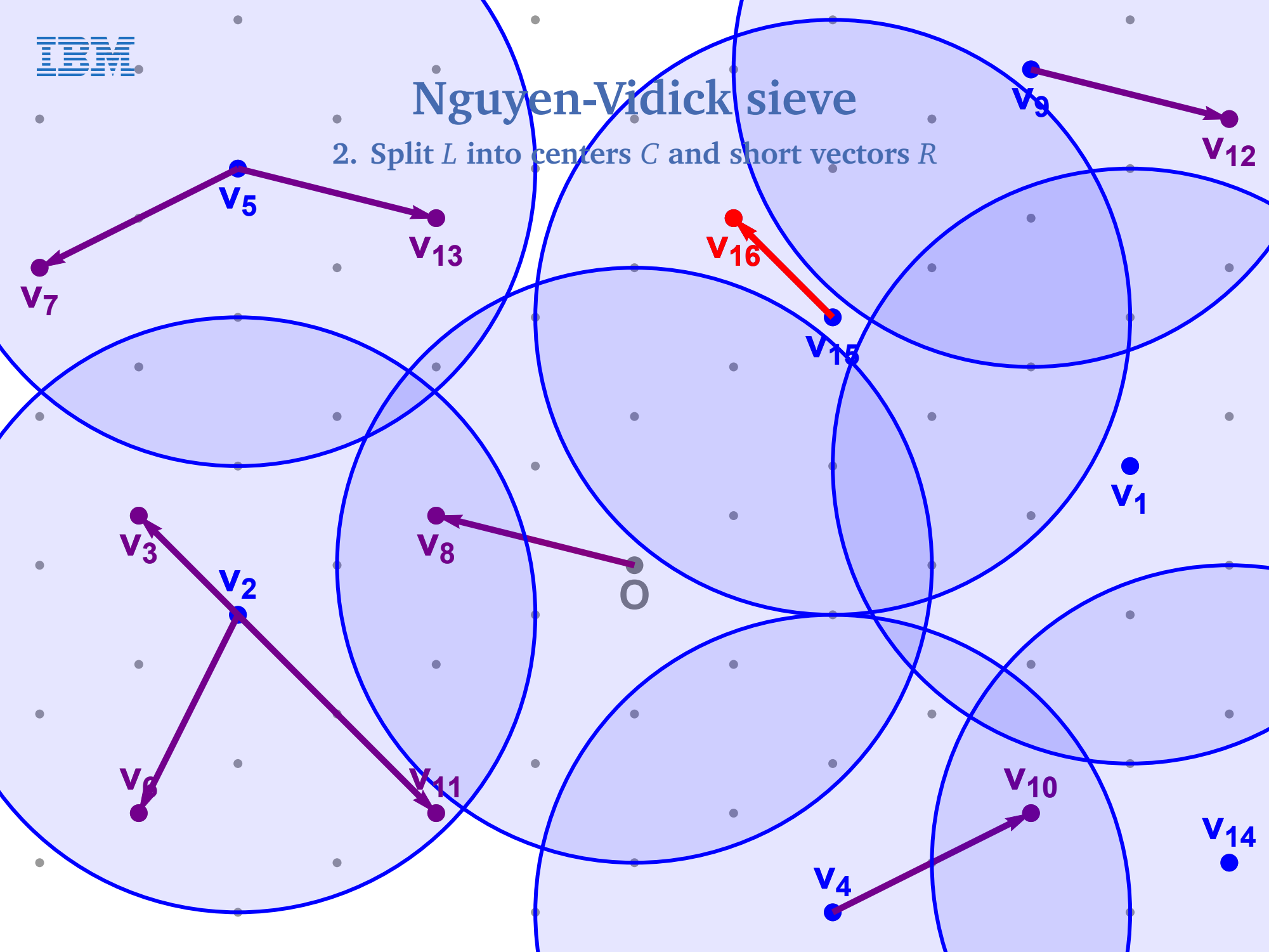

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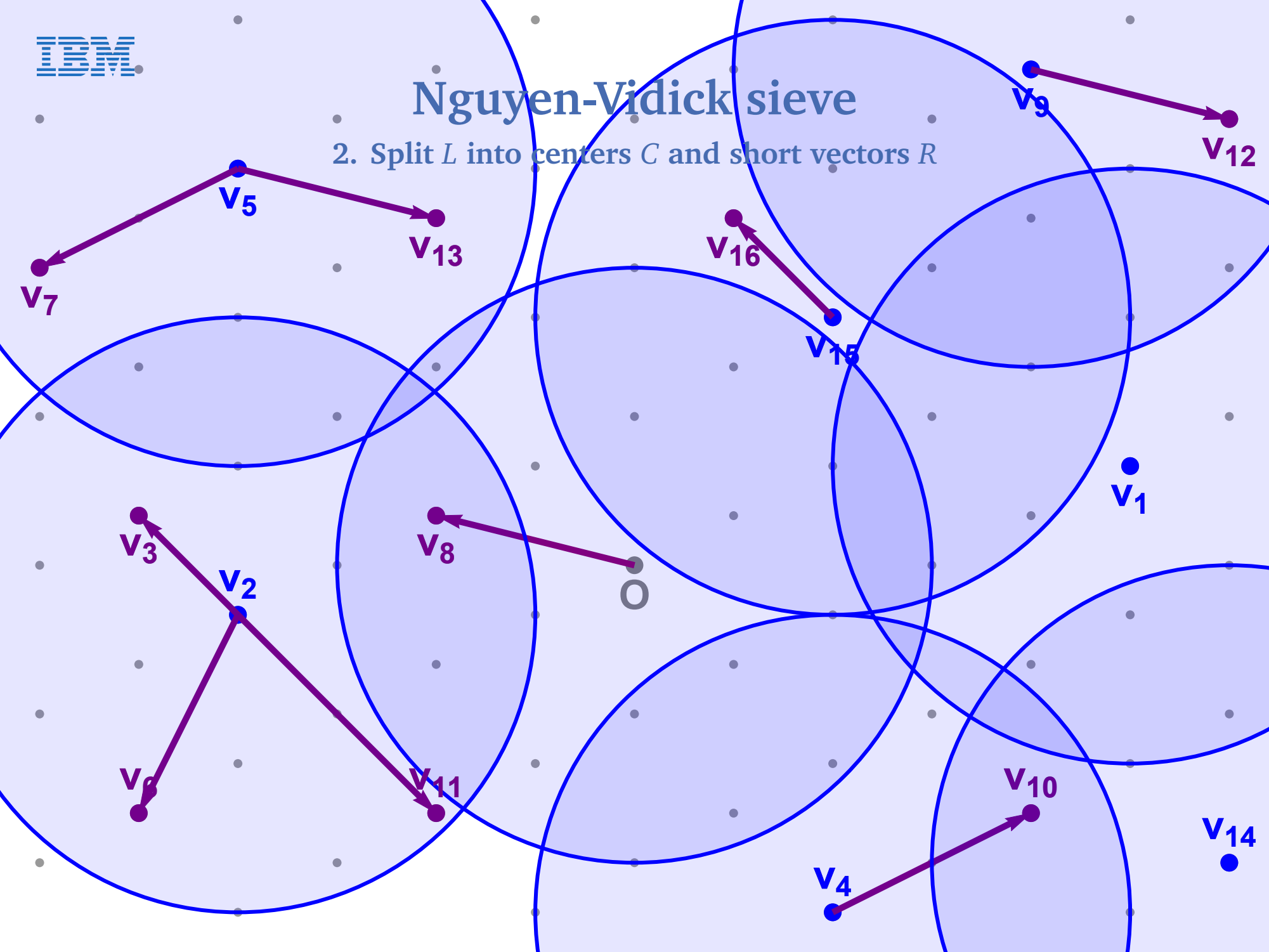
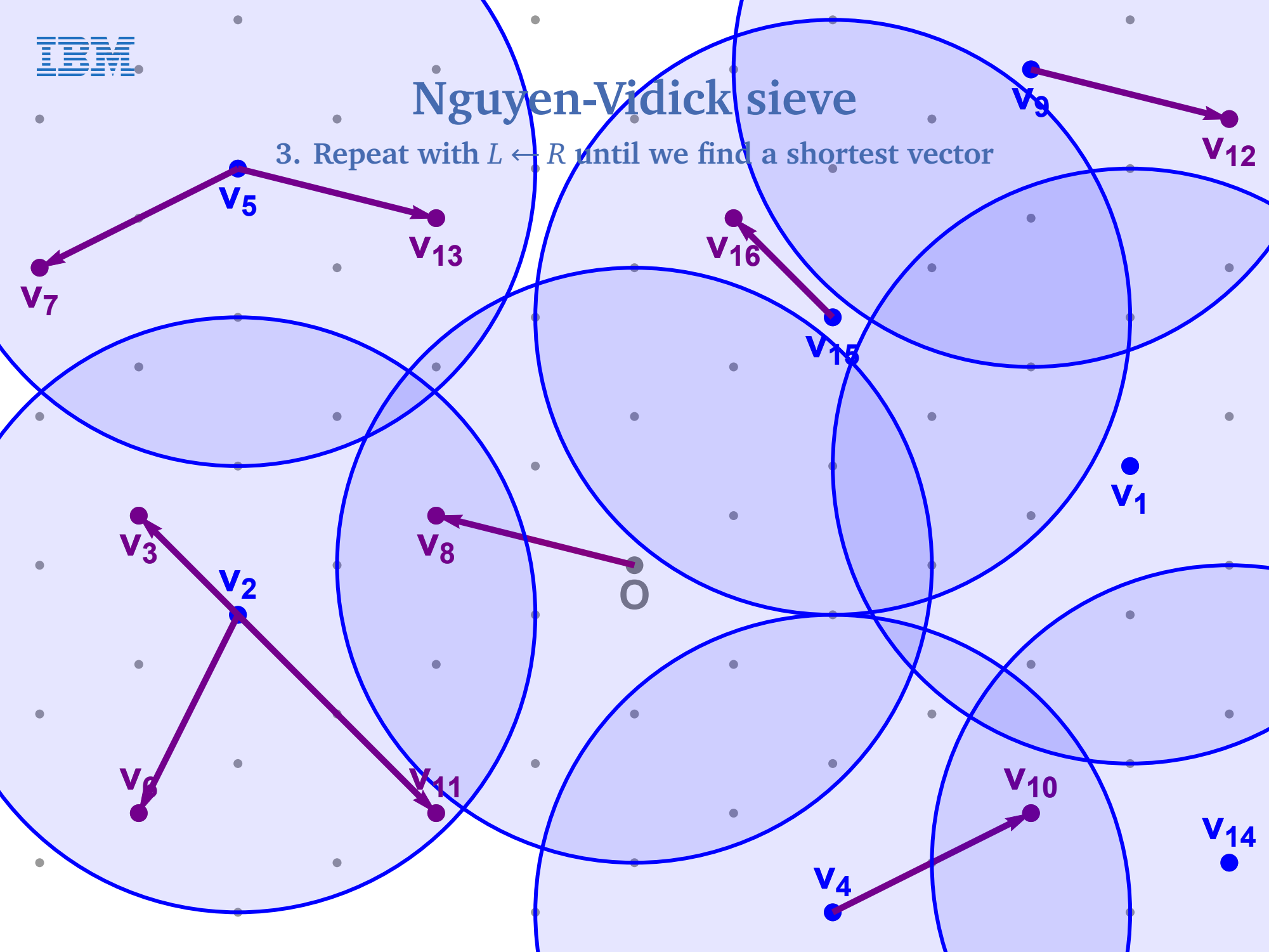


Nguyen-Vidick sieve

3. Repeat with $L \hookleftarrow R$ until we find a shortest vector



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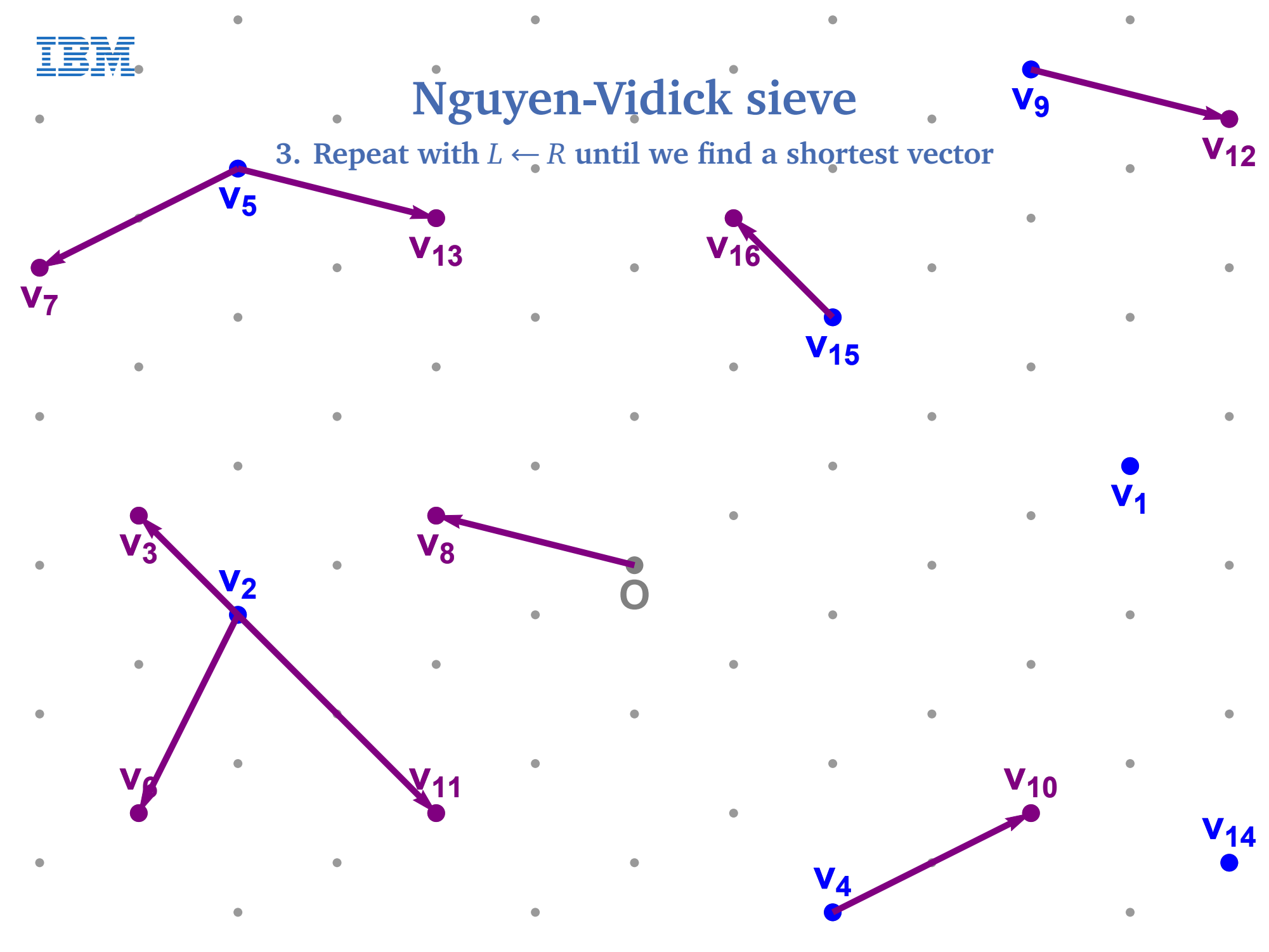

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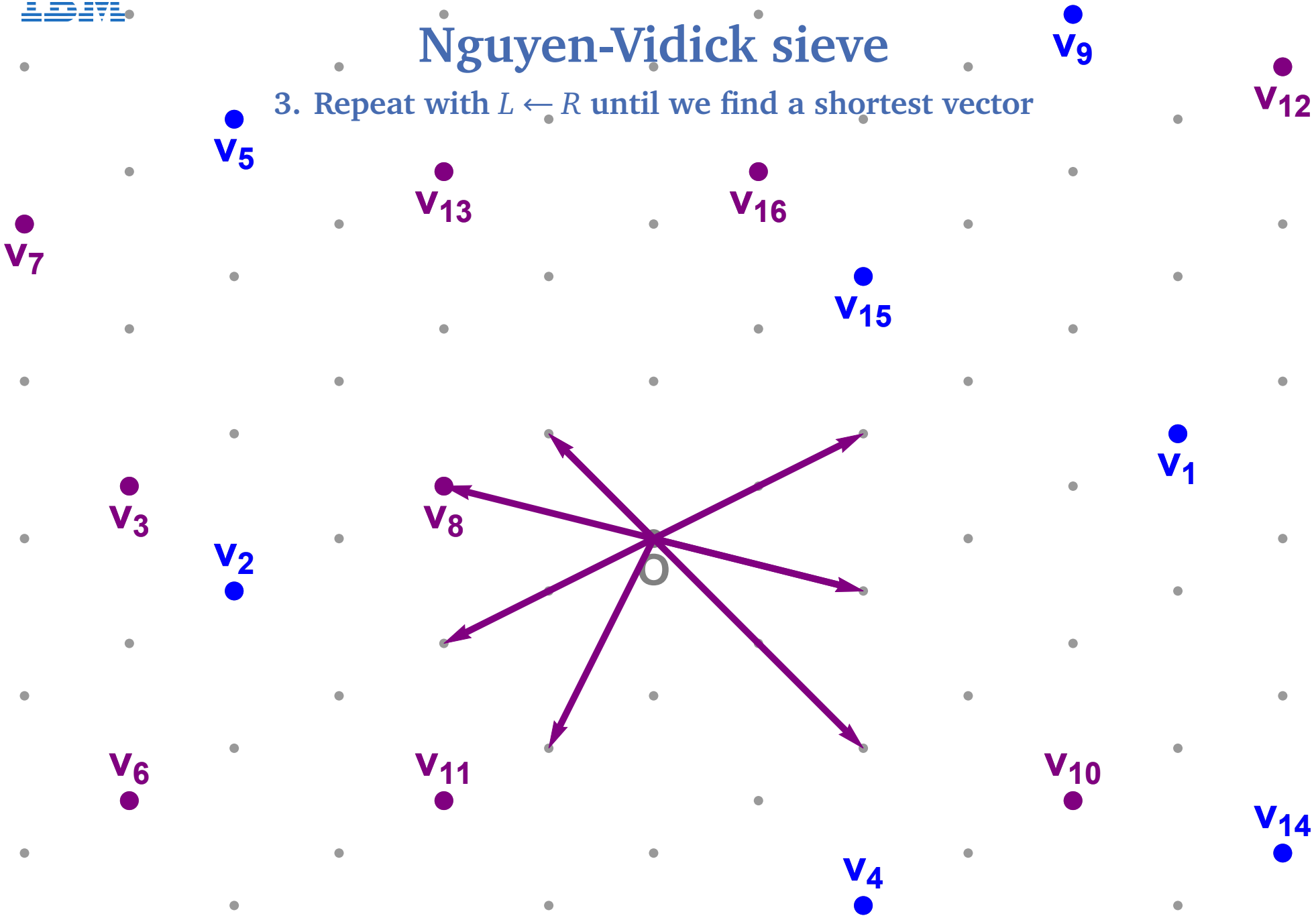

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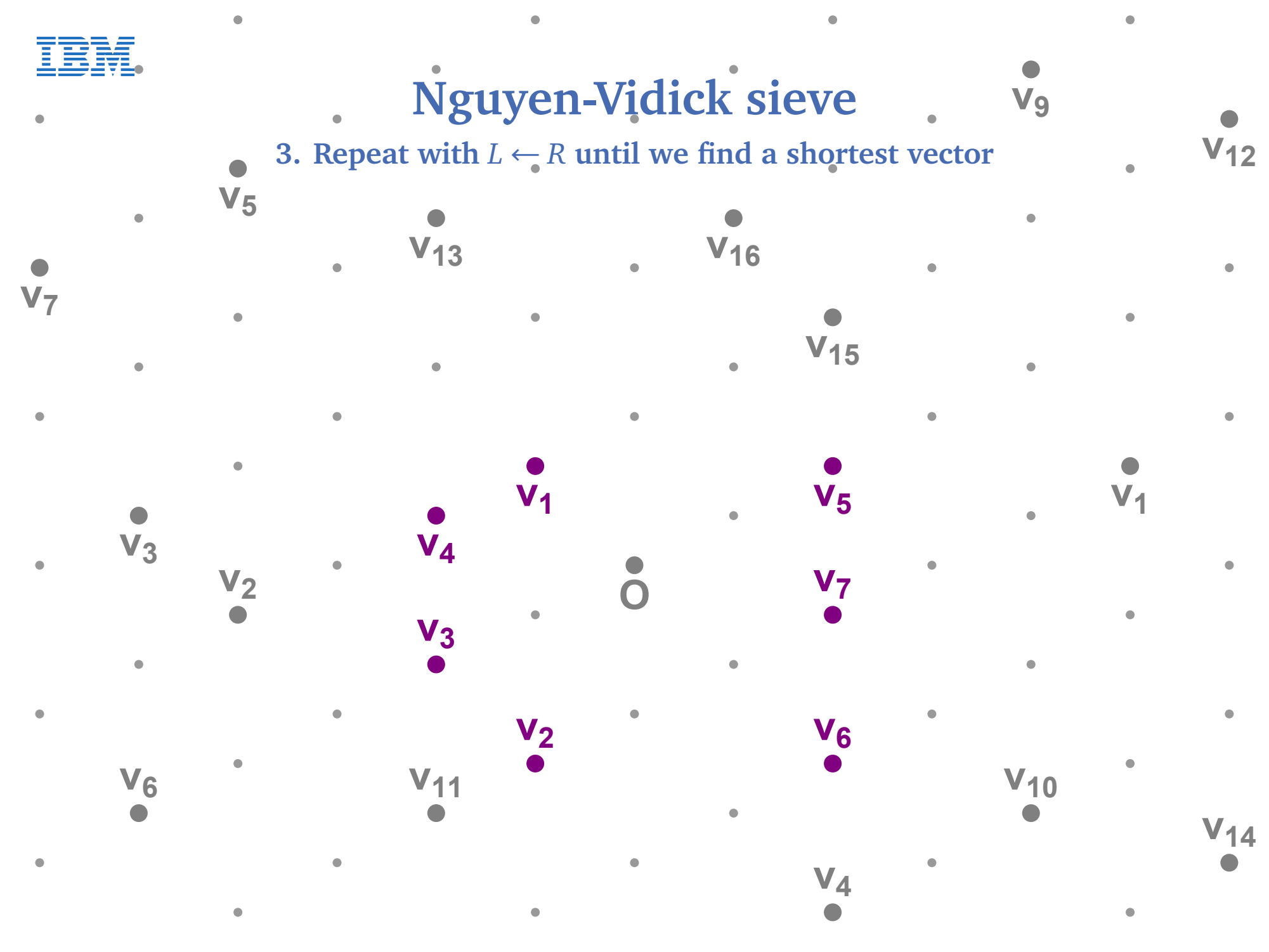

Nguyen-Vidick sieve

Overview

Nguyen-Vidick sieve

Overview

Heuristic (Nguyen and Vidick, J. Math. Crypt. '08)

The Nguyen-Vidick sieve runs in time $(4/3)^n$ *and space* $\sqrt{4/3}^n$.

Two-level sieve

1. Sample a list L of random lattice vectors

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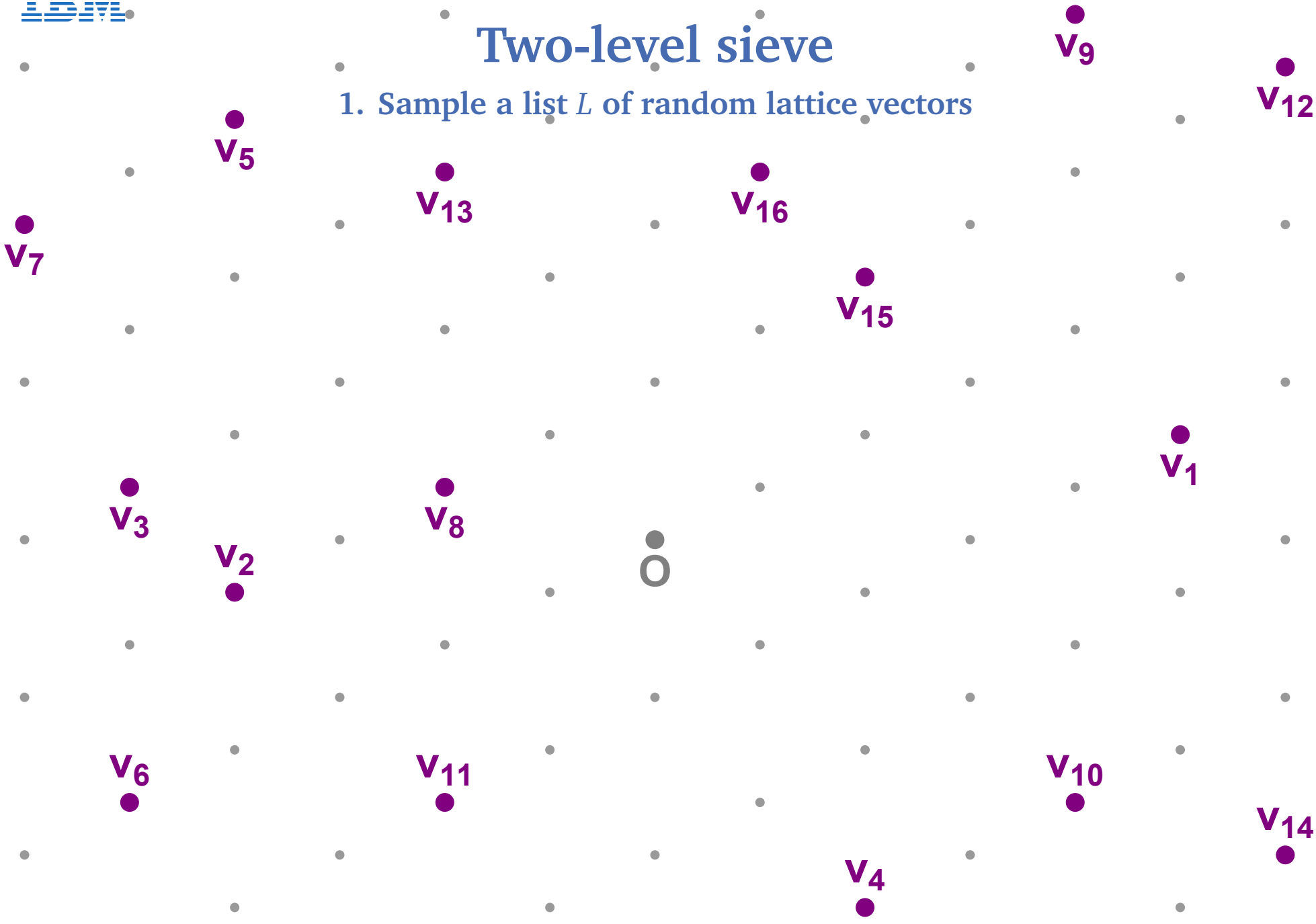
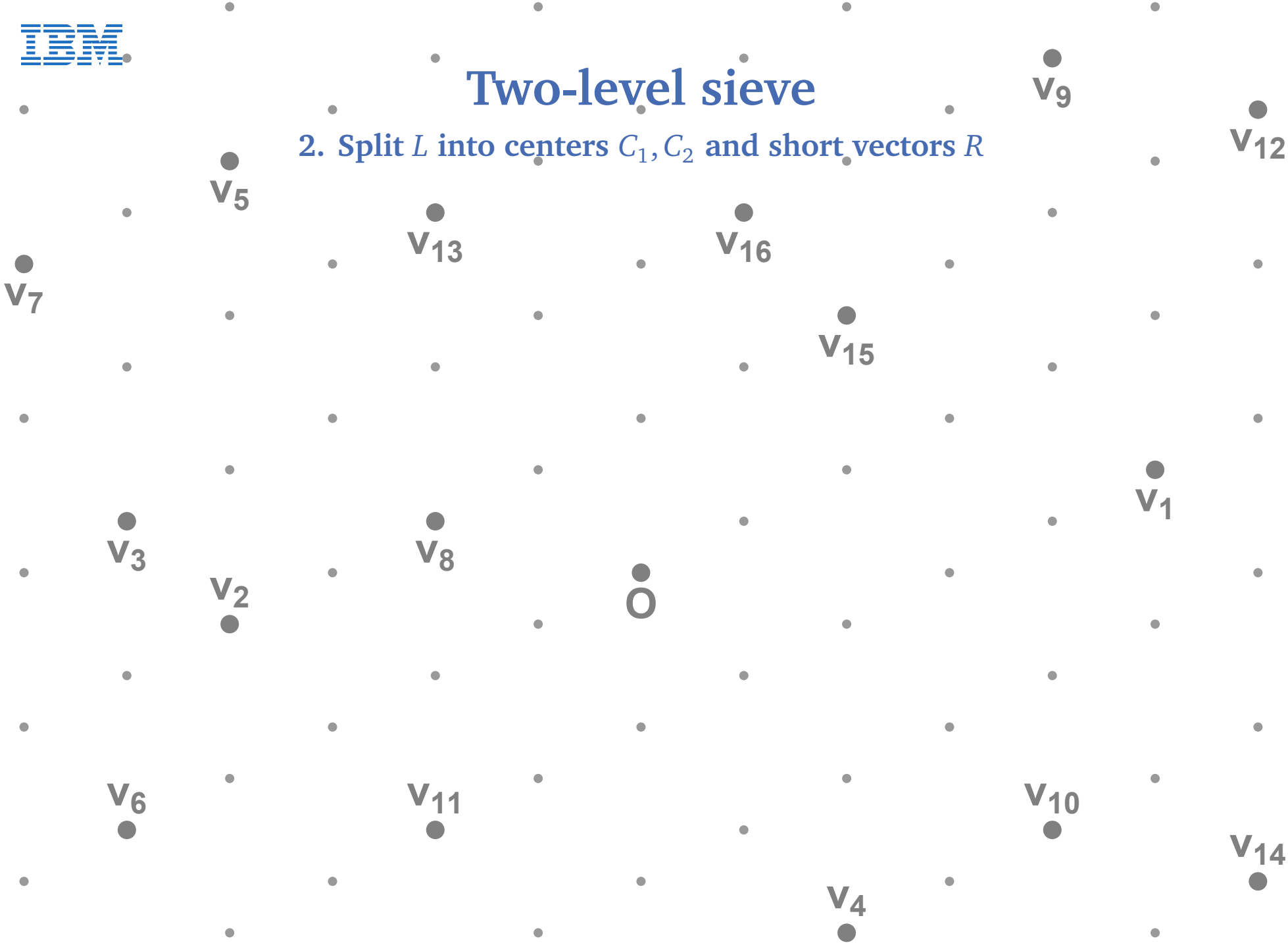


Two-level sieve

2. Split L into centers C_1, C_2 and short vectors R



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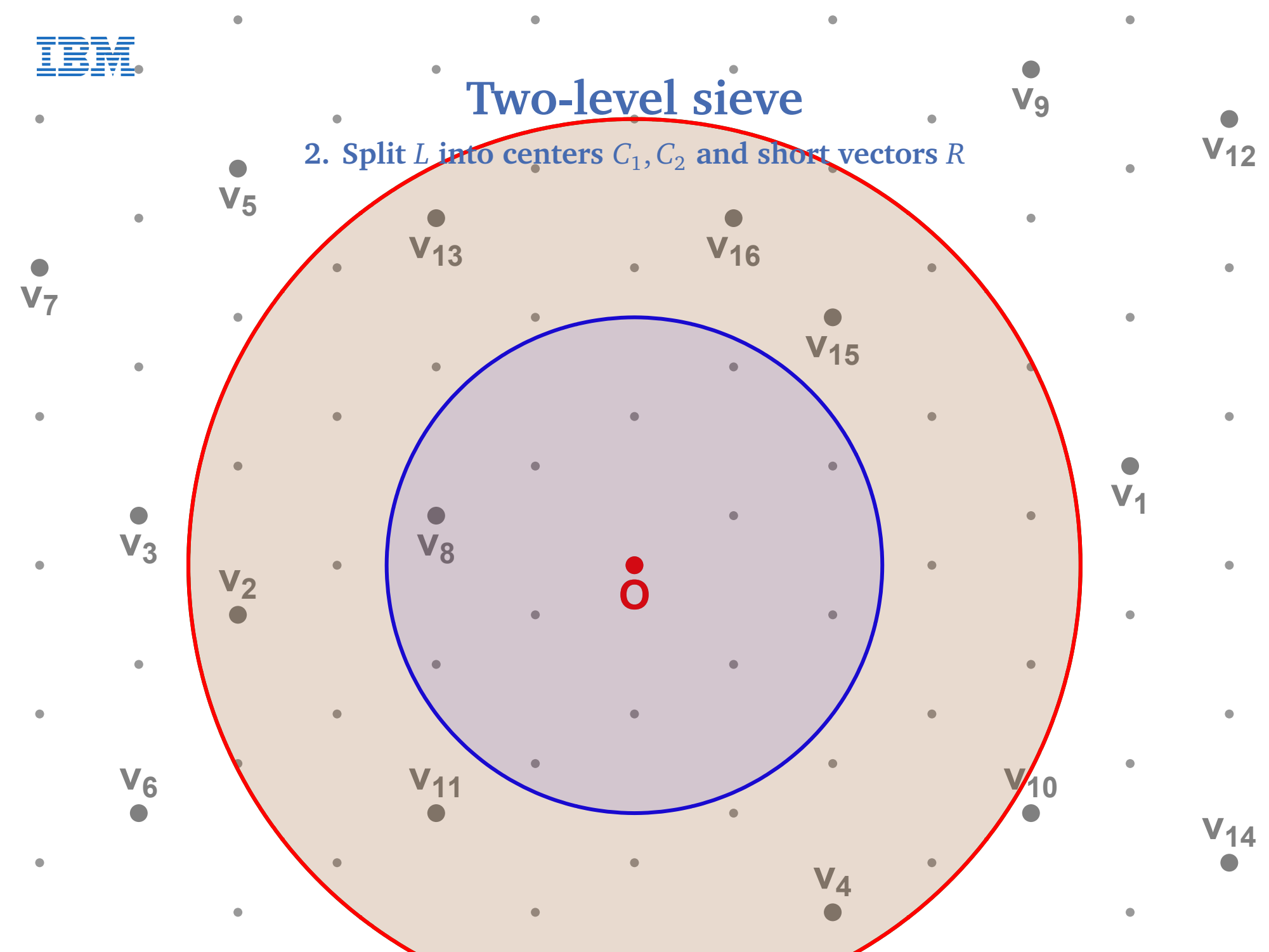

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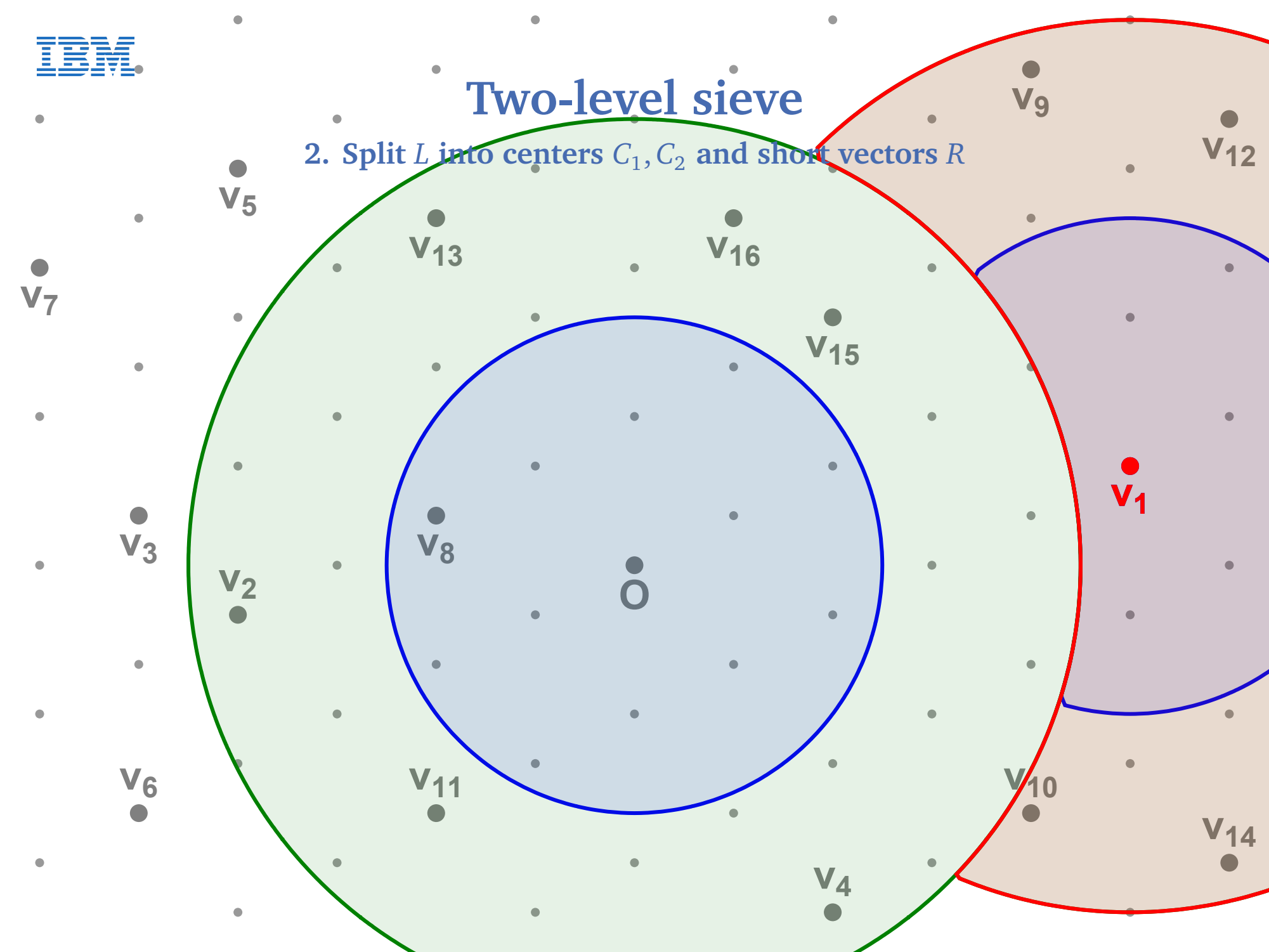

