IBM Research

Algorithms for finding shortest lattice vectors

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Outline

Lattices

Enumeration algorithms
- Fincke–Pohst enumeration
- Kannan enumeration
- (Extreme) pruning

Constructing the Voronoi cell

Sieving algorithms
- Basic sieving
- Leveled sieving
- Nearest neighbor searching

Conclusion

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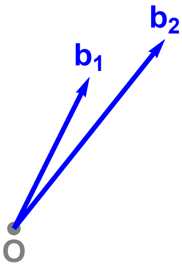

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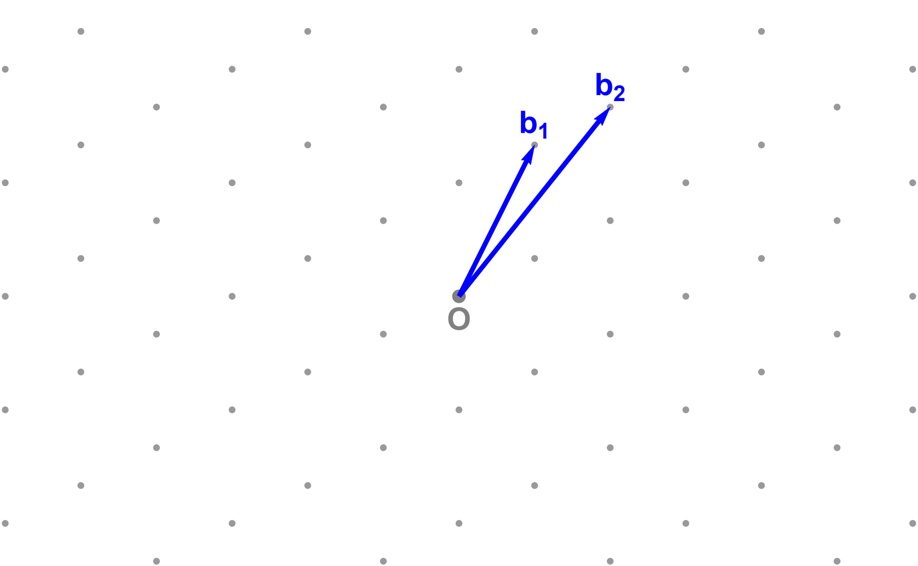
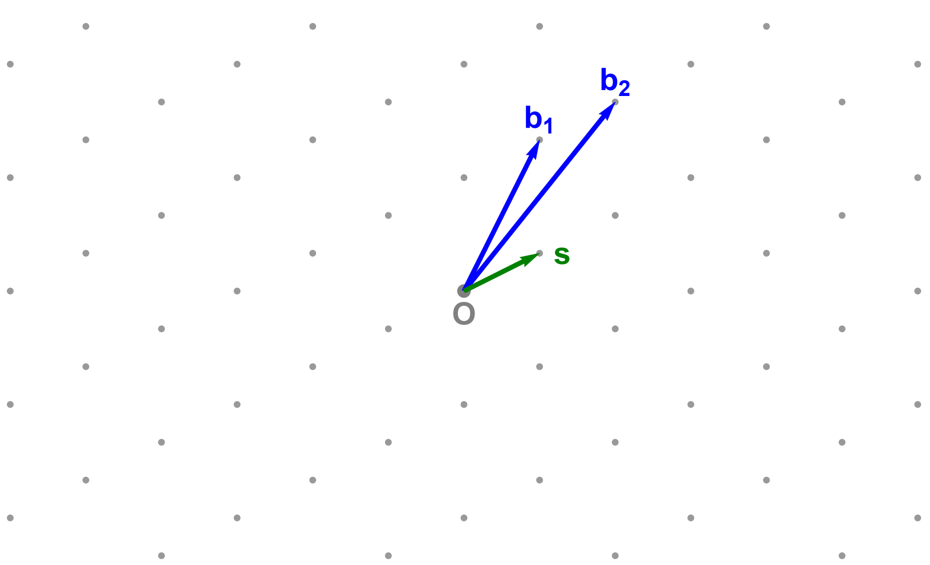


Lattices

Shortest Vector Problem (SVP)



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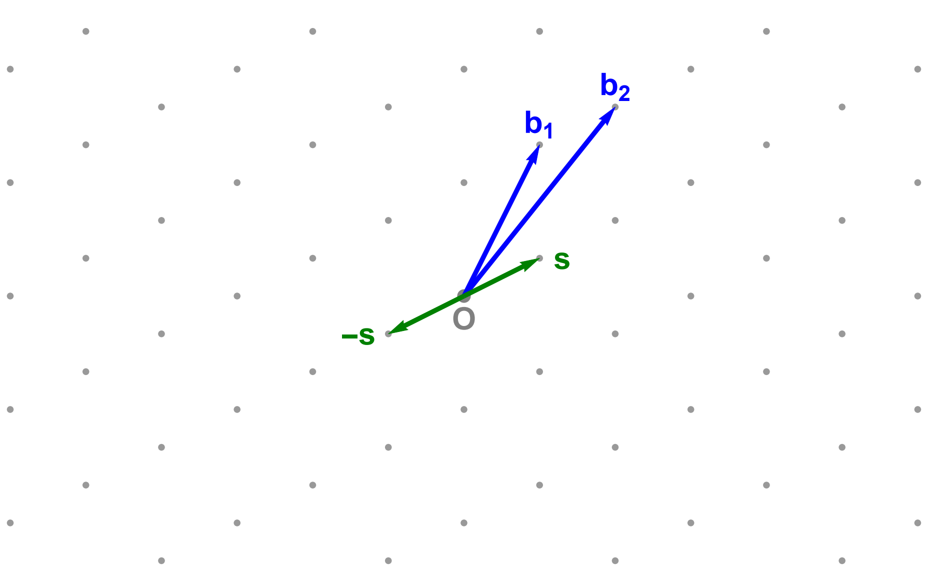

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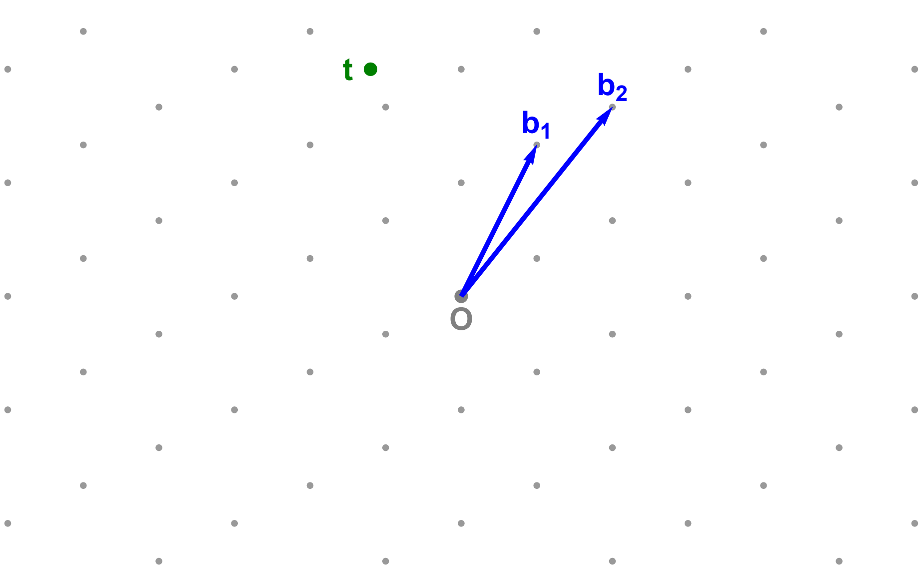

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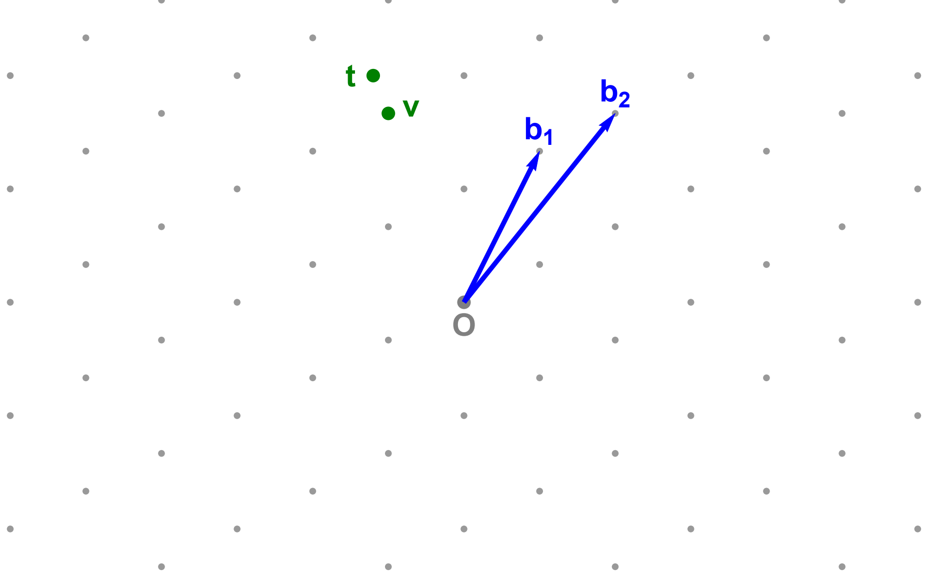

Lattices

Exact SVP algorithms

	Algorithm	$\log_2(\text{Time})$	$\log_2(\text{Space})$
Provable SVP	Enumeration [Poh81, Kan83, . . . , MW15]	$\Omega(n \log n)$	$O(\log n)$
	AKS-sieve [AKS01, NV08, MV10, HPS11]	$3.398n$	$1.985n$
	ListSieve [MV10, MDB14]	$3.199n$	$1.327n$
	AKS-sieve-birthday [PS09, HPS11]	$2.648n$	$1.324n$
	ListSieve-birthday [PS09]	$2.465n$	$1.233n$
	Voronoi cell algorithm [AEVZ02, MV10b]	$2.000n$	$1.000n$
	Discrete Gaussians [ADRS15, ADS15, Ste16]	$1.000n$	$1.000n$
Heuristic SVP	Nguyen–Vidick sieve [NV08]	$0.415n$	$0.208n$
	GaussSieve [MV10, . . . , IKMT14, BNvdP14]	$0.415n$	$0.208n$
	Two-level sieve [WLTB11]	$0.384n$	$0.256n$
	Three-level sieve [ZPH13]	$0.3778n$	$0.283n$
	Overlattice sieve [BGJ14]	$0.3774n$	$0.293n$
	Hyperplane LSH [Laa15, MLB15, Mar15]	$0.337n$	$0.208n$
	May and Ozerov's NNS method [BGJ15]	$0.311n$	$0.208n$
	Spherical LSH [LdW15]	$0.298n$	$0.208n$
	Cross-polytope LSH [BL15]	$0.298n$	$0.208n$
	Spherical filtering [BDGL16, Laa15, ML15]	$0.293n$	$0.208n$

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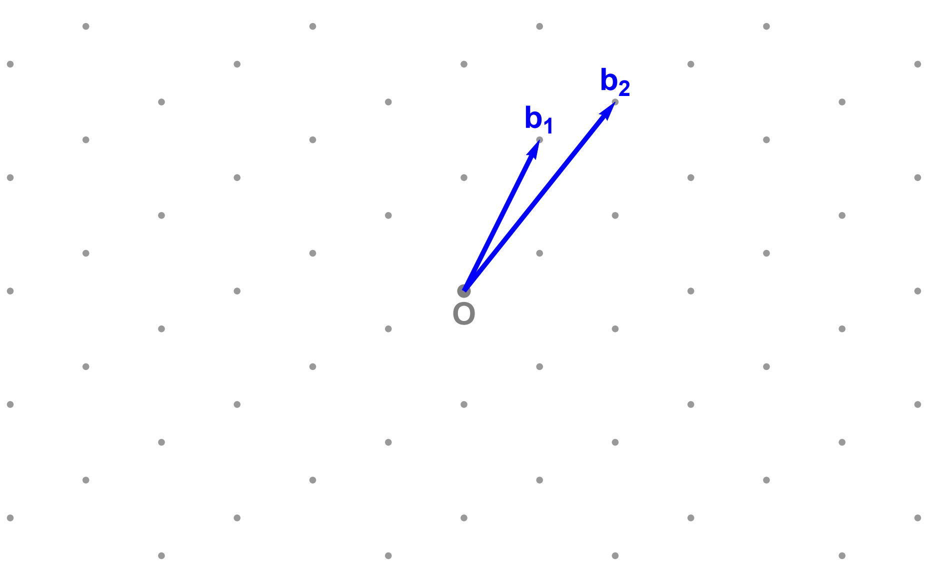
Fincke-Pohst enumeration

Determine possible coefficients of b_2



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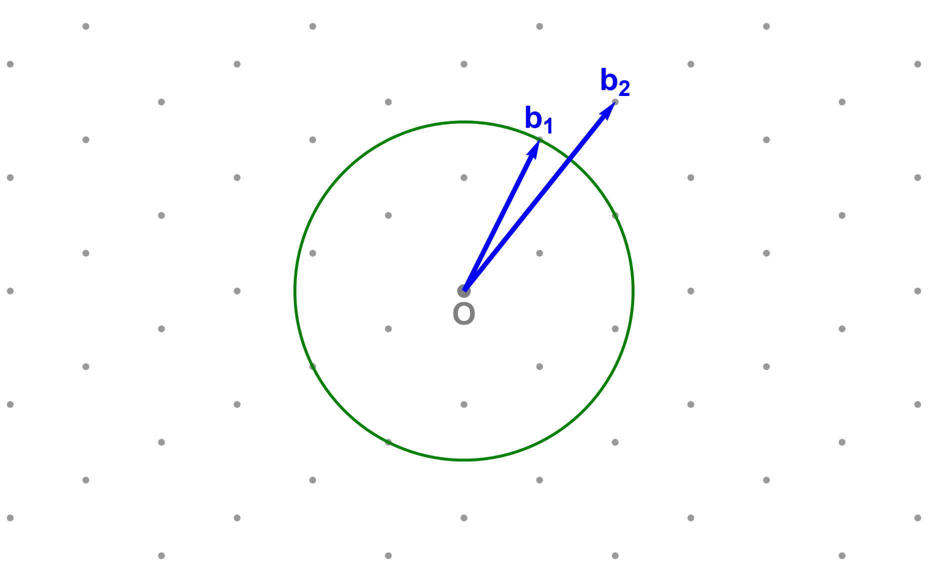

