IBM Research

Lattice-based cryptography (I)

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PQCrypto Summer School 2017
(June 20, 2017)

IBM Research

Part 1: Lattices, cryptography, and lattice basis reduction

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Lattices

What is a lattice?

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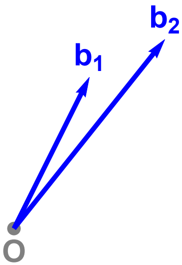

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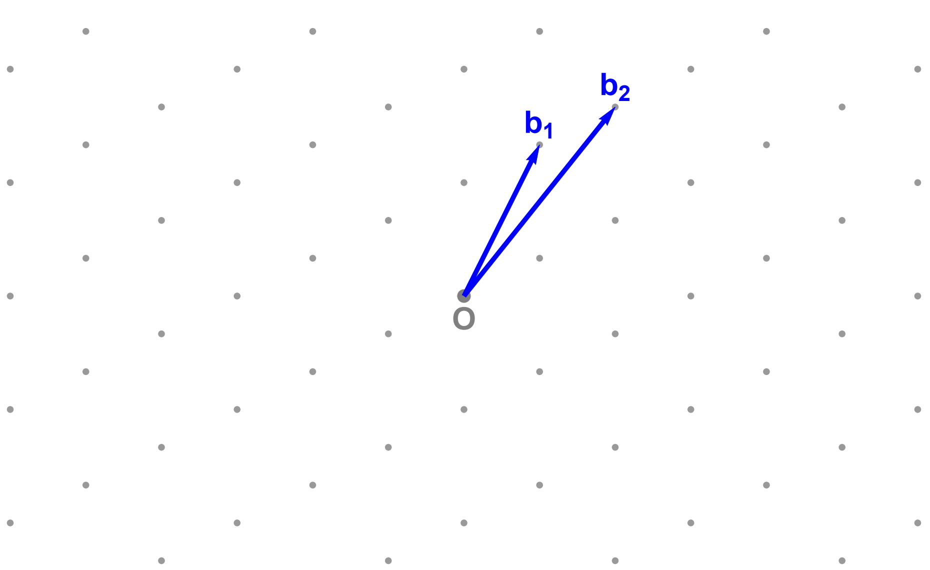
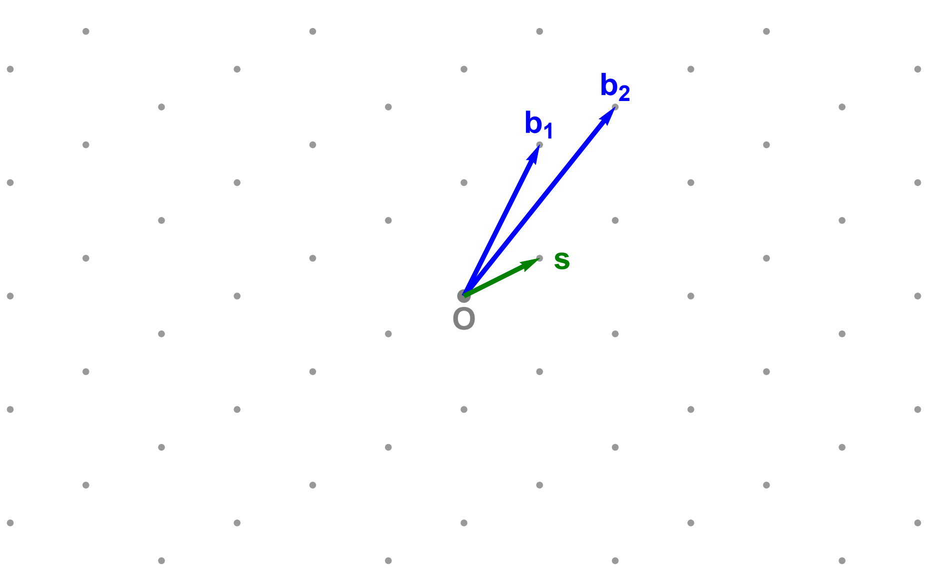


Lattices

Shortest Vector Problem (SVP)



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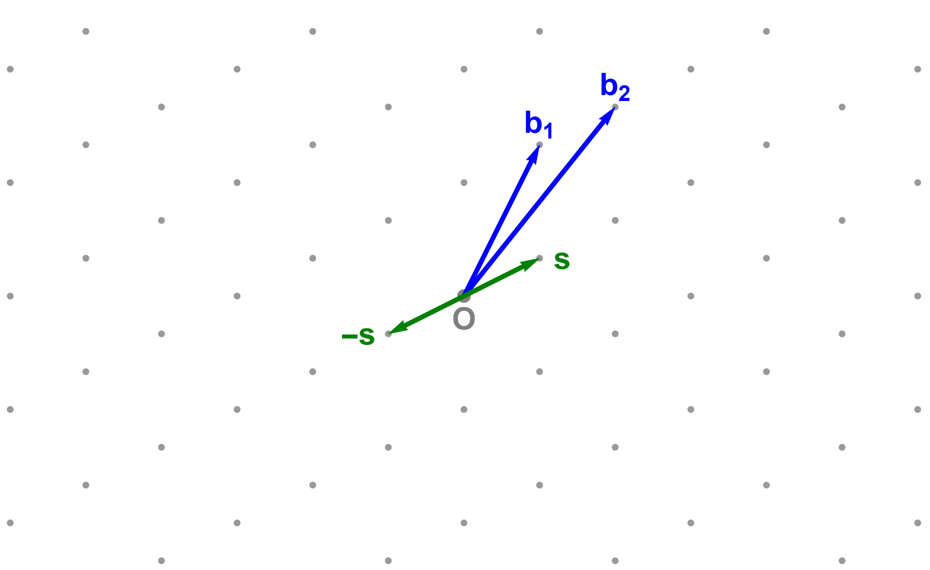