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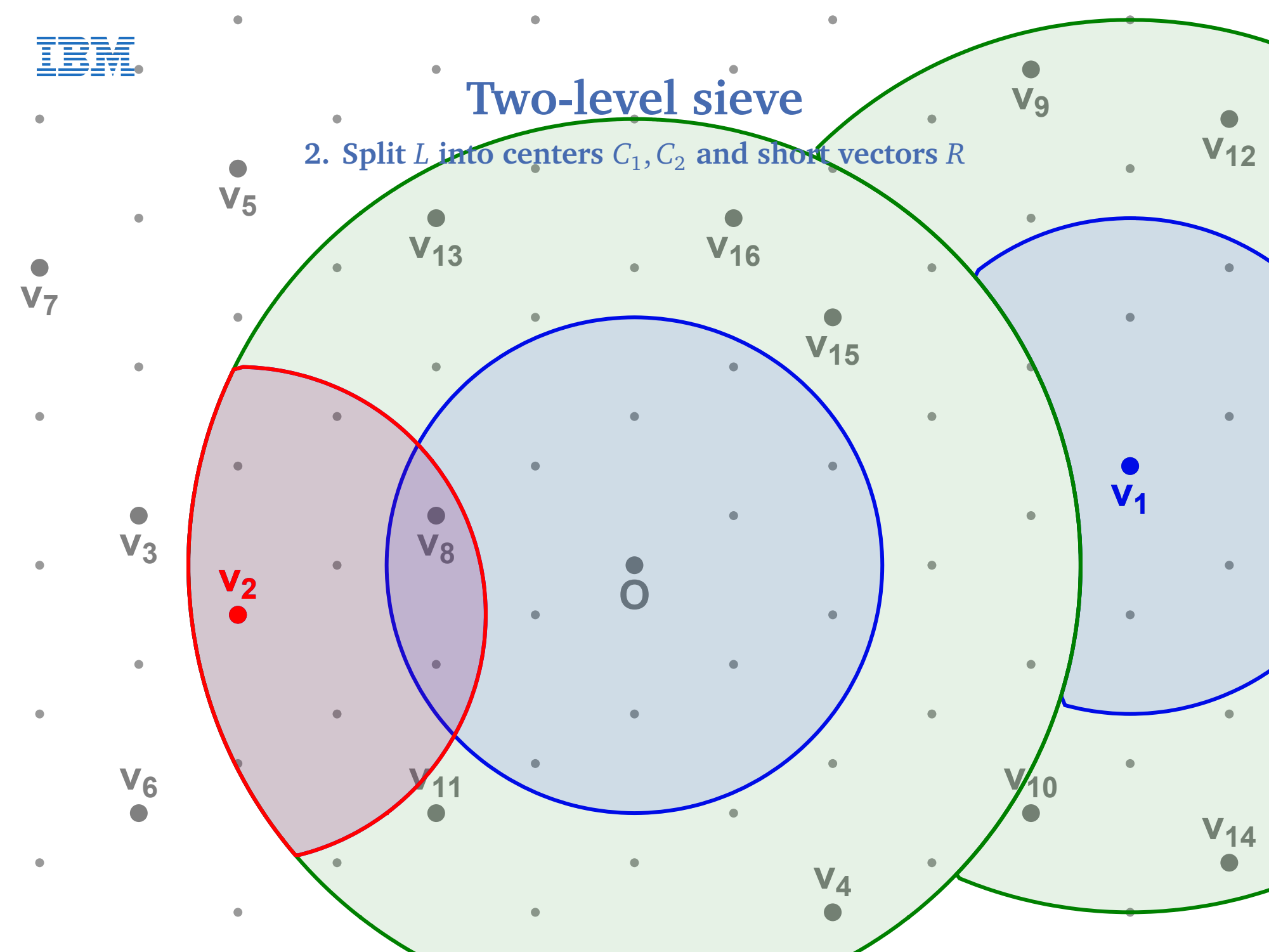

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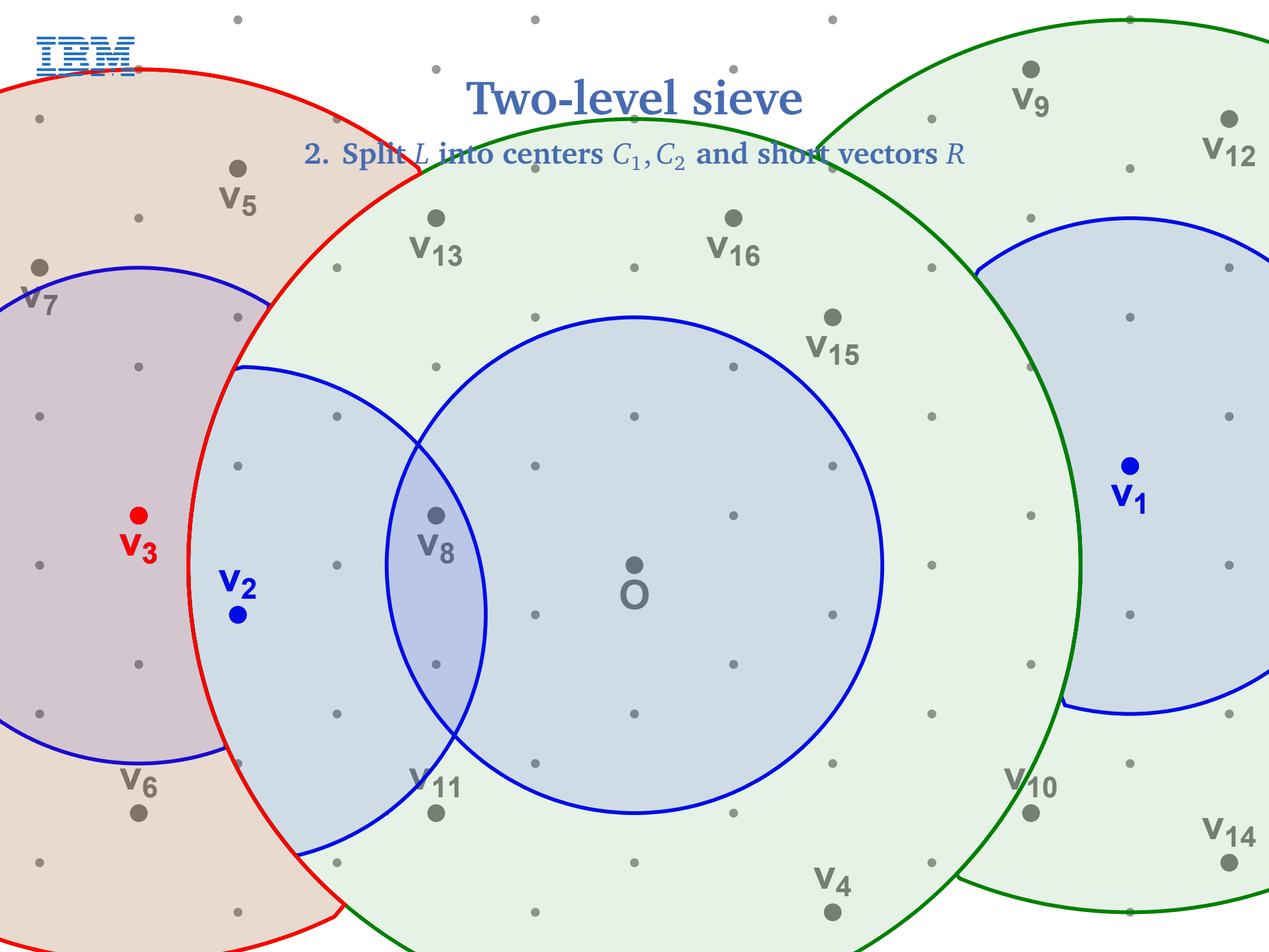

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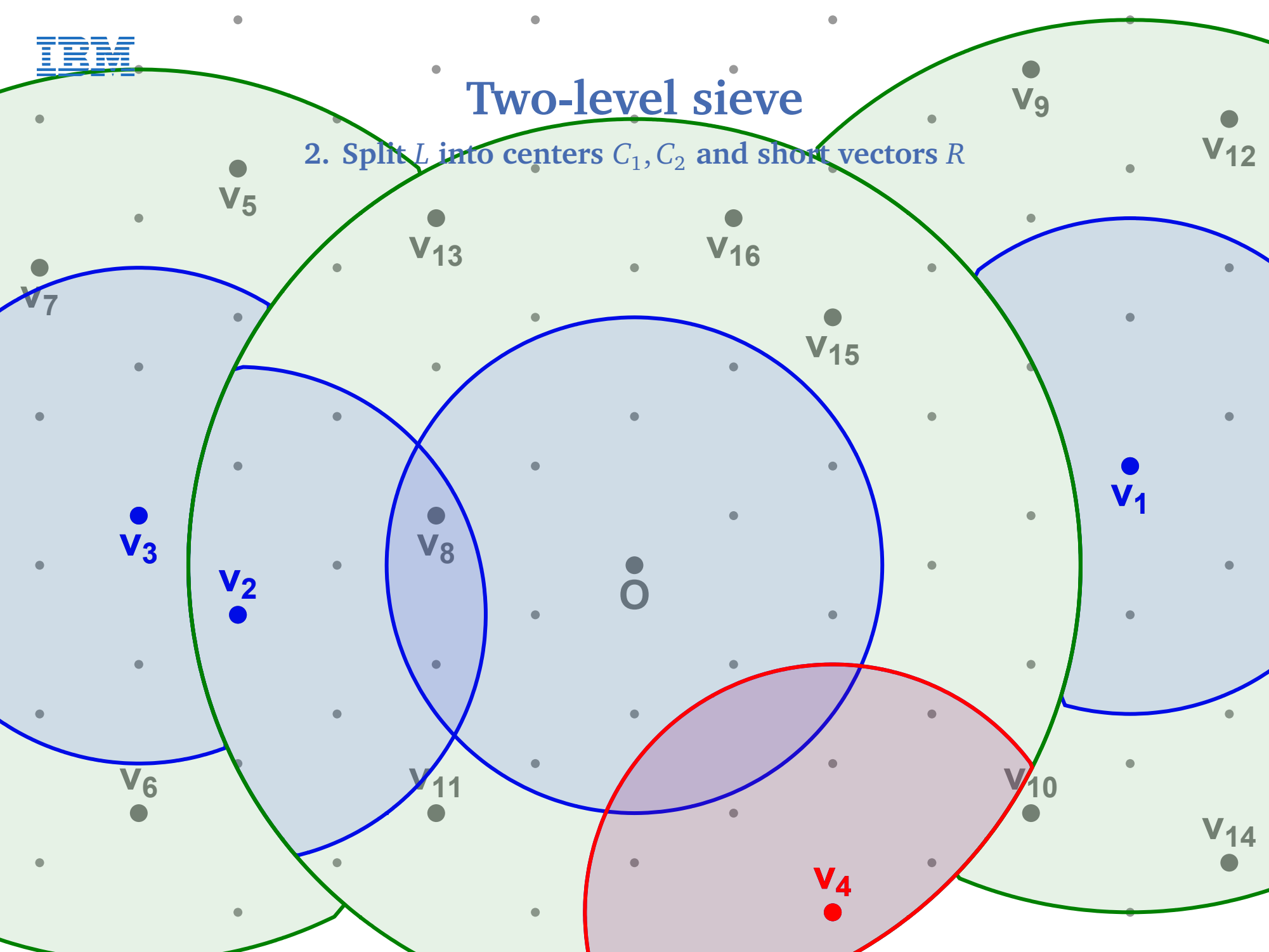

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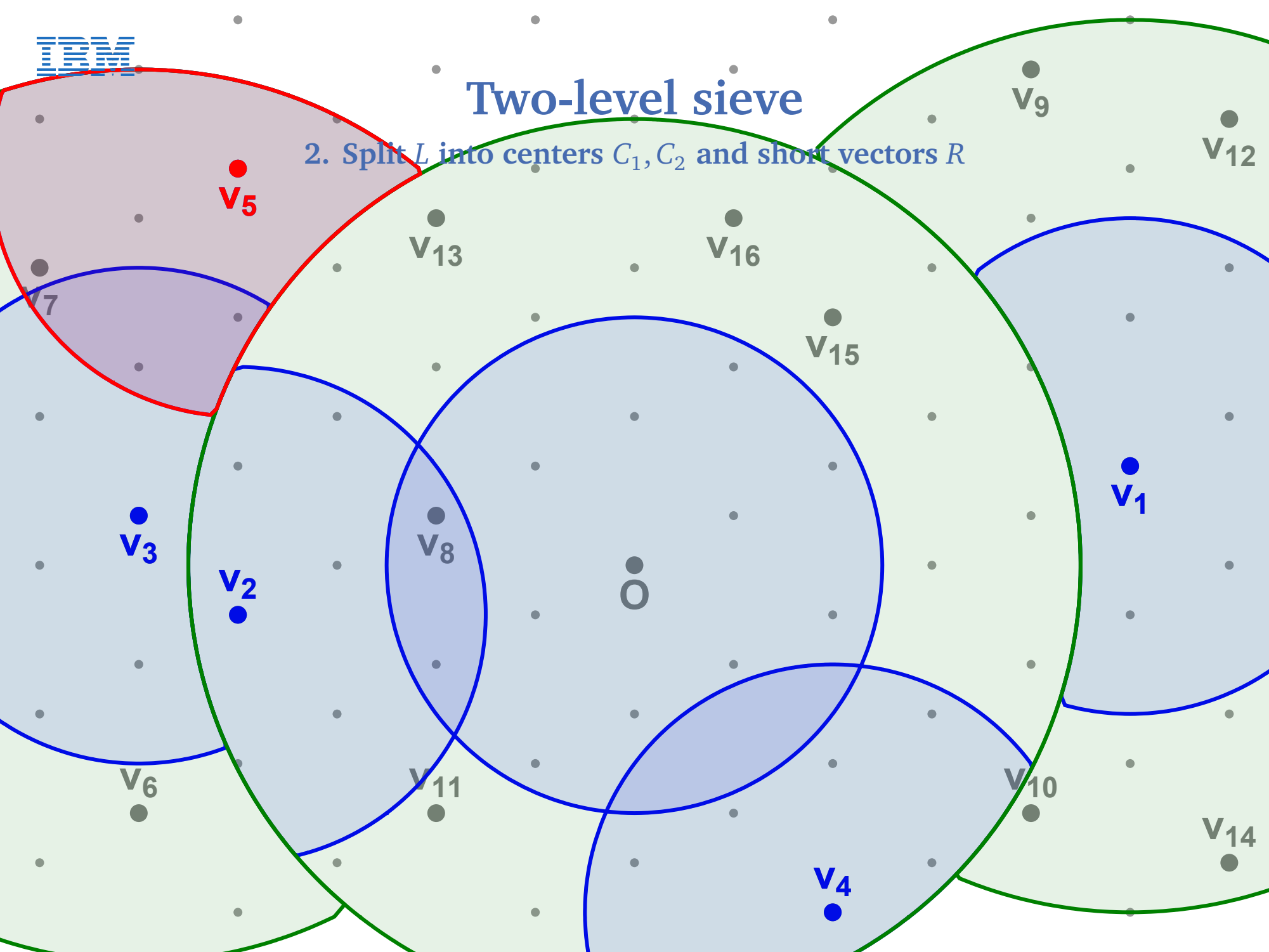

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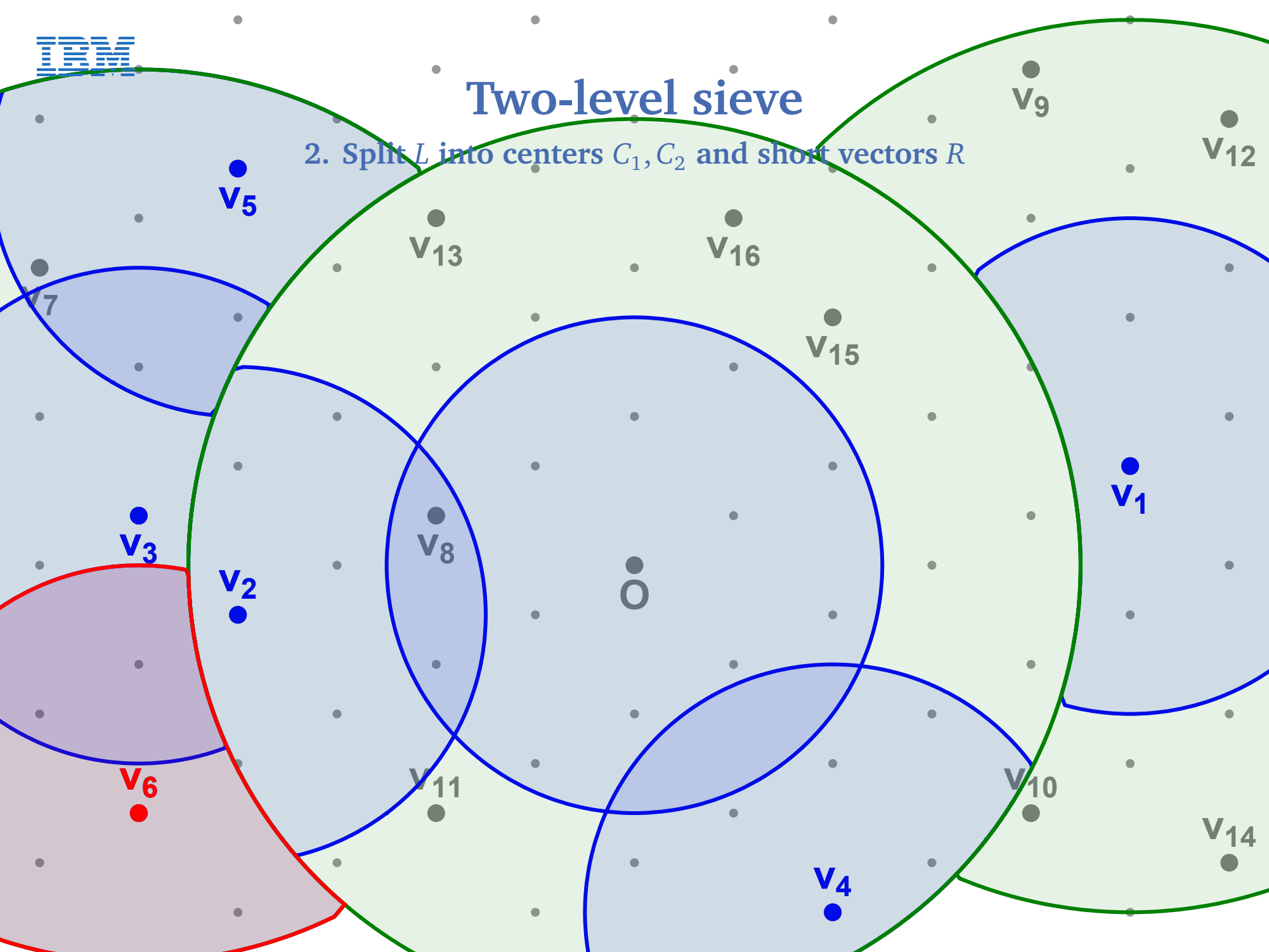

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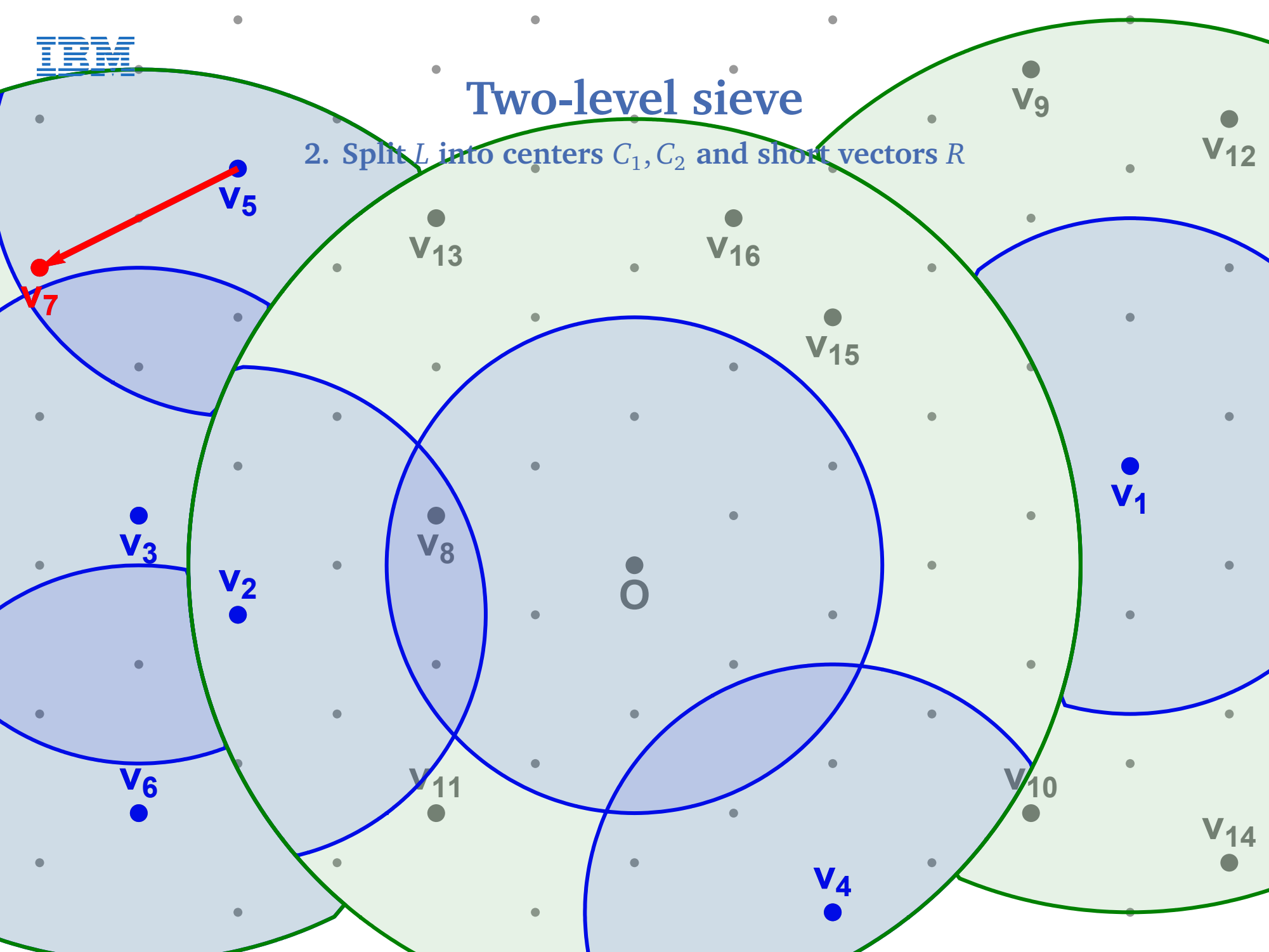

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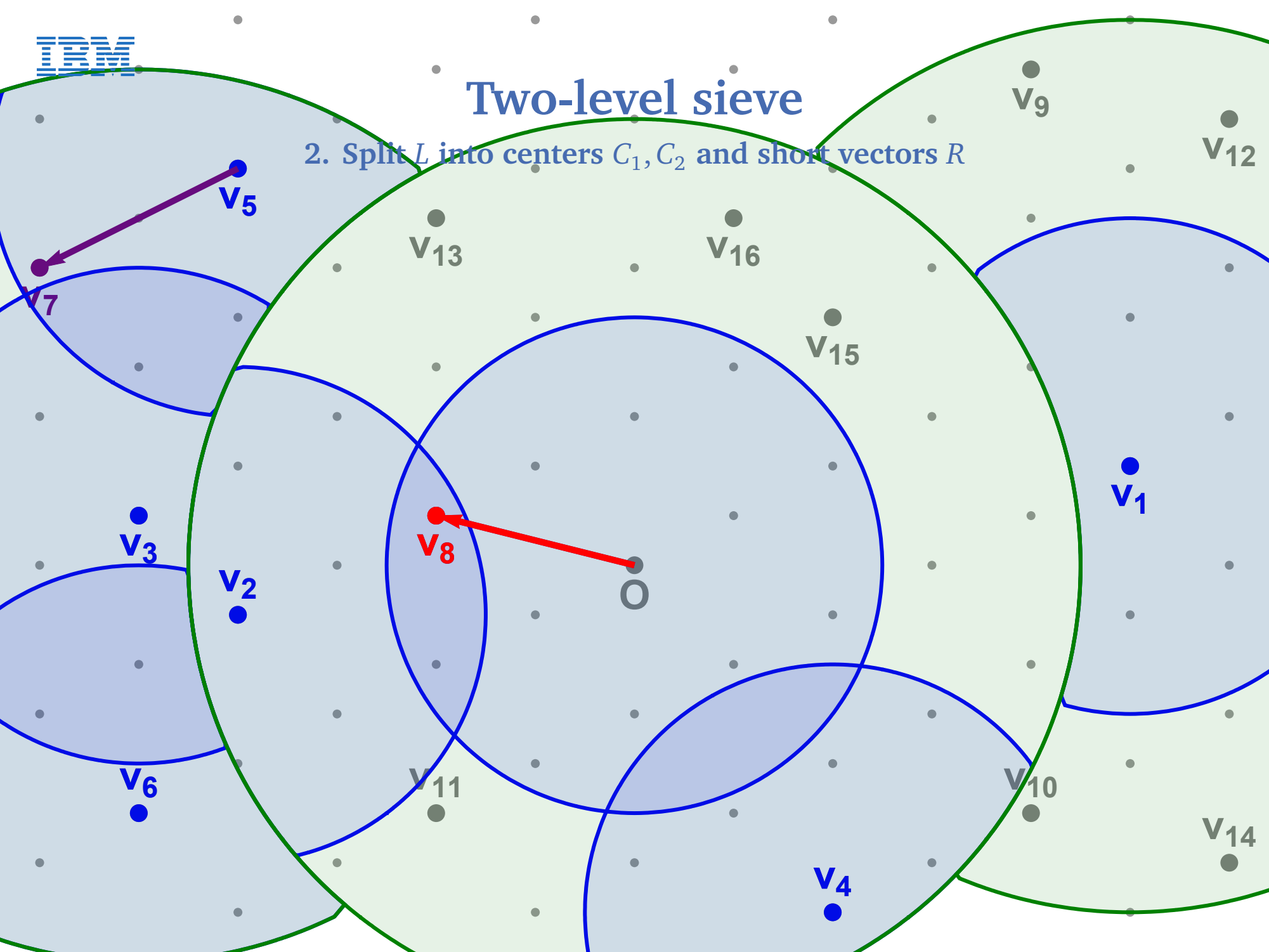

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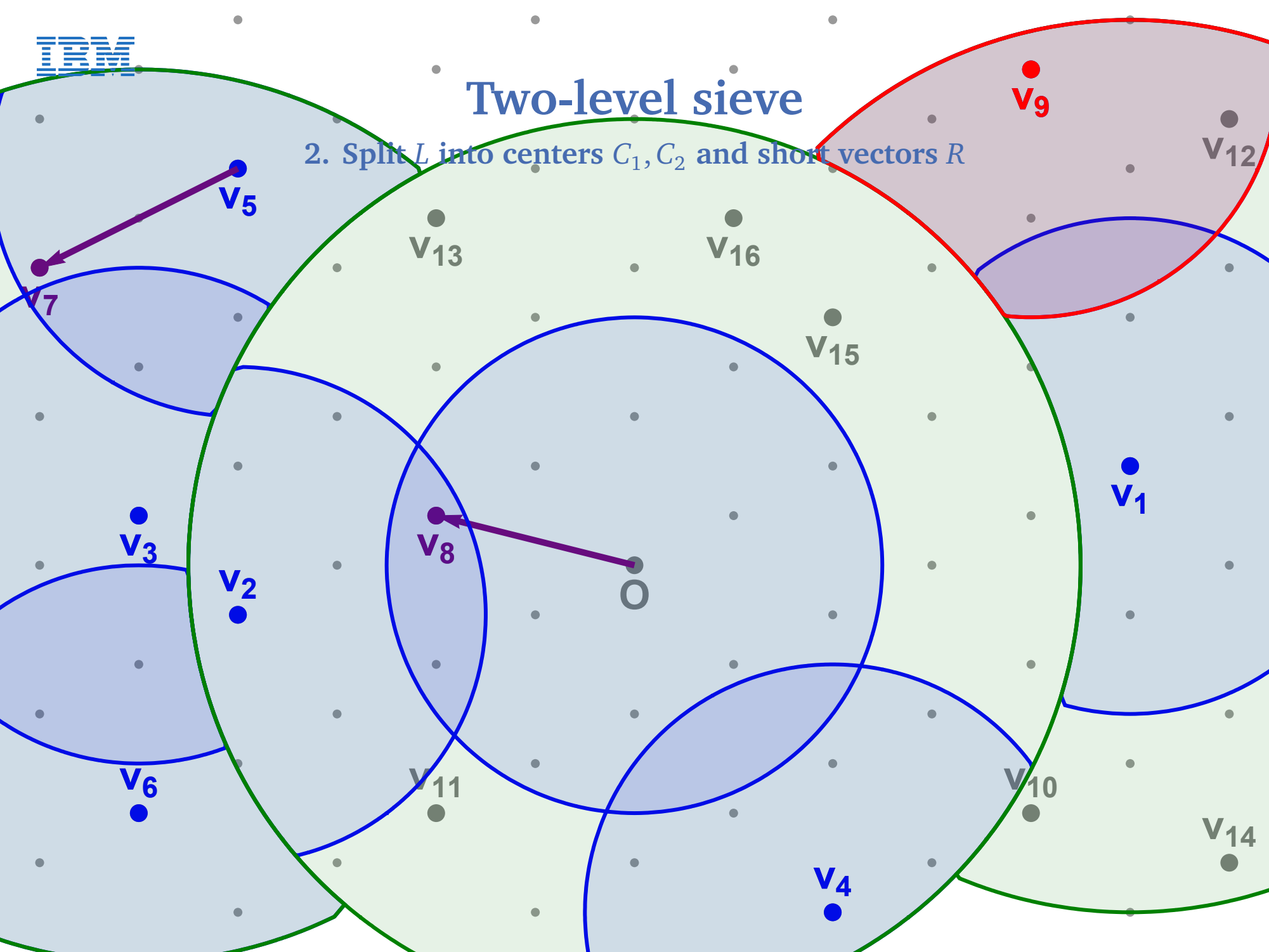

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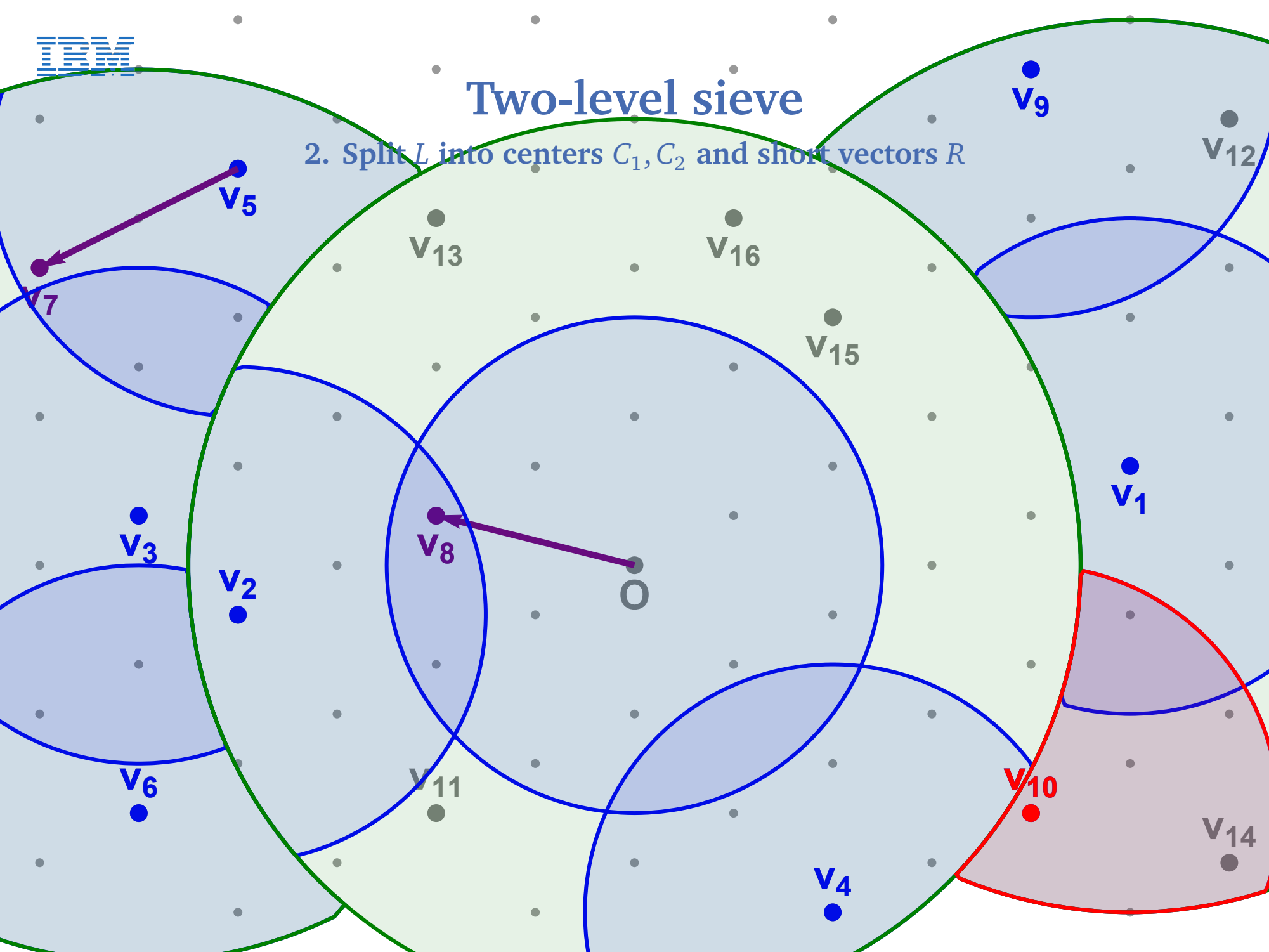

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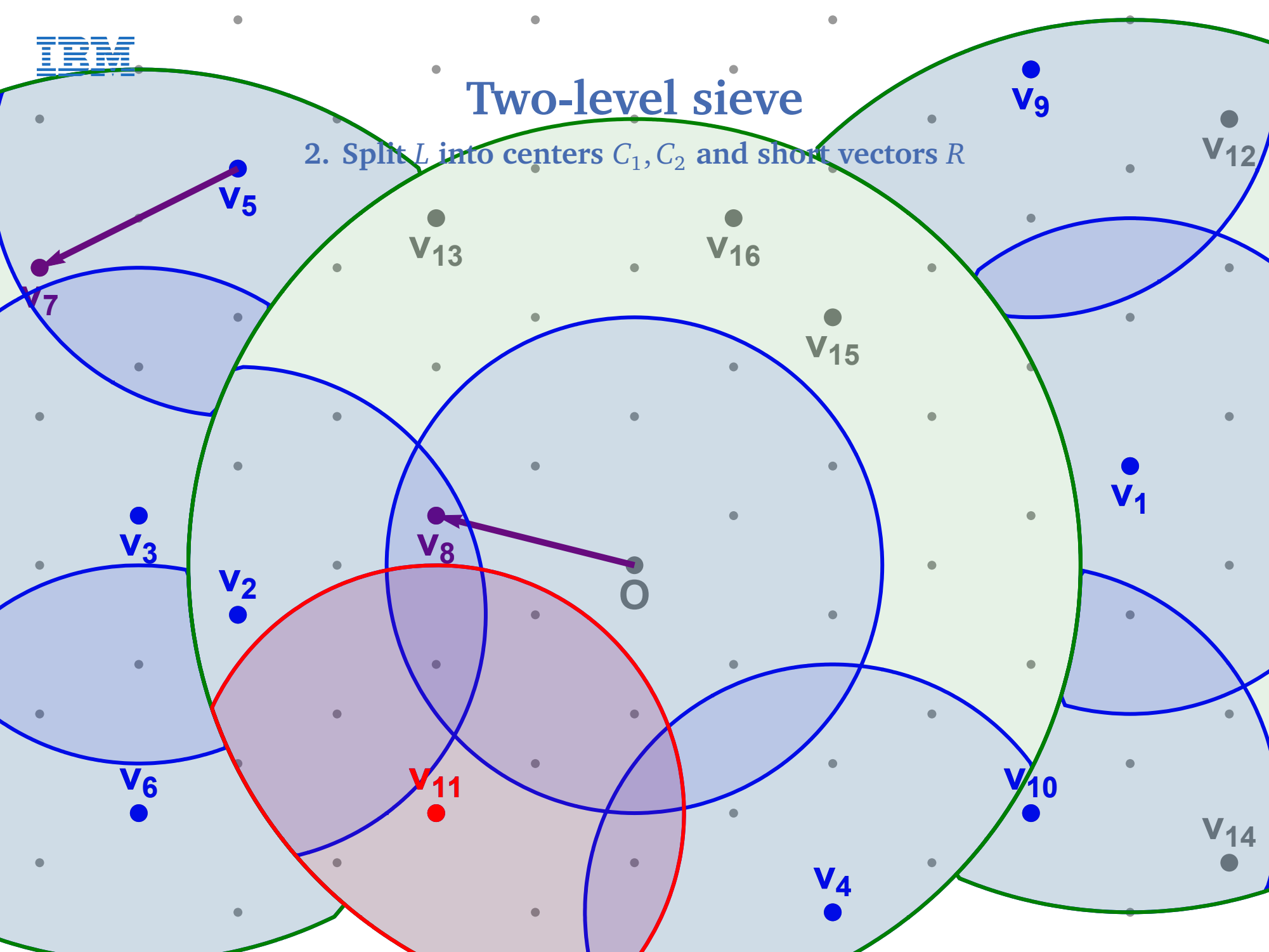