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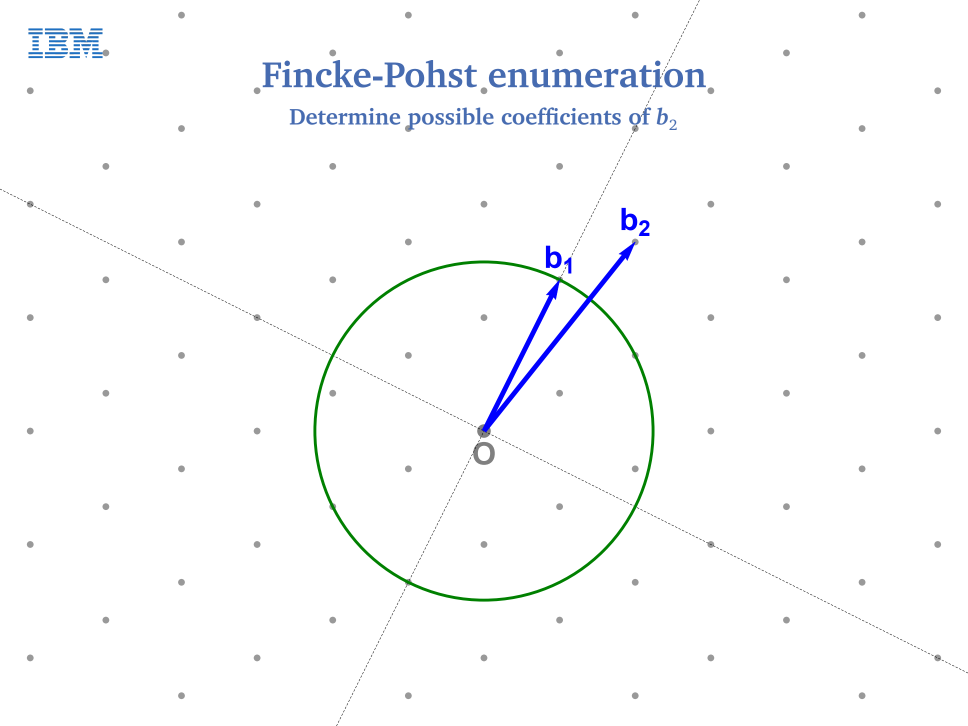

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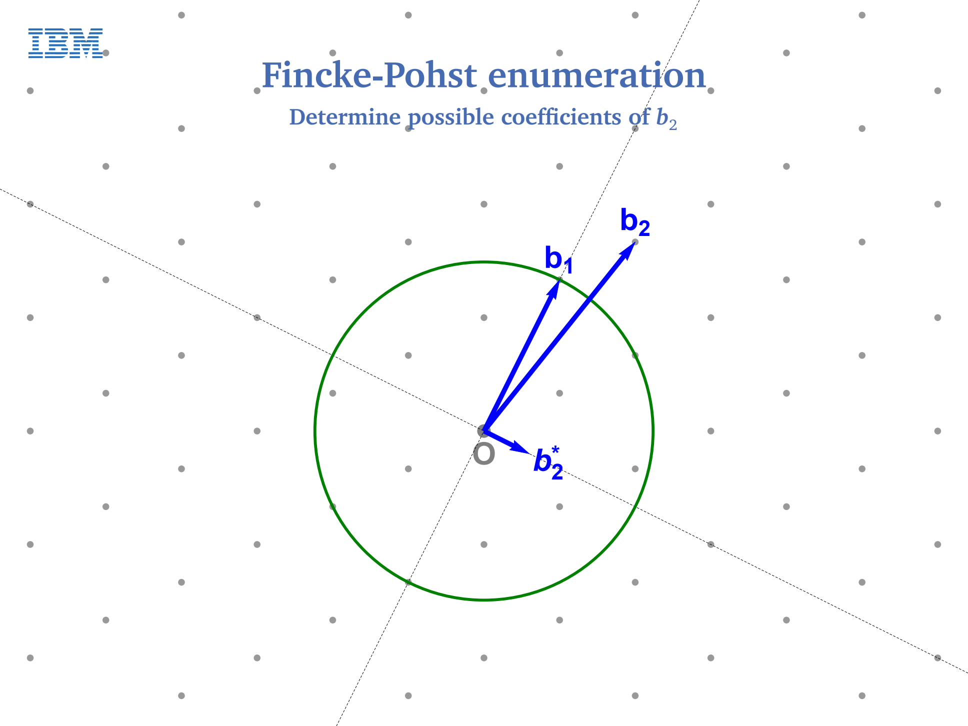

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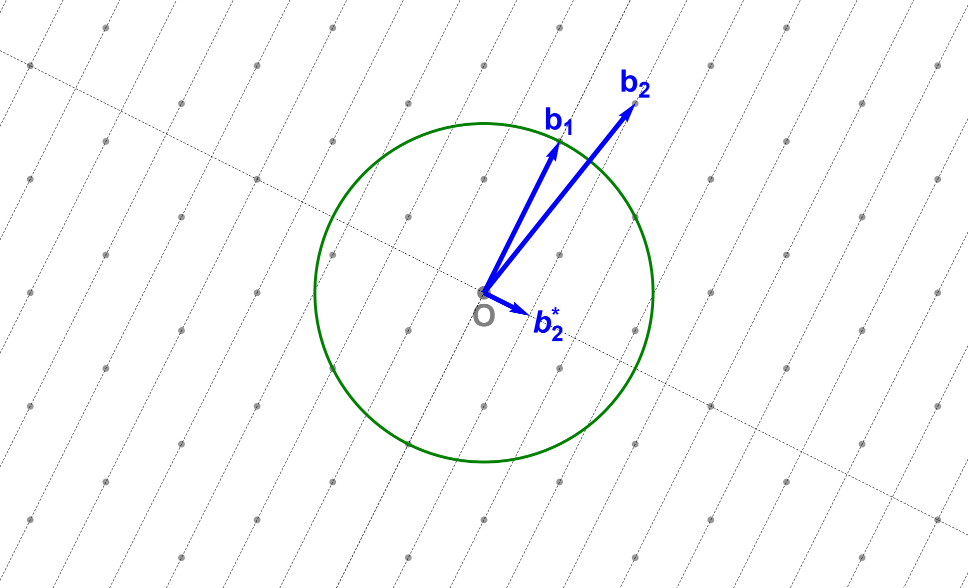
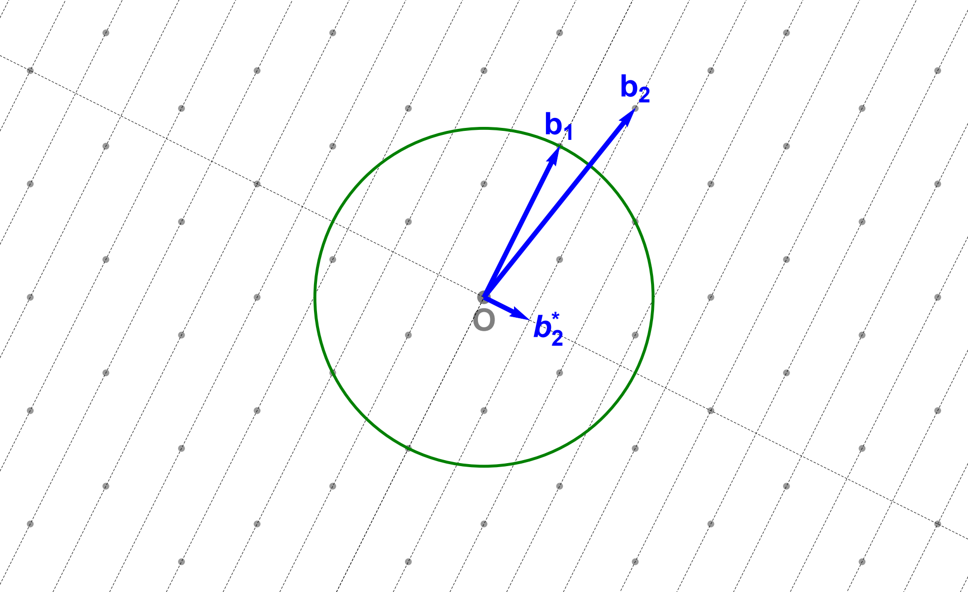


Fincke-Pohst enumeration

Find short vectors for each coefficient of b_2



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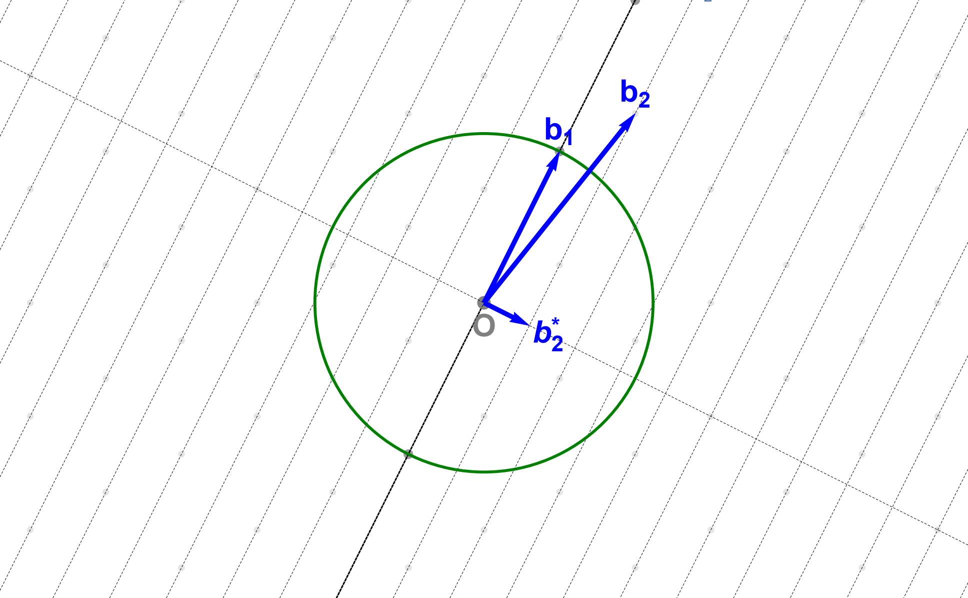

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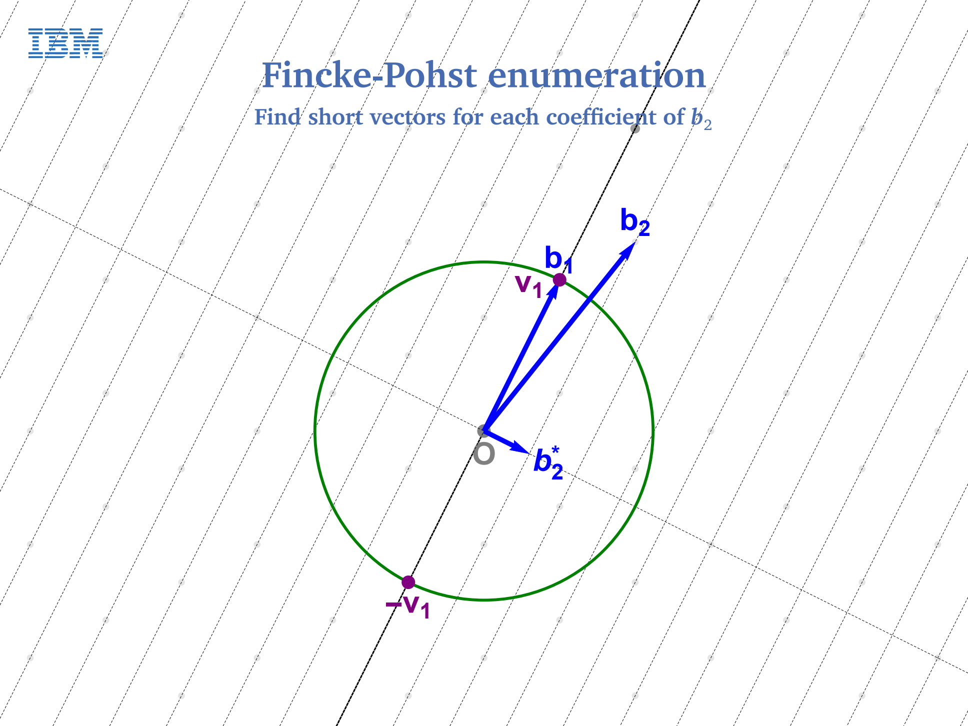

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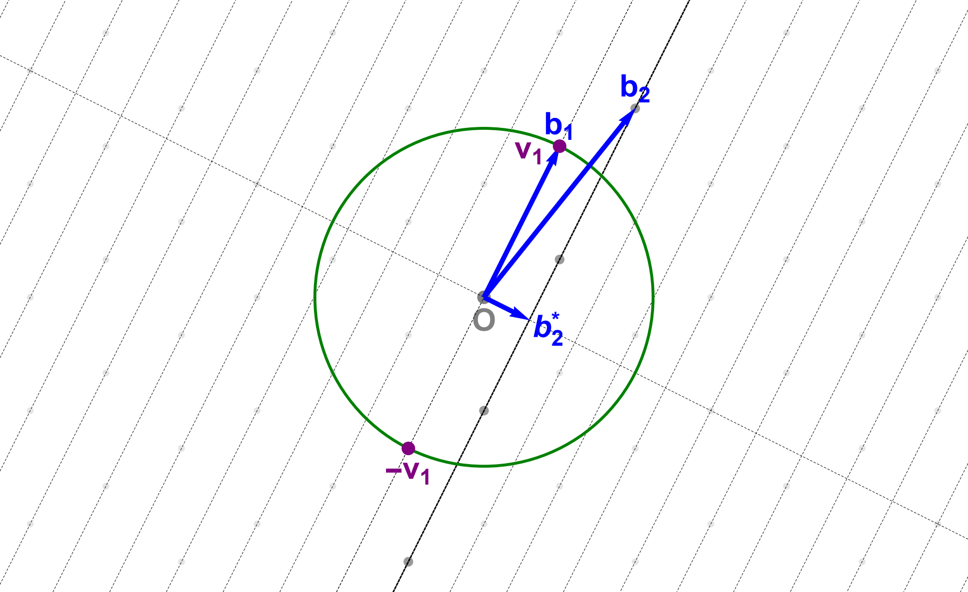

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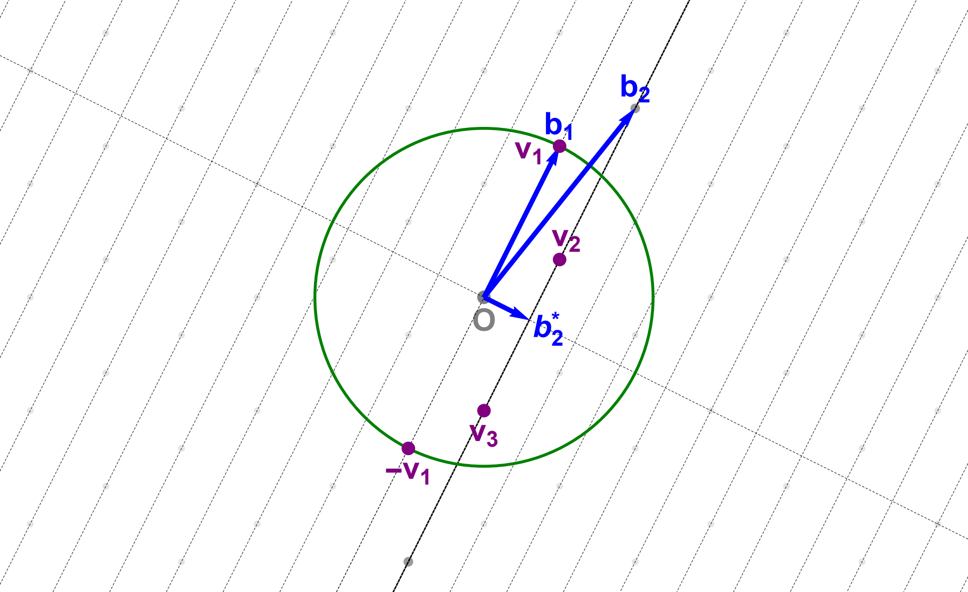