Lattices

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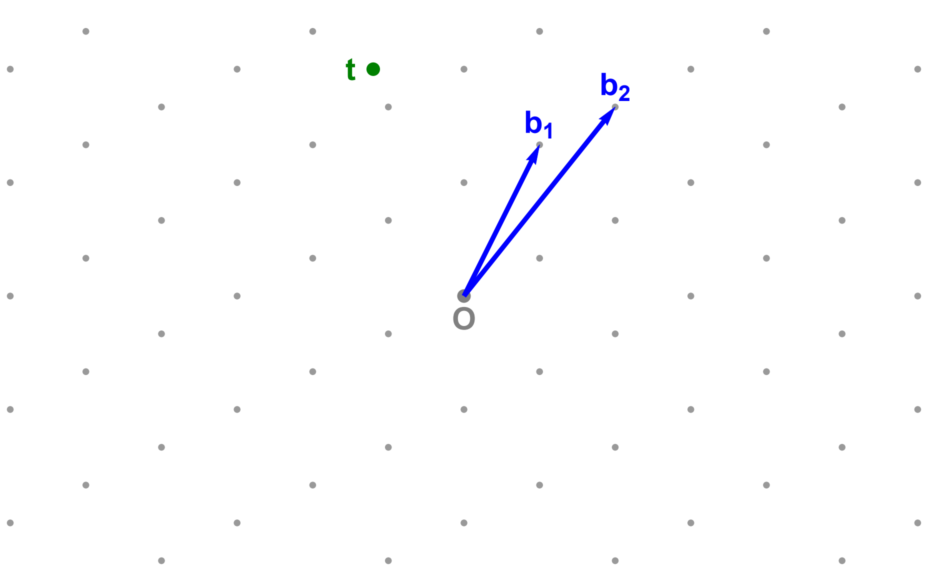

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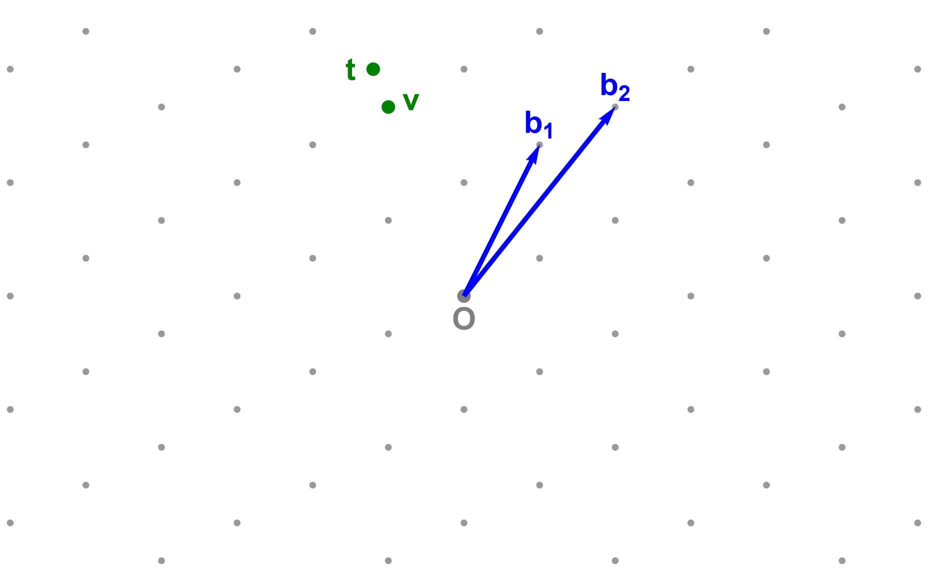
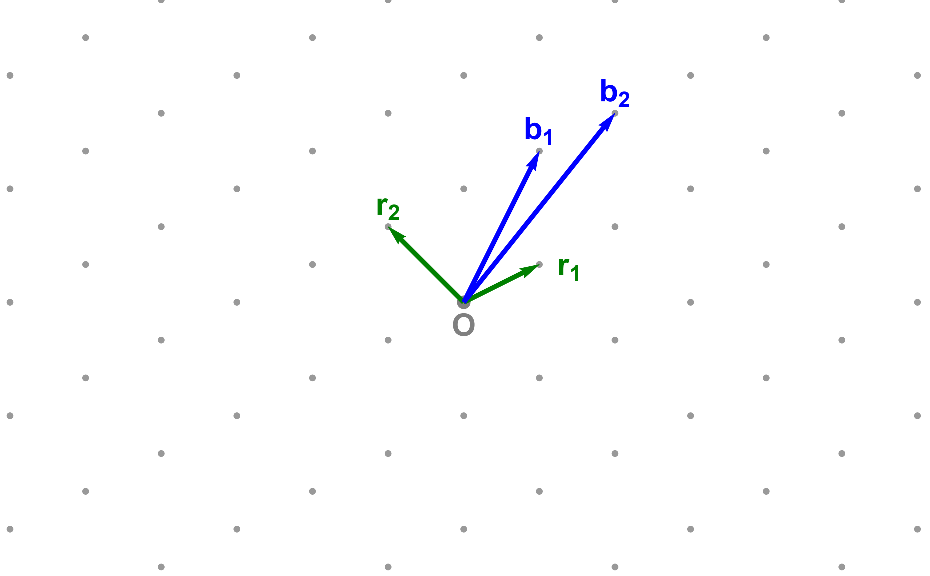