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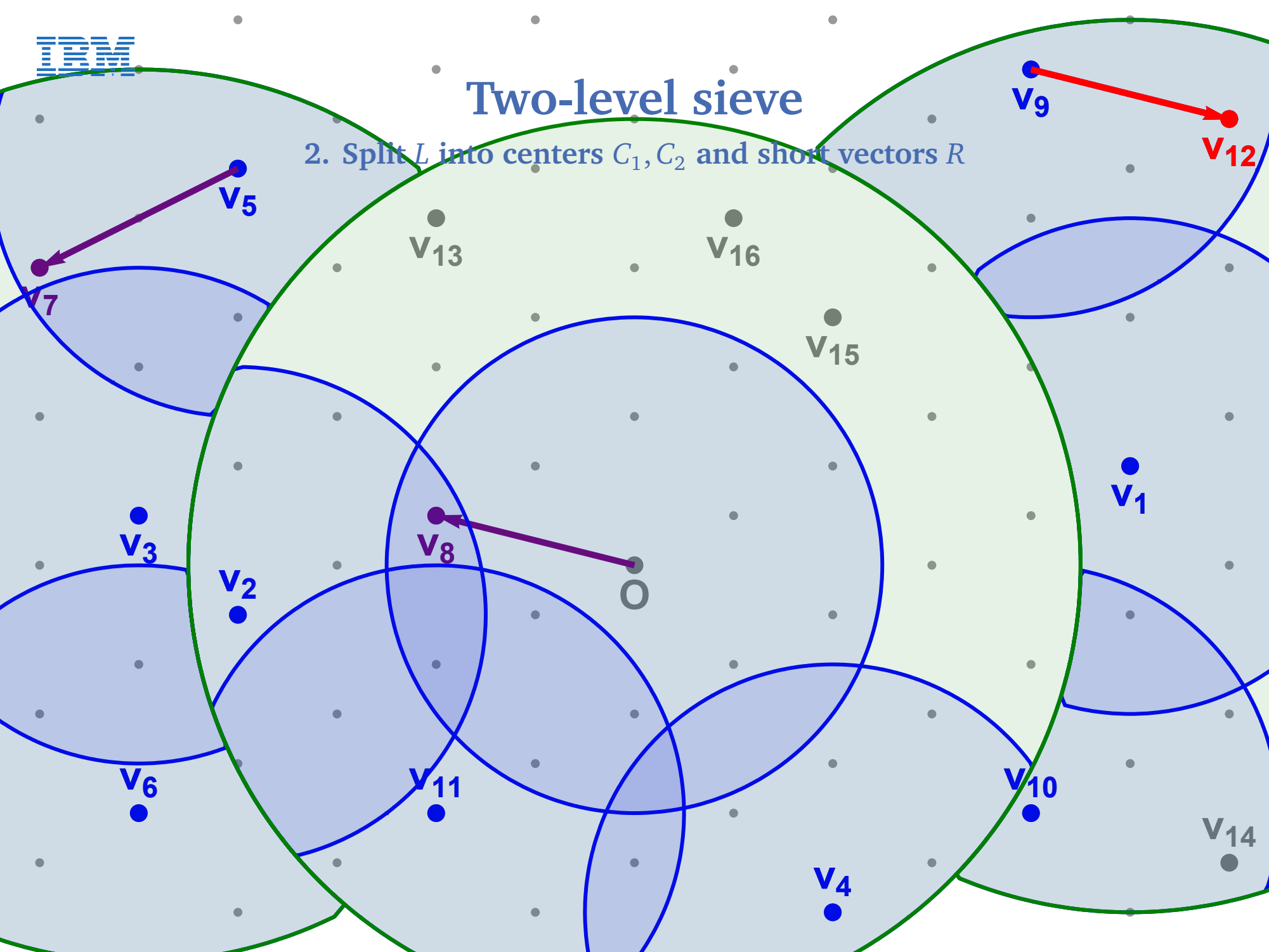

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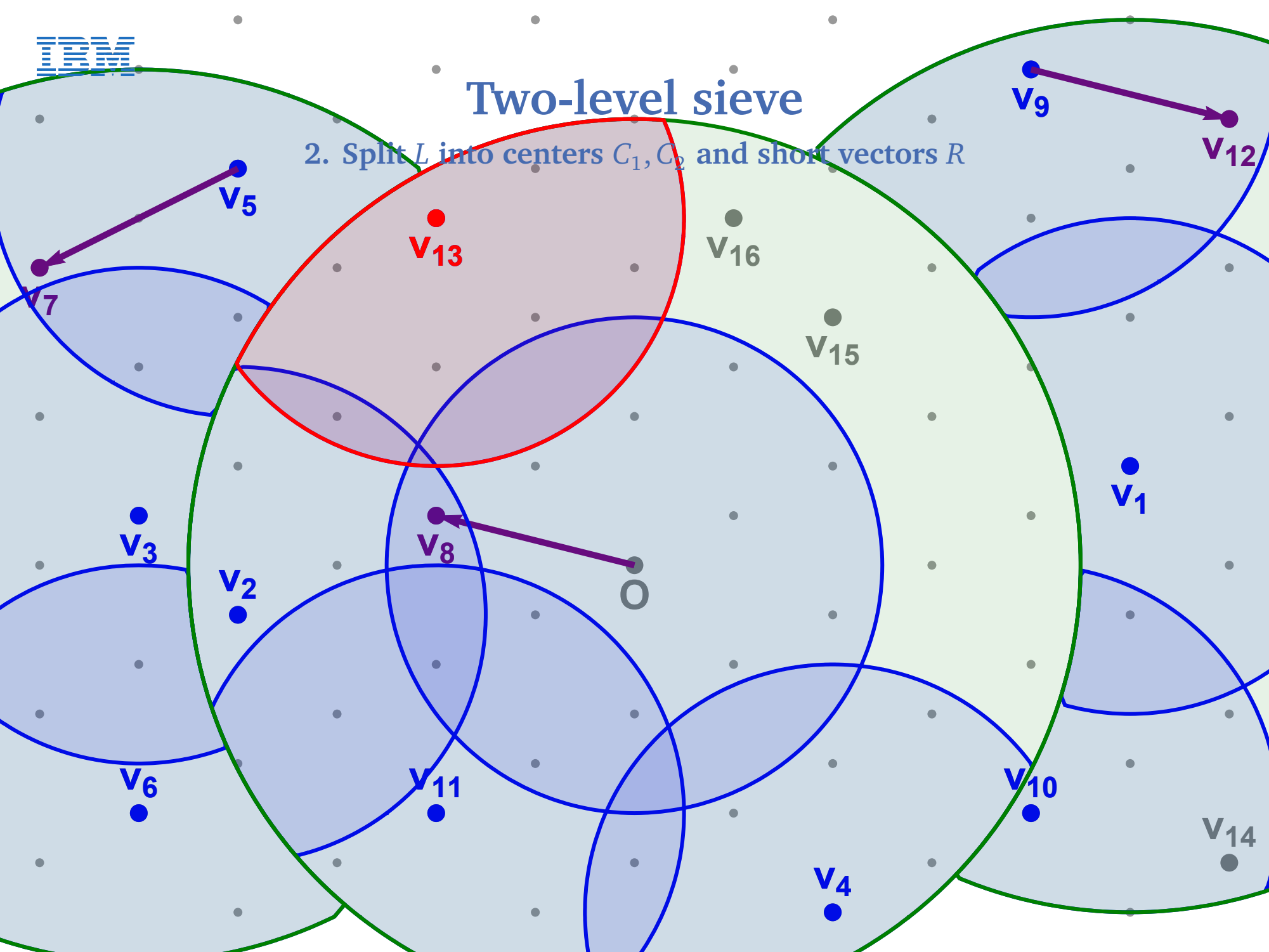

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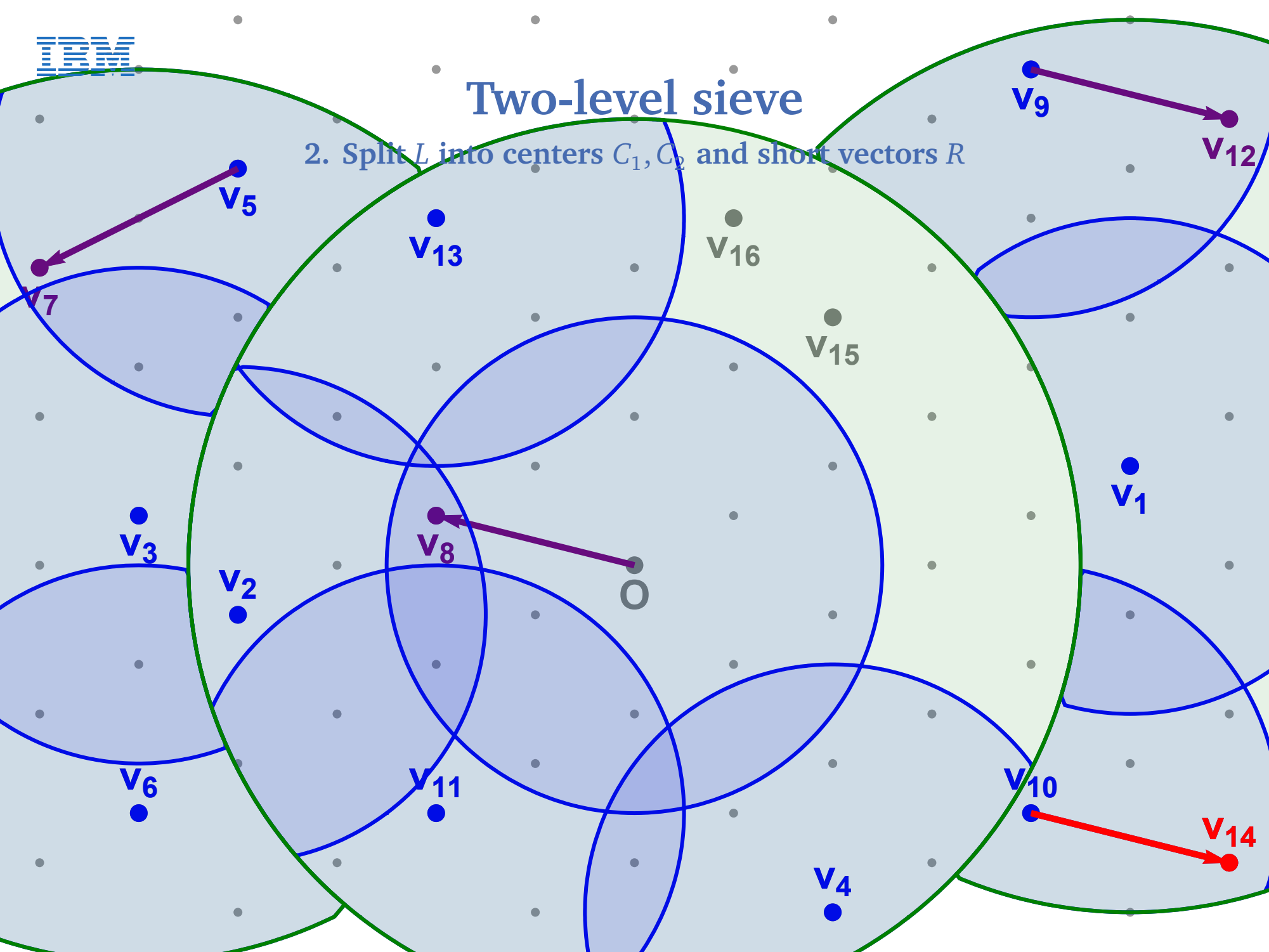

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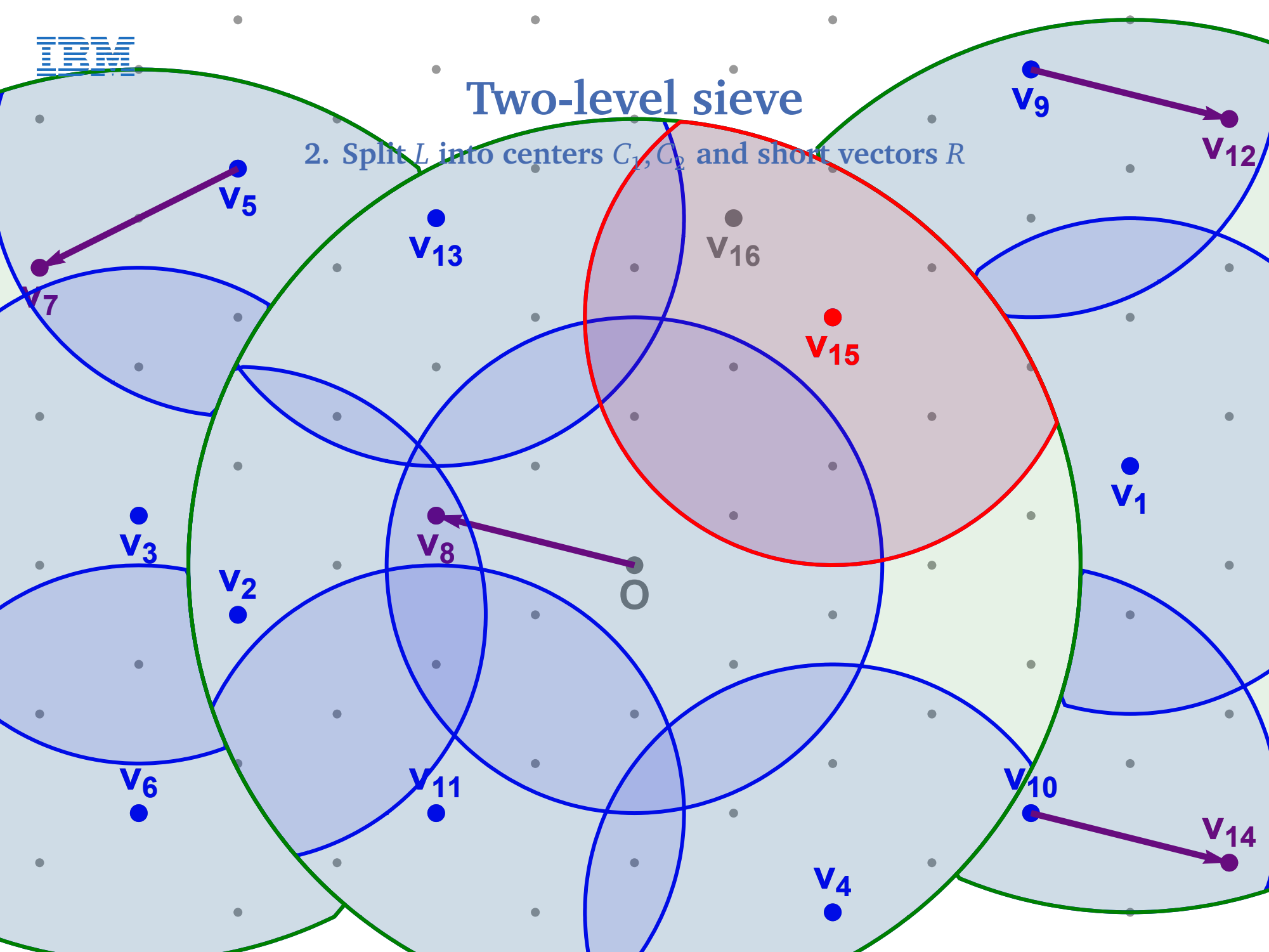

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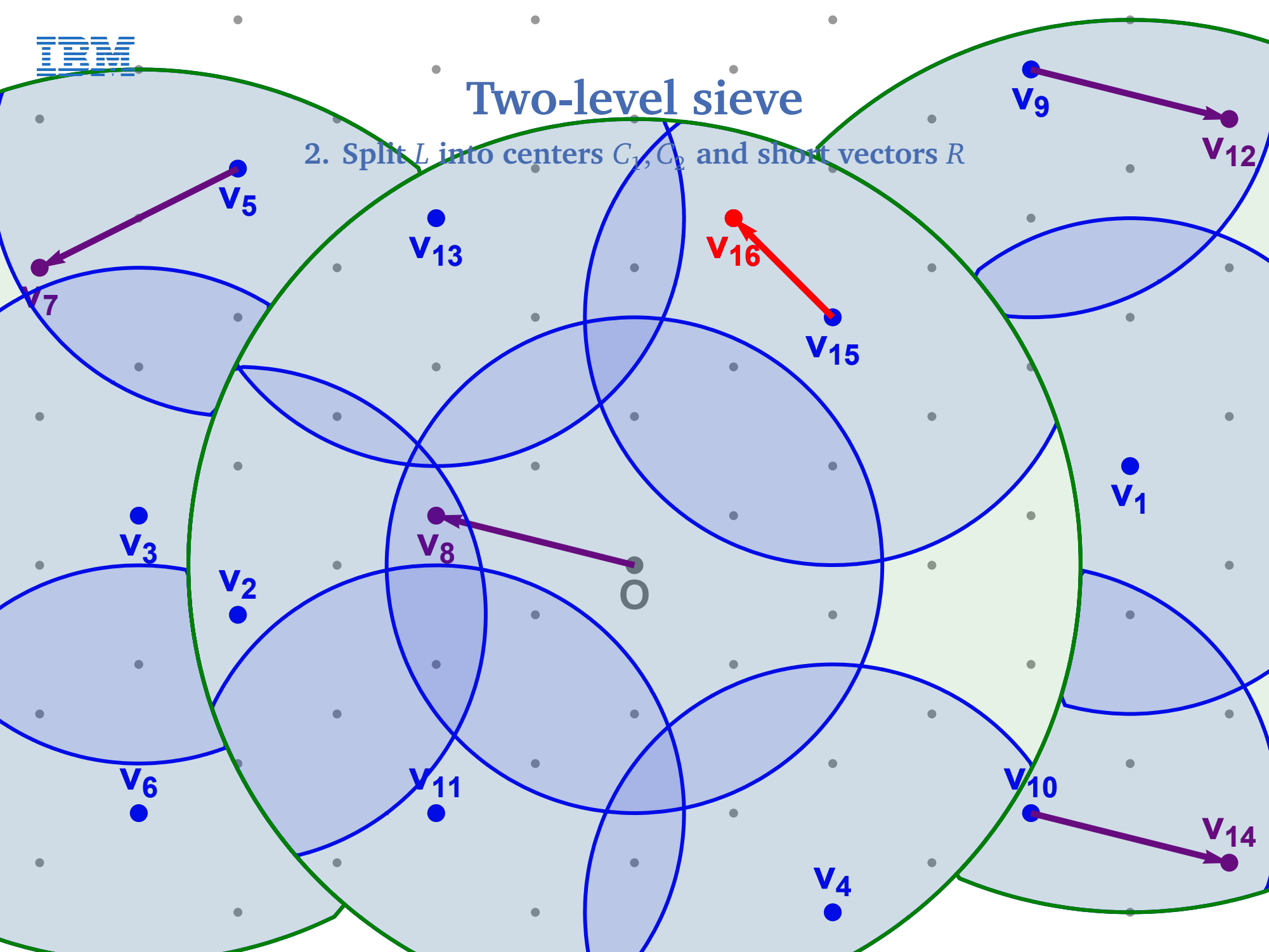

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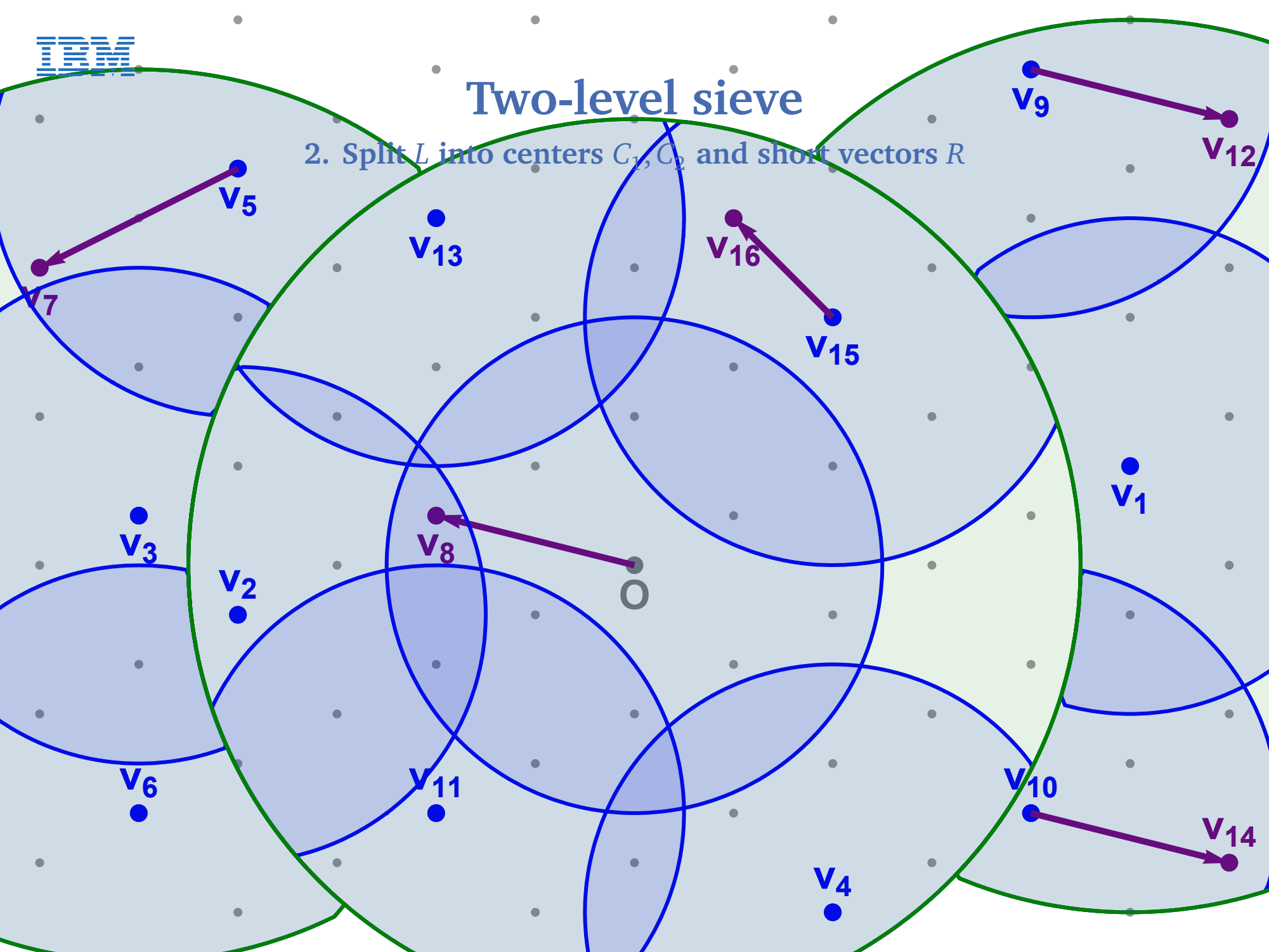
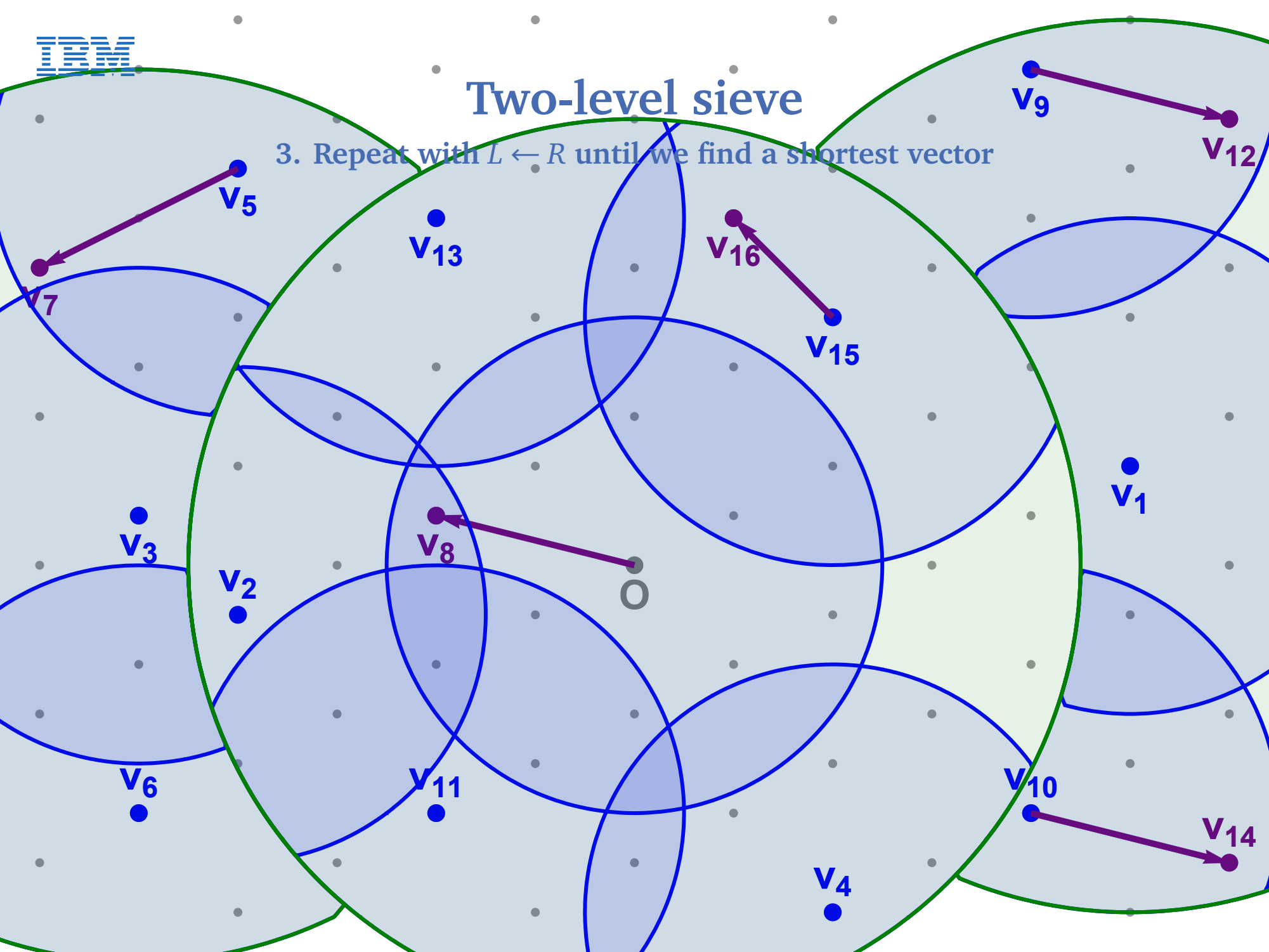


Two-level sieve

3. Repeat with $L \leftarrow R$ until we find a shortest vector



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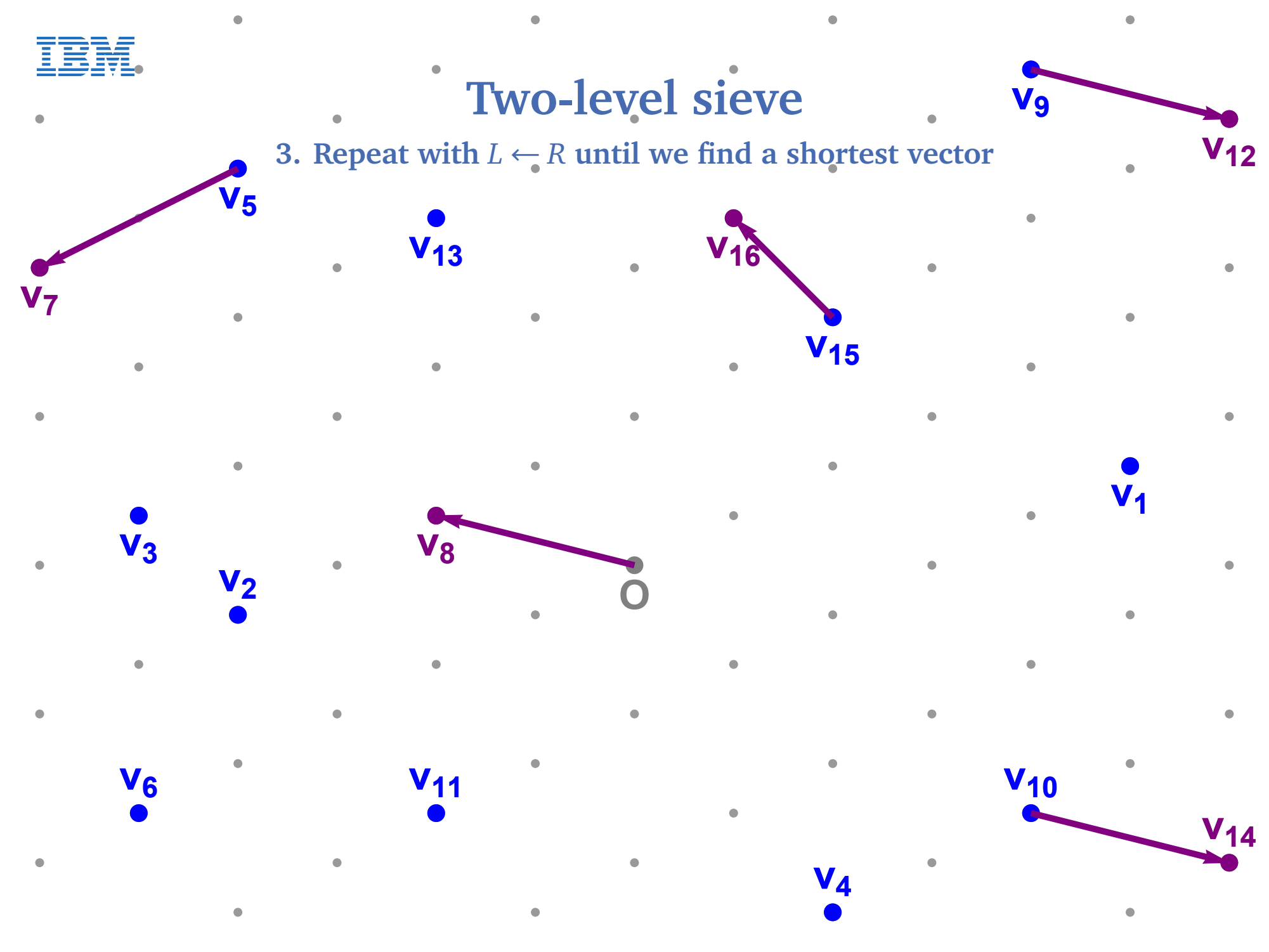

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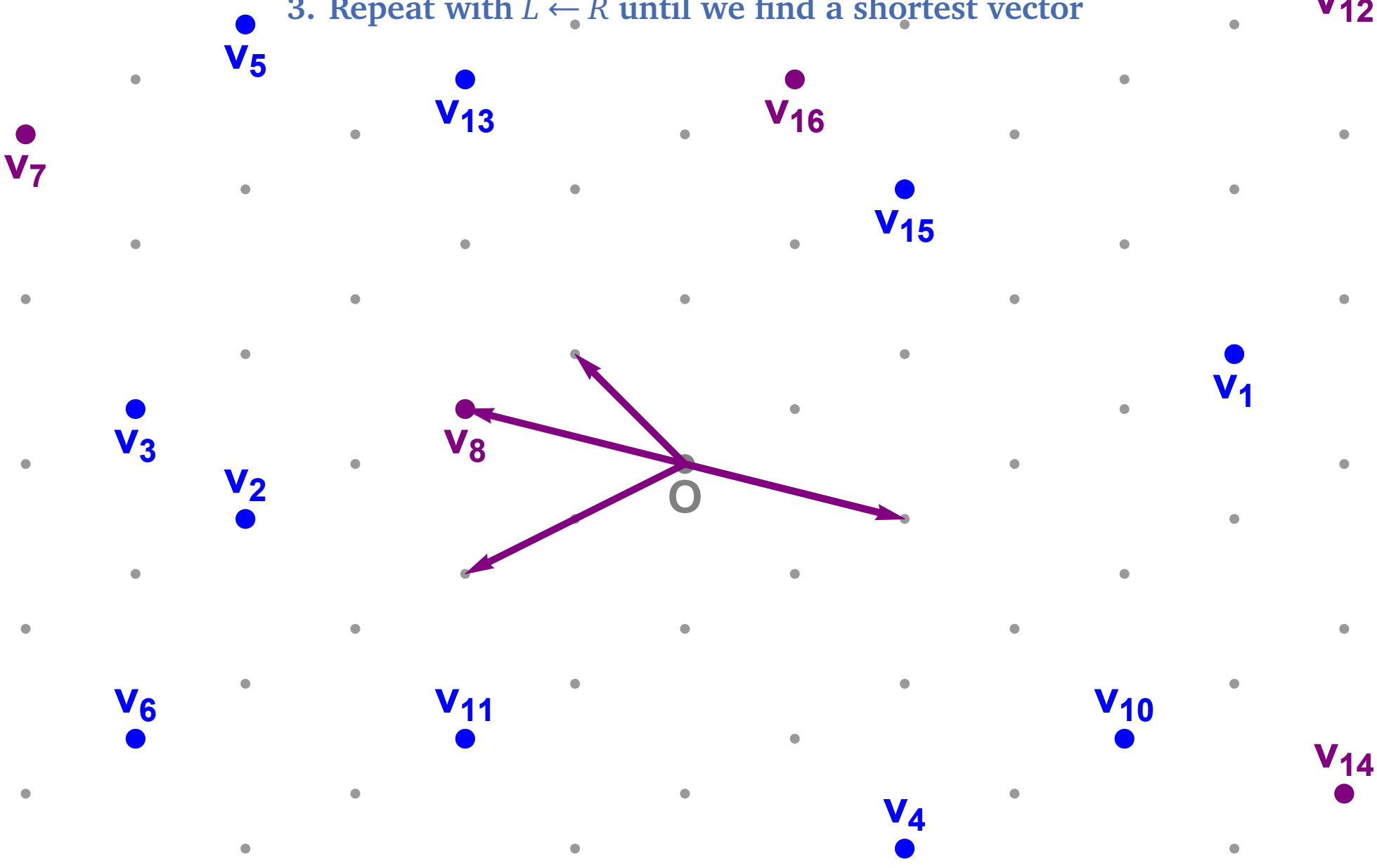

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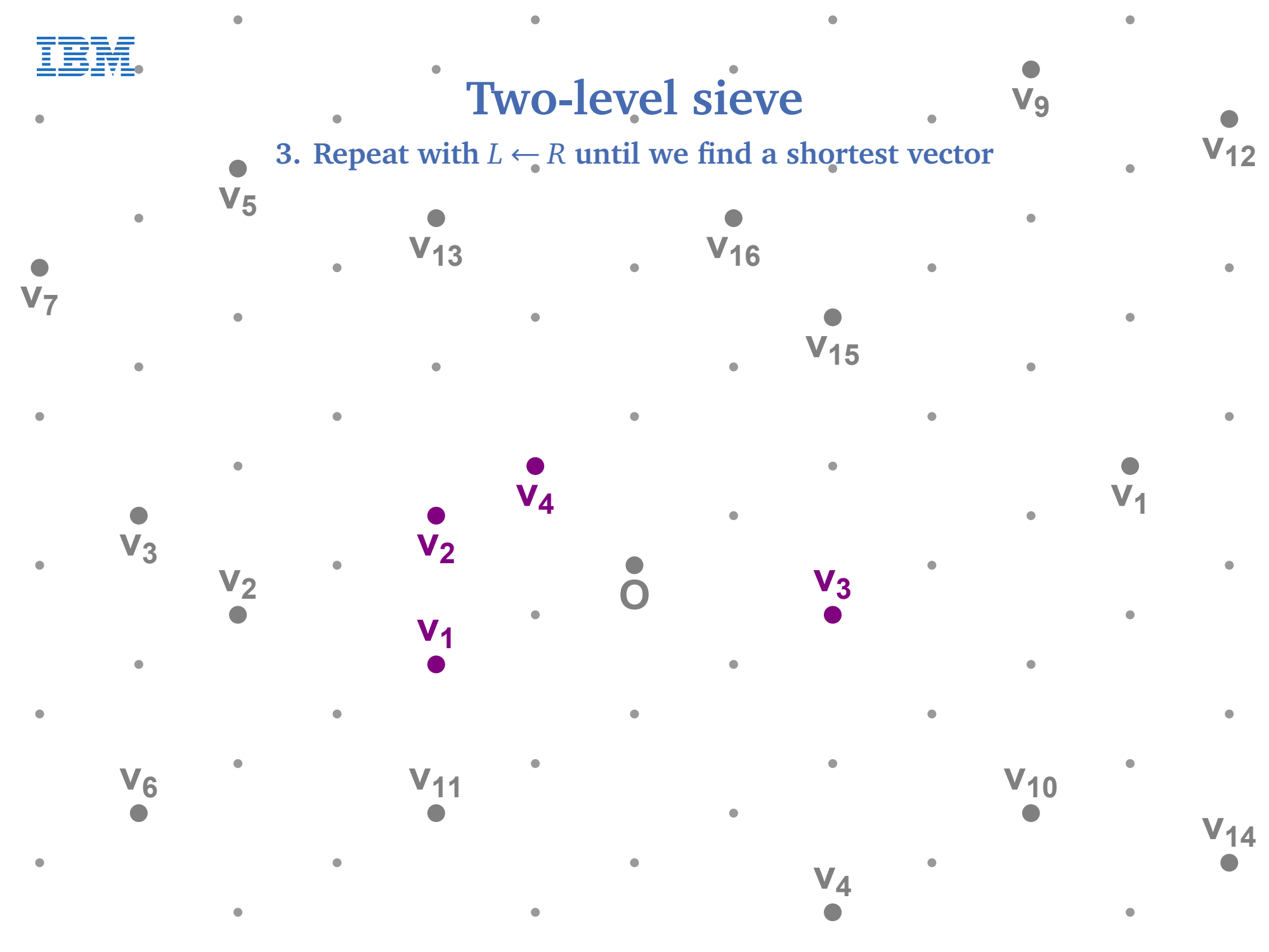

Multiple levels

Overview

Multiple levels

Overview

Heuristic (Nguyen and Vidick, J. Math. Crypt. '08)

The one-level sieve runs in time $2^{0.4150n}$ *and space* $2^{0.2075n}$.