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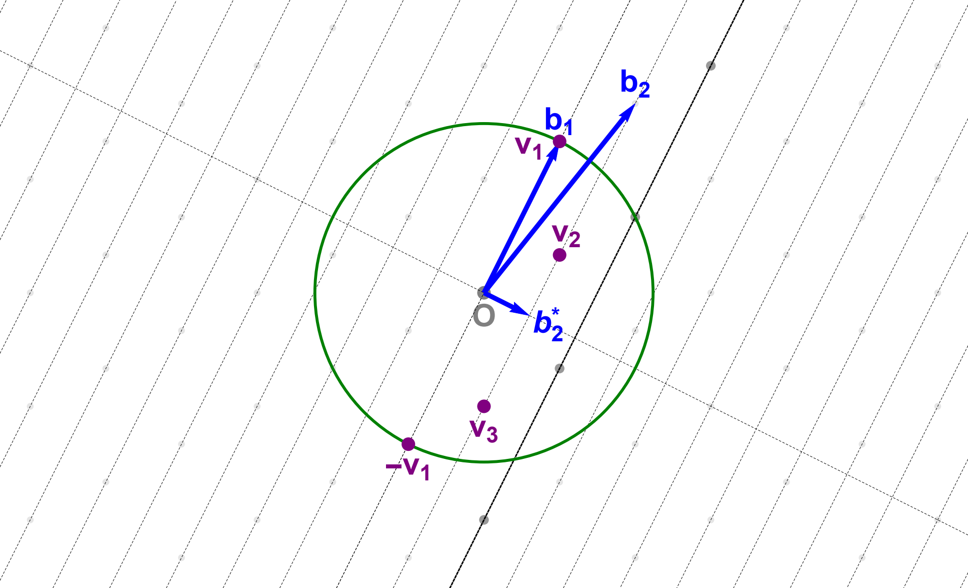

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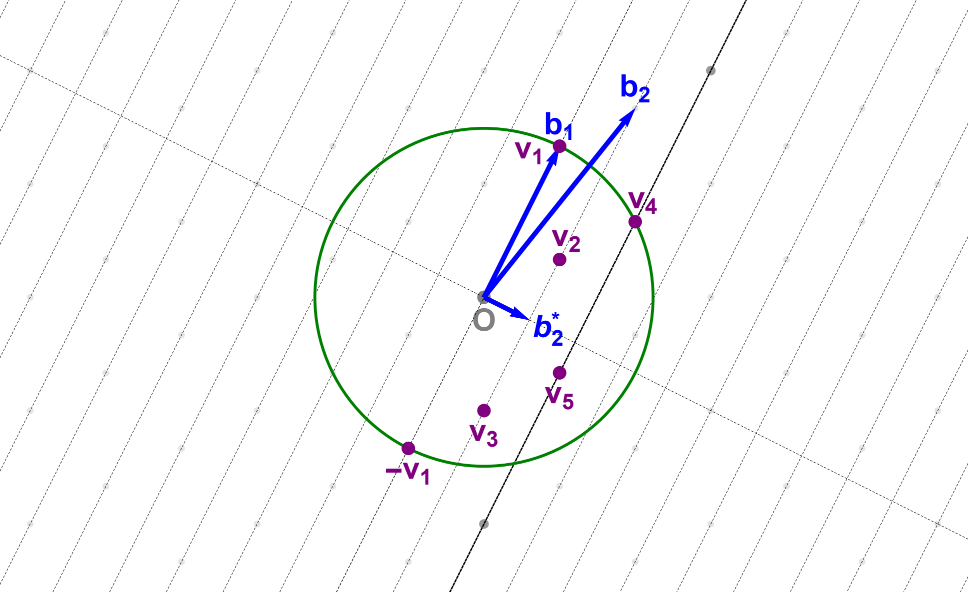

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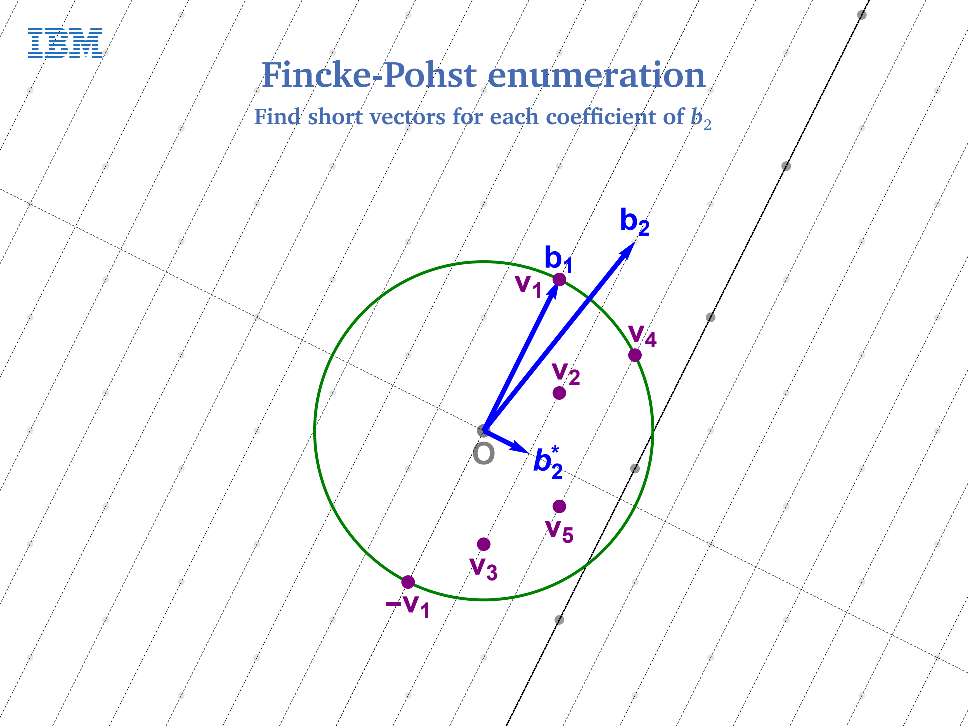

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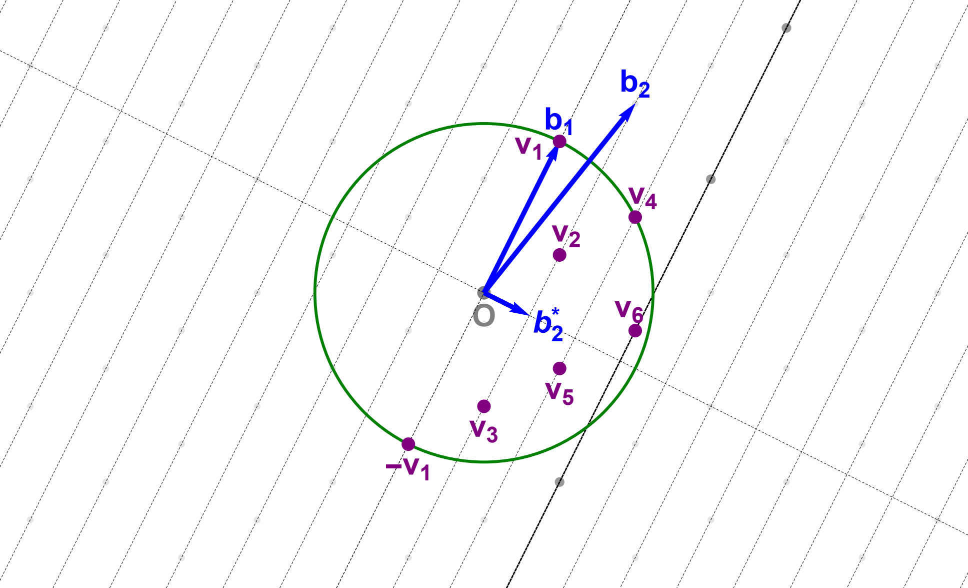

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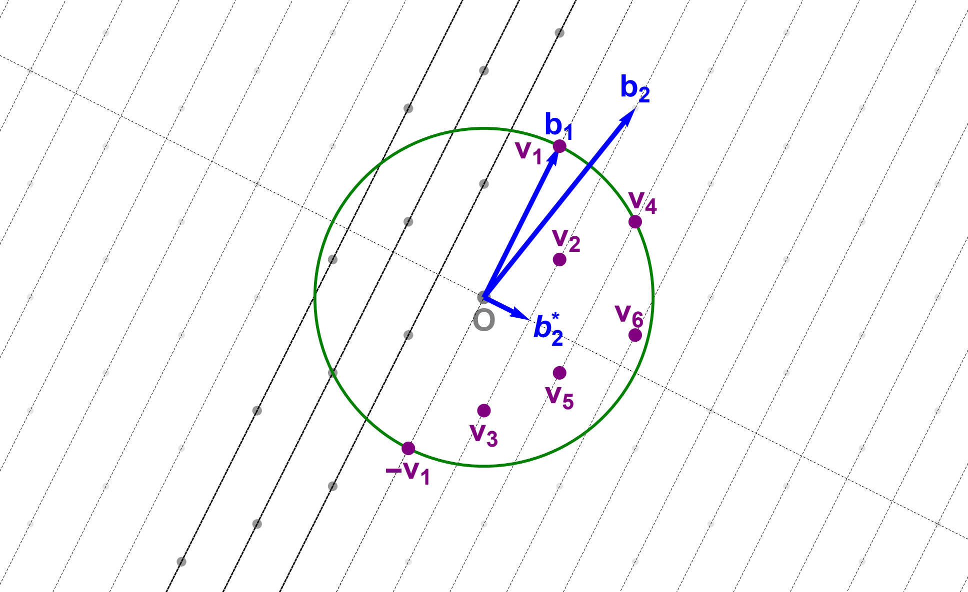

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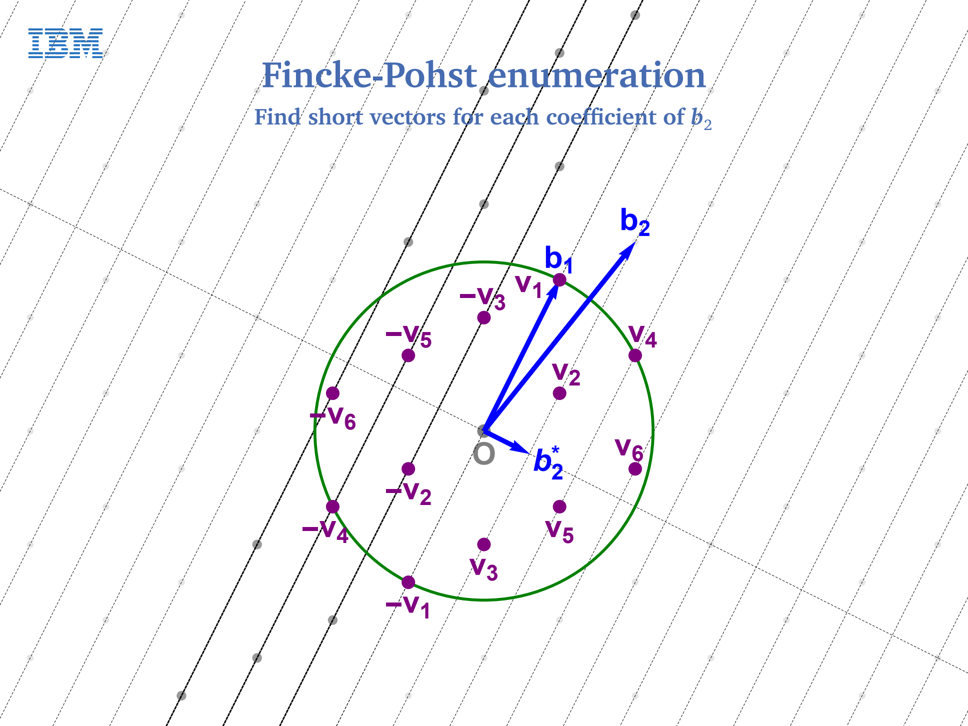

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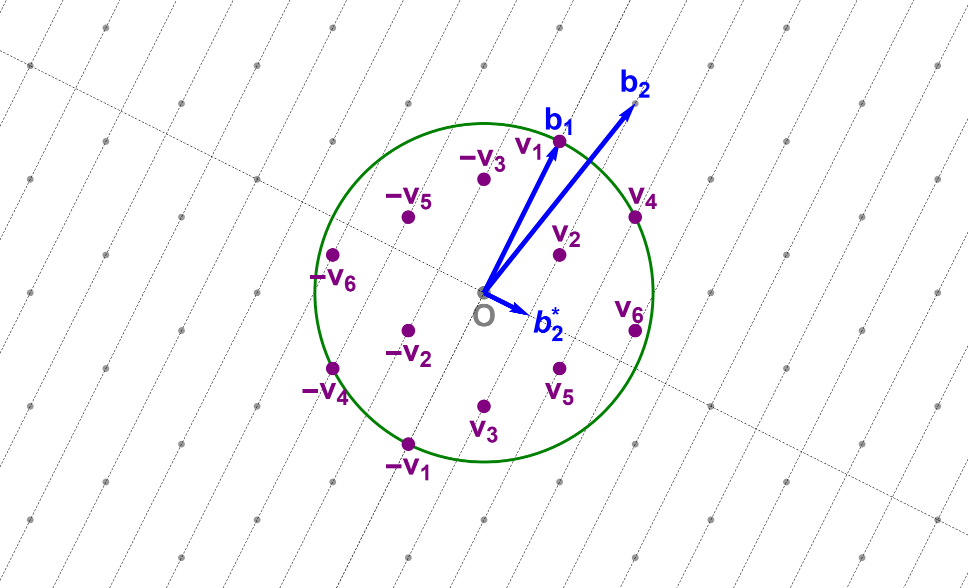
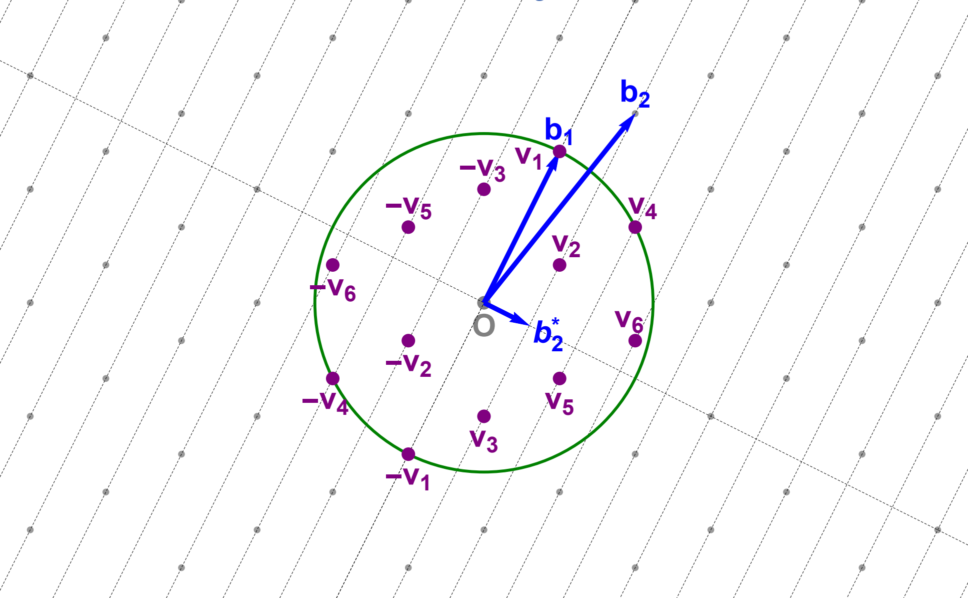