Lattices

Lattice basis reduction



Outline

Motivation: GGH encryption

Lattice basis reduction

- Gauss reduction
- LLL reduction
- BKZ reduction

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GGH cryptosystem

Overview

Private key: $R = \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \end{pmatrix}$

Public key: $B = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}$

Encrypt $\boldsymbol{m}$:

$$\boldsymbol{v} = \boldsymbol{m}B$$
$$\boldsymbol{c} = \boldsymbol{v} + \boldsymbol{e}$$

Decrypt $\boldsymbol{c}$:

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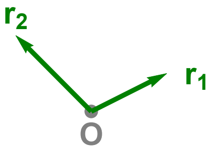

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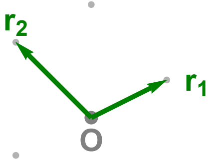

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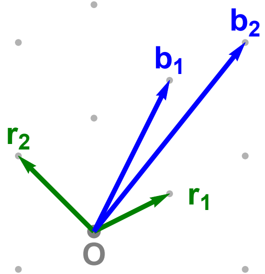

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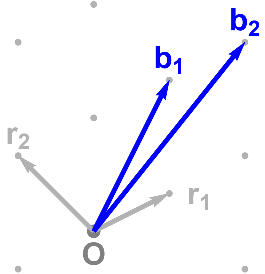

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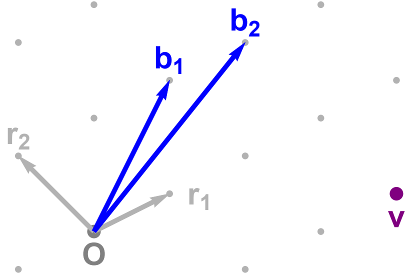

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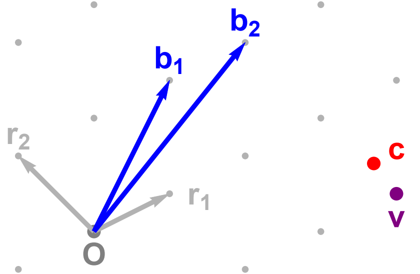

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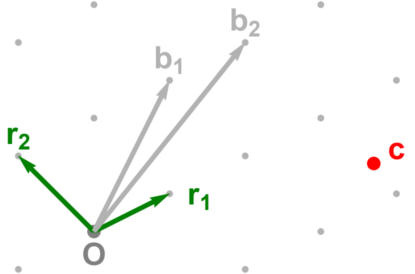

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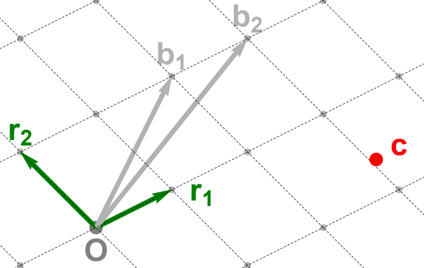

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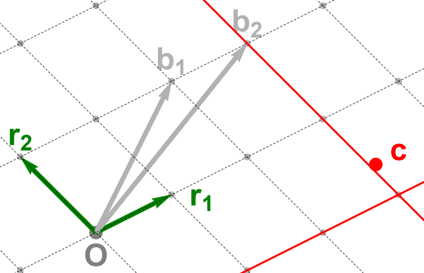

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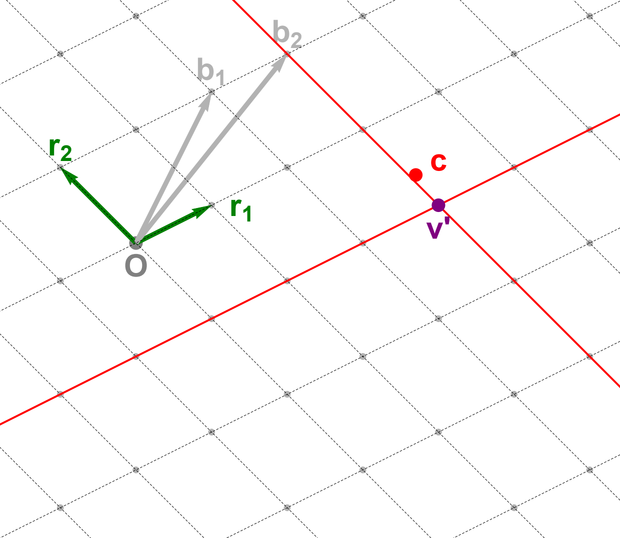