Multiple levels

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The one-level sieve runs in time $2^{0.4150n}$ *and space* $2^{0.2075n}$.

Heuristic (Wang et al., ASIACCS'11)

The two-level sieve runs in time $2^{0.3836n}$ *and space* $2^{0.2557n}$.

Multiple levels

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Heuristic (Nguyen and Vidick, J. Math. Crypt. '08)

The one-level sieve runs in time $2^{0.4150n}$ *and space* $2^{0.2075n}$.

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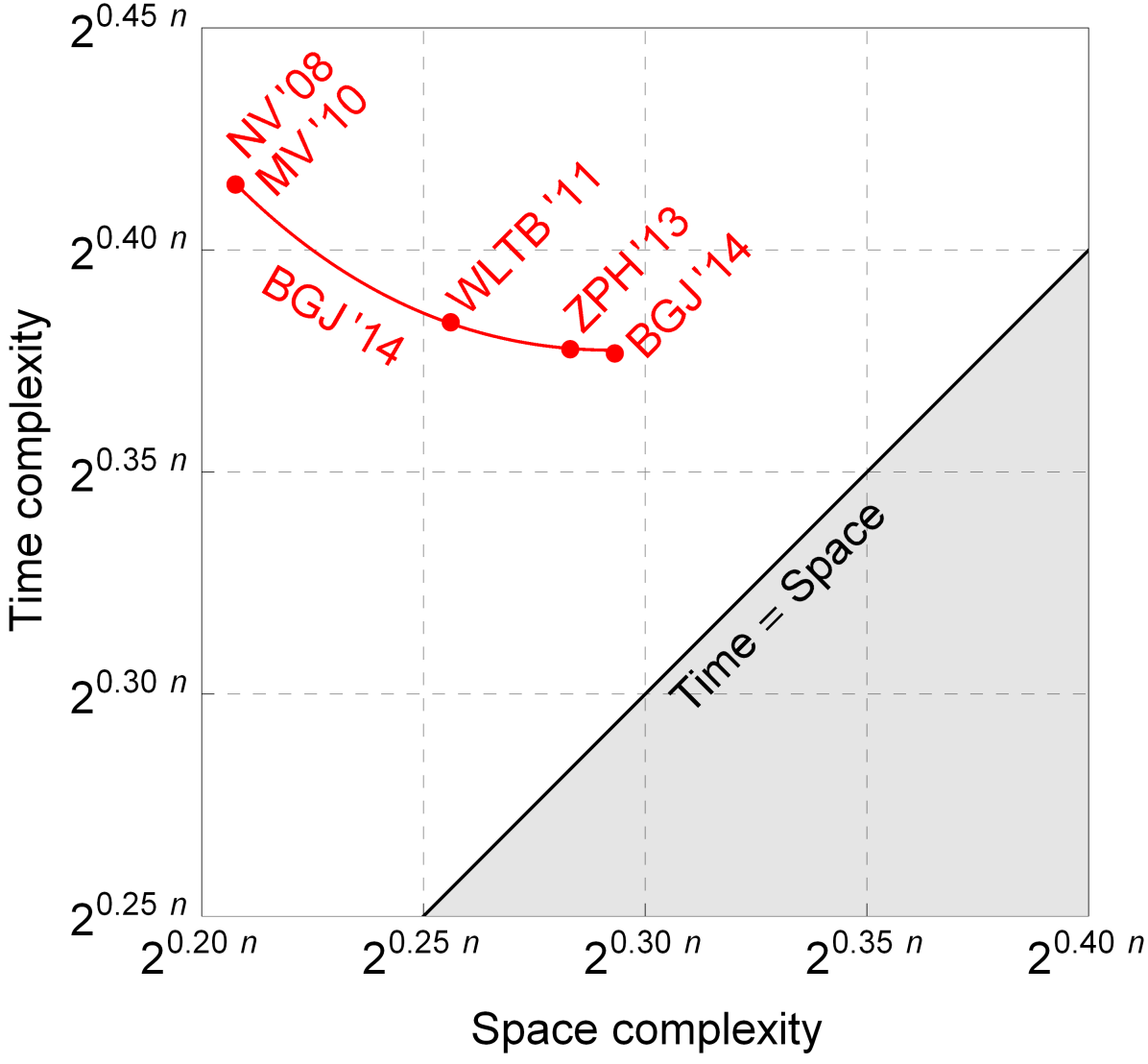
The two-level sieve runs in time $2^{0.3836n}$ *and space* $2^{0.2557n}$.

Heuristic (Zhang et al., SAC'13)

The three-level sieve runs in time $2^{0.3778n}$ *and space* $2^{0.2833n}$.

Sieving

Space/time trade-off



Hyperplane LSH

1. Sample a list L of random lattice vectors

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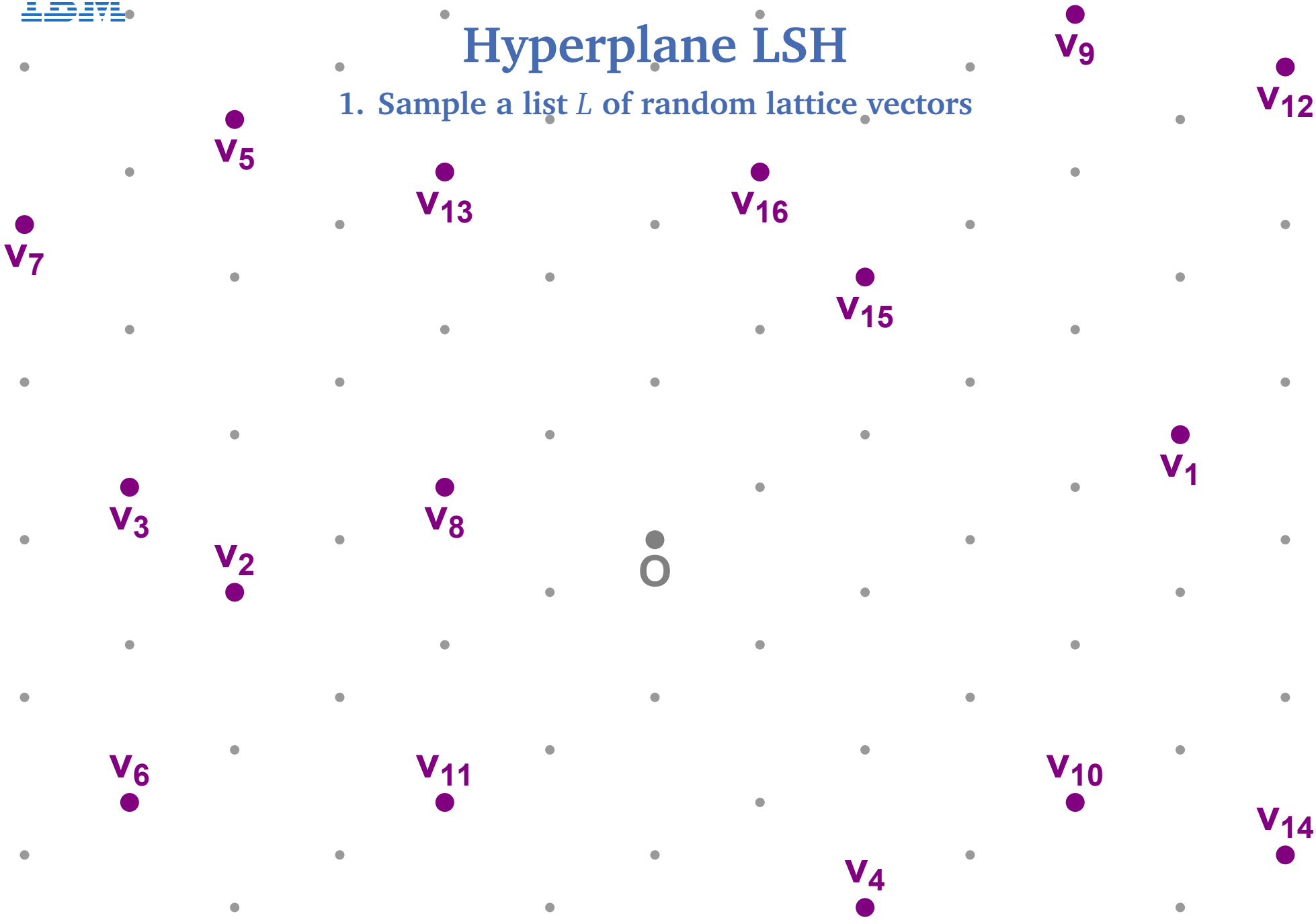
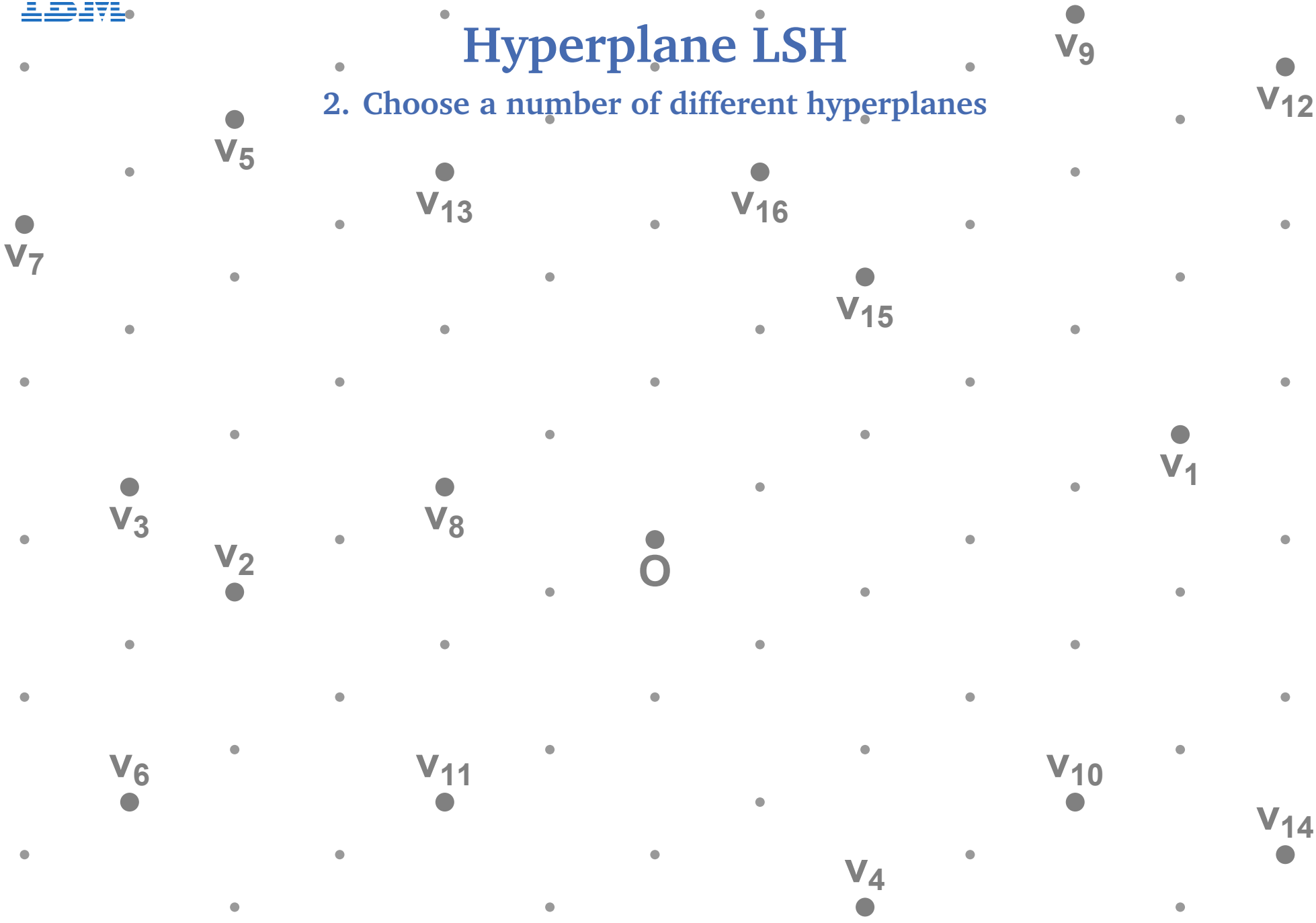


Hyperplane LSH

2. Choose a number of different hyperplanes



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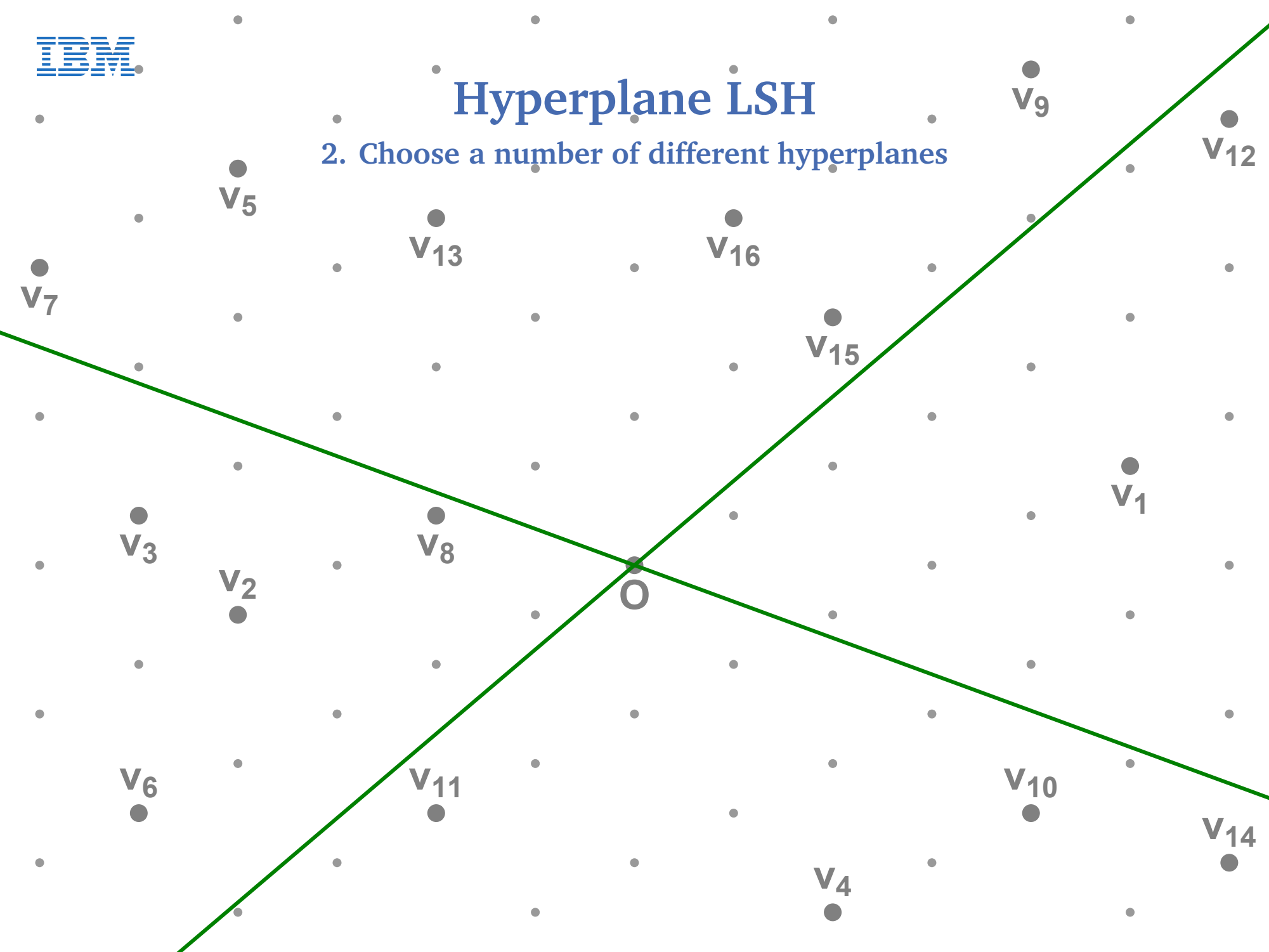
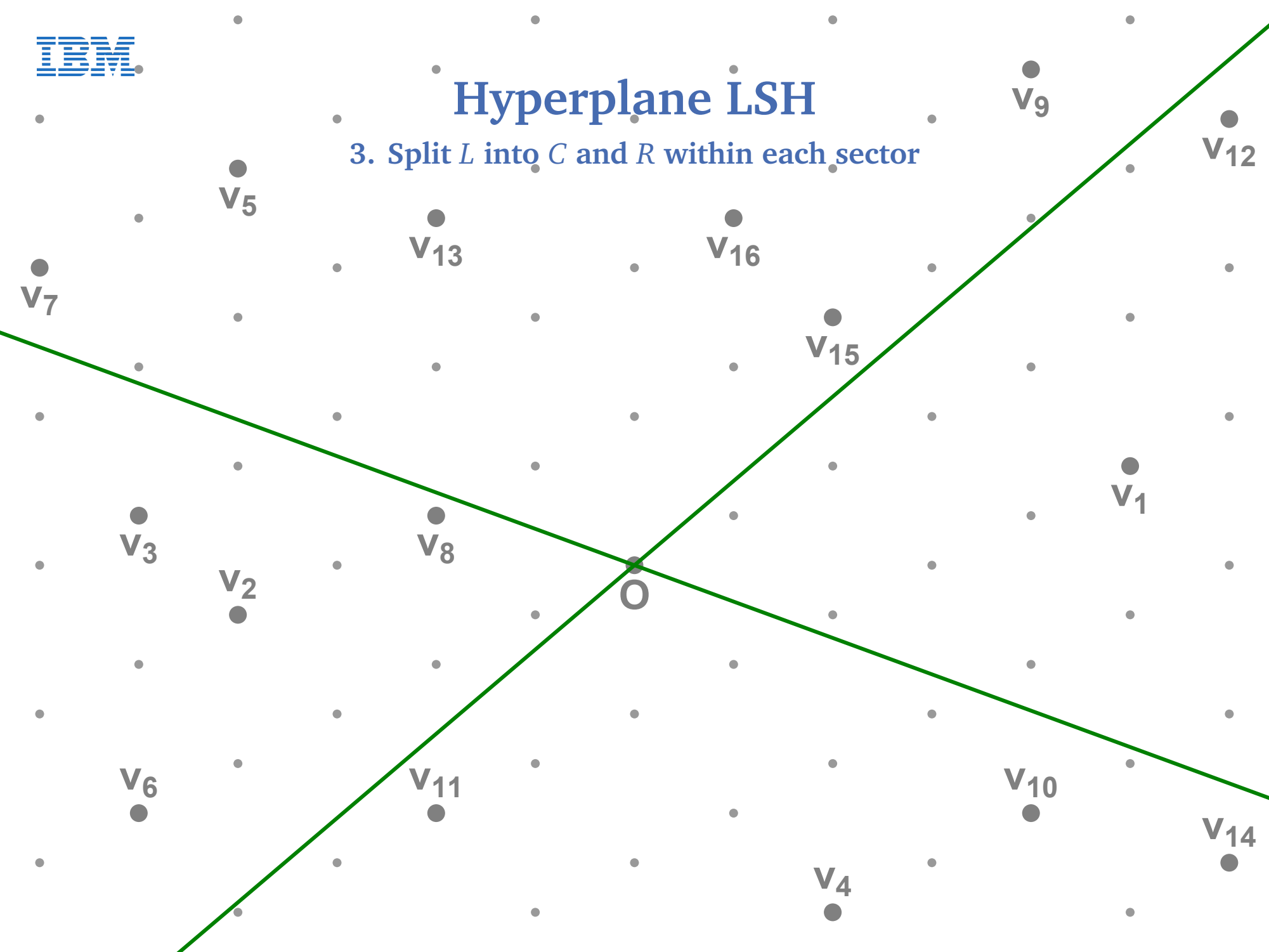


Hyperplane LSH

3. Split L into C and R within each sector



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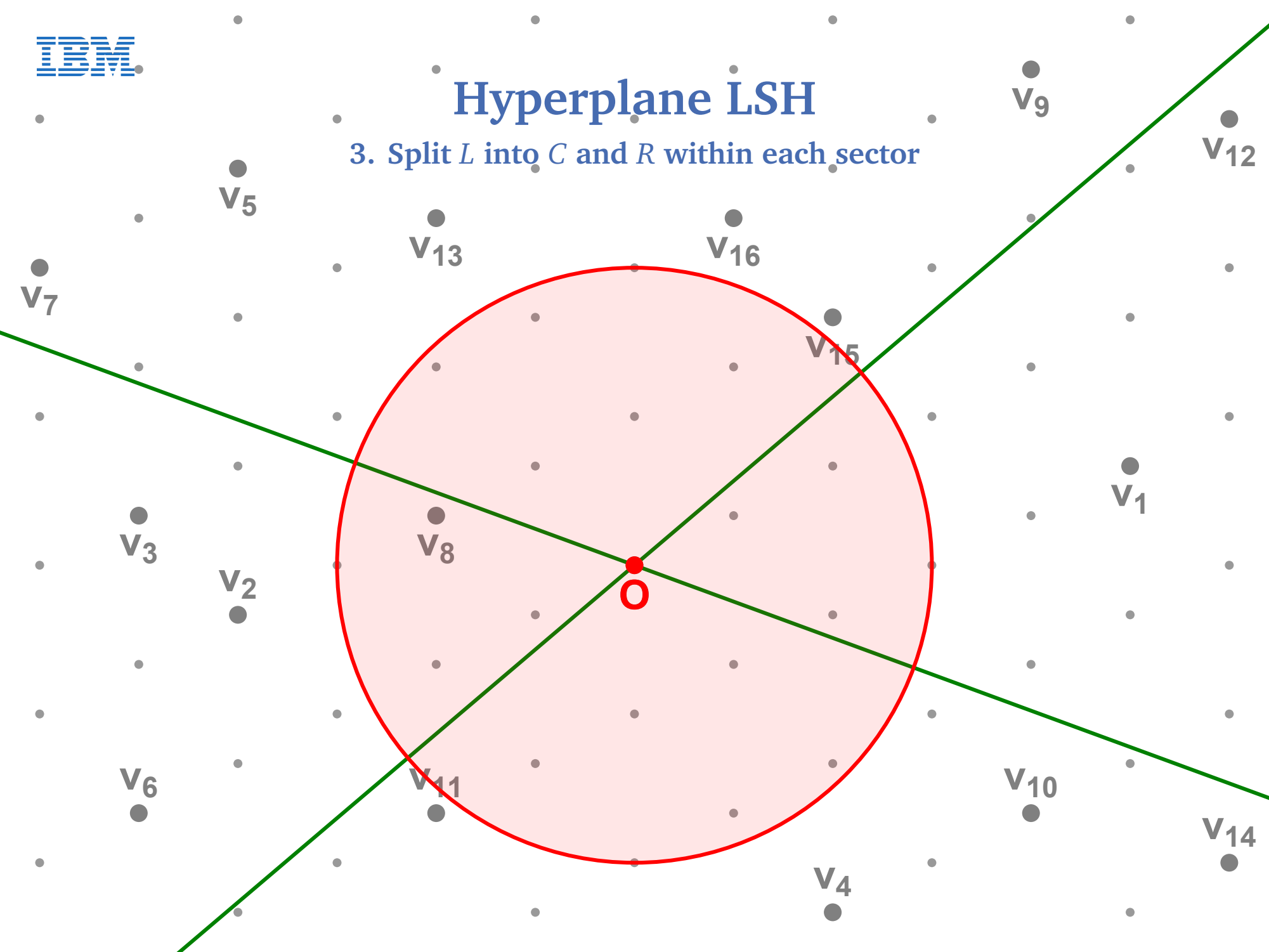
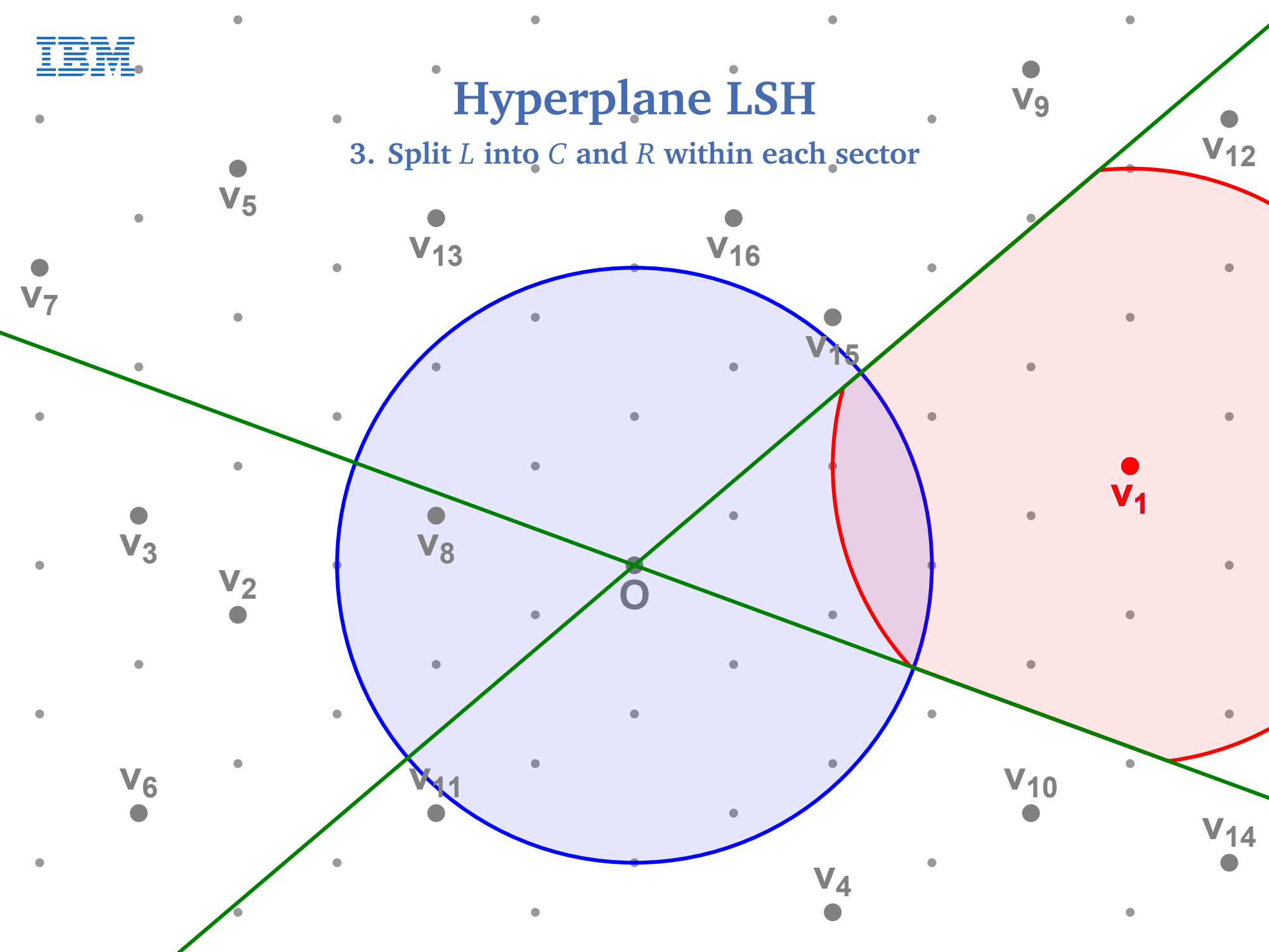


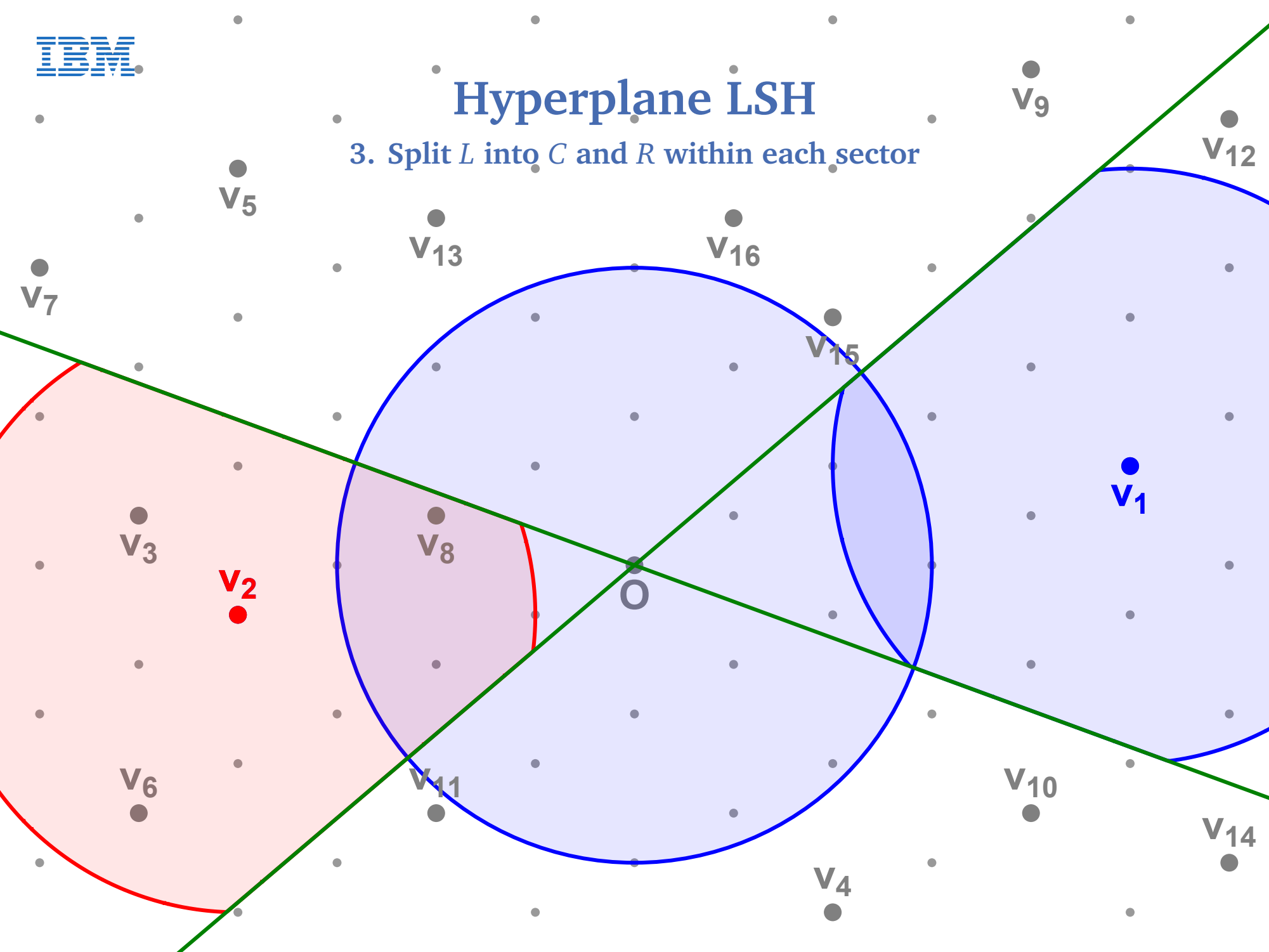
Hyperplane LSH

3. Split L into C and R within each sector



Hyperplane LSH

3. Split L into C and R within each sector



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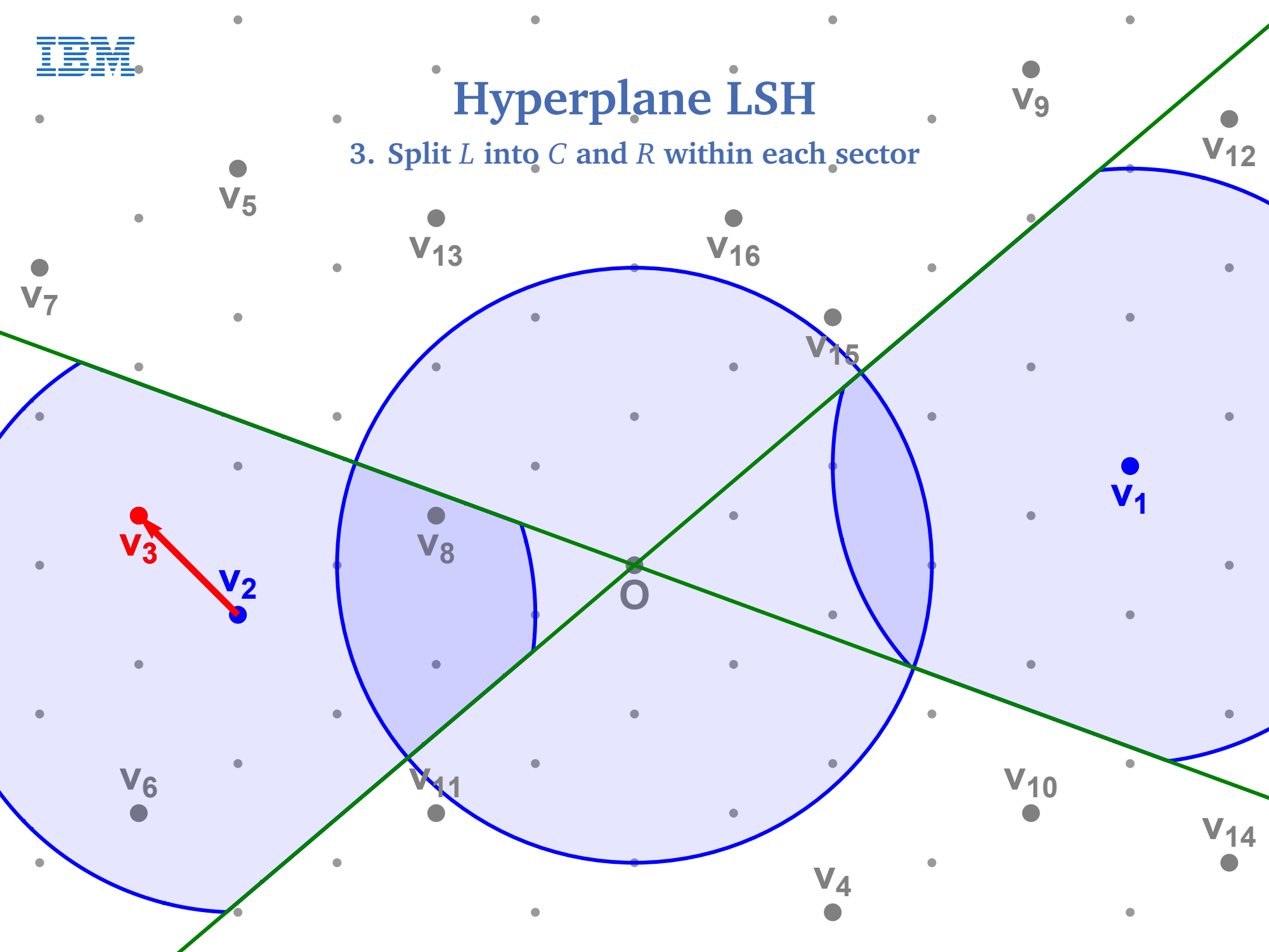

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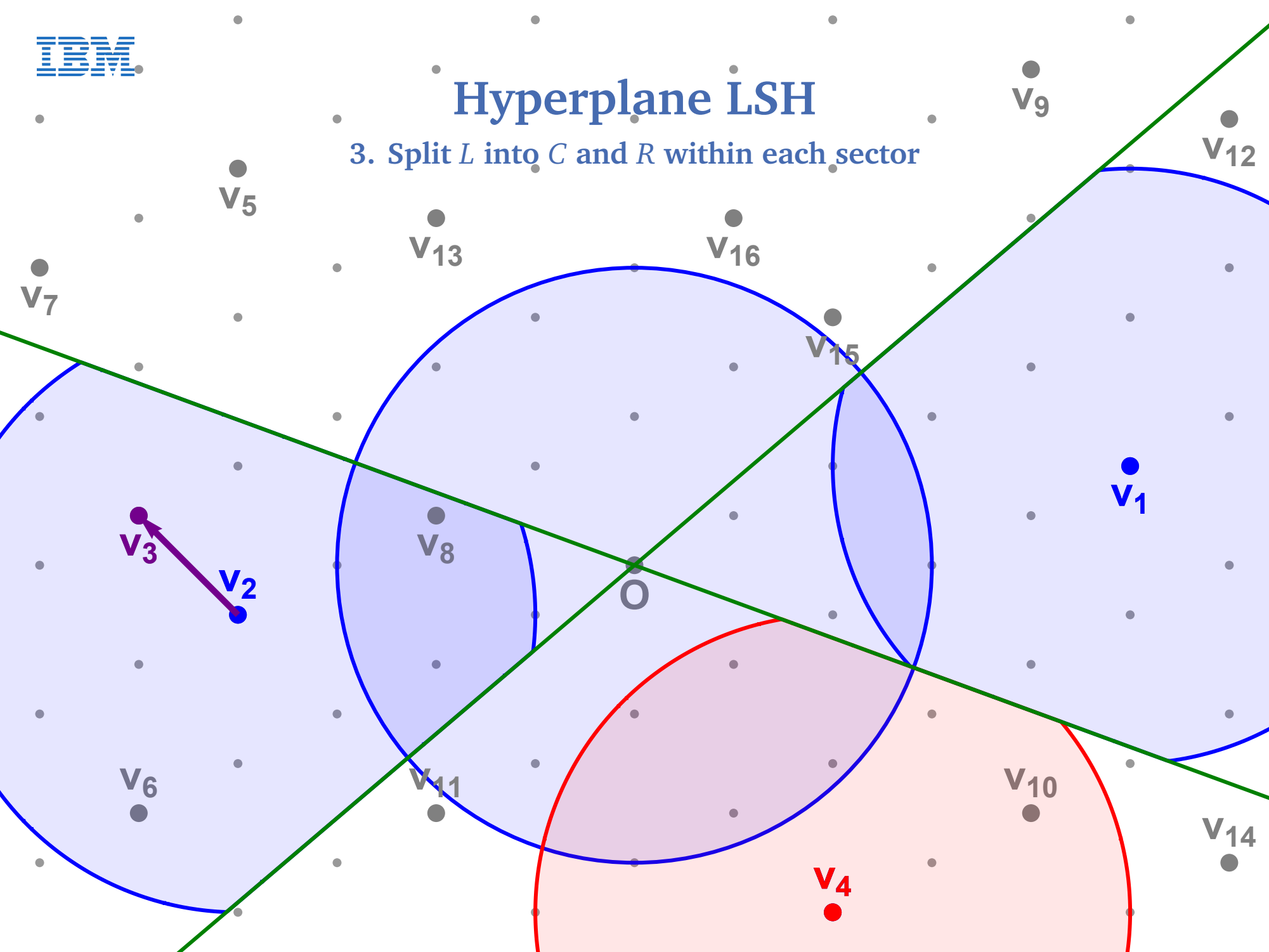