Fincke-Pohst enumeration

Find a shortest vector among all found vectors



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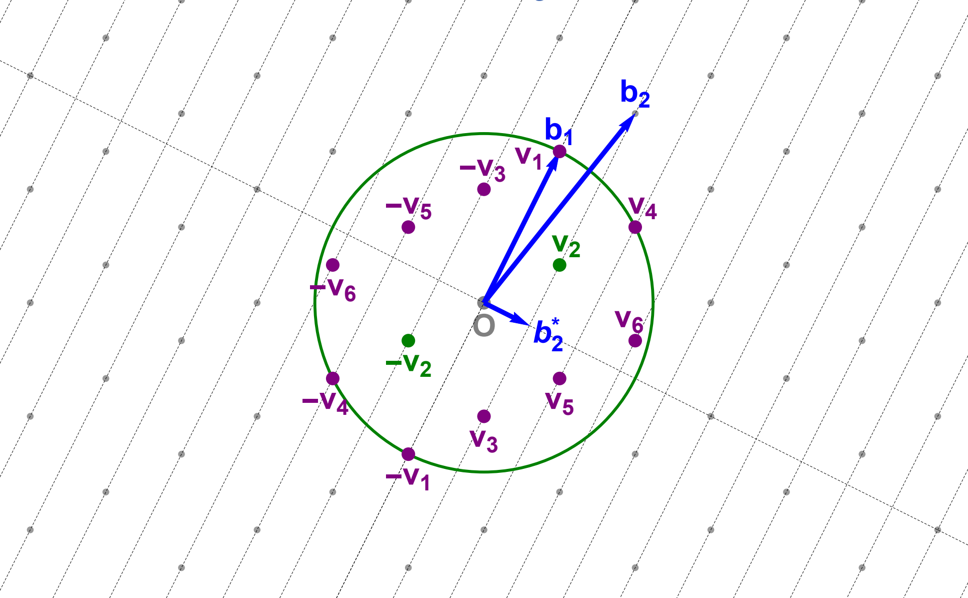

Fincke-Pohst enumeration

Overview

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Theorem (Fincke–Pohst, Math. of Comp. '85)

Fincke-Pohst enumeration runs in time $2^{O(n^2)}$ *and space* $\mathrm{poly}(n)$.

Fincke-Pohst enumeration

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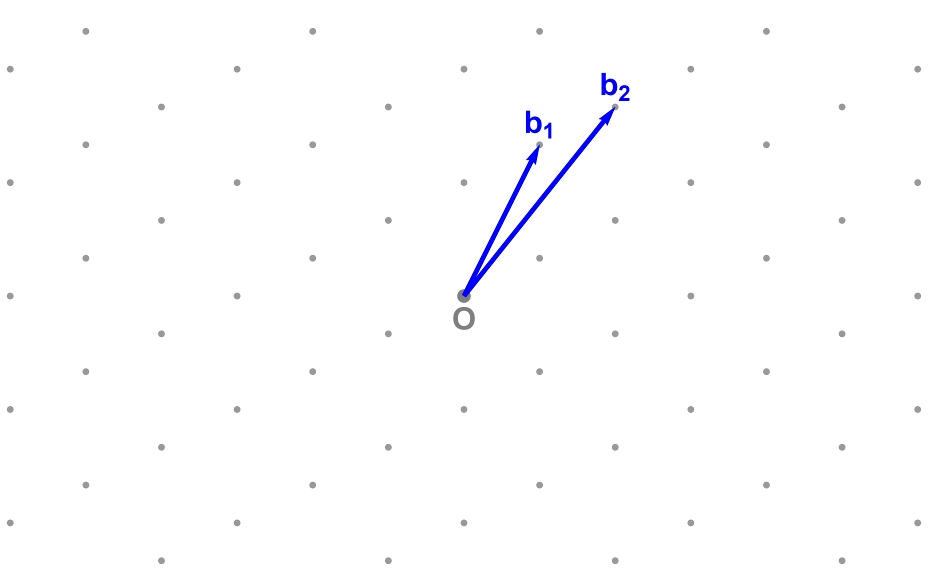
Theorem (Fincke–Pohst, Math. of Comp. '85)

Fincke-Pohst enumeration runs in time $2^{O(n^2)}$ *and space* $\text{poly}(n)$.

Essentially reduces SVP_n (CVP_n) to $2^{O(n)}$ instances of CVP_{n-1}

Kannan enumeration

Better bases



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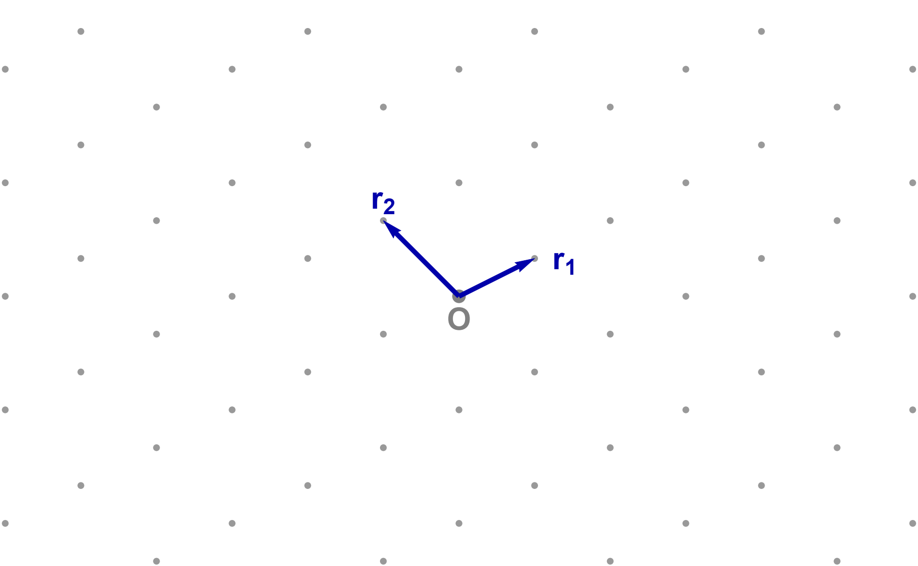

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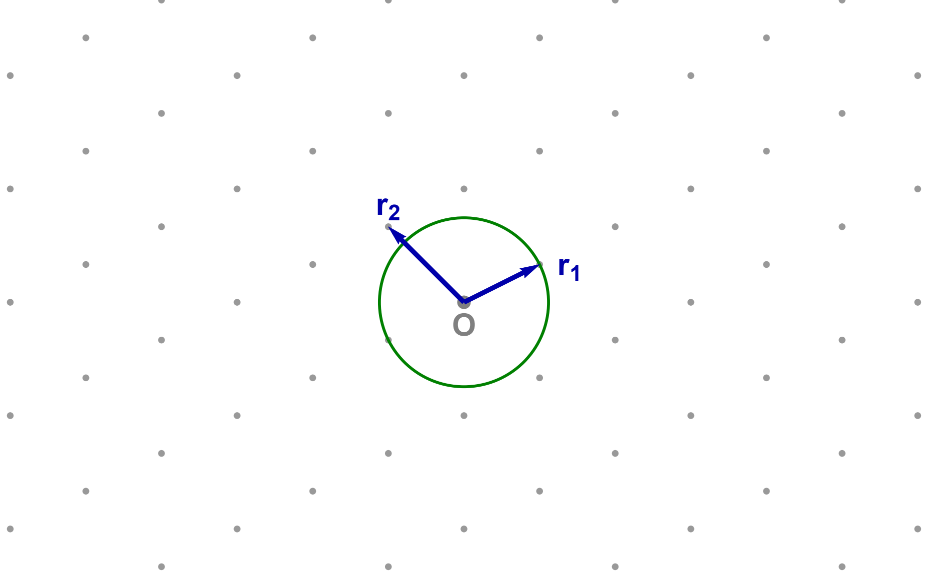

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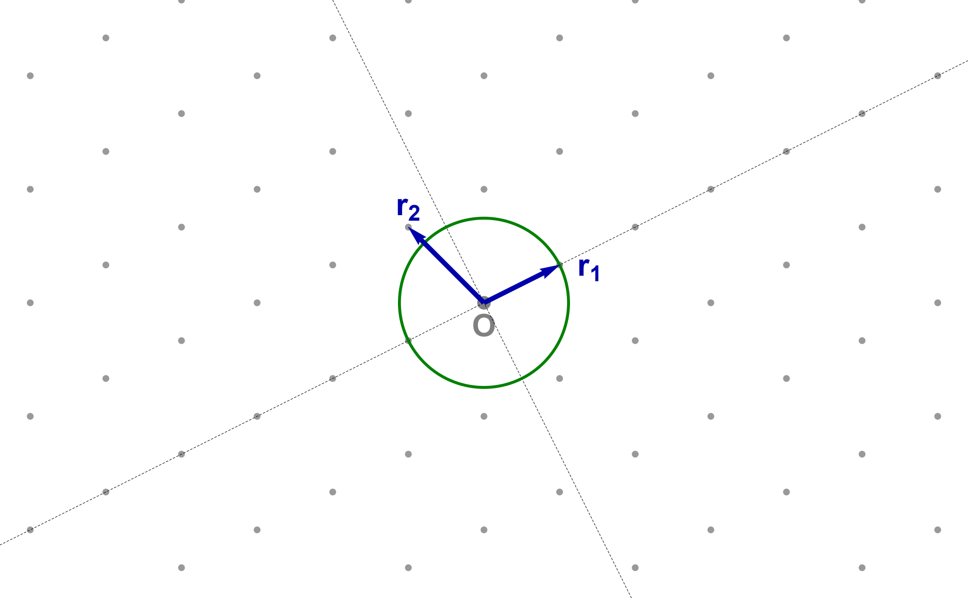

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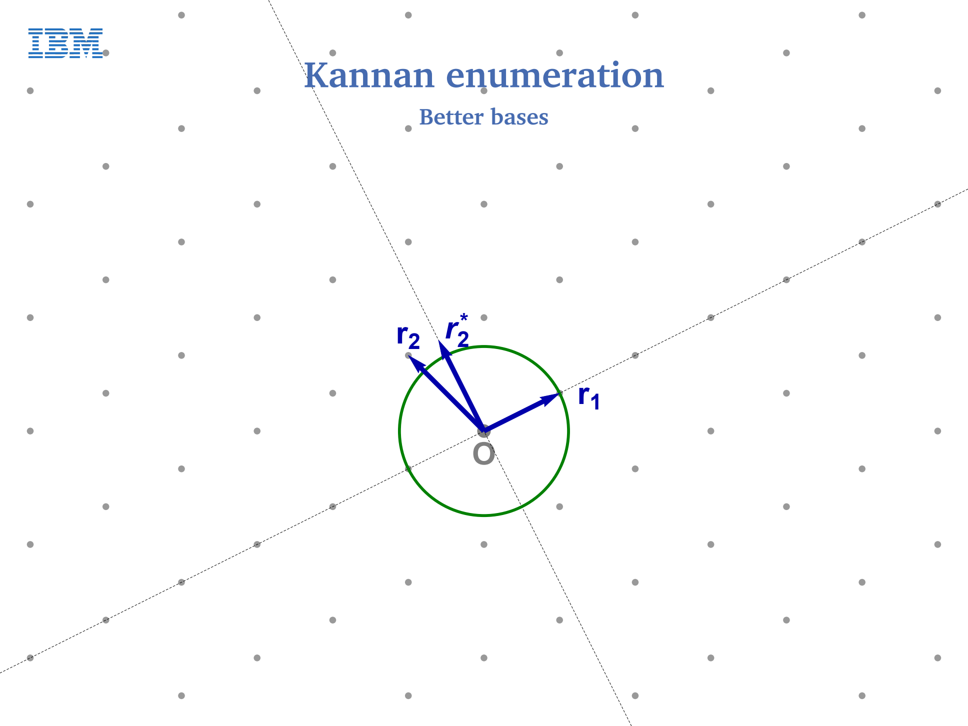

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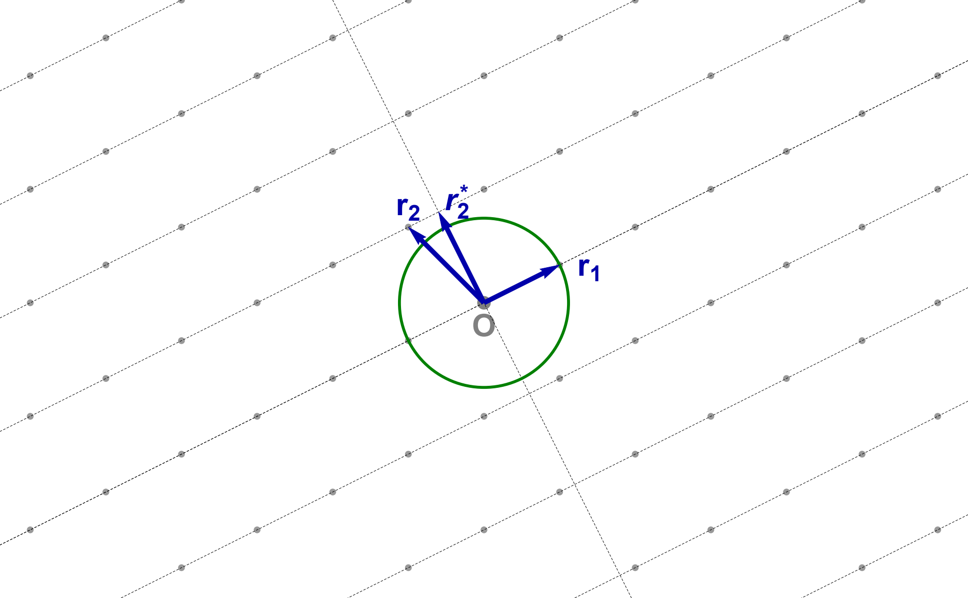