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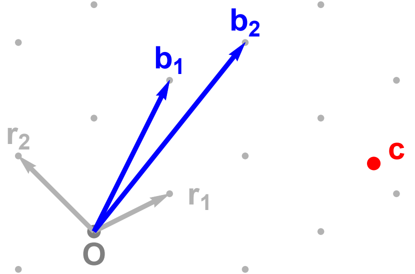

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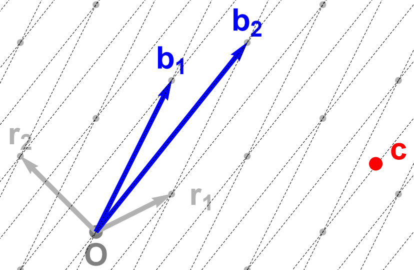

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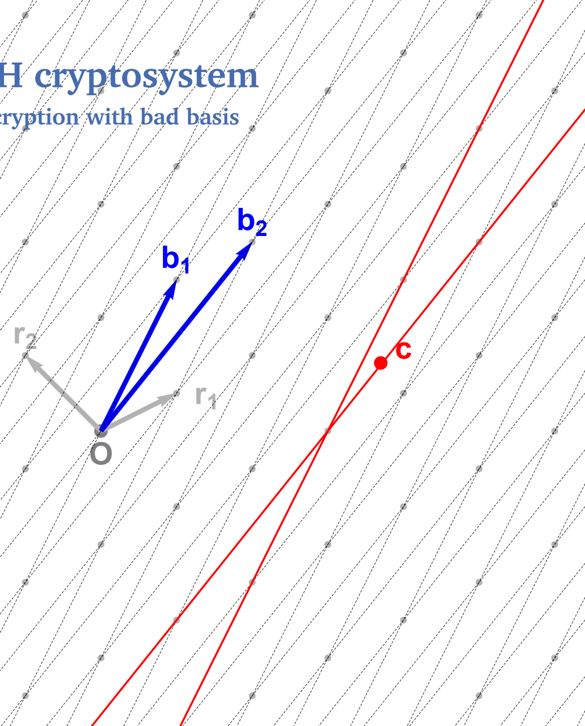

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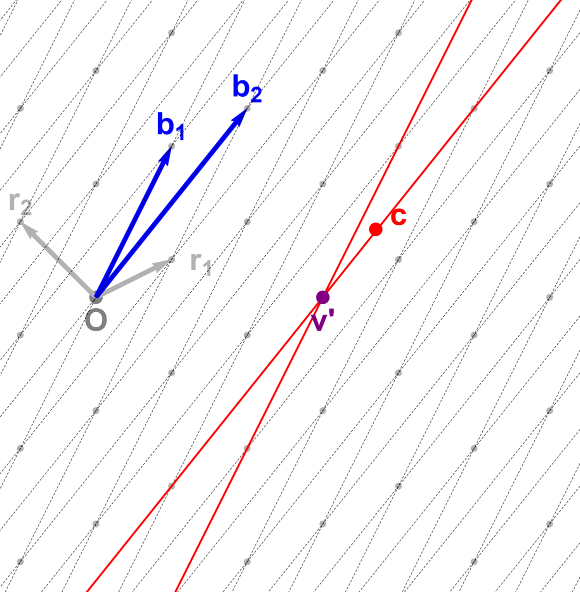

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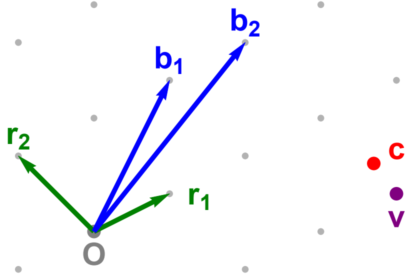

Outline

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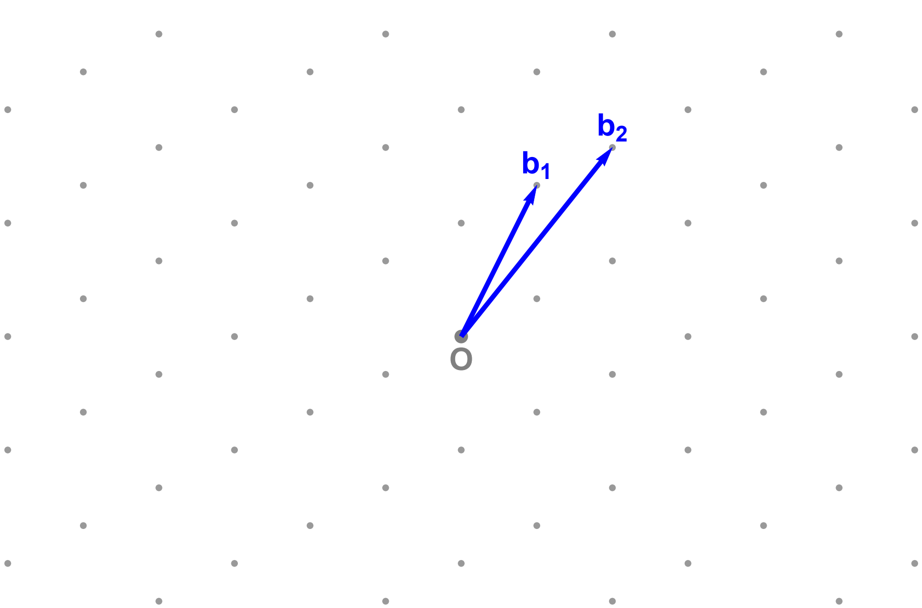
Lattice basis reduction