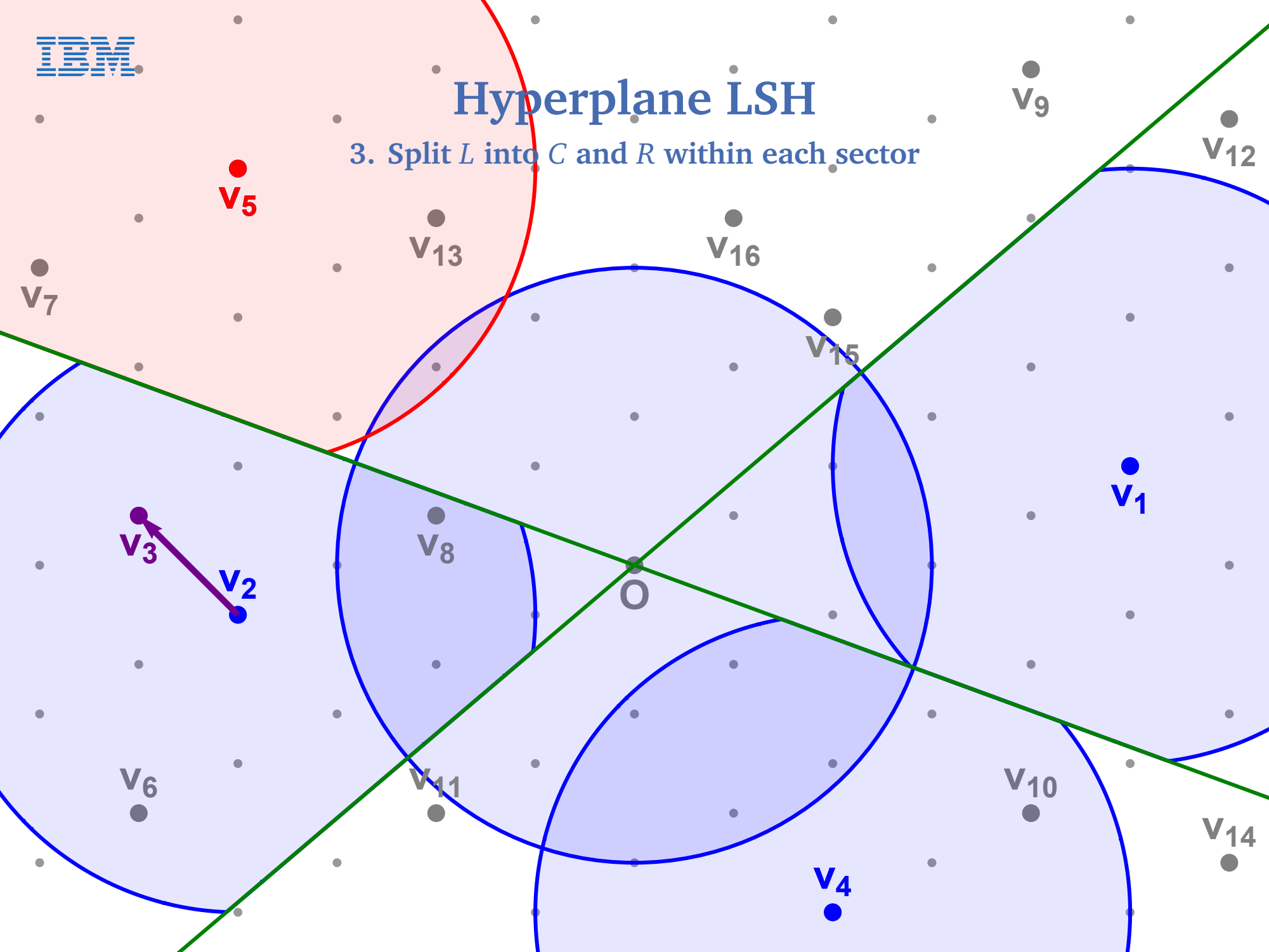
IBM
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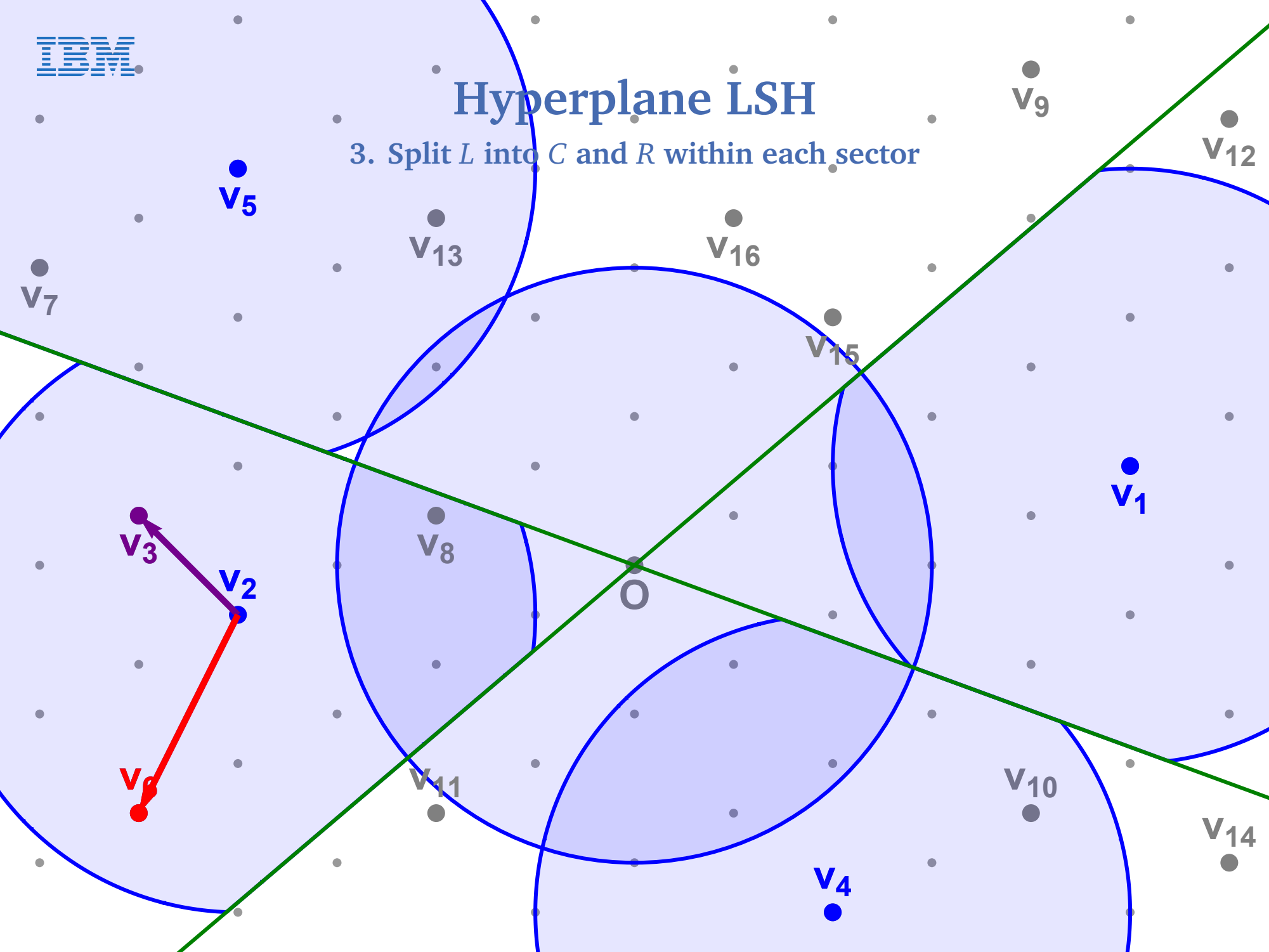

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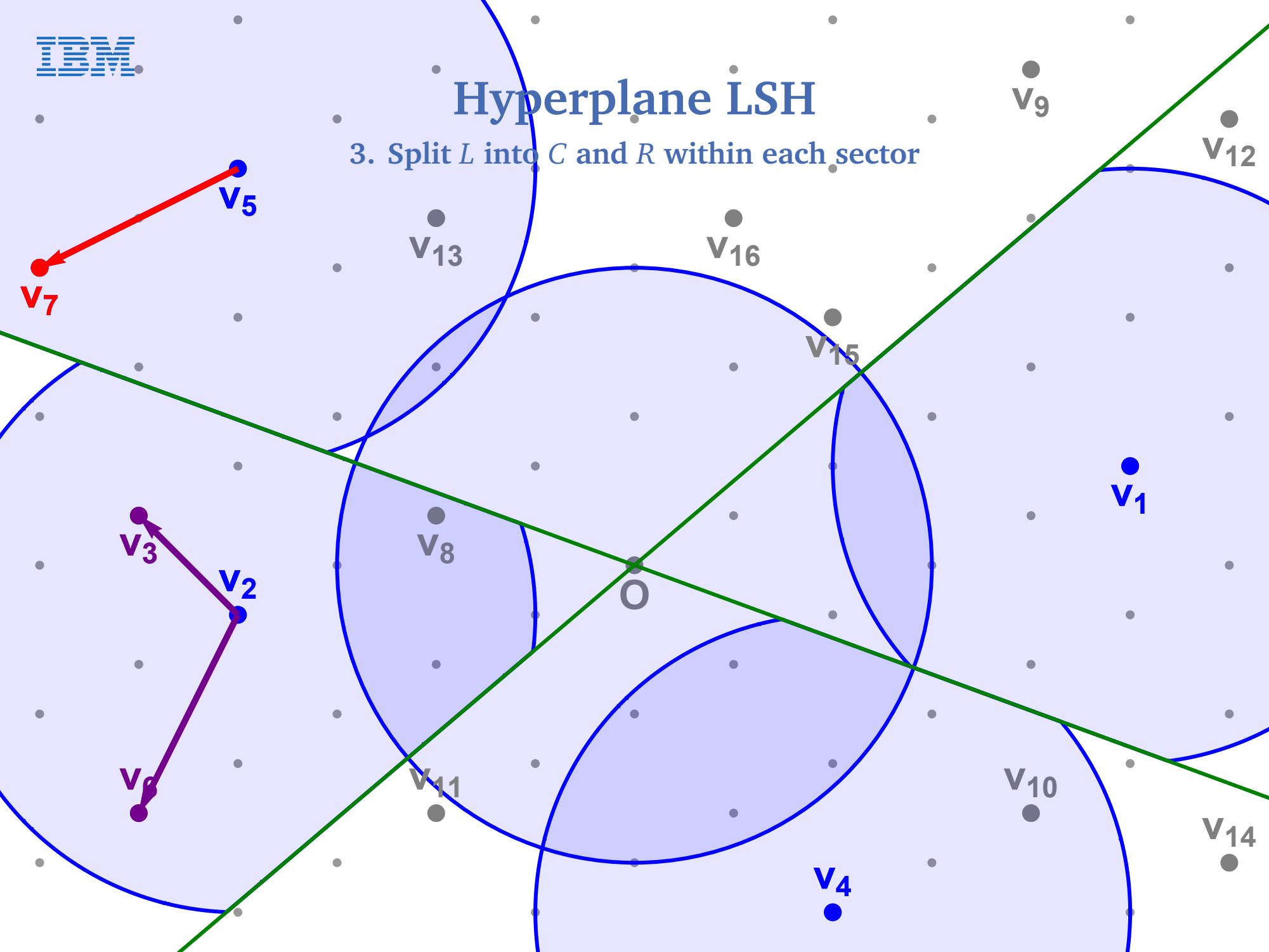

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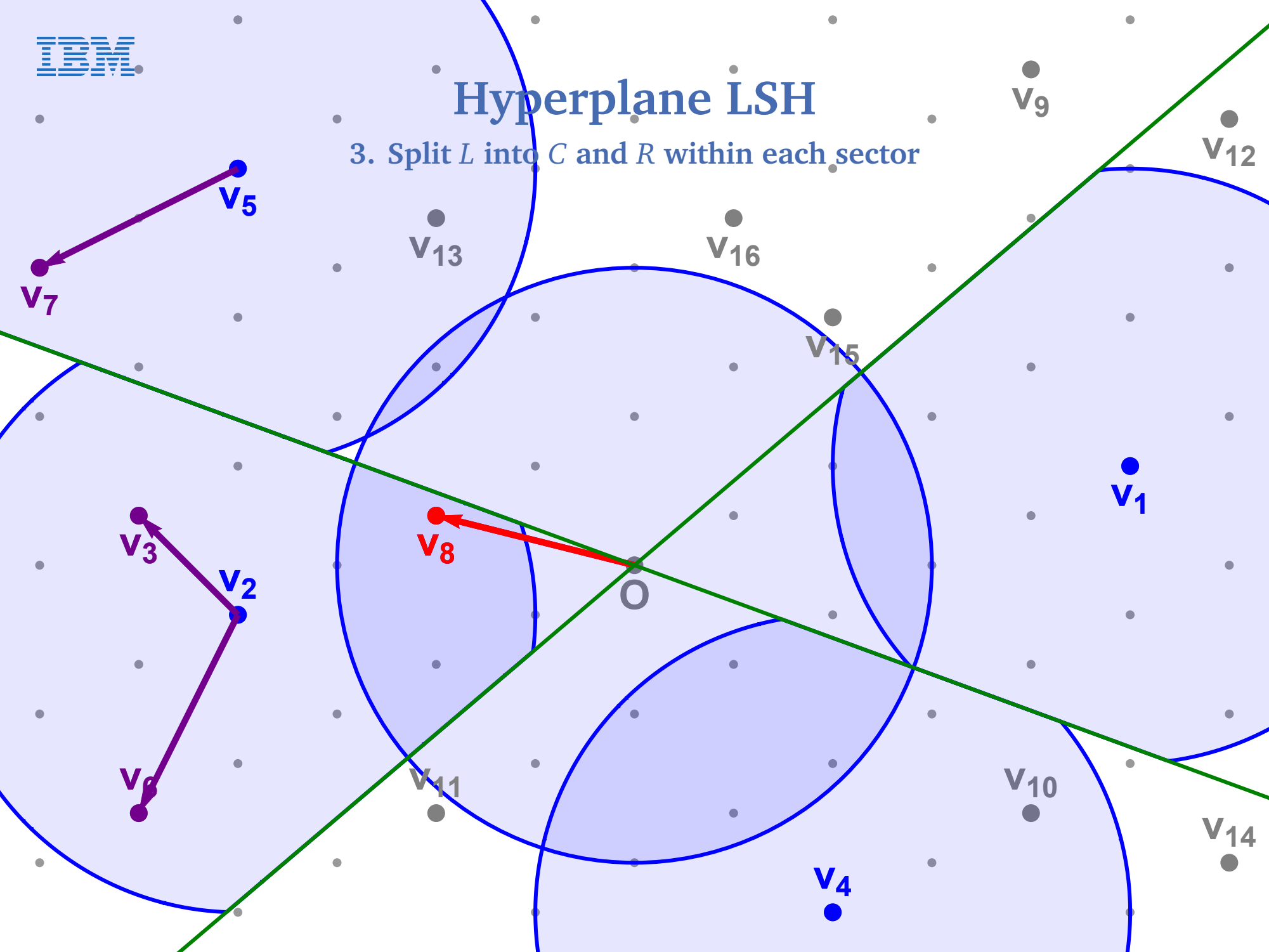

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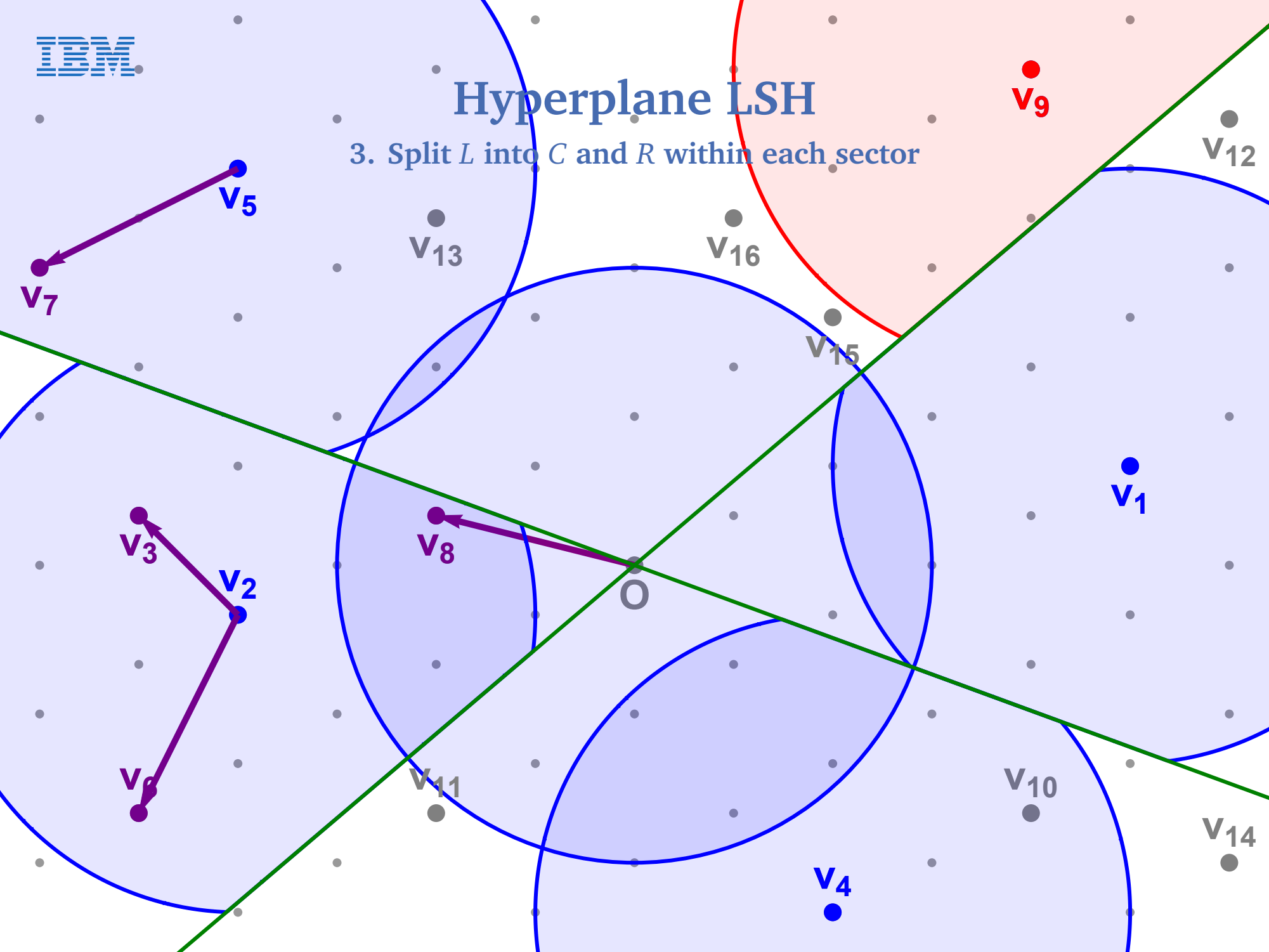

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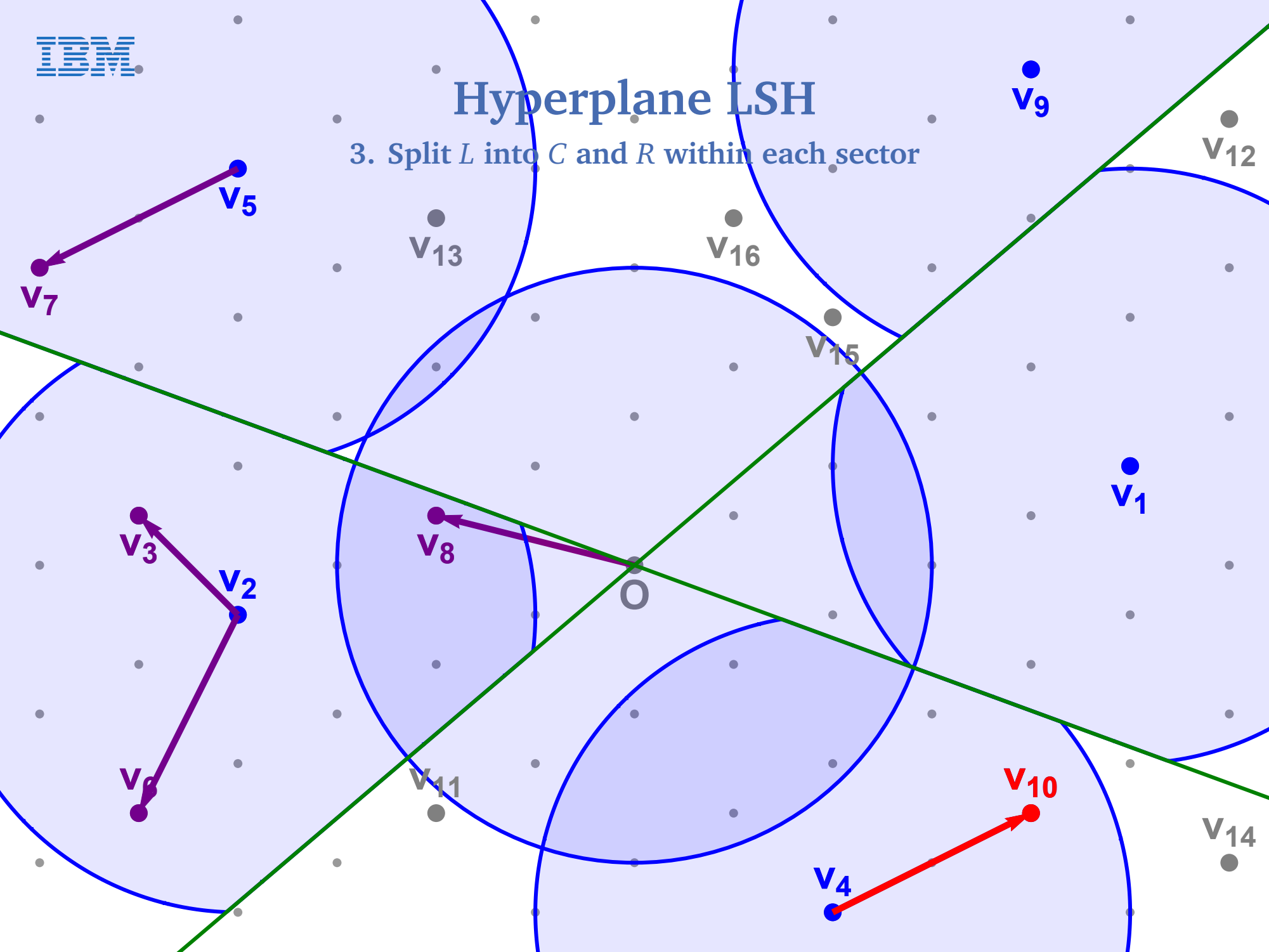

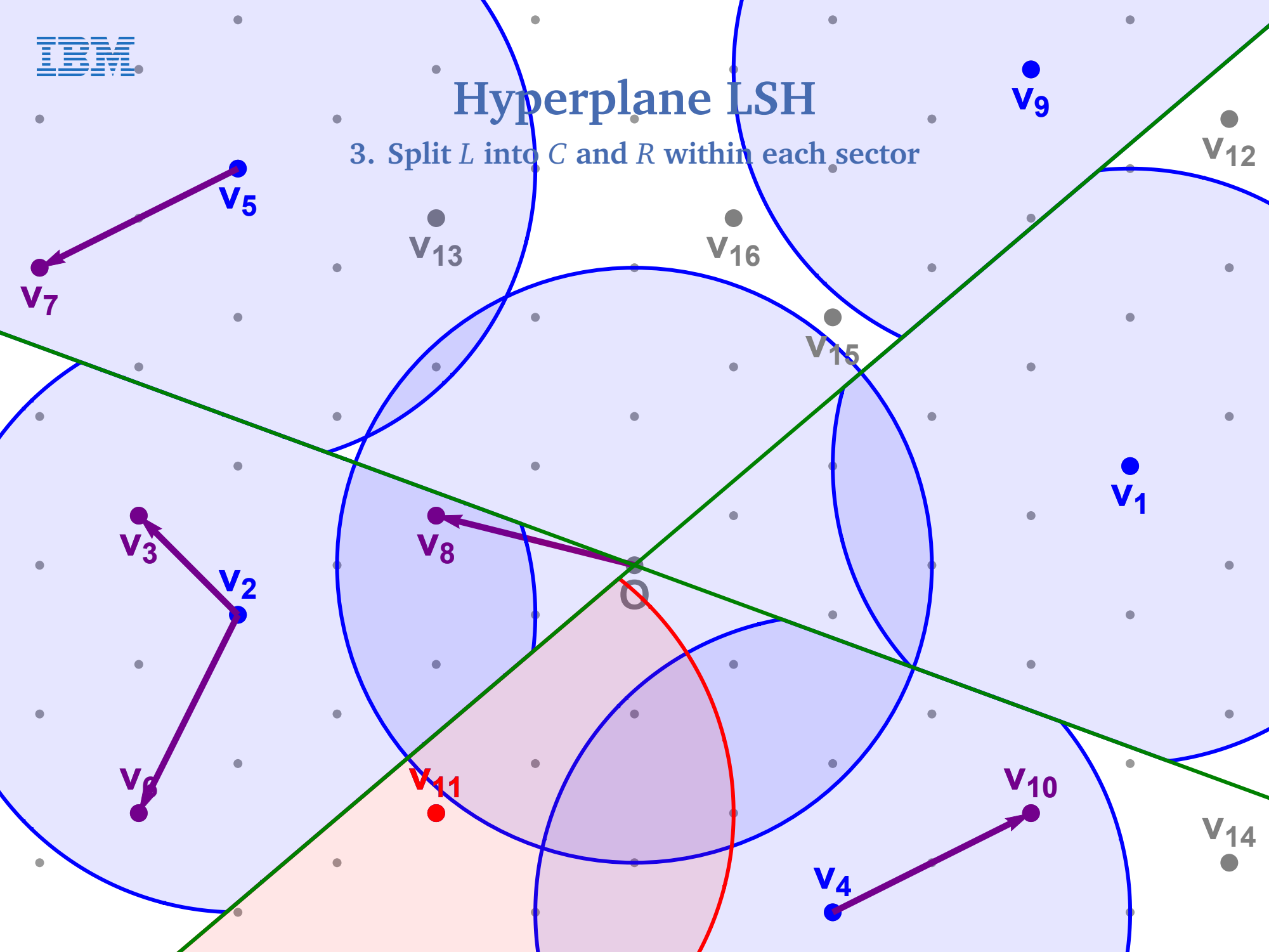
IBM
Hyperplane LSH
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Hyperplane LSH

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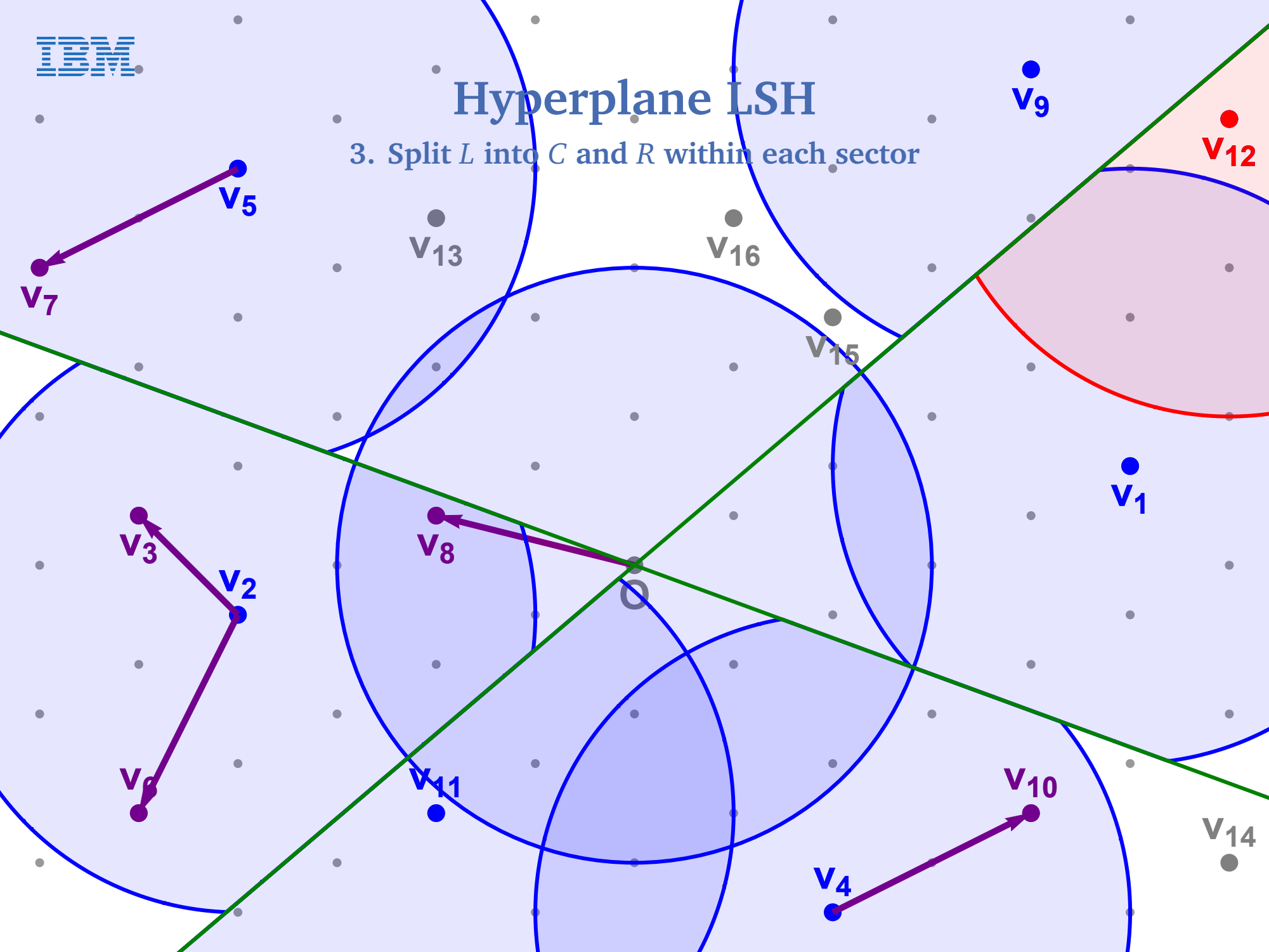

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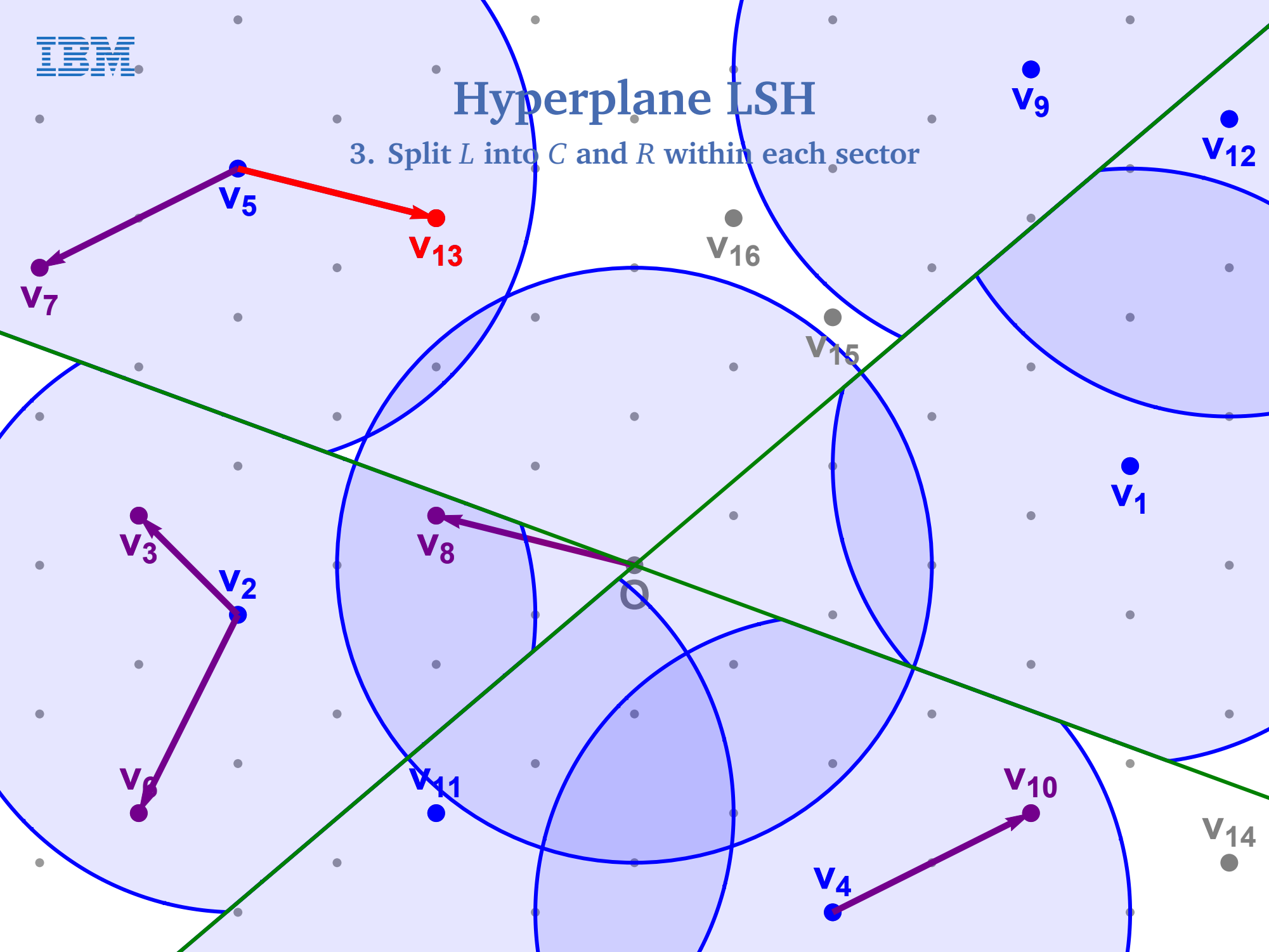

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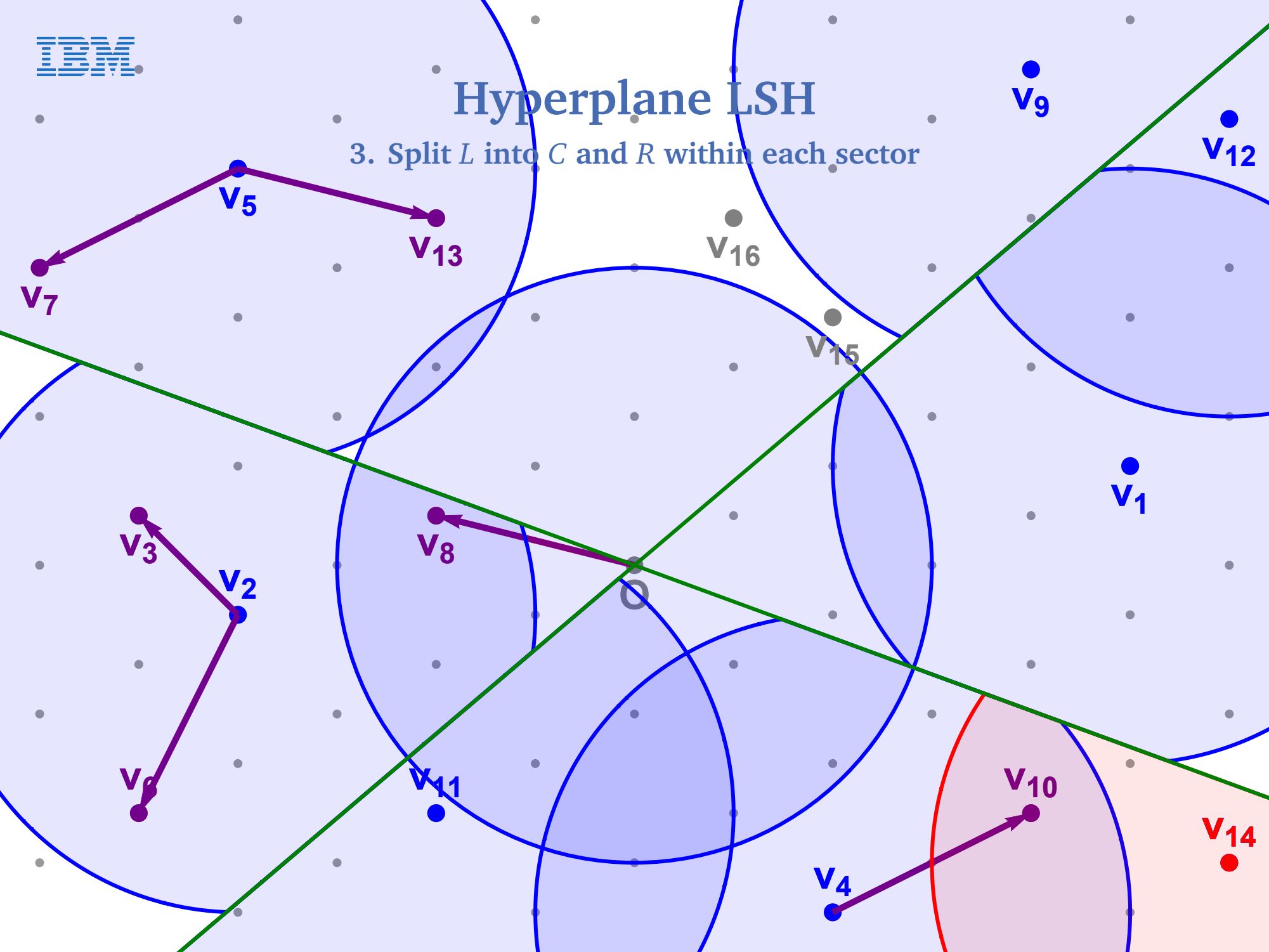