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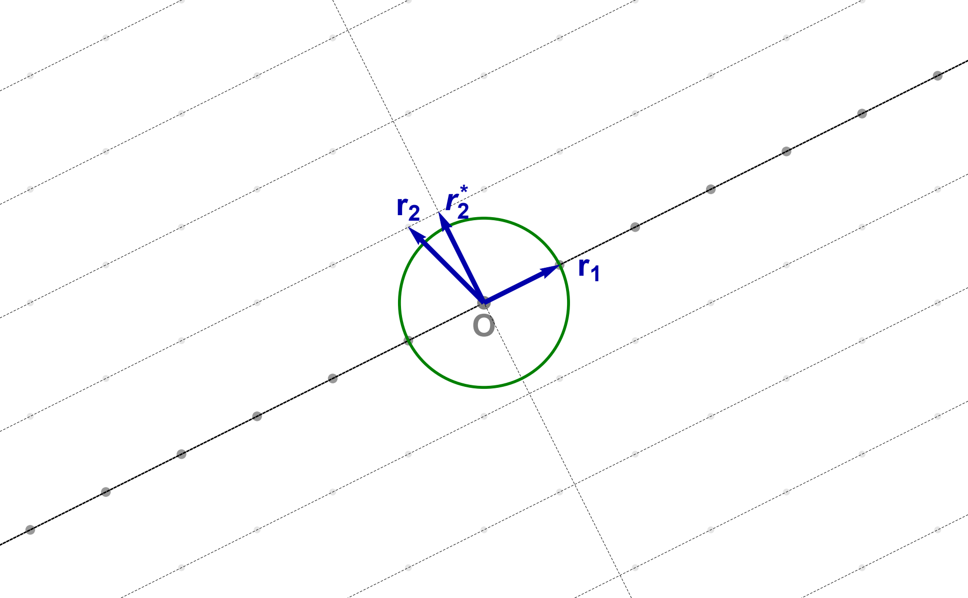

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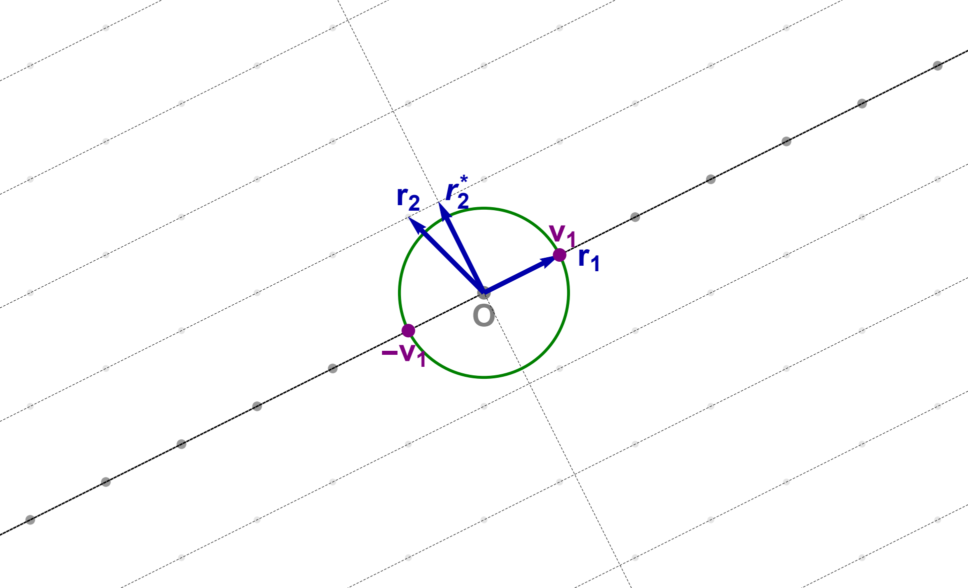

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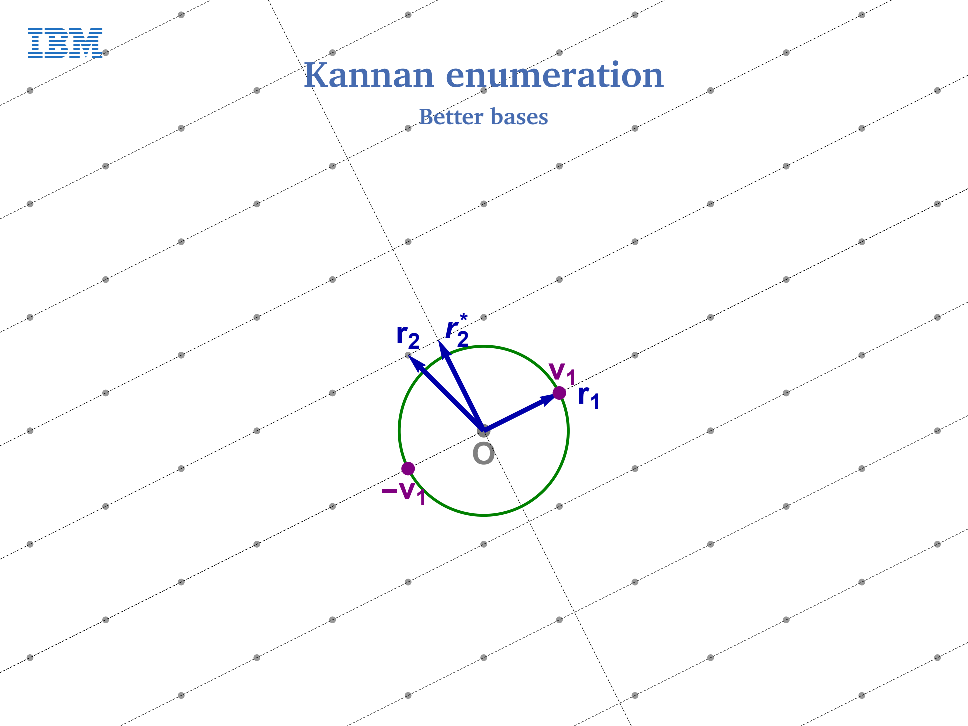

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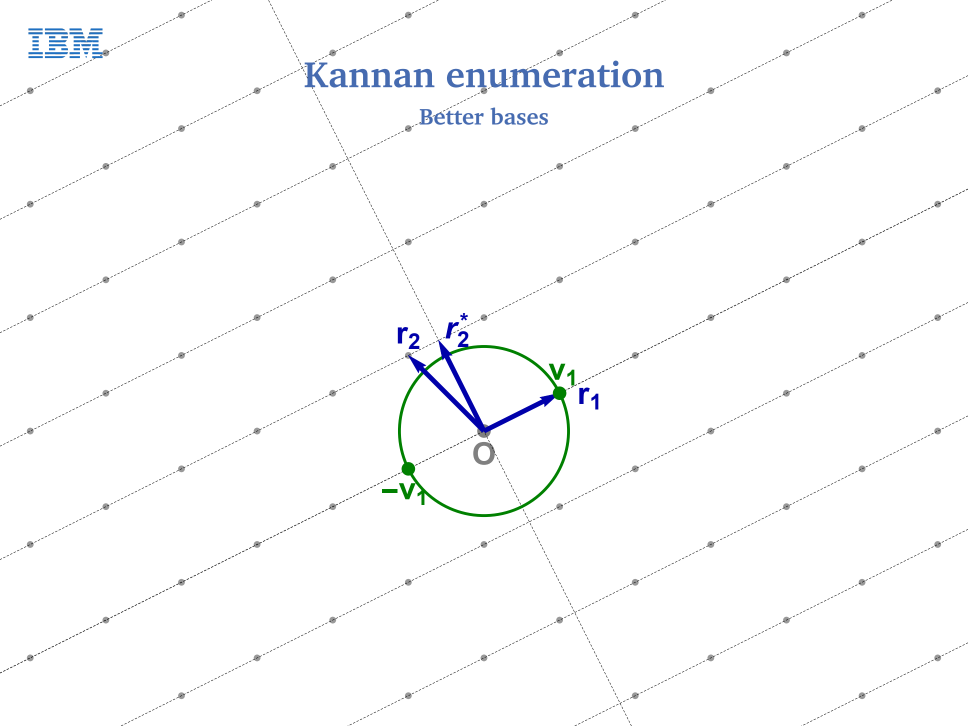

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Theorem (Kannan, STOC'83)

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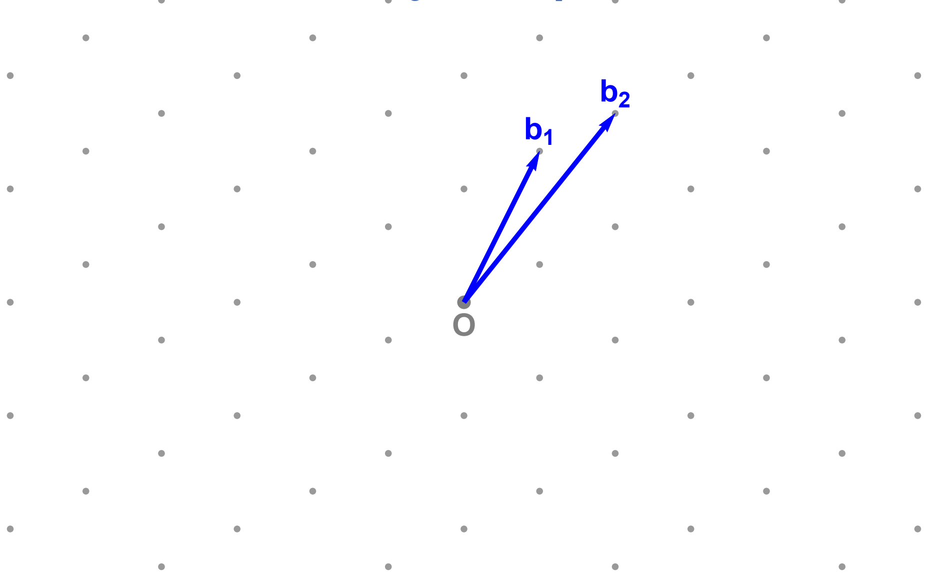
Theorem (Kannan, STOC'83)

Kannan enumeration runs in time $2^{O(n \log n)}$ *and space* $\mathrm{poly}(n)$.

> *"Our algorithm reduces an n-dimensional problem to polynomially many (instead of* $2^{O(n)}$*)* $(n-1)$*-dimensional problems. [...] The algorithm we propose, first finds a more orthogonal basis for a lattice in time* $2^{O(n \log n)}$*."*
>
> — Kannan, *STOC'83*

Pruned enumeration

Reducing the search space



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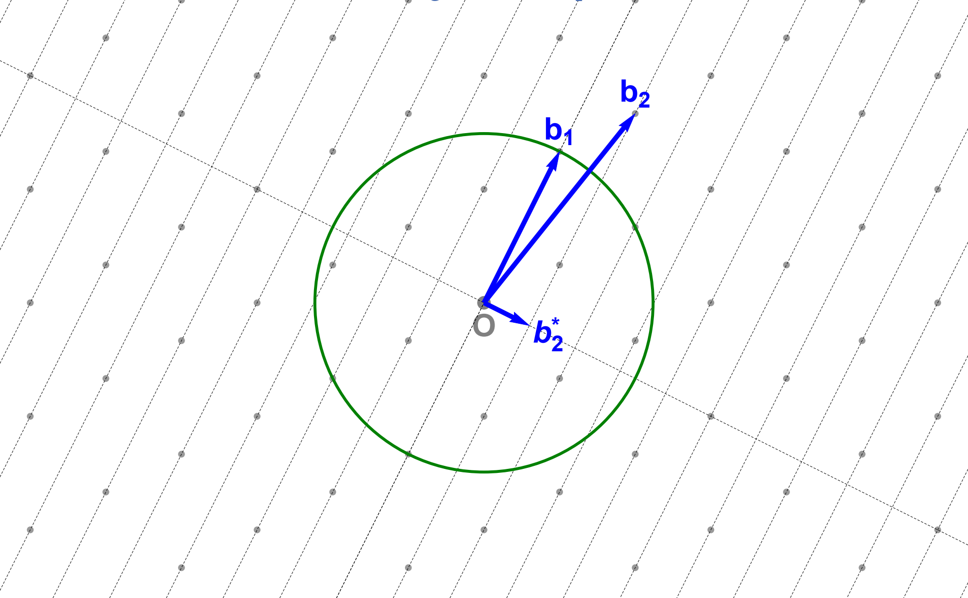

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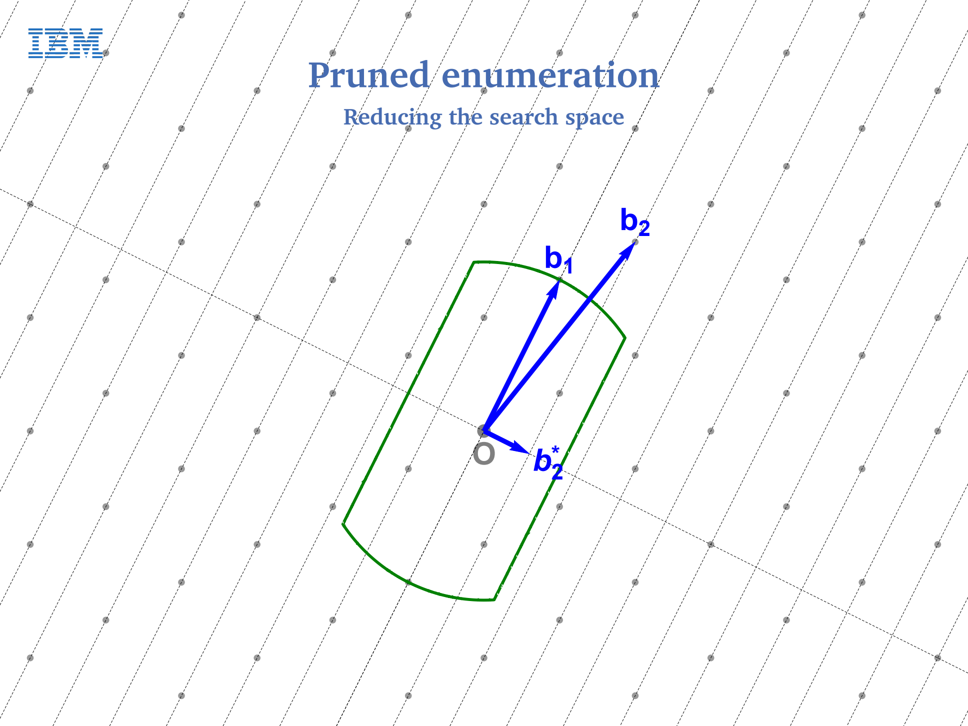

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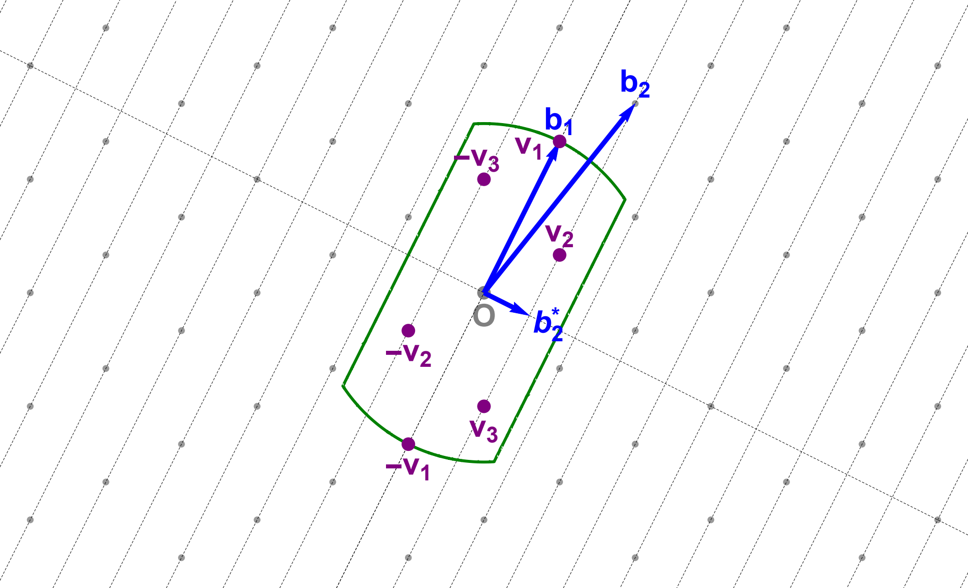