- Gauss reduction
- LLL reduction
- BKZ reduction

Gauss reduction



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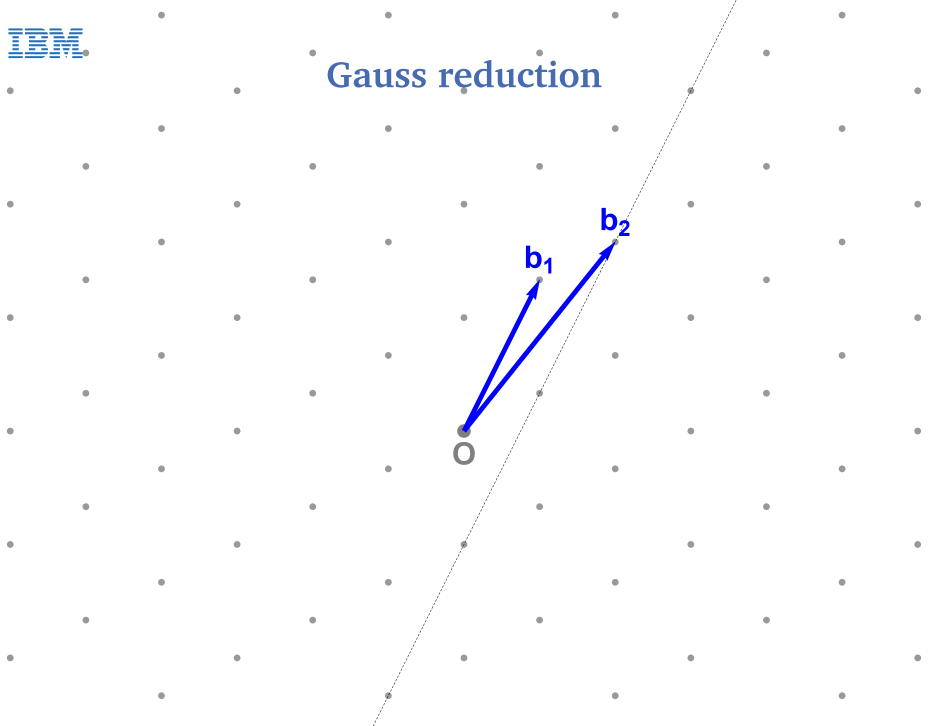

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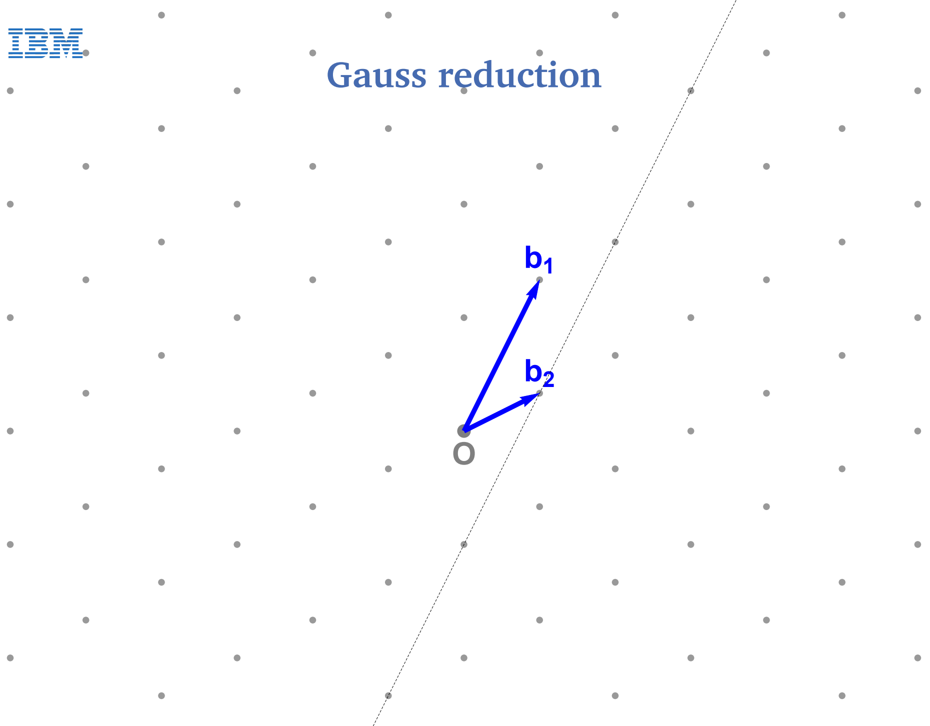