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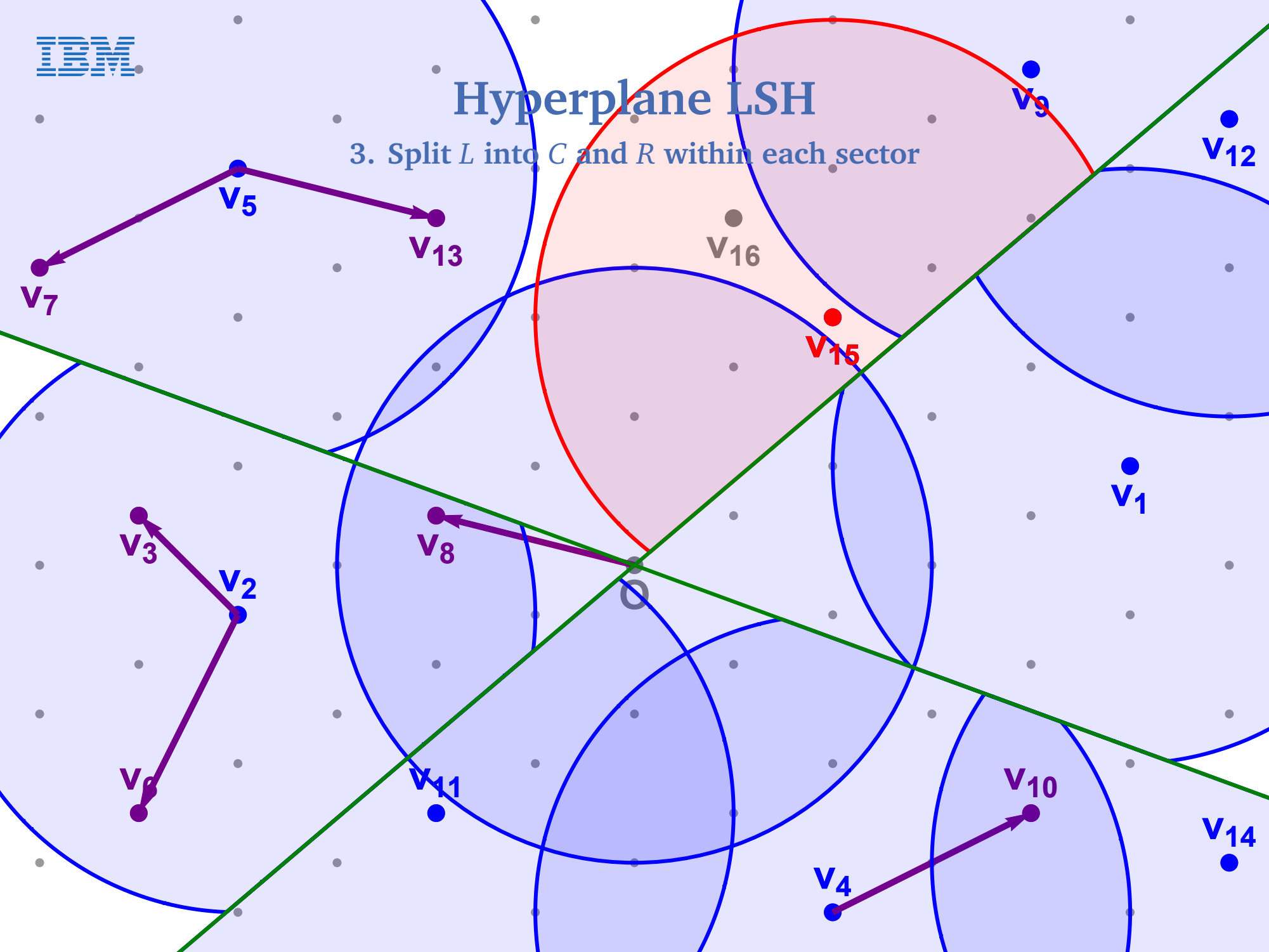

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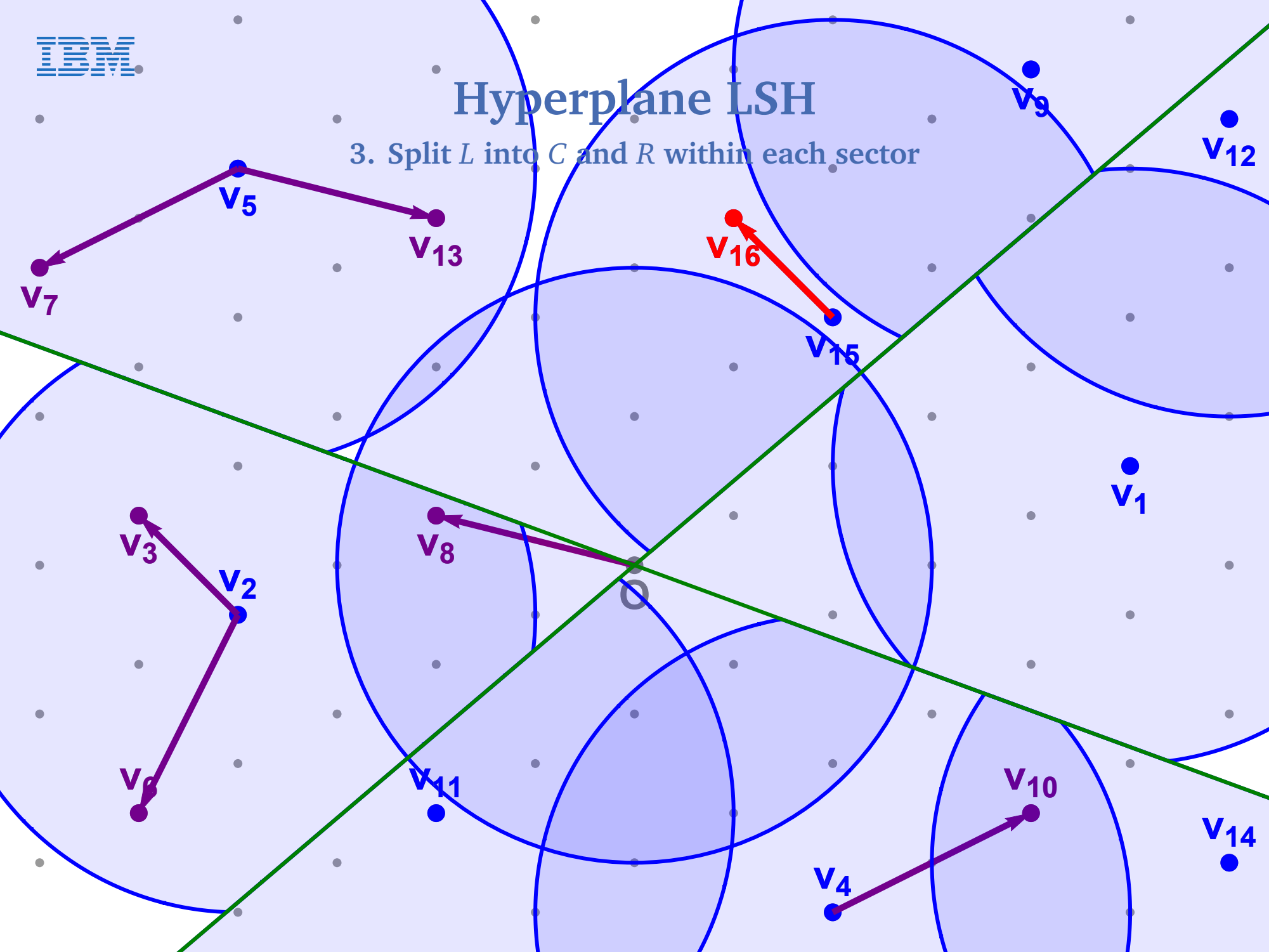

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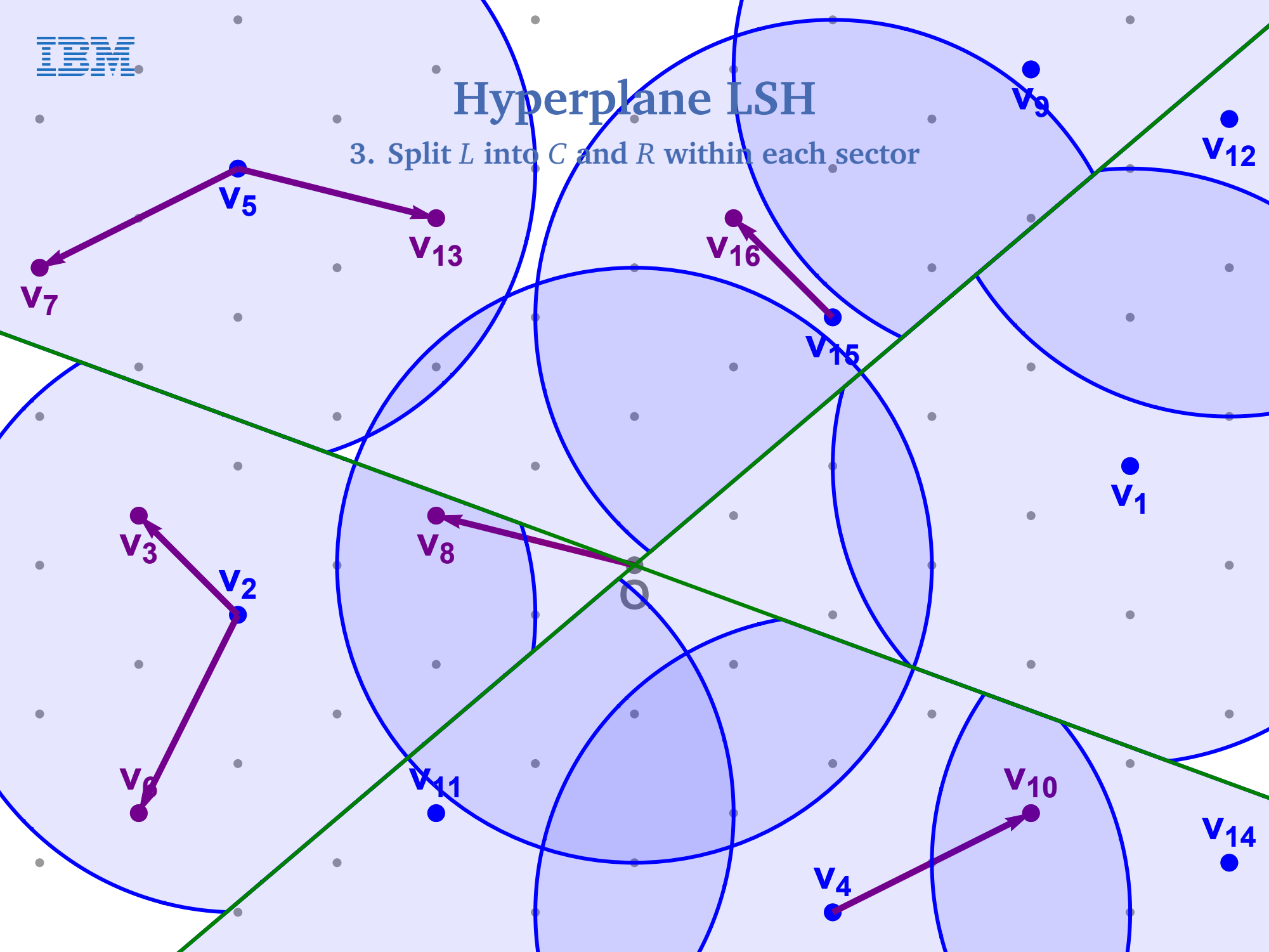
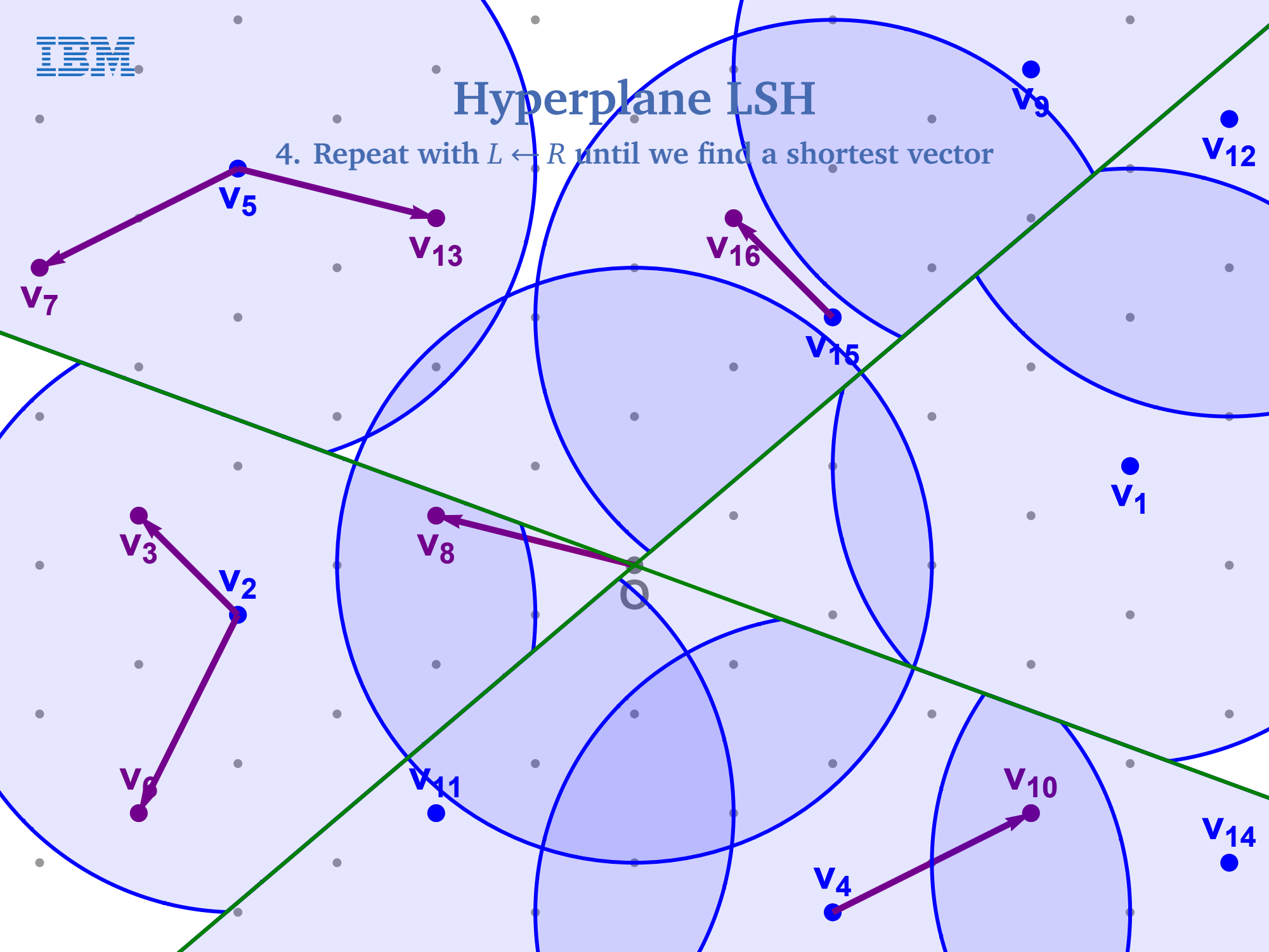


Hyperplane LSH

4. Repeat with $L \leftarrow R$ until we find a shortest vector



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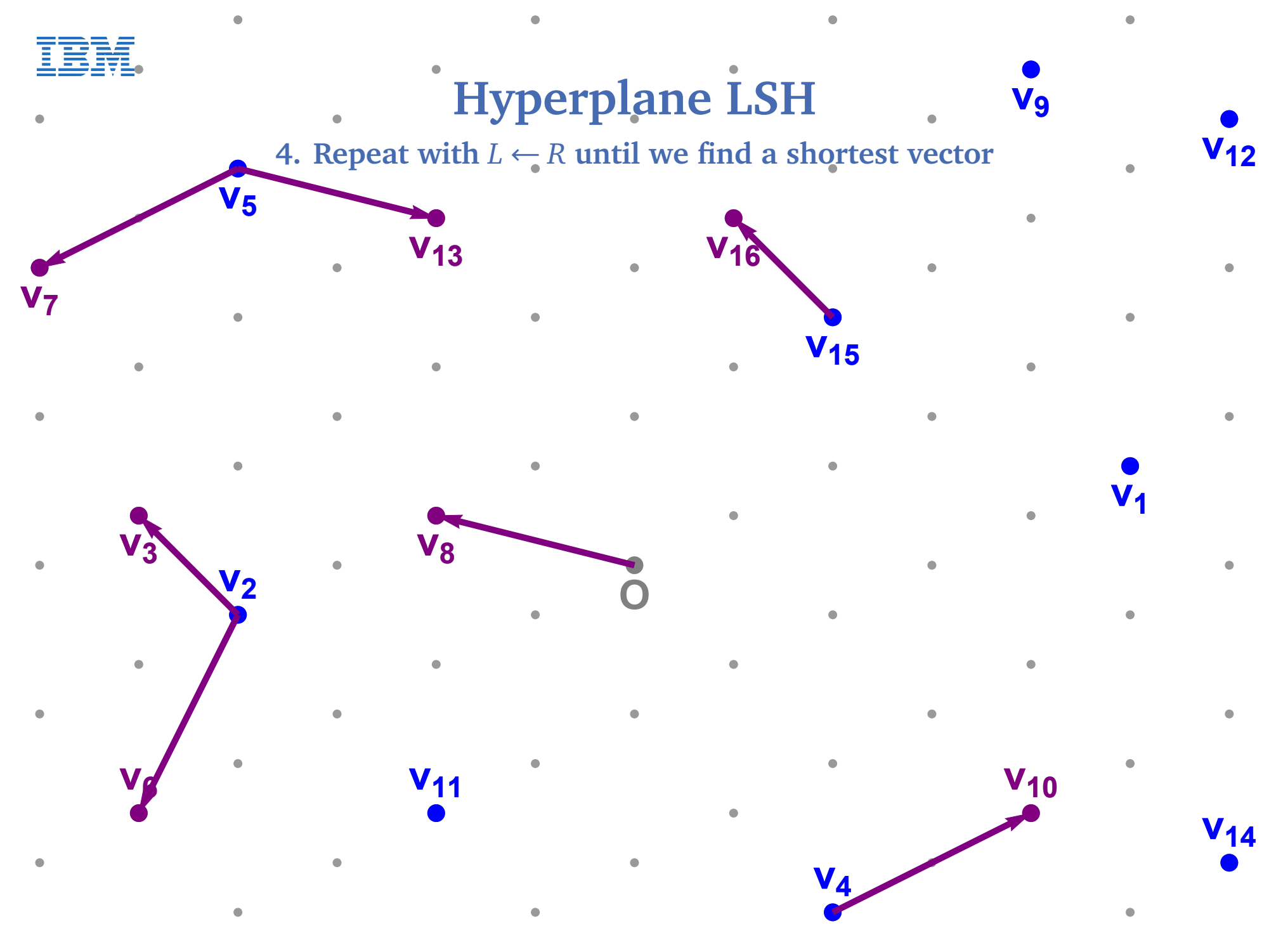

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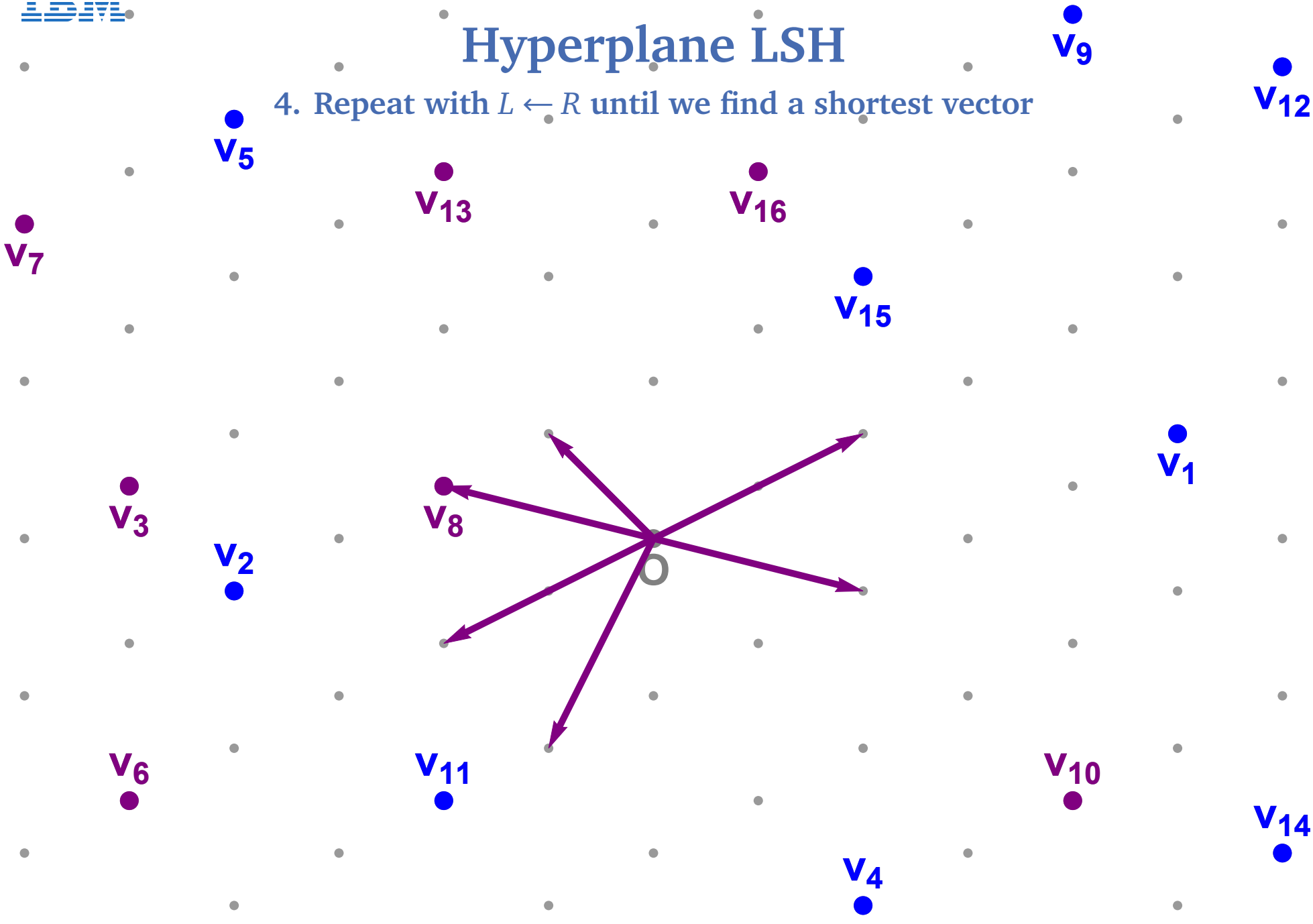

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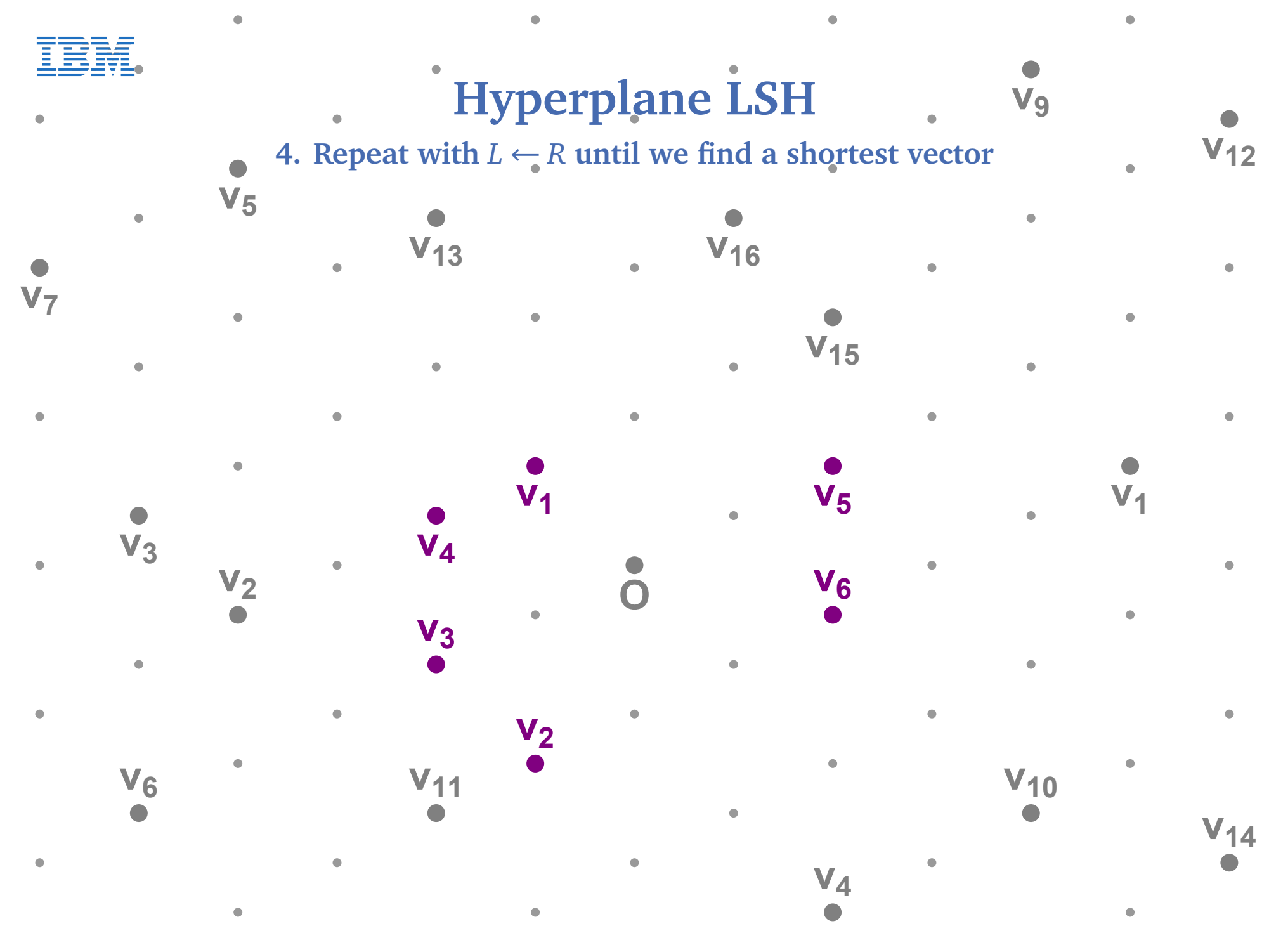
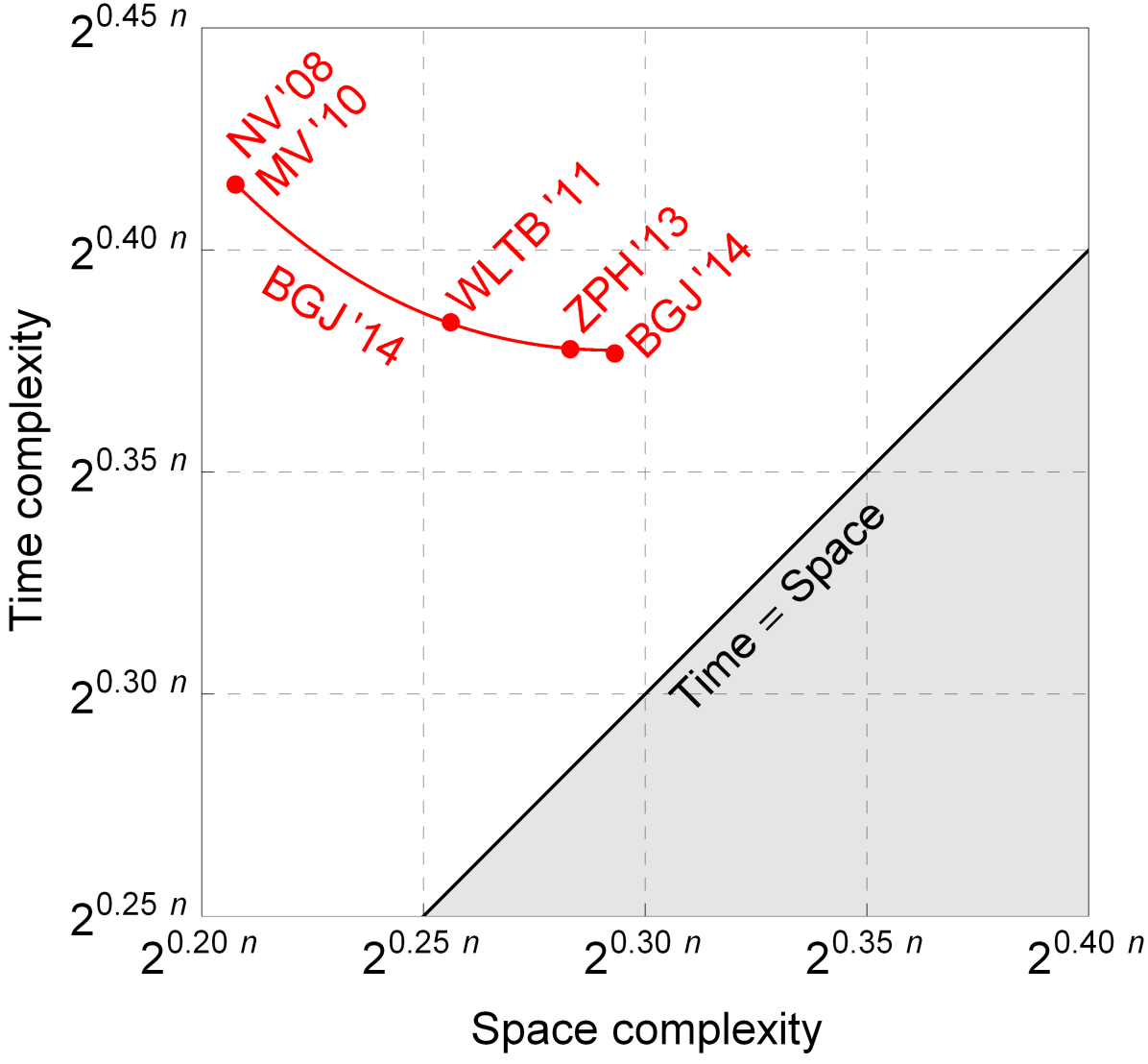


Hyperplane LSH

Space/time trade-off



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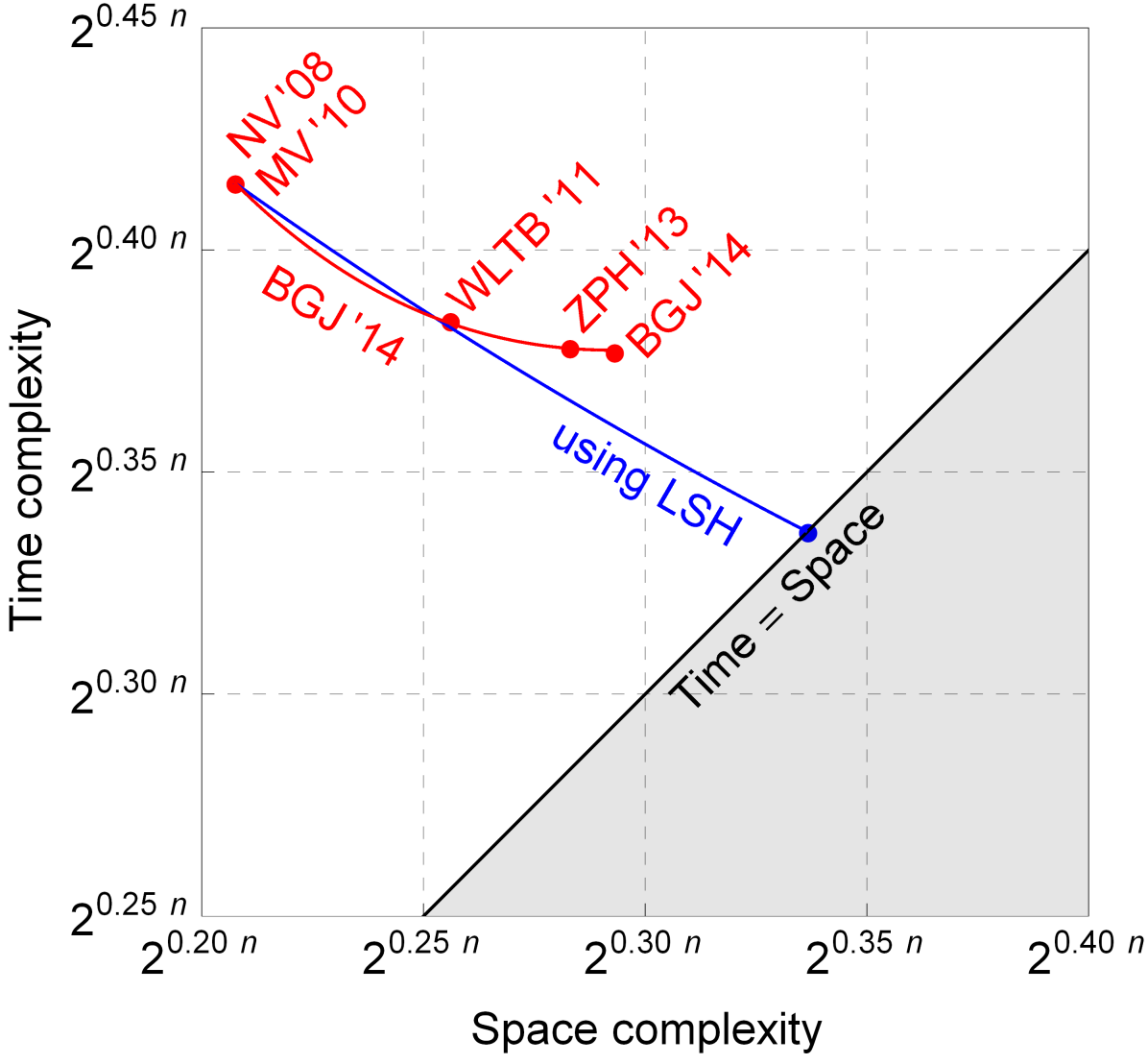