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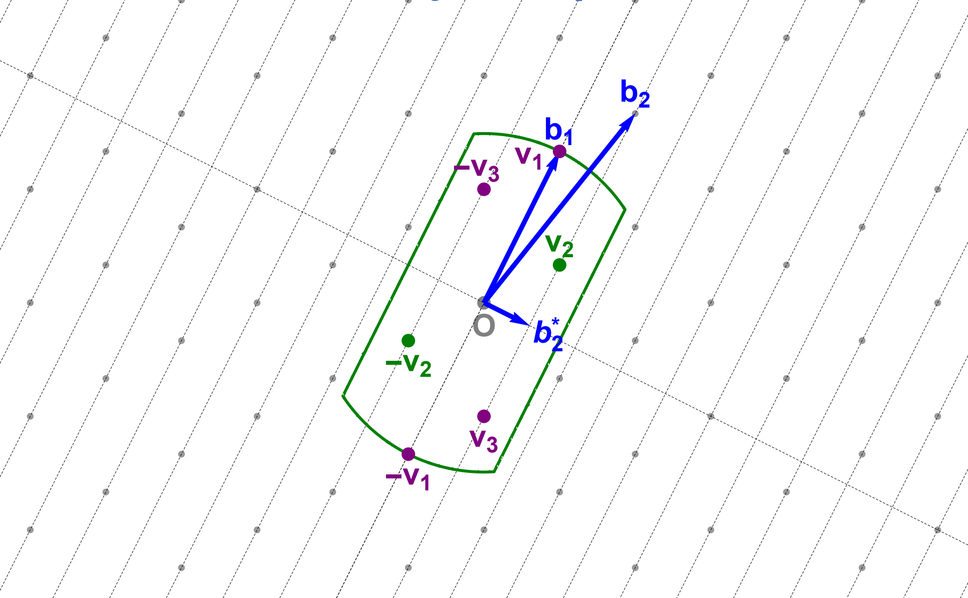

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"Well-chosen bounding functions lead asymptotically to an exponential speedup of about $2^{n/4}$ over basic enumeration, maintaining a success probability $\geq$ 95%."

— Gama–Nguyen–Regev, *EUROCRYPT'10*

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"Well-chosen bounding functions lead asymptotically to an exponential speedup of about $2^{n/4}$ *over basic enumeration, maintaining a success probability* $\geq 95\%$*."*

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"With extreme pruning, the probability of finding the desired vector is actually rather low (say, 0.1%), but surprisingly, the running time of the enumeration is reduced by a much more significant factor (say, much more than 1000)."

— Gama–Nguyen–Regev, *EUROCRYPT'10*

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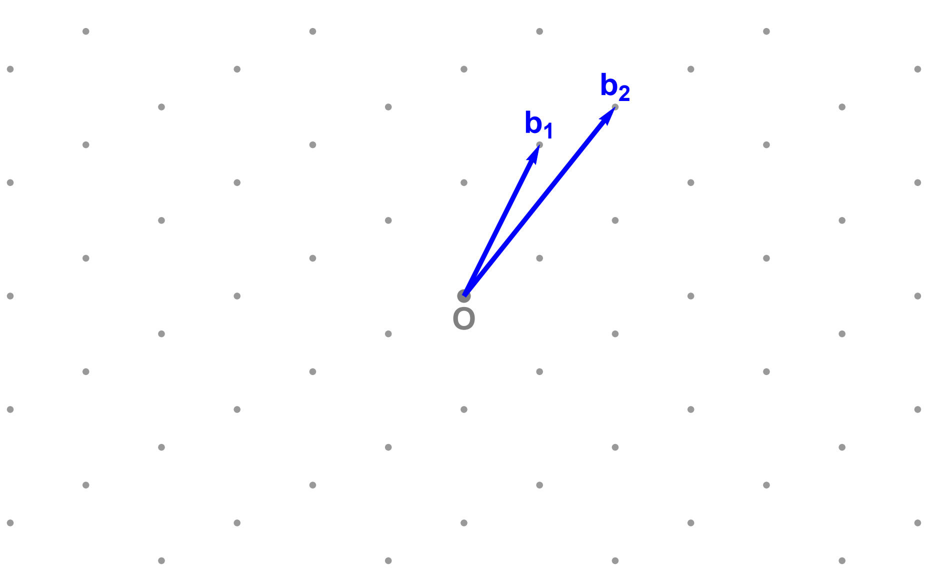
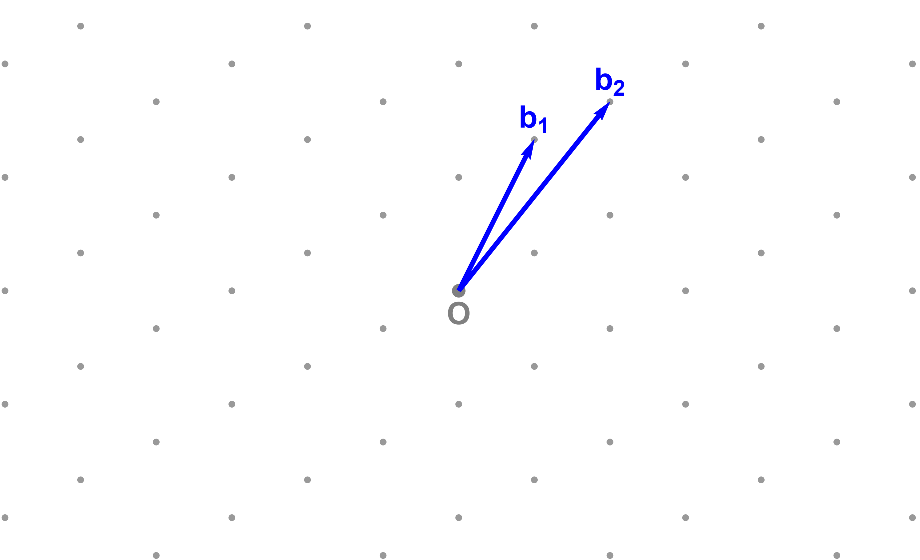
The Voronoi cell algorithm

Constructing the Voronoi cell of L



The Voronoi cell algorithm

Solve CVP in $2L$ for $\{0,1\}\mathbf{b}_1 + \cdots + \{0,1\}\mathbf{b}_n$



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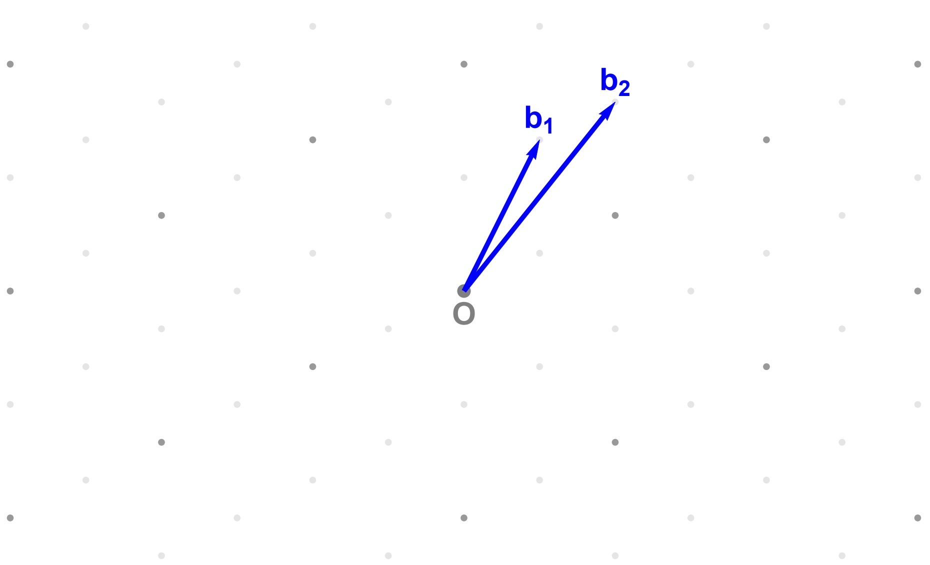

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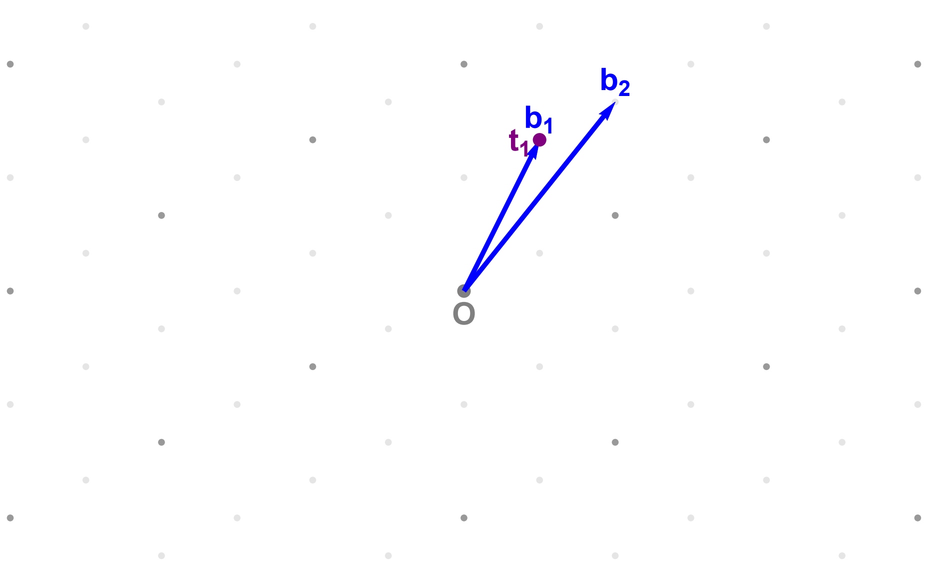

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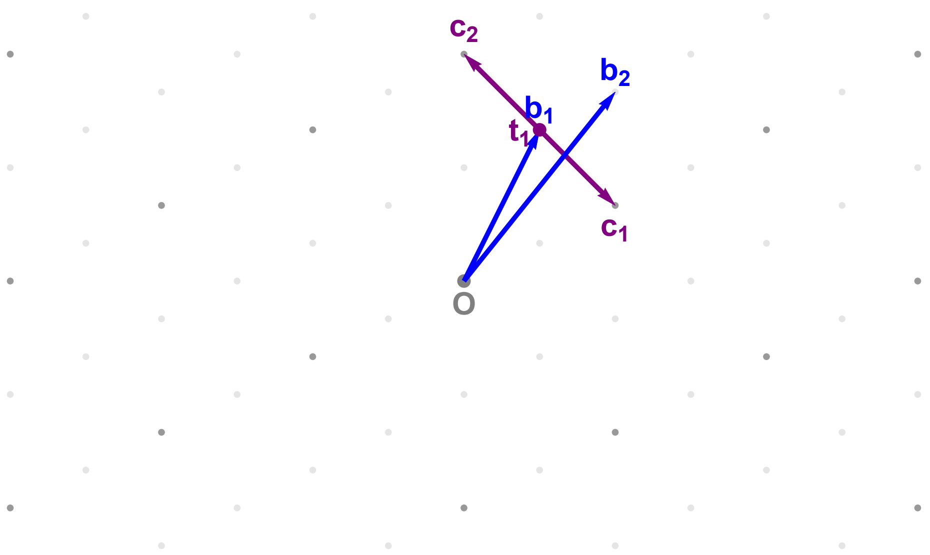

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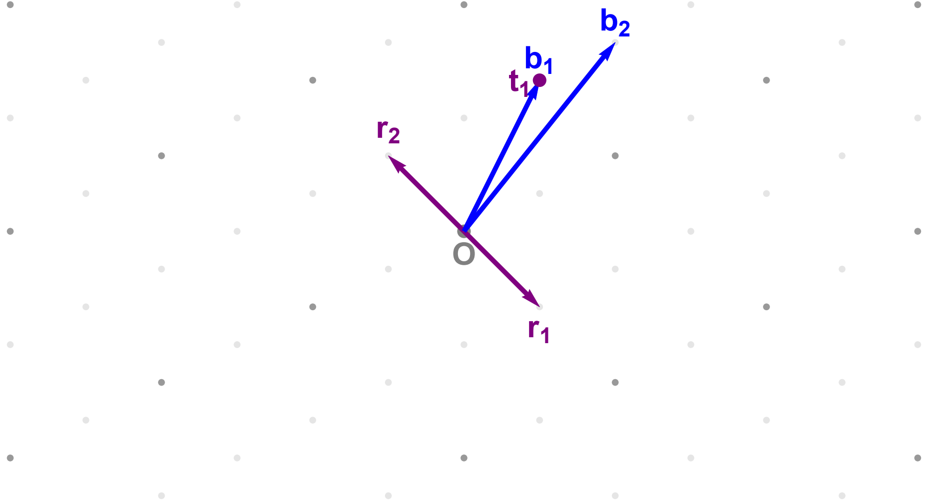

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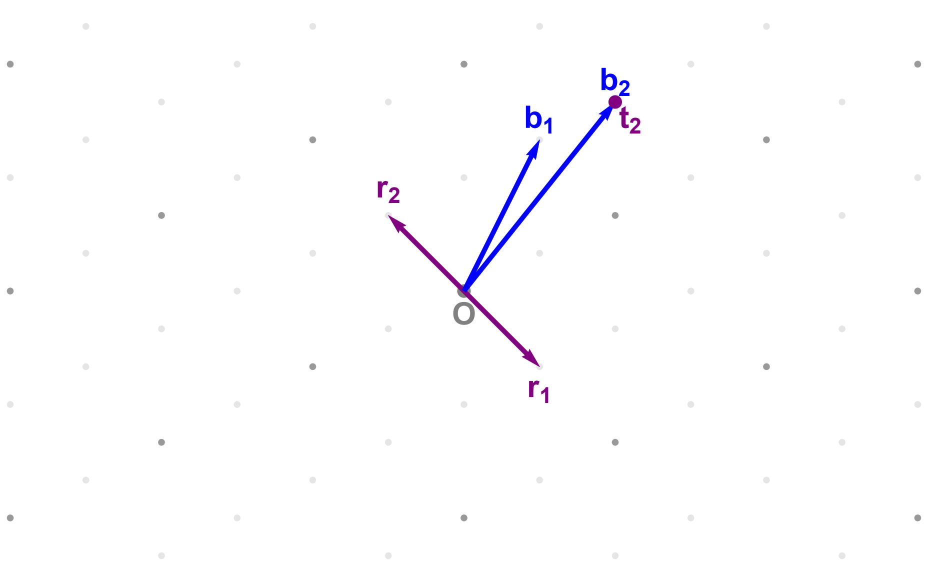