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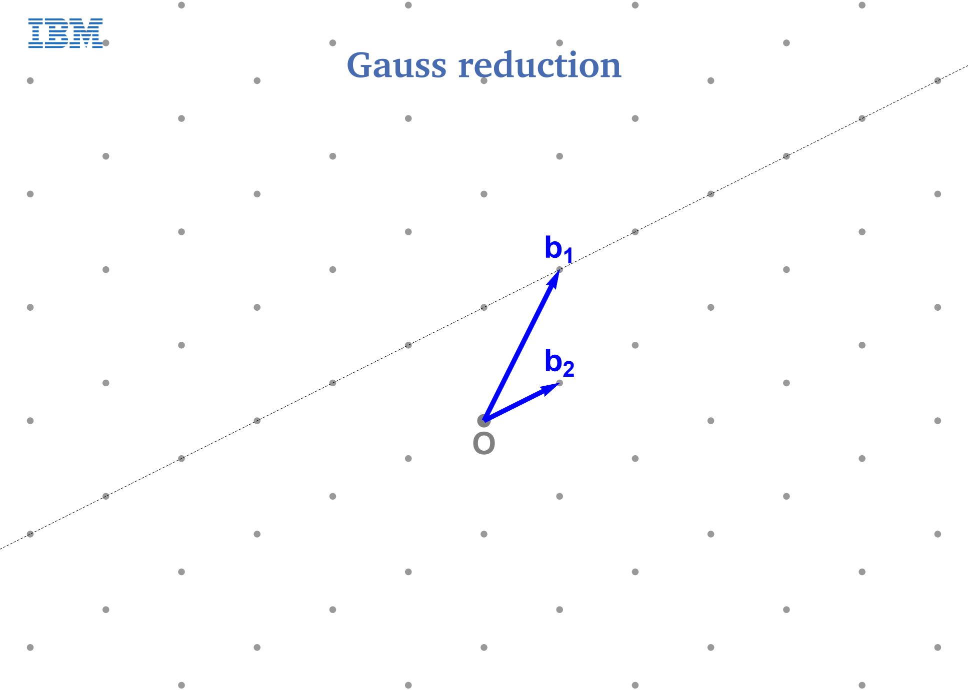

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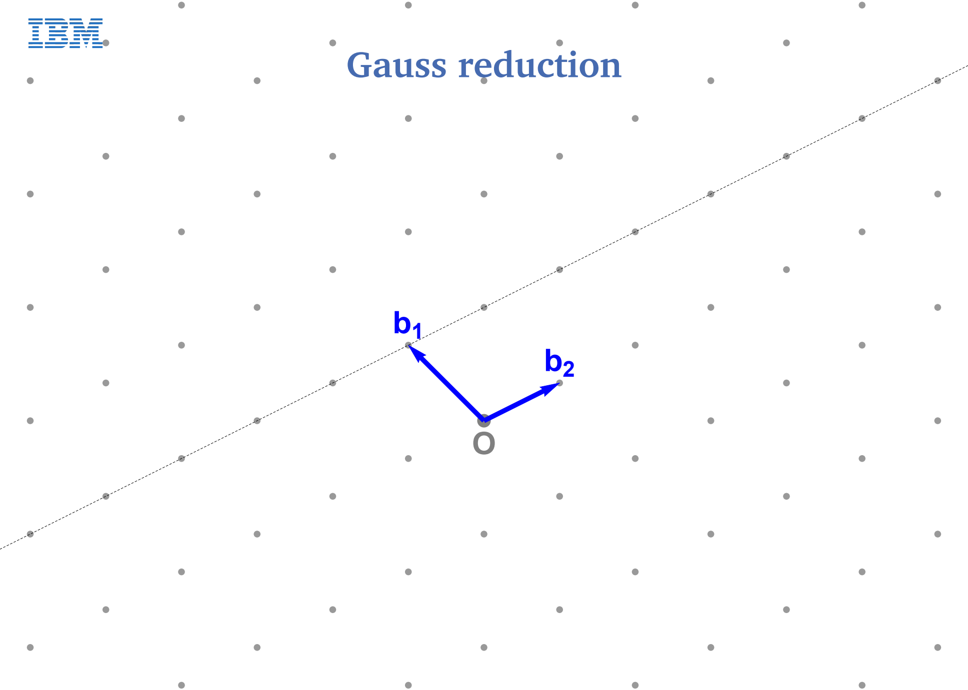