Spherical LSH

Sample a list L of random lattice vectors

Spherical LSH

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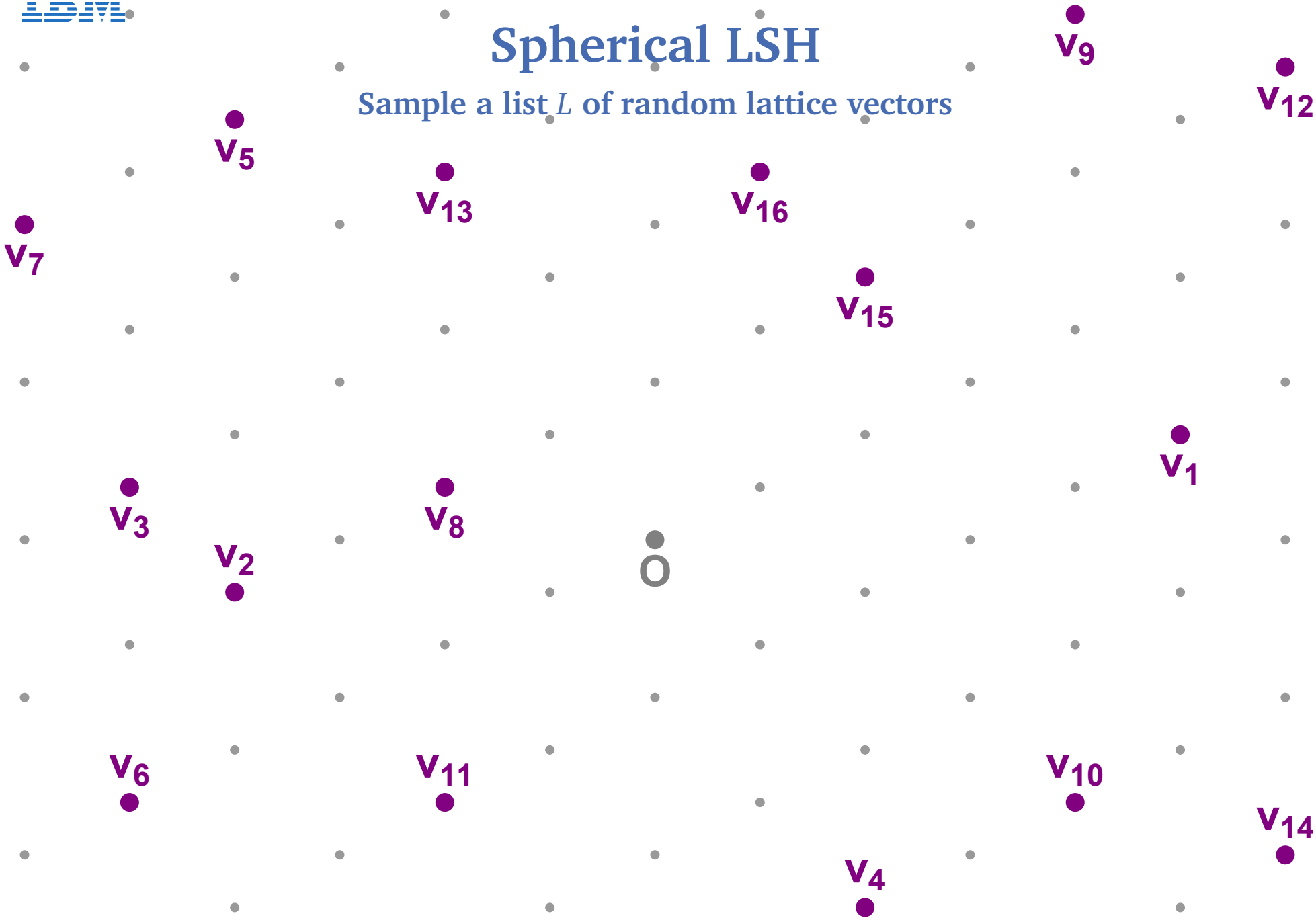
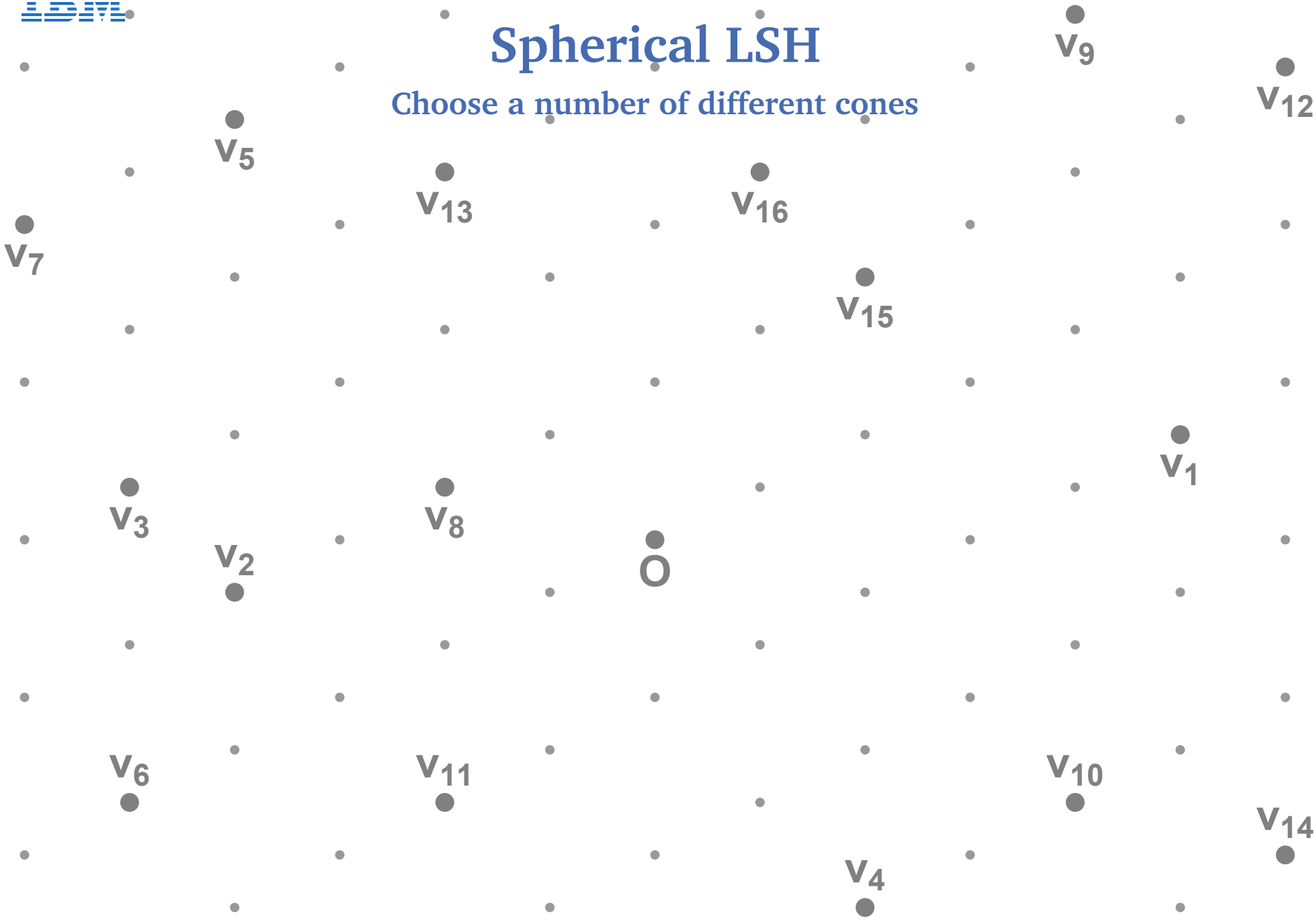


Spherical LSH

Choose a number of different cones



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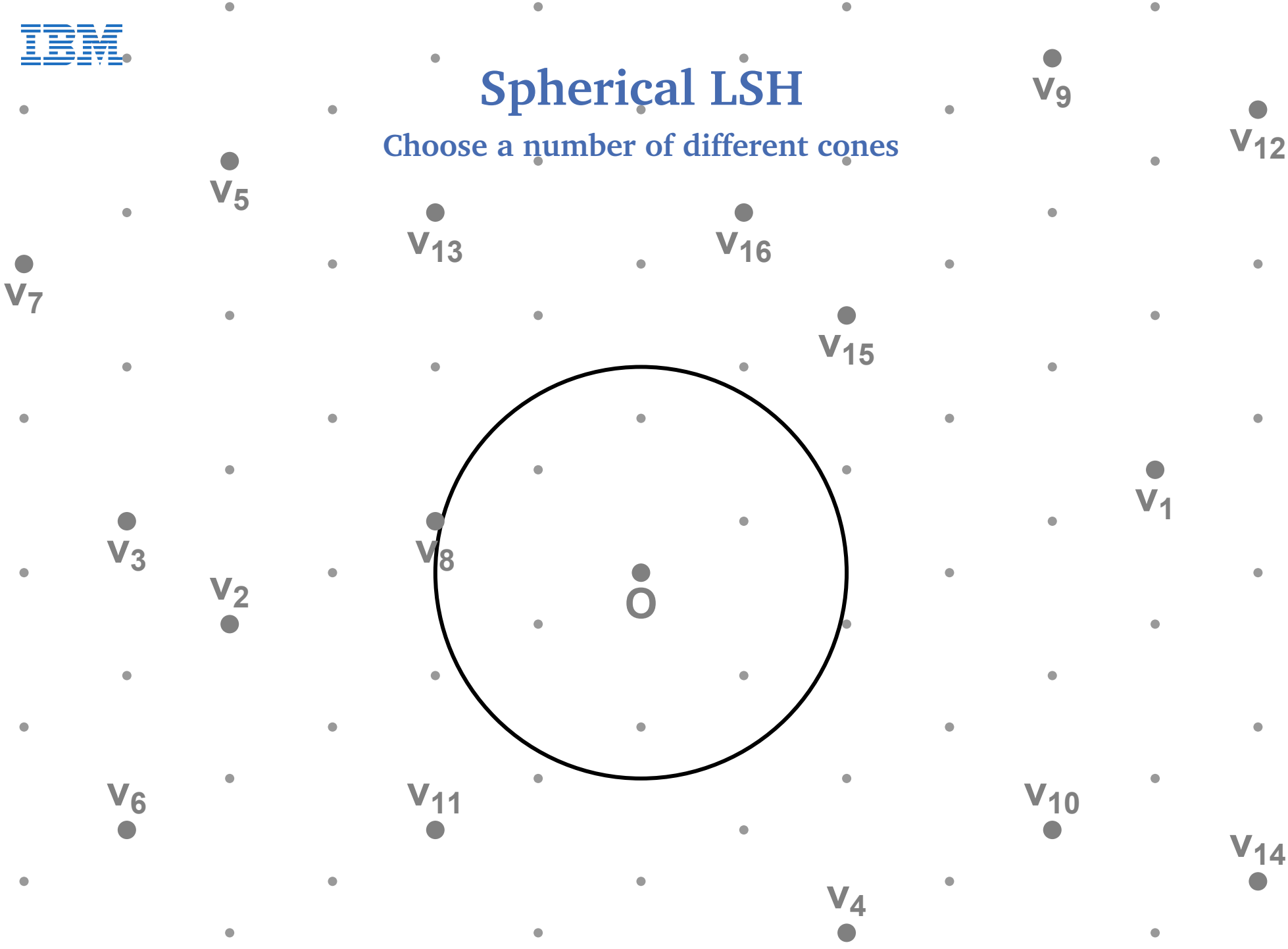

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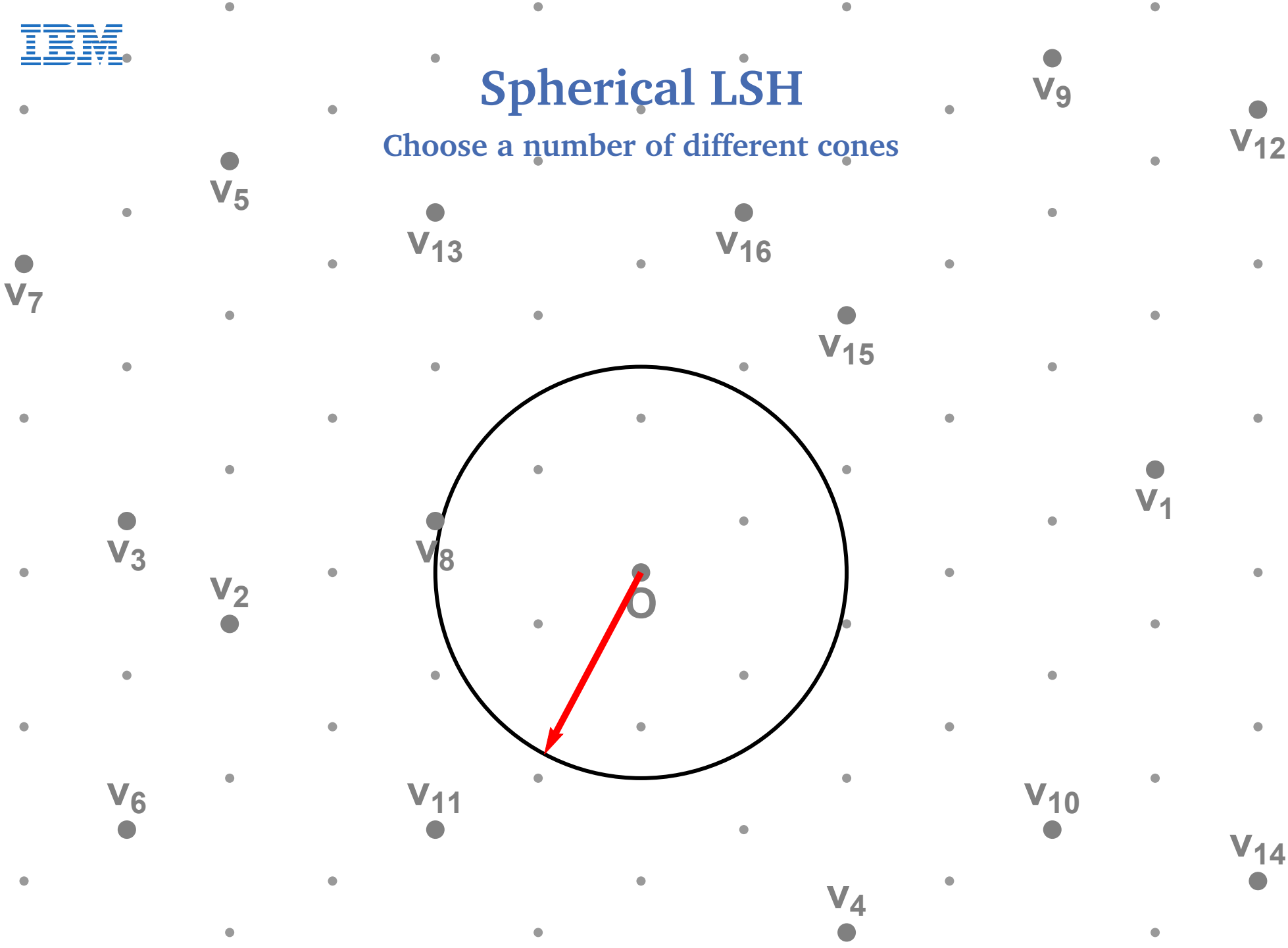

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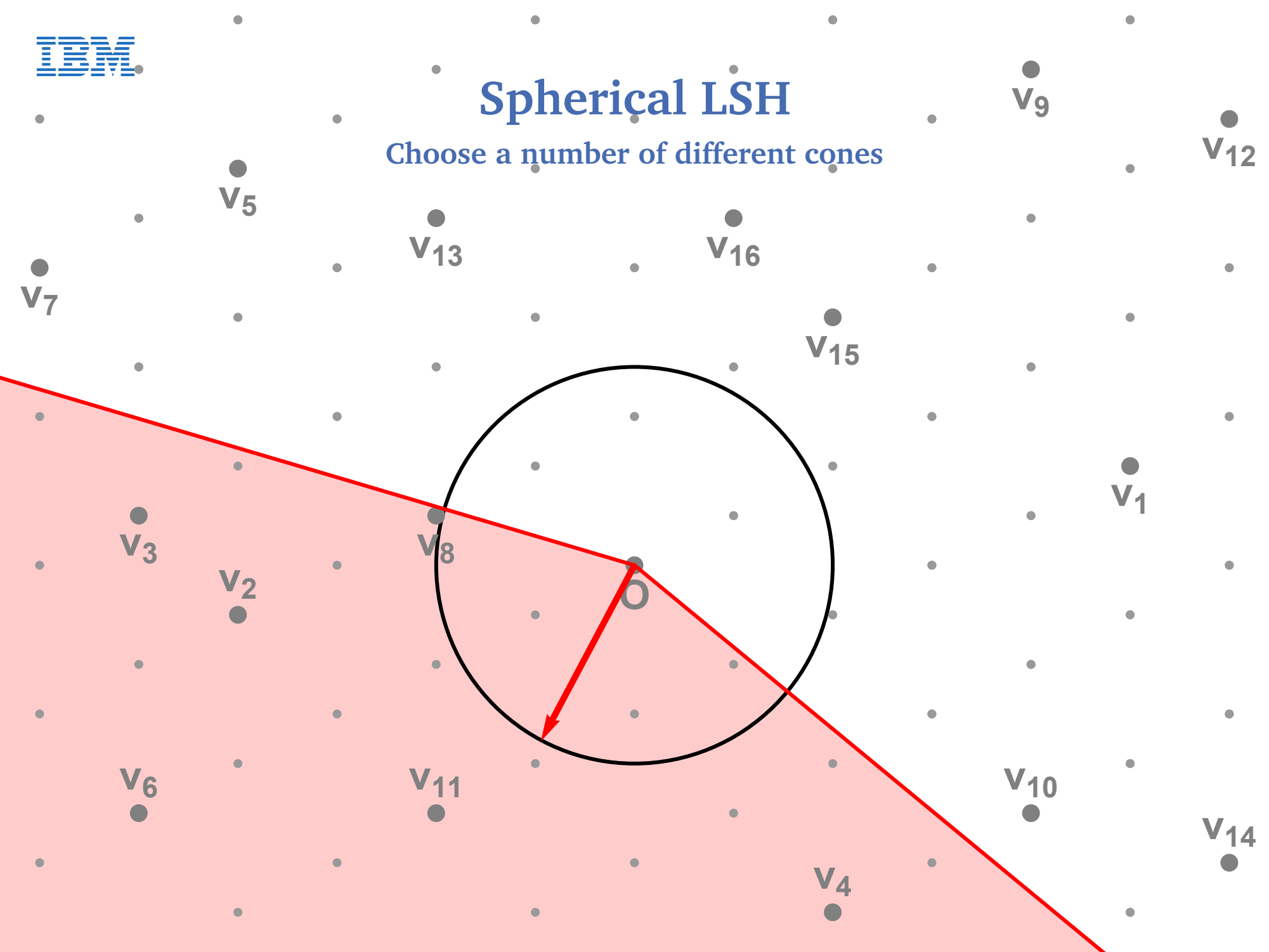

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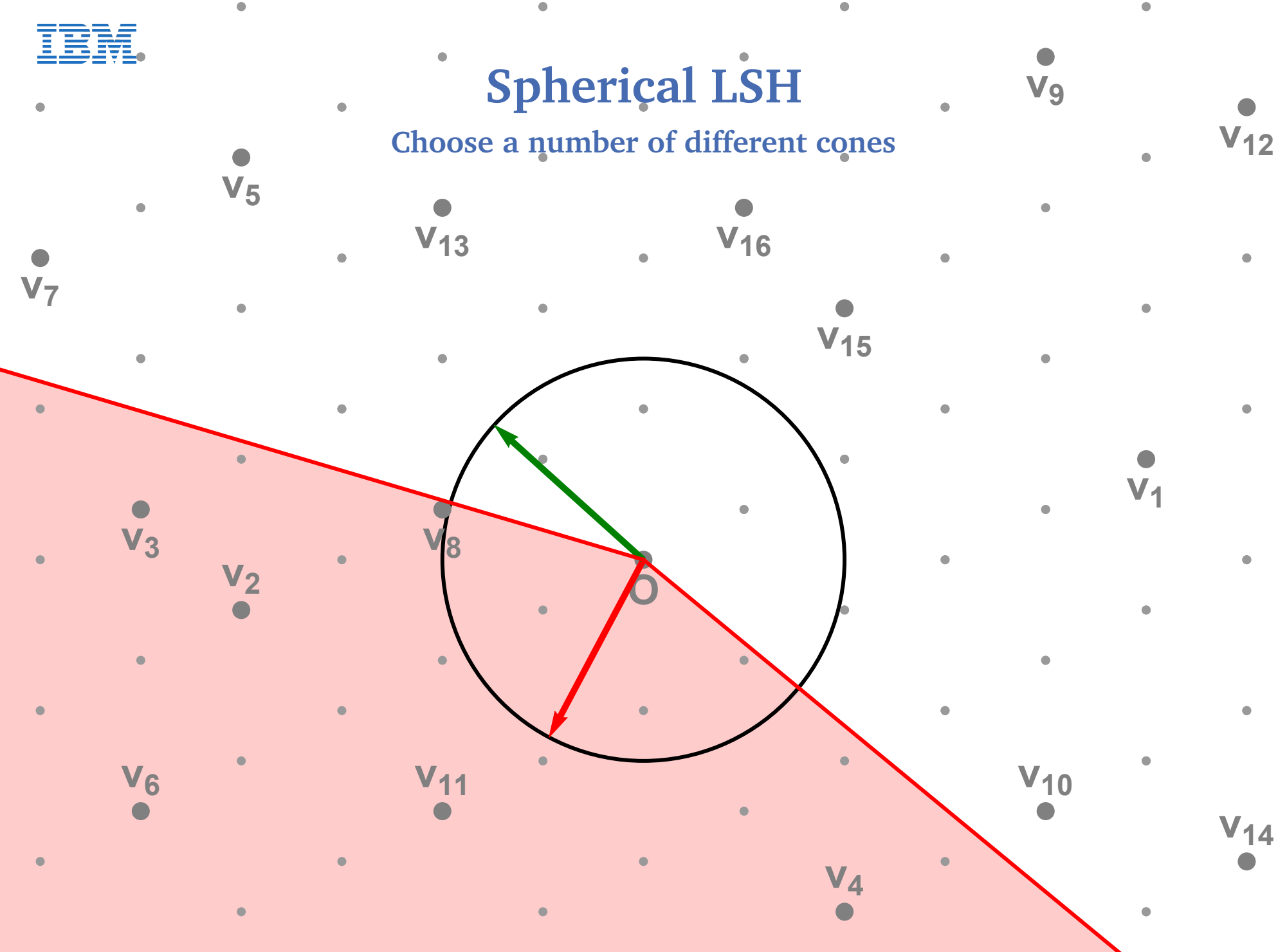

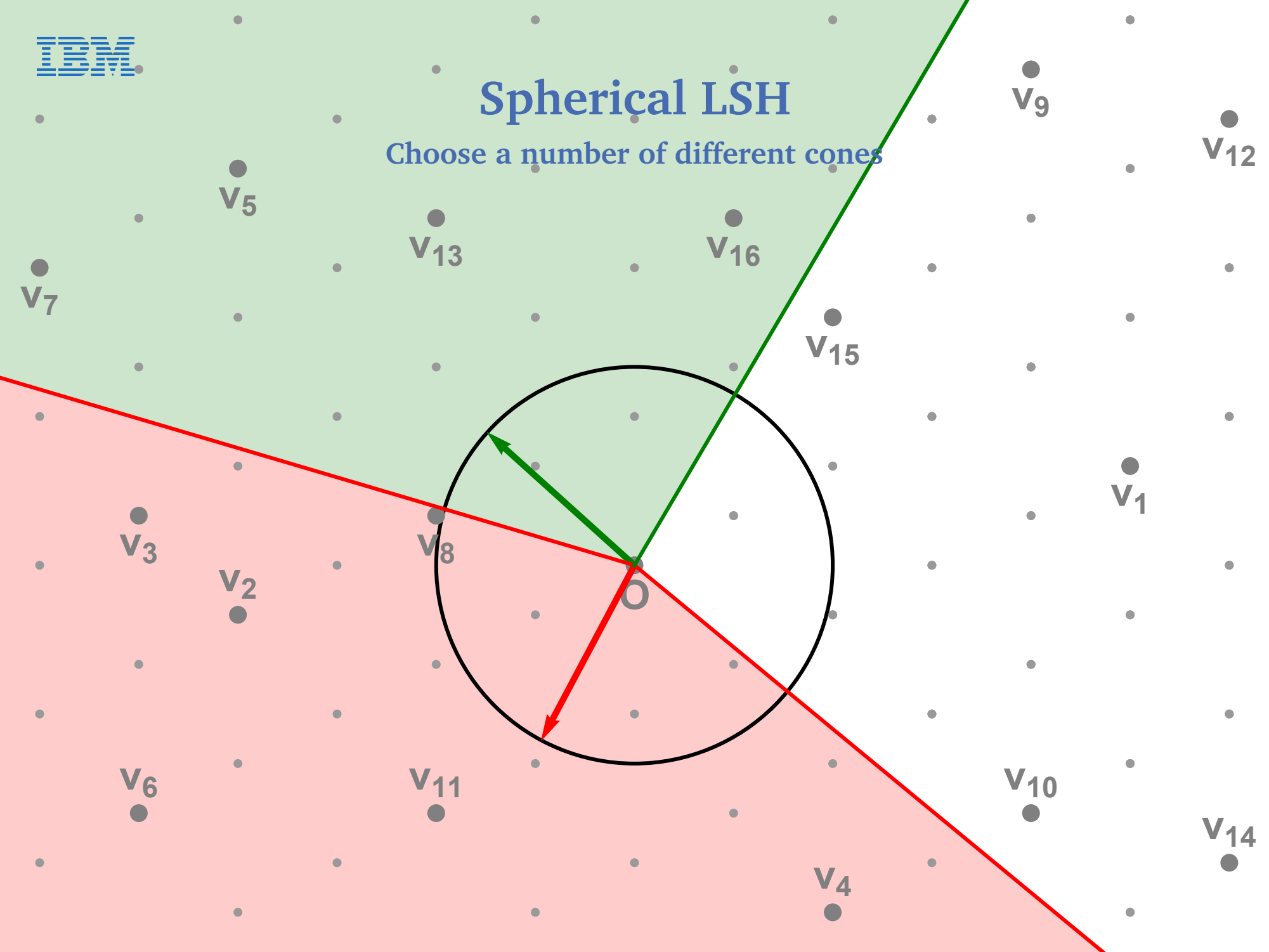
IBM
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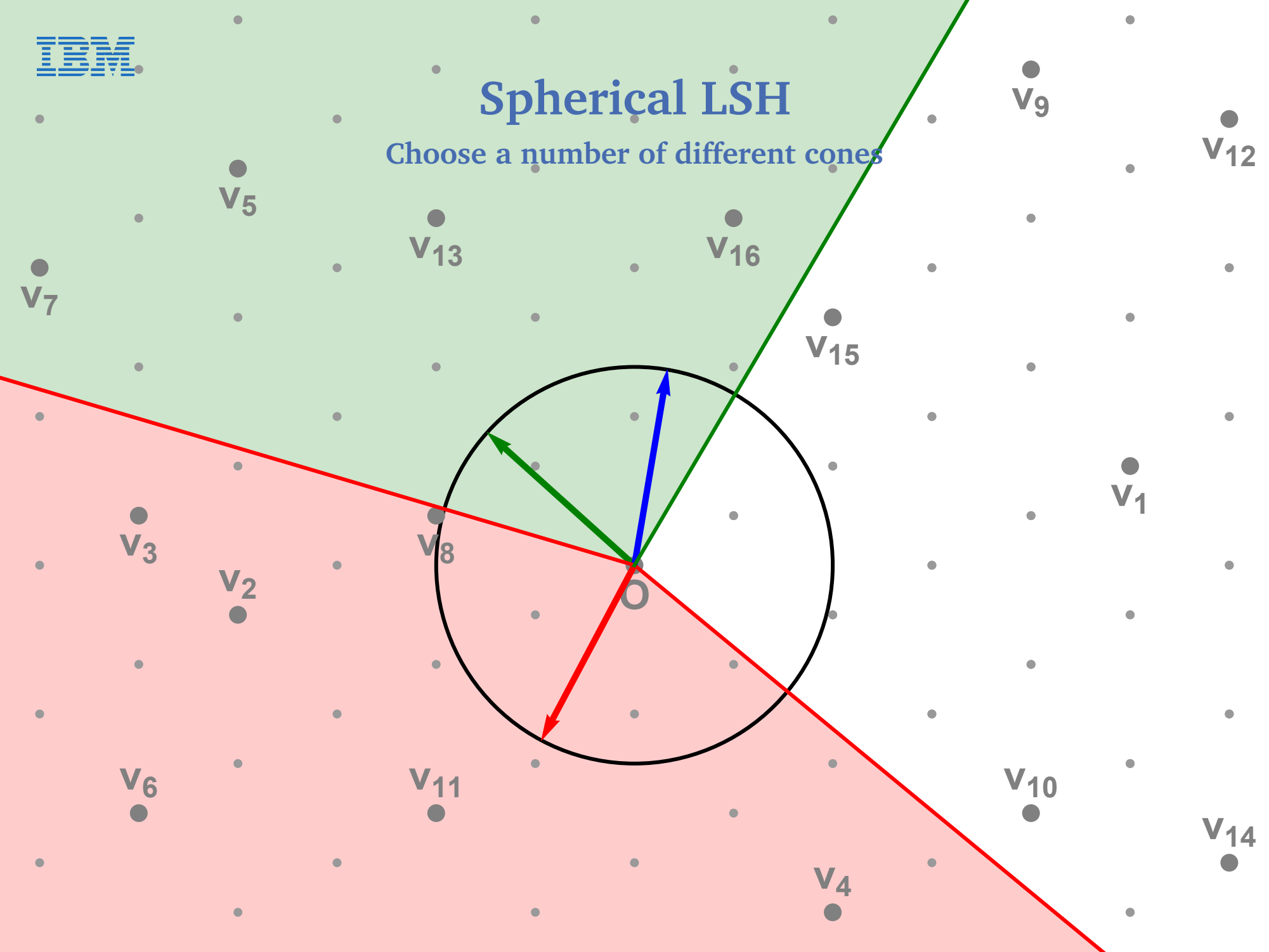
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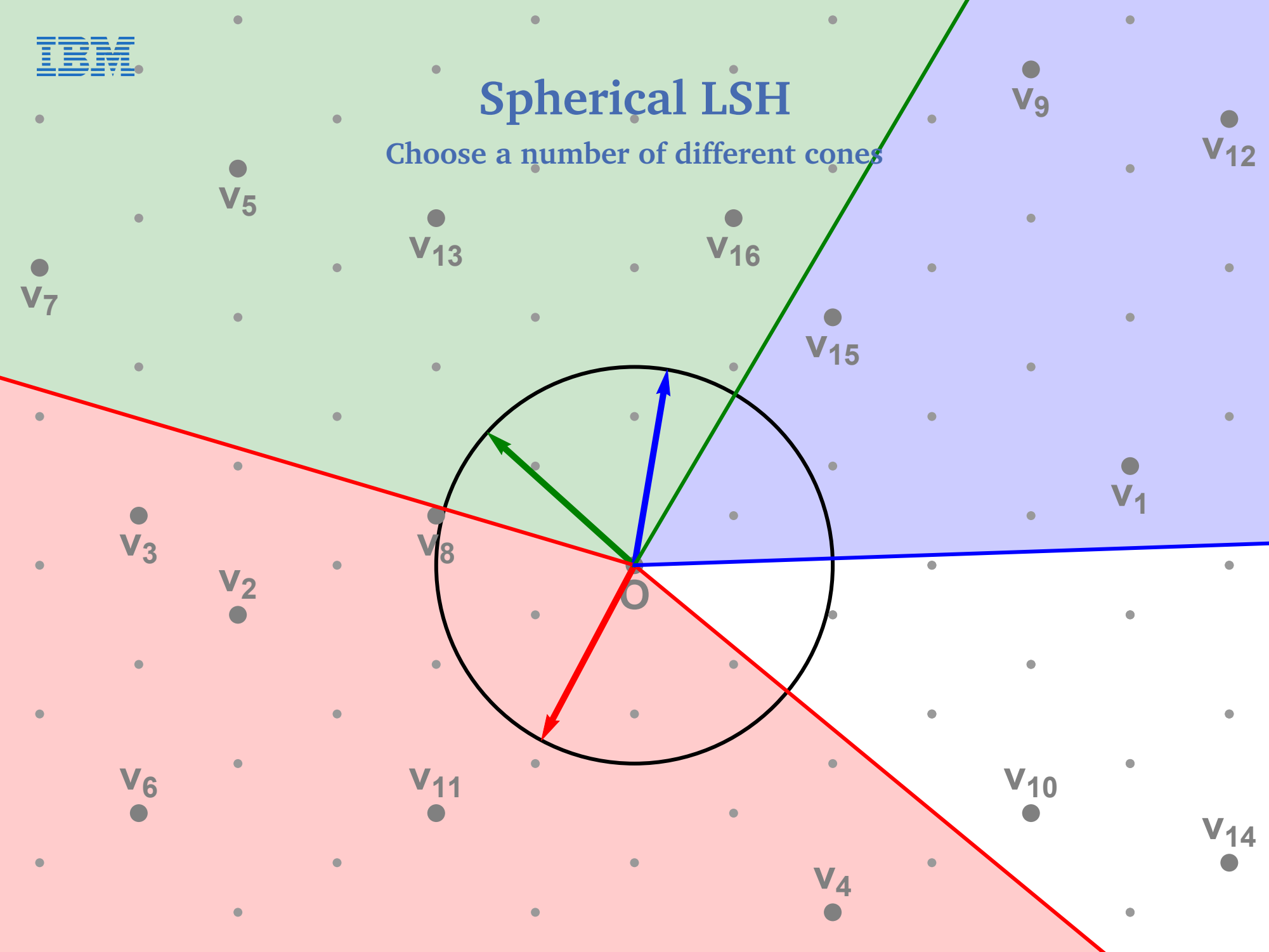
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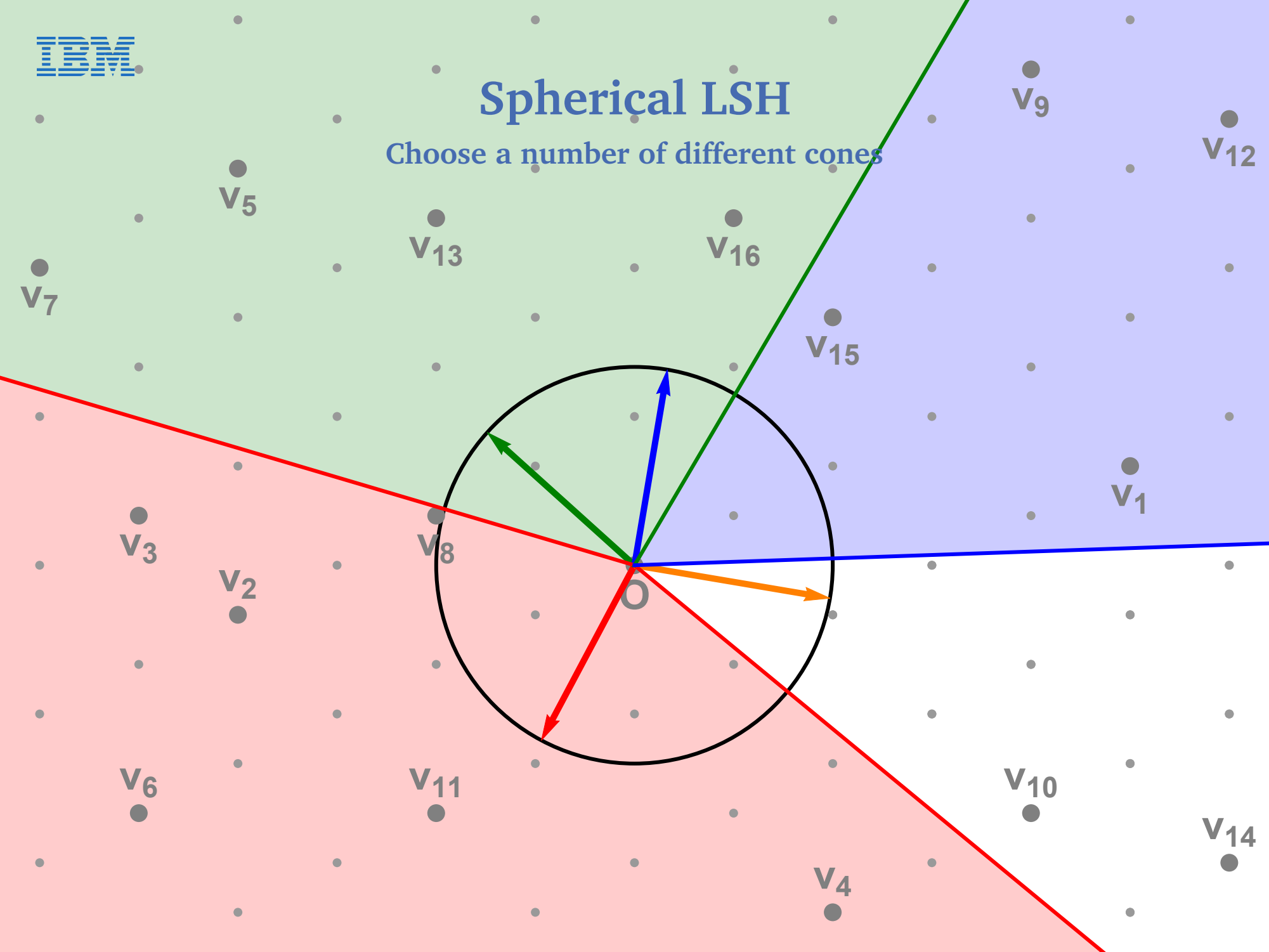
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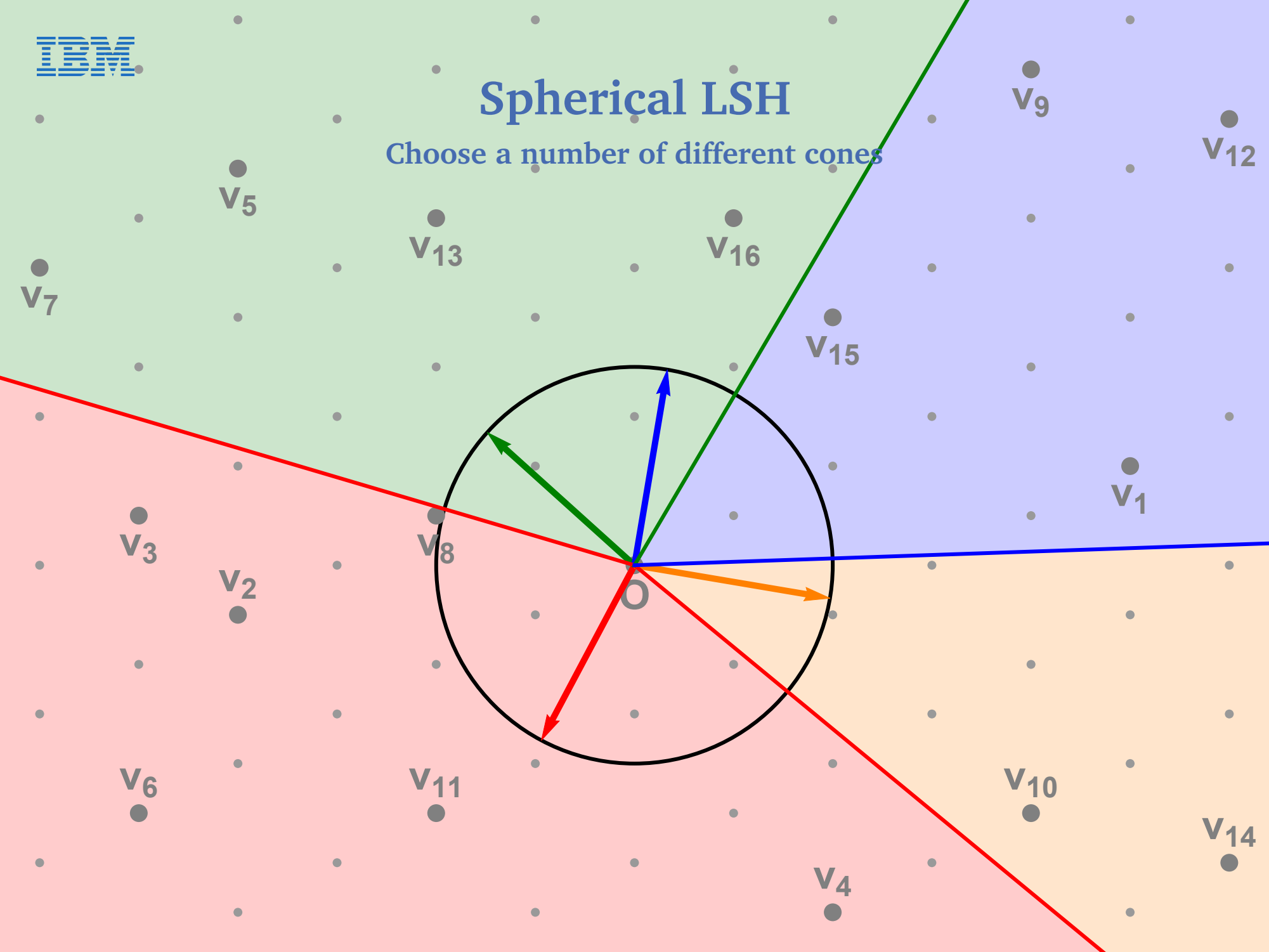
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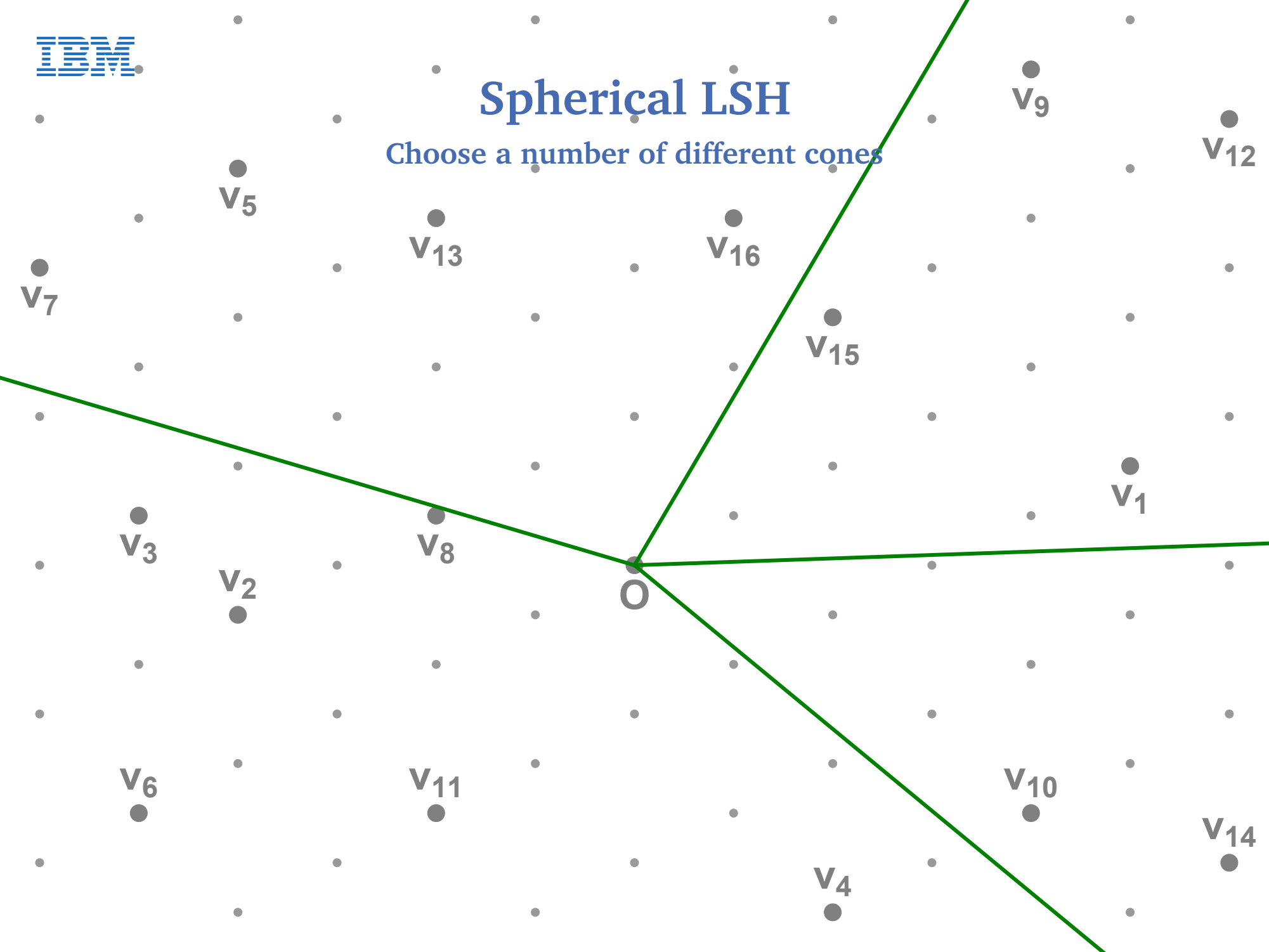
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Cross-Polytope LSH