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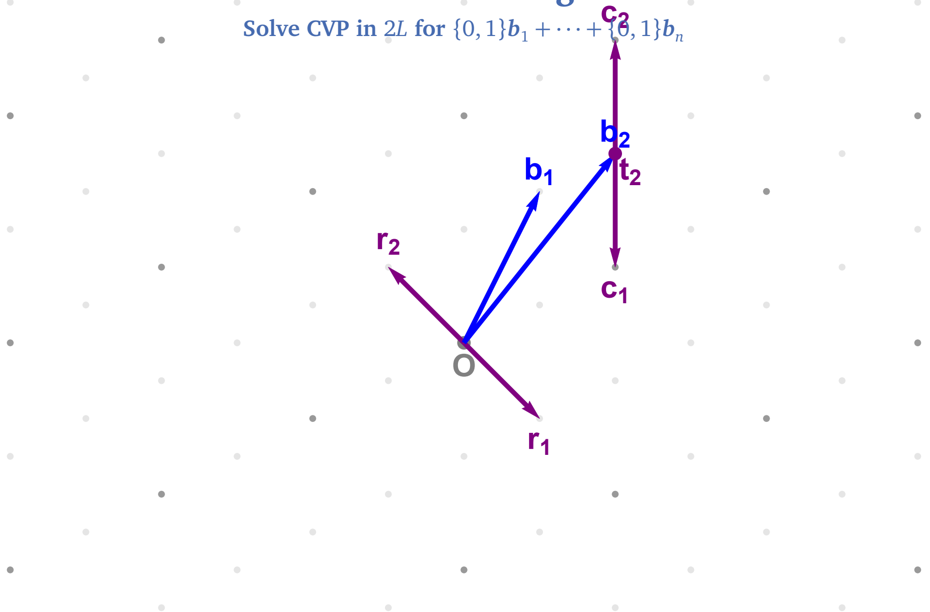

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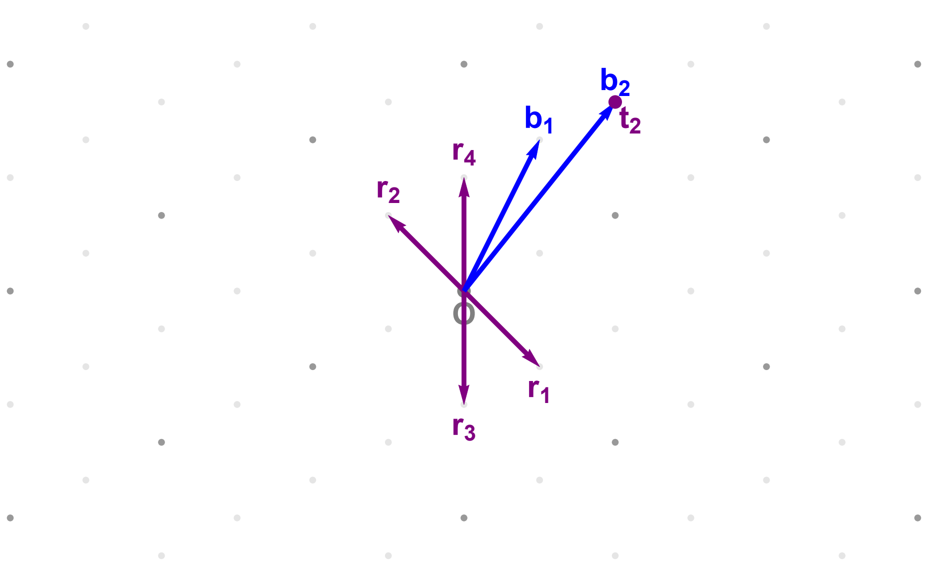

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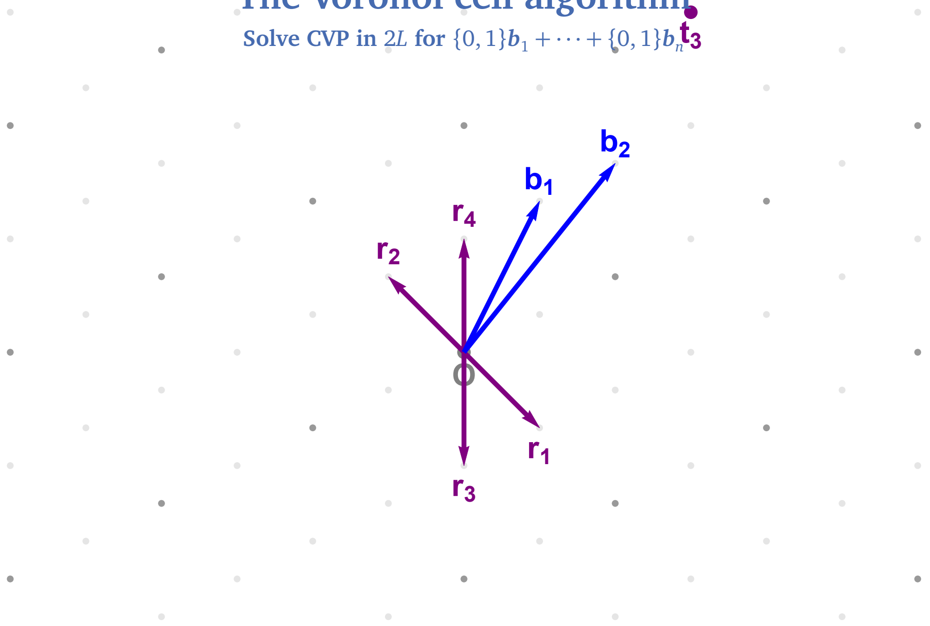

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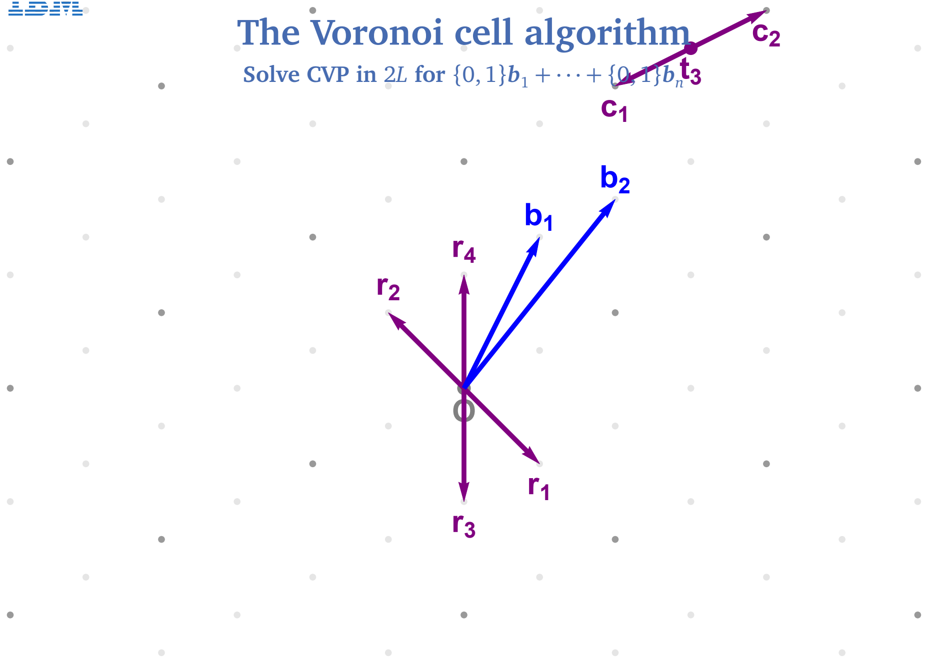

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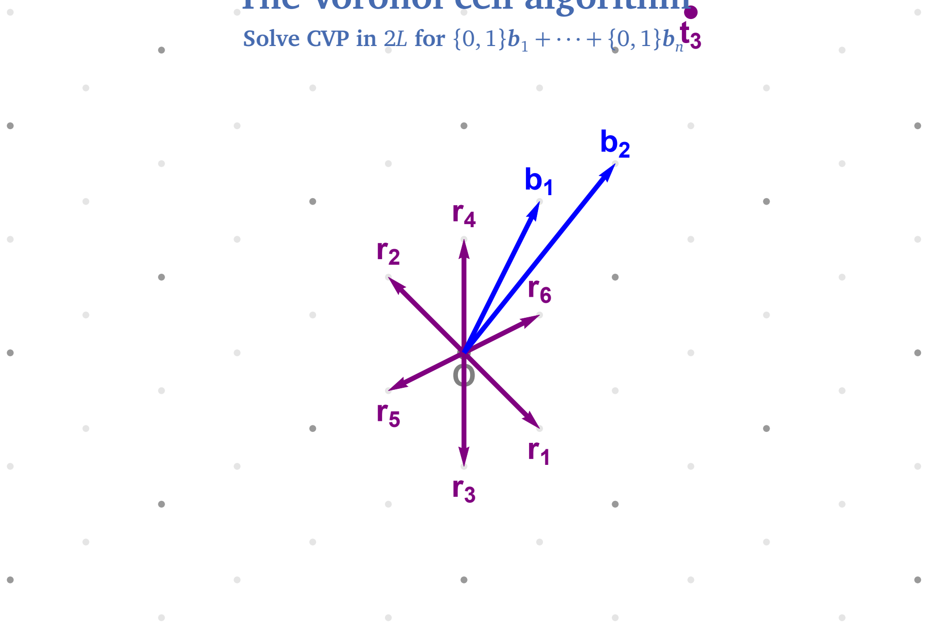
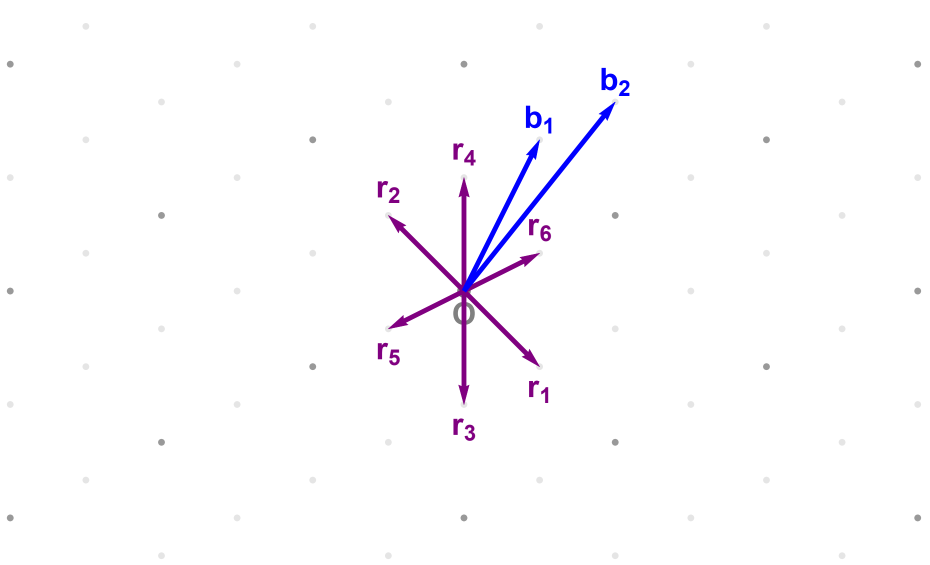


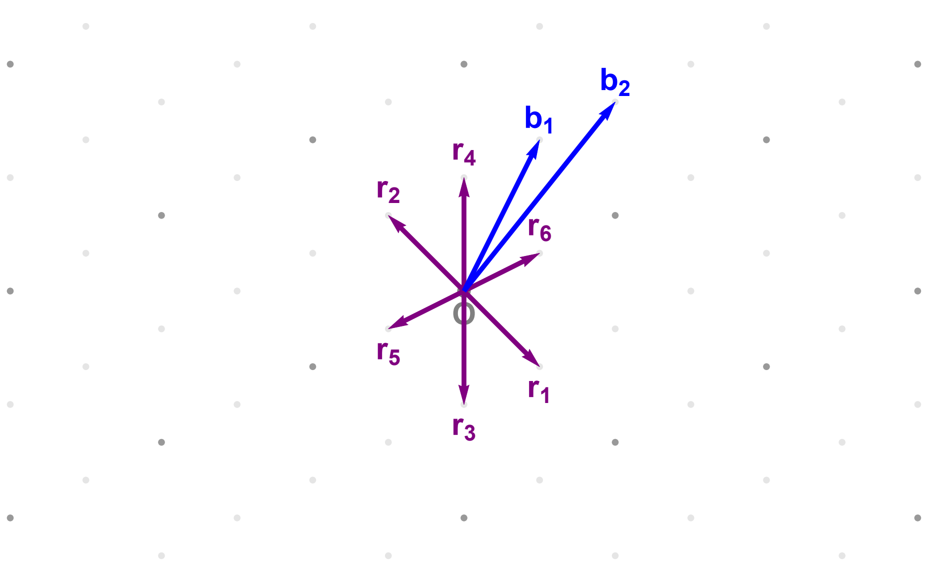
The Voronoi cell algorithm

Solve CVP in $2L$ for $\{0,1\}b_1 + \cdots + \{0,1\}b_n$



The Voronoi cell algorithm

Resulting vectors define Voronoi cell of L



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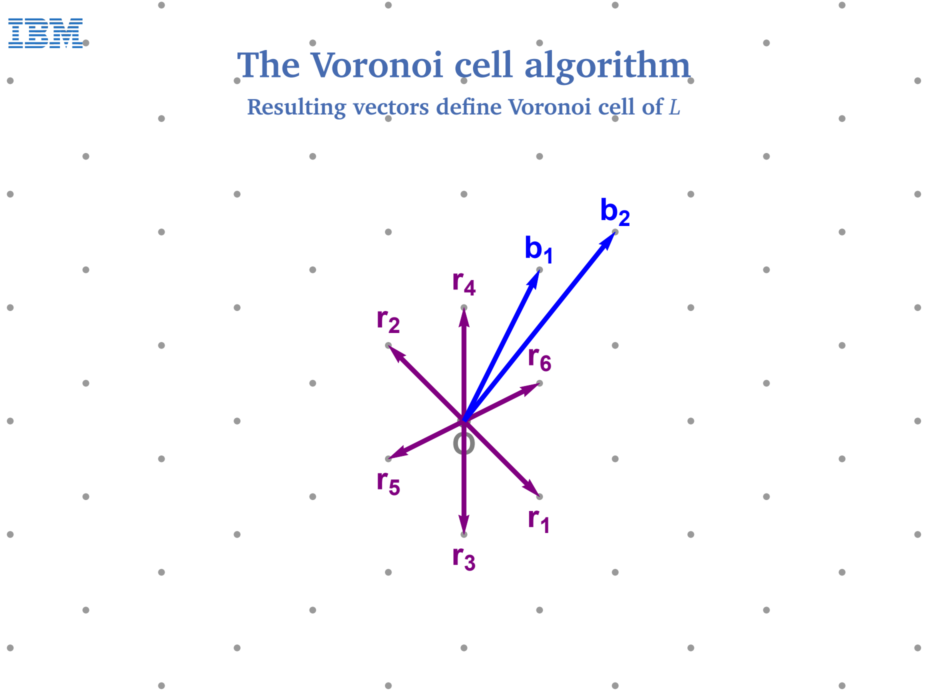

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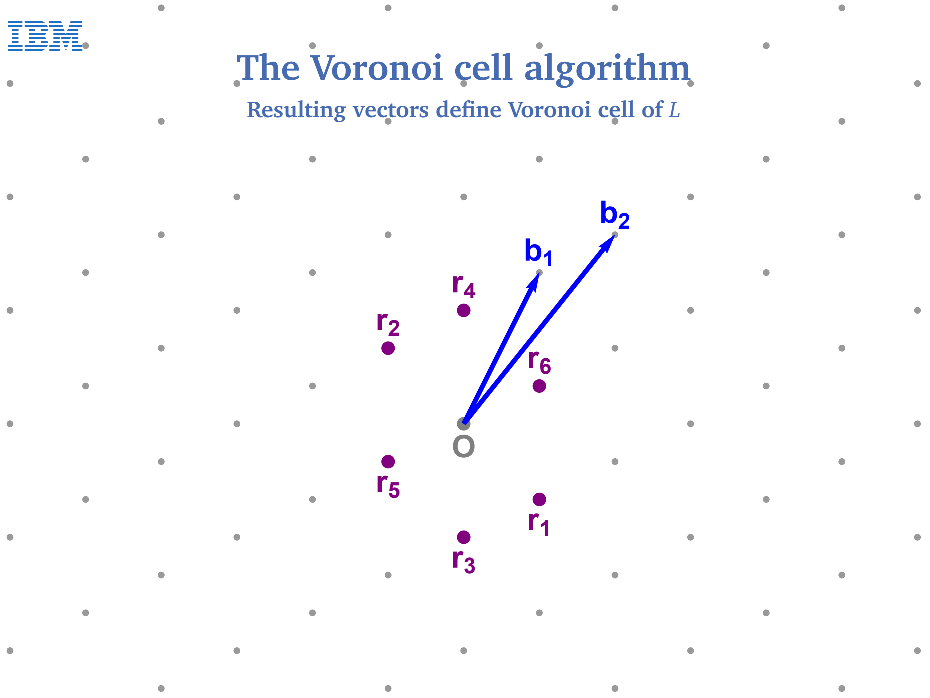

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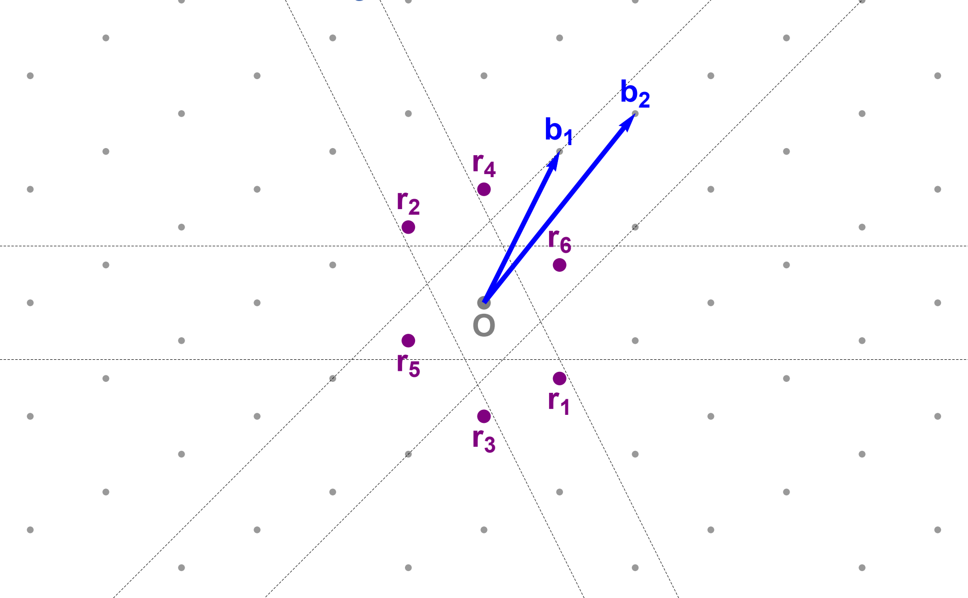

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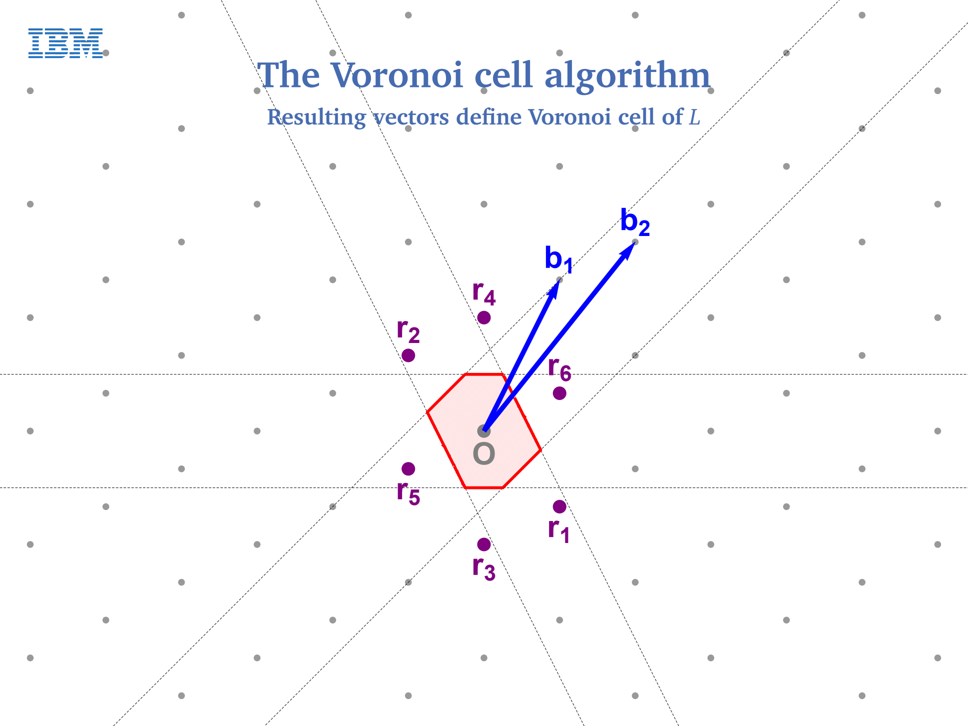
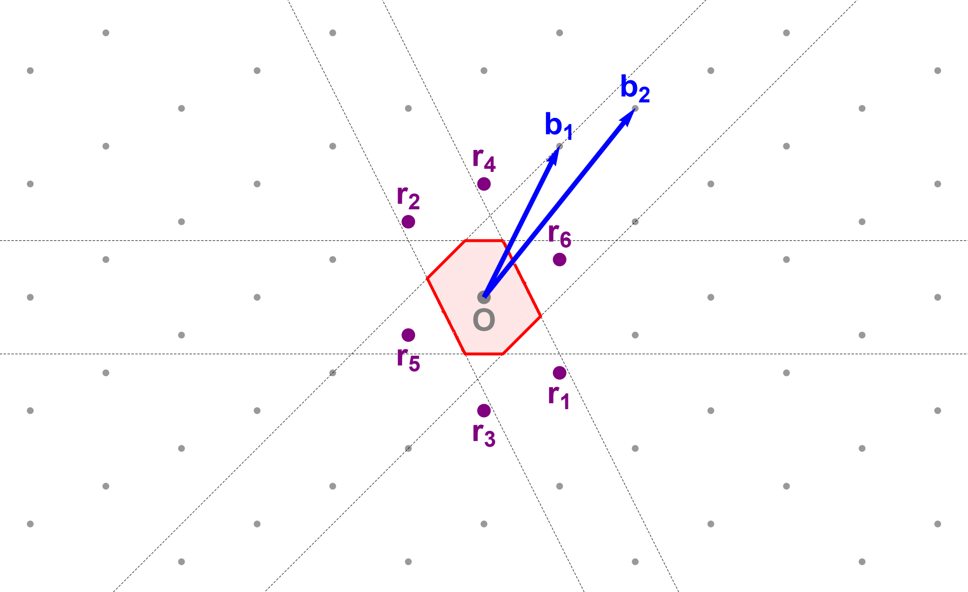


The Voronoi cell algorithm

Find a shortest vector among these vectors



IBM

The Voronoi cell algorithm

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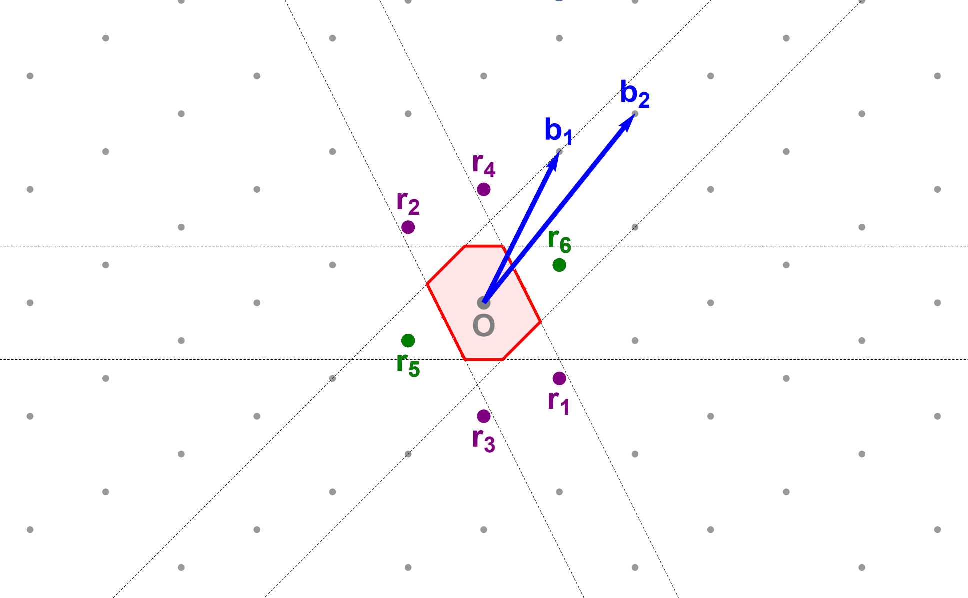


The Voronoi cell algorithm

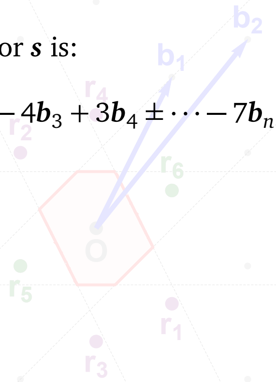
Analysis

The Voronoi cell algorithm

Analysis

Suppose the shortest vector $\boldsymbol{s}$ is:

$$\boldsymbol{s} = 5\boldsymbol{b}_1 + 2\boldsymbol{b}_2 - 4\boldsymbol{b}_3 + 3\boldsymbol{b}_4 \pm \cdots - 7\boldsymbol{b}_n$$



The Voronoi cell algorithm

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$$\boldsymbol{s} = 5\boldsymbol{b}_1 + 2\boldsymbol{b}_2 - 4\boldsymbol{b}_3 + 3\boldsymbol{b}_4 \pm \cdots - 7\boldsymbol{b}_n$$

Let the vector $\boldsymbol{t} \in \{0,1\}^n \cdot B$ be such that $\boldsymbol{s} + \boldsymbol{t} \in 2L$:

$$\begin{aligned} \boldsymbol{t} &= 1\boldsymbol{b}_1 + 0\boldsymbol{b}_2 + 0\boldsymbol{b}_3 + 1\boldsymbol{b}_4 \pm \cdots + 1\boldsymbol{b}_n \\ \boldsymbol{s} + \boldsymbol{t} &= 6\boldsymbol{b}_1 + 2\boldsymbol{b}_2 - 4\boldsymbol{b}_3 + 4\boldsymbol{b}_4 \pm \cdots - 6\boldsymbol{b}_n \qquad (\in 2L) \end{aligned}$$

The Voronoi cell algorithm

Analysis

Suppose the shortest vector $\boldsymbol{s}$ is:

$$\boldsymbol{s} = 5\boldsymbol{b}_1 + 2\boldsymbol{b}_2 - 4\boldsymbol{b}_3 + 3\boldsymbol{b}_4 \pm \cdots - 7\boldsymbol{b}_n$$

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The vectors closest to $\boldsymbol{t}$ in the sublattice $2L$ are $\boldsymbol{t} \pm \boldsymbol{s}$.

The Voronoi cell algorithm

Overview

Theorem (Micciancio–Voulgaris, SODA'10)

The Voronoi cell algorithm can solve SVP in time $2^{2n+o(n)}$ *and space* $2^{n+o(n)}$.

The Voronoi cell algorithm

Overview

Theorem (Micciancio–Voulgaris, SODA'10)

The Voronoi cell algorithm can solve SVP in time $2^{2n+o(n)}$ *and space* $2^{n+o(n)}$.

Essentially reduces SVP_n (CVP_n) to CVP_{n-1} with $2^{O(n)}$ overhead

Outline

Lattices

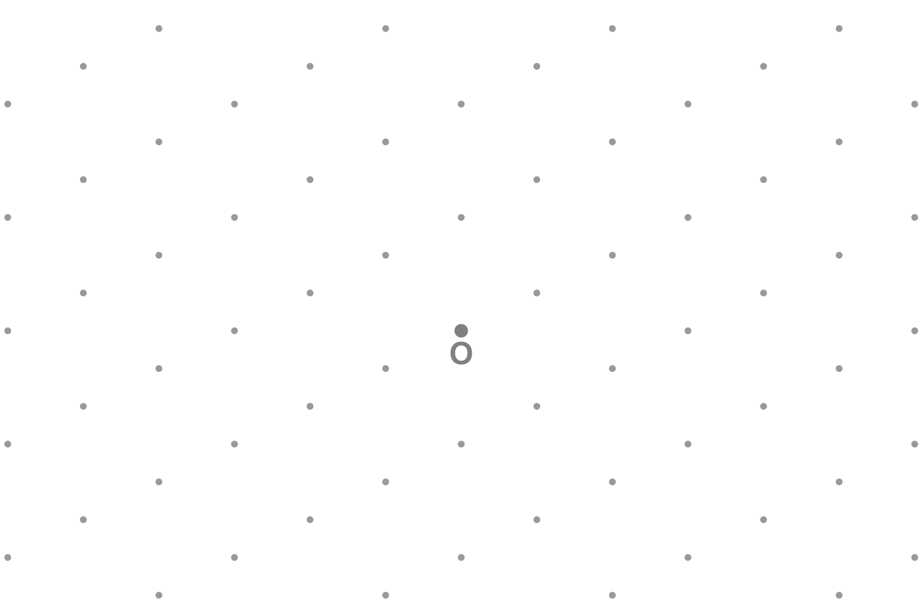
Enumeration algorithms
- Fincke–Pohst enumeration
- Kannan enumeration
- (Extreme) pruning

Constructing the Voronoi cell

Sieving algorithms
- Basic sieving
- Leveled sieving
- Nearest neighbor searching