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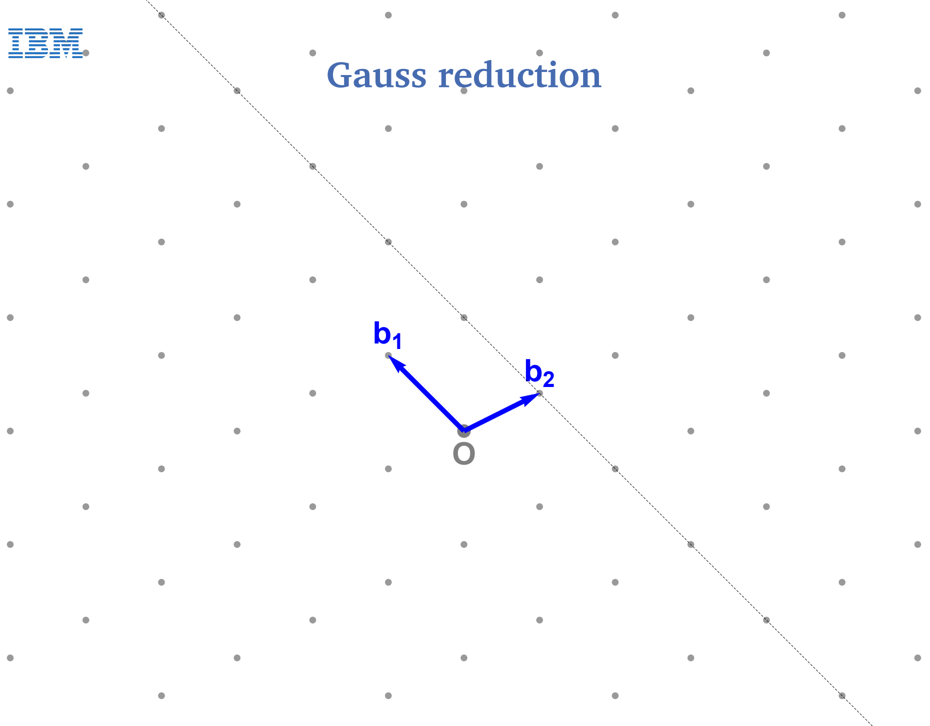

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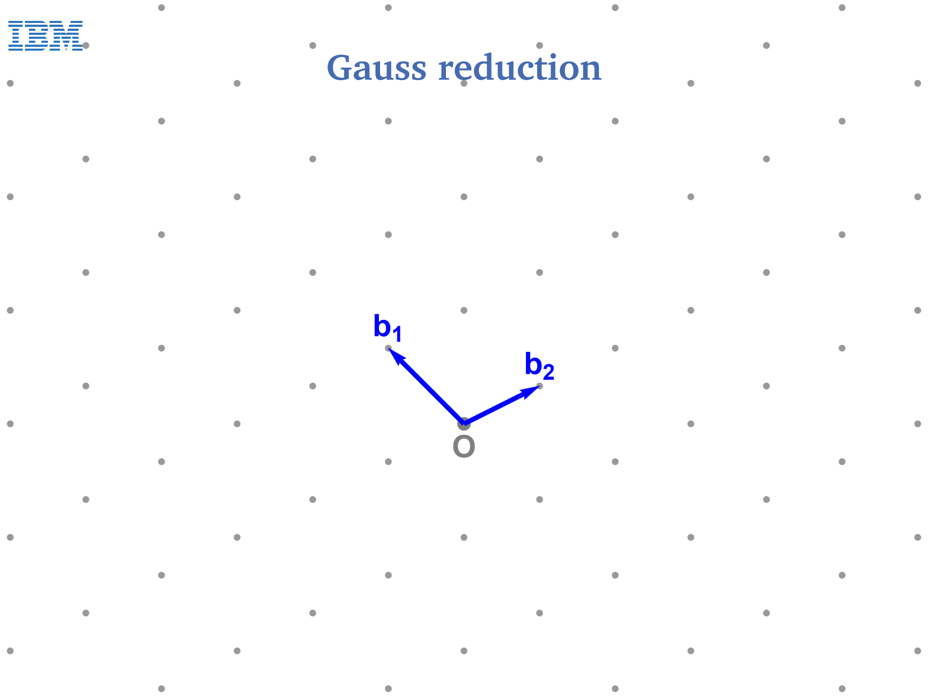

Gauss reduction

Given $\mathbf{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$, repeat two steps:

- **Swap**: If $\|\boldsymbol{b}_1\| > \|\boldsymbol{b}_2\|$, then swap $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$.
- **Reduce**: While $\|\boldsymbol{b}_2 \pm \boldsymbol{b}_1\| < \|\boldsymbol{b}_2\|$, replace $\boldsymbol{b}_2 \leftarrow \boldsymbol{b}_2 \pm \boldsymbol{b}_1$.

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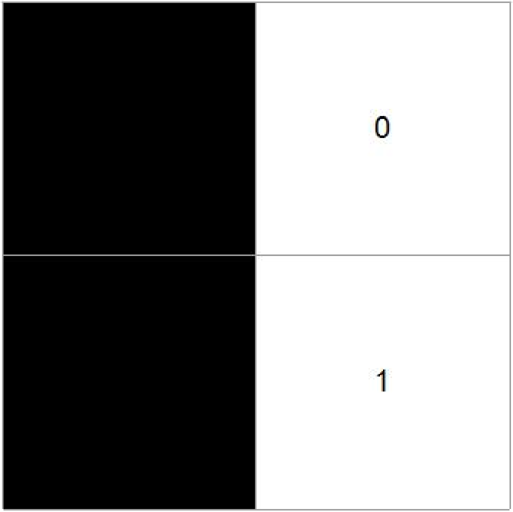
At the end, $\boldsymbol{b}_1$ is a shortest (non-zero) lattice vector and $\boldsymbol{b}_2$ a "second shortest" (non-zero) lattice vector.