Sample a list L of random lattice vectors

Cross-Polytope LSH

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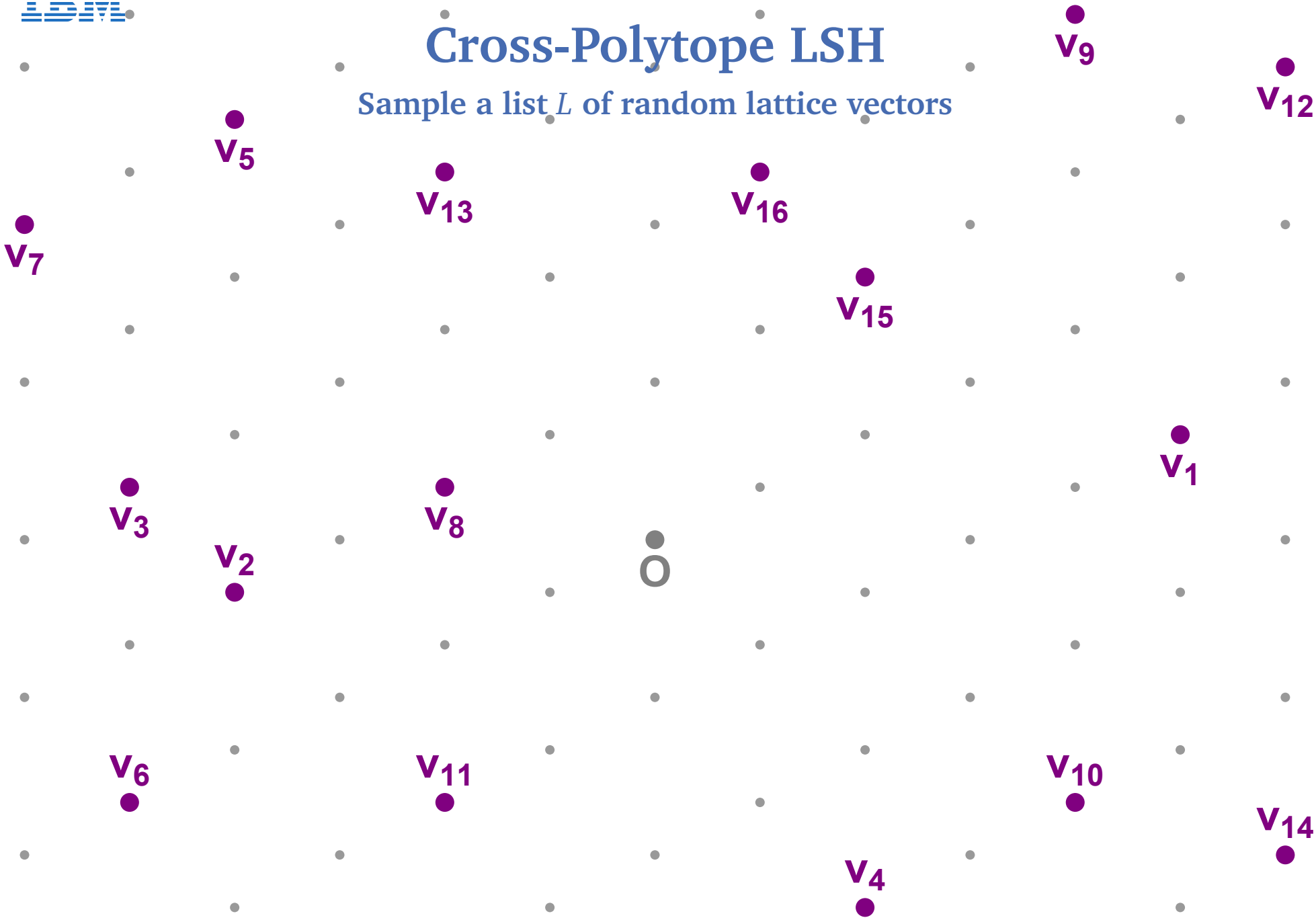
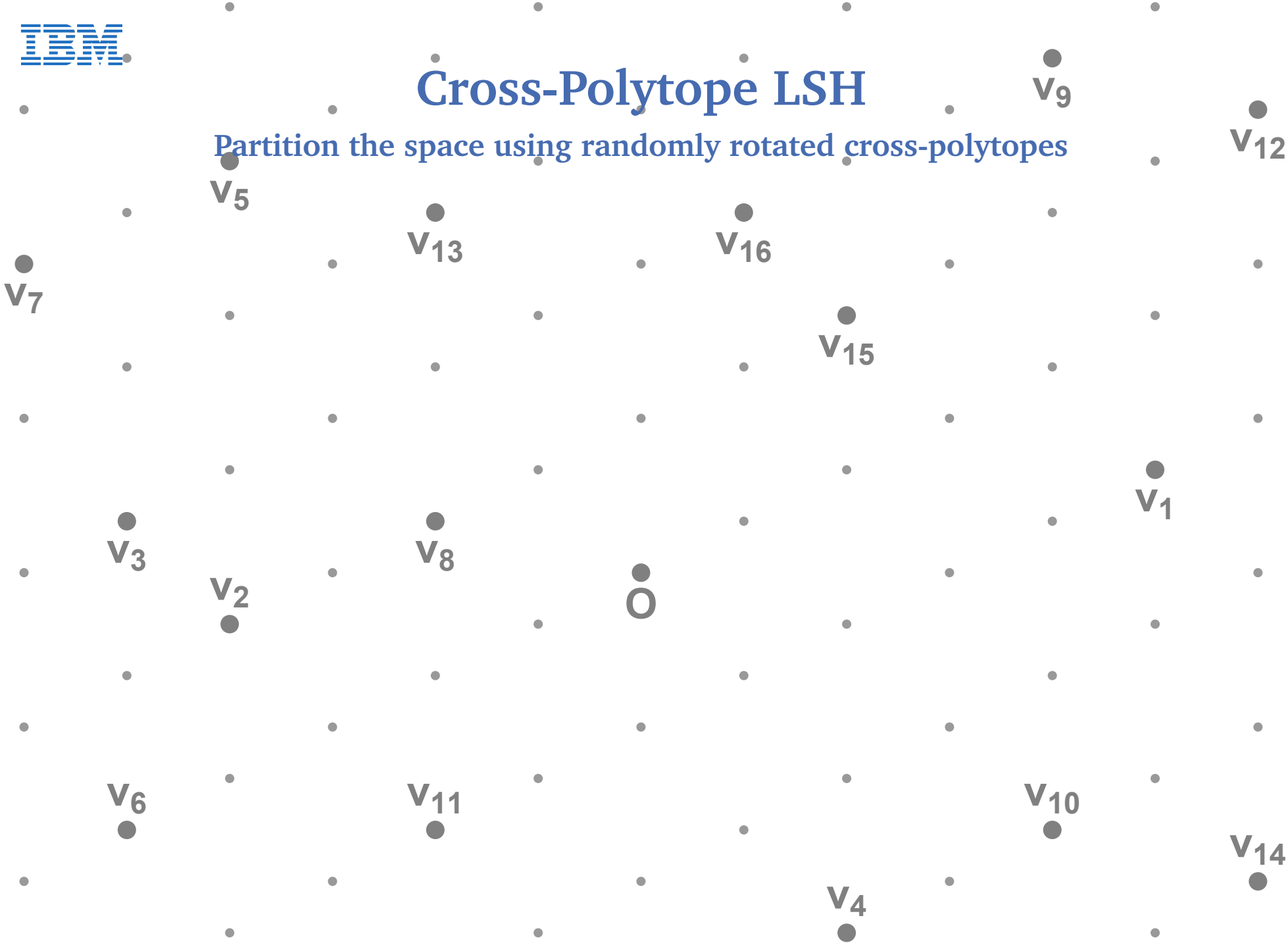


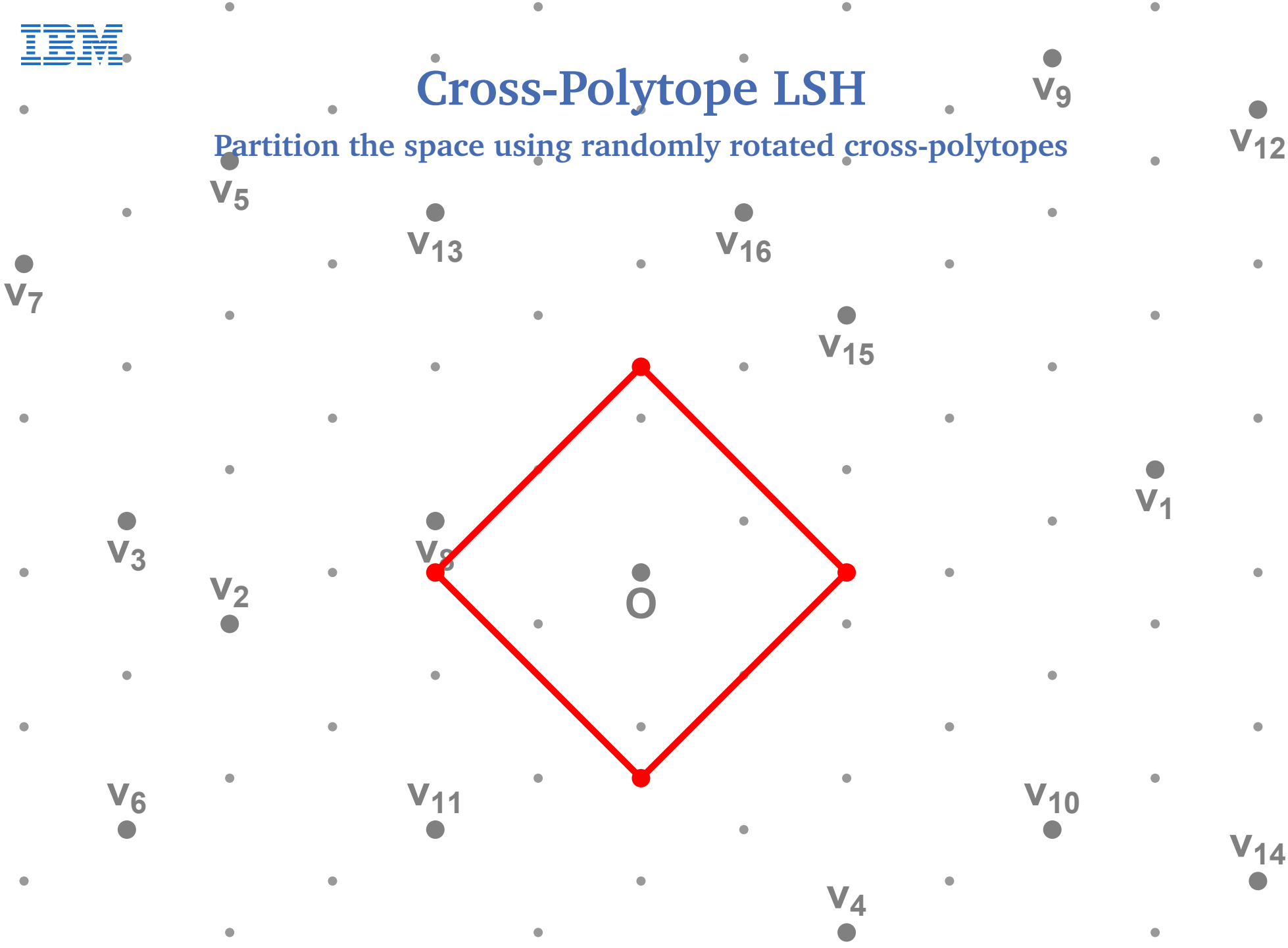
Cross-Polytope LSH

Partition the space using randomly rotated cross-polytopes



Cross-Polytope LSH

Partition the space using randomly rotated cross-polytopes



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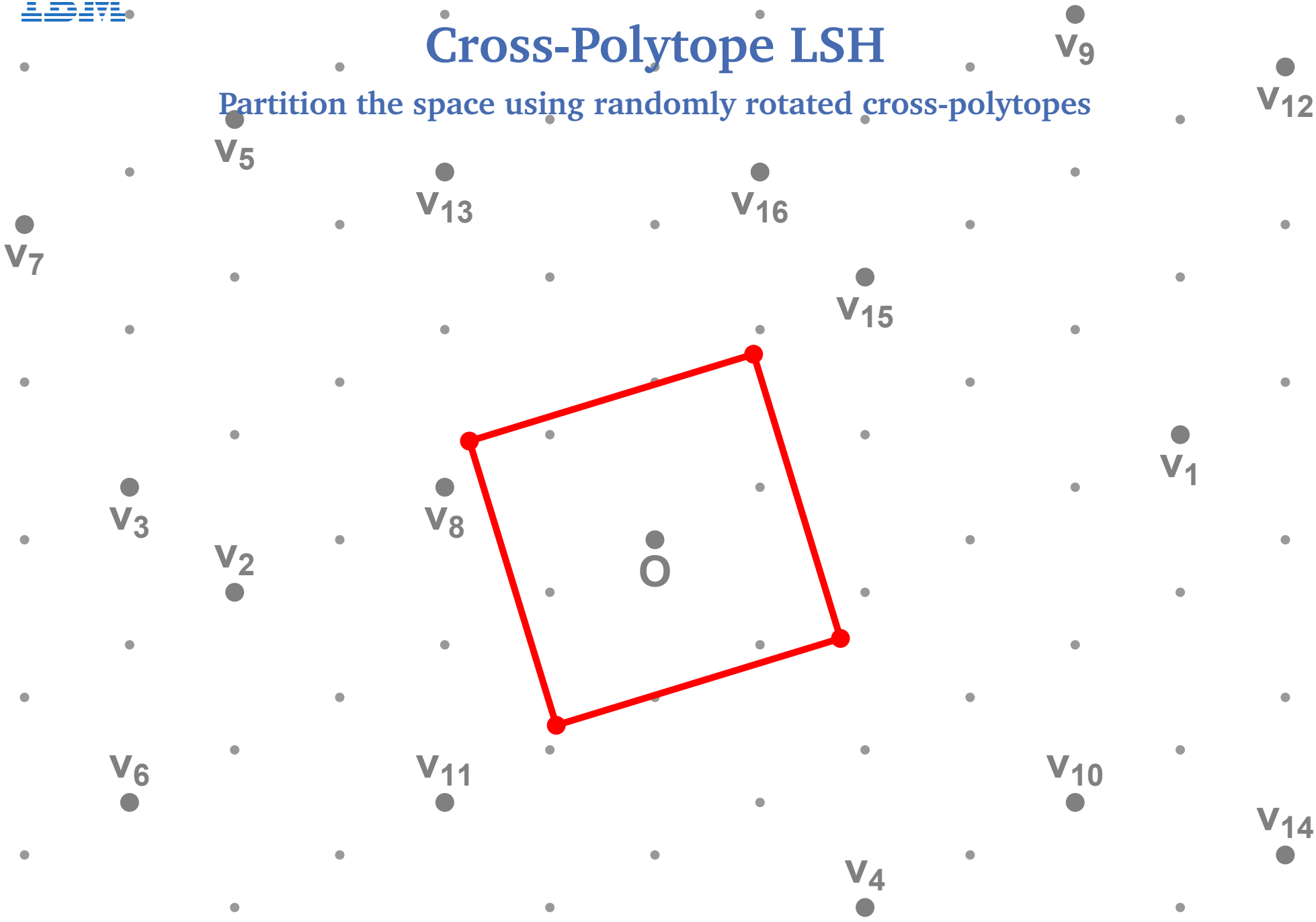

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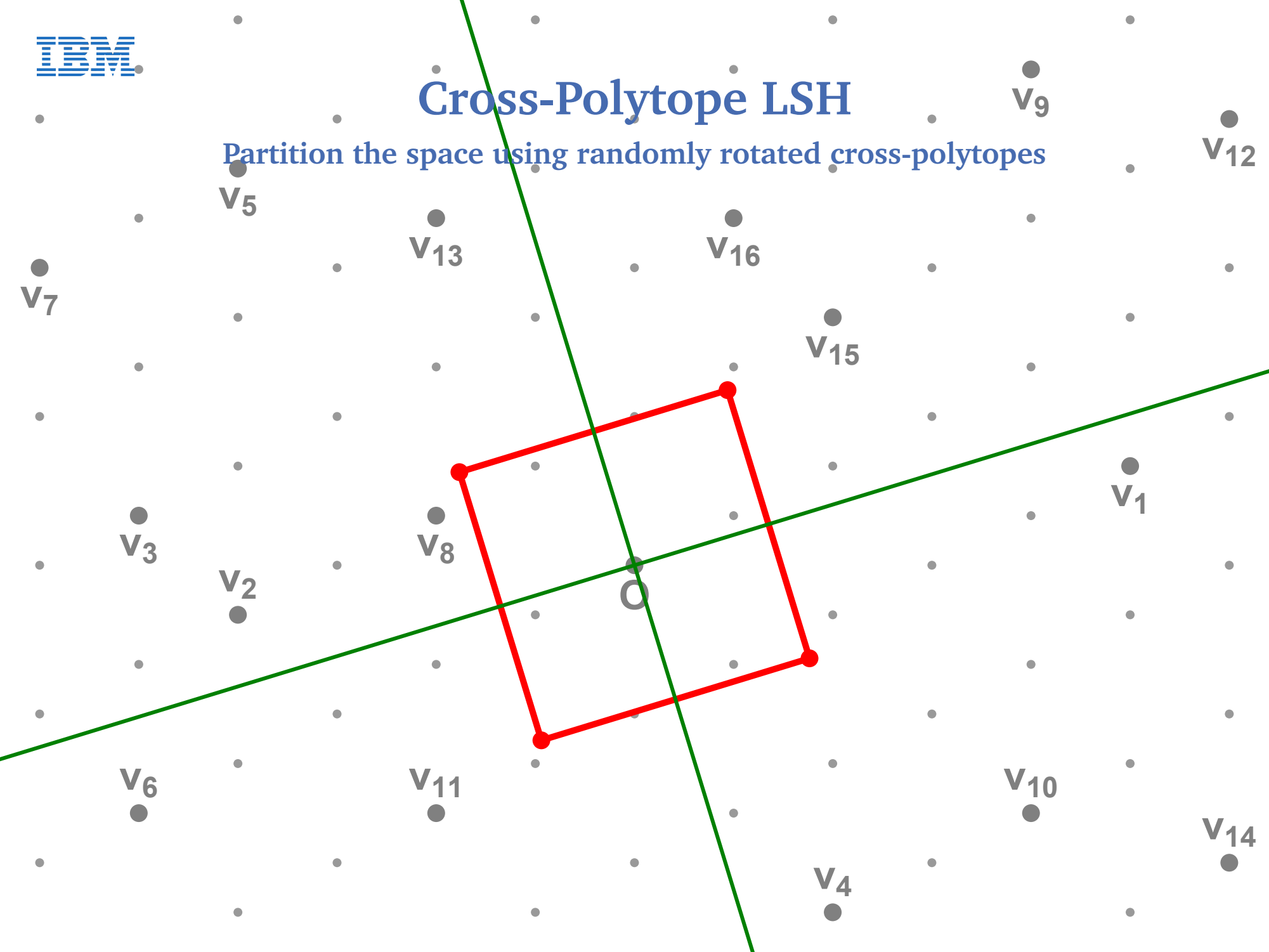

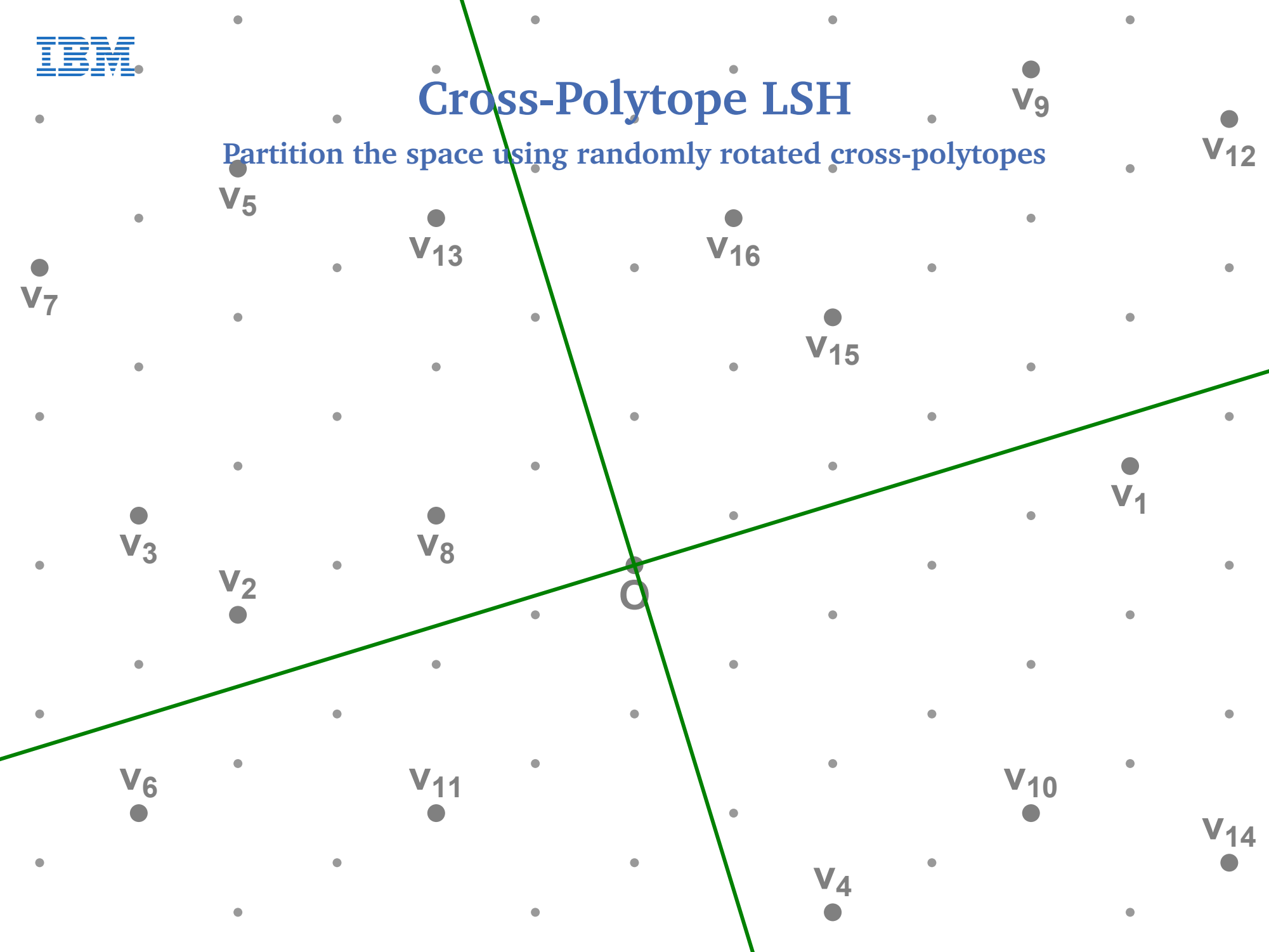
IBM
Cross-Polytope LSH
Partition the space using randomly rotated cross-polytopes
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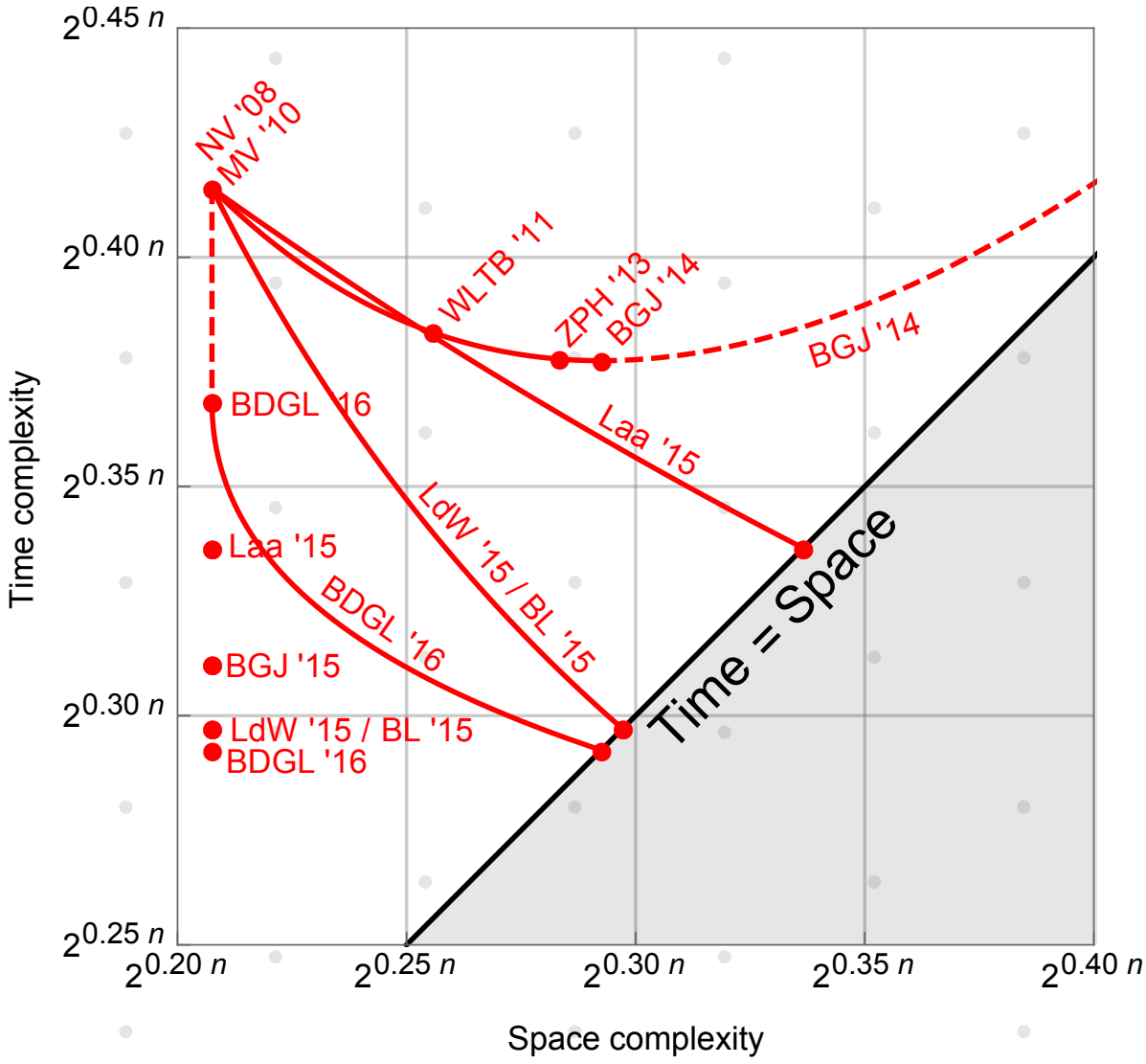
Sieving algorithms

Overview



Outline

Enumeration algorithms
- Fincke-Pohst enumeration
- Kannan enumeration
- Pruning the enumeration tree

Sieving algorithms
- Nguyen-Vidick sieve
- Multiple levels
- Near neighbor techniques

Practical comparison

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SVP in practice

"We expect our [enumeration] algorithm to be more efficient than lattice sieving up to dimension $n = 1895$."

– Micciancio–Walter, *SODA'15*

SVP in practice

"We expect our [enumeration] algorithm to be more efficient than lattice sieving up to dimension $n = 1895$."

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"As far as I know, everyone who has tried sieving as a BKZ subroutine in place of enumeration has concluded that sieving is much too slow to be useful—the cutoff is beyond cryptographically relevant sizes."

– Bernstein, *Google groups '16*

SVP in practice

"We expect our [enumeration] algorithm to be more efficient than lattice sieving up to dimension $n = 1895$."

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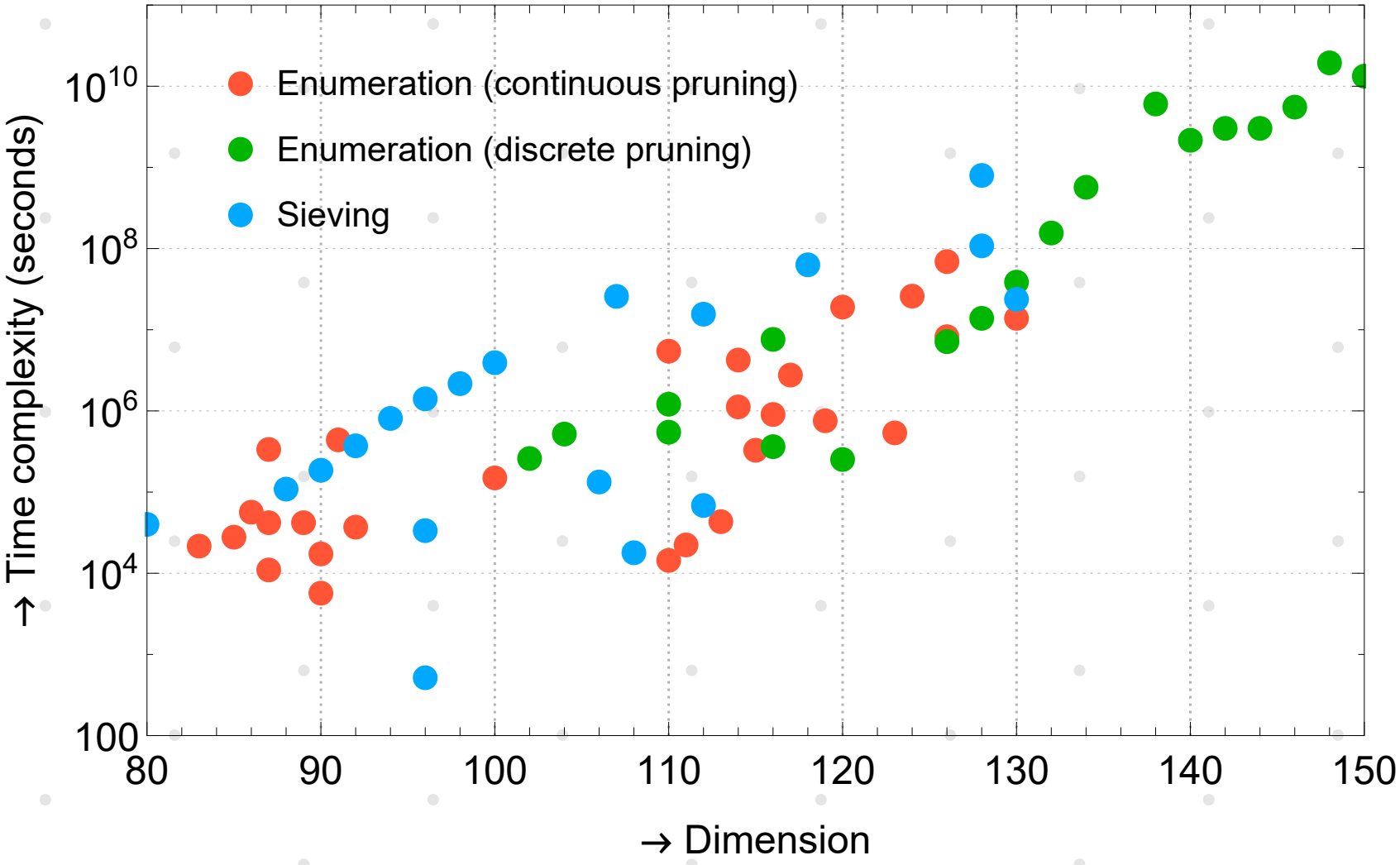
"As far as I know, everyone who has tried sieving as a BKZ subroutine in place of enumeration has concluded that sieving is much too slow to be useful—the cutoff is beyond cryptographically relevant sizes."

– Bernstein, *Google groups '16*

"I compute a cross-over point between enumeration and the HashSieve at dimension $b = 217$."

– Ducas, *Google groups '16*

SVP in practice



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Summary

- Lattice-based crypto relies on hardness of finding short bases
- State-of-the-art basis reduction: BKZ with fast SVP subroutine
- Enumeration for SVP:
  - Memory-efficient
  - Best in low dimensions
  - Fast pruning heuristics
- Sieving for SVP:
  - Large memory requirement
  - Fastest in high dimensions
  - Practical near neighbor speedups
- Enumeration still leading, but sieving is catching up!

Questions?