Gauss reduction

3141592653	0
5897932384	1

Gauss reduction



LLL algorithm

Lenstra-Lenstra-Lovasz (LLL) algorithm [LLL82]

- Blockwise generalization of Gauss reduction
- Do reductions/swaps on $(\boldsymbol{b}_i, \boldsymbol{b}_{i+1})$ for $i = 1, \ldots, n-1$

LLL algorithm

	0	0	0	0	0	0	0
	1	0	0	0	0	0	0
	0	1	0	0	0	0	0
	0	0	1	0	0	0	0
	0	0	0	1	0	0	0
	0	0	0	0	1	0	0
	0	0	0	0	0	1	0
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LLL algorithm

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	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
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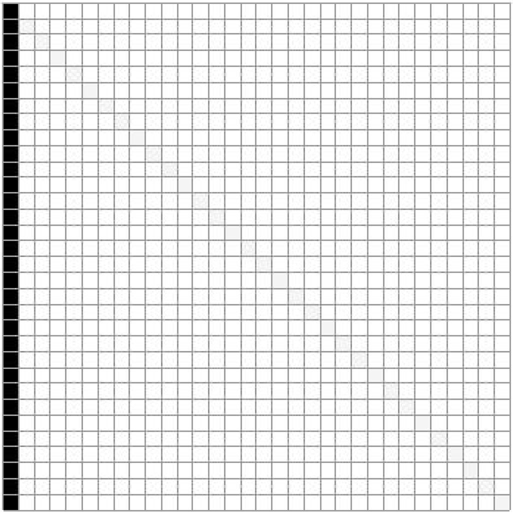

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  - Theoretically: $\|\boldsymbol{b}_1\| \leq 1.075^n \cdot \det(\mathscr{L})$
  - Experimentally: $\|\boldsymbol{b}_1\| \approx 1.022^n \cdot \det(\mathscr{L})$

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Blockwise Korkine-Zolotarev (BKZ) reduction [Sch87, SE94]

- Blockwise generalization of Korkine-Zolotarev reduction
- Do reductions/swaps on $(\boldsymbol{b}_i, \ldots, \boldsymbol{b}_{i+k-1})$ for $i = 1, \ldots, n-k+1$
- Blocksize k offers time-quality tradeoff

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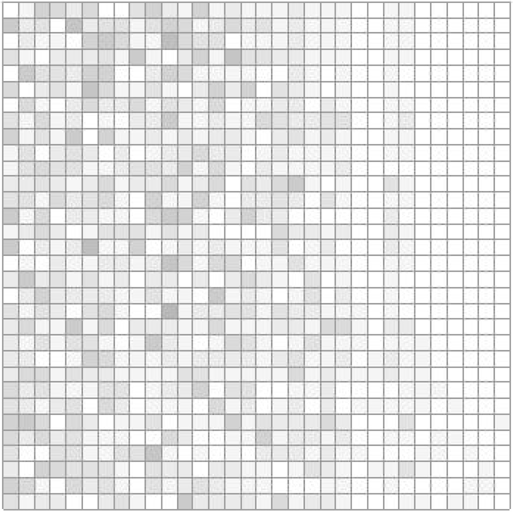

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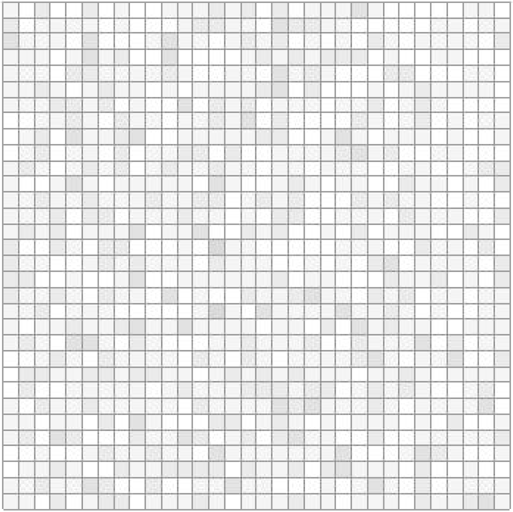

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- Basis quality deteriorates with the dimension n
  - Theoretically: $\|\boldsymbol{b}_1\| \leq 1.075^n \cdot \det(\mathscr{L})$
  - Experimentally: $\|\boldsymbol{b}_1\| \approx 1.022^n \cdot \det(\mathscr{L})$

Blockwise Korkine-Zolotarev (BKZ) reduction [Sch87, SE94]

- Blockwise generalization of Korkine-Zolotarev reduction
- Do reductions/swaps on $(\boldsymbol{b}_i, \ldots, \boldsymbol{b}_{i+k-1})$ for $i = 1, \ldots, n-k+1$
- Blocksize k offers time-quality tradeoff

BKZ uses **exact SVP** algorithm in dimension k as subroutine

BKZ algorithm

Lenstra-Lenstra-Lovasz (LLL) algorithm [LLL82]

- Blockwise generalization of Gauss reduction
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- Basis quality deteriorates with the dimension n
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Next hour: How to solve exact SVP in high dimensions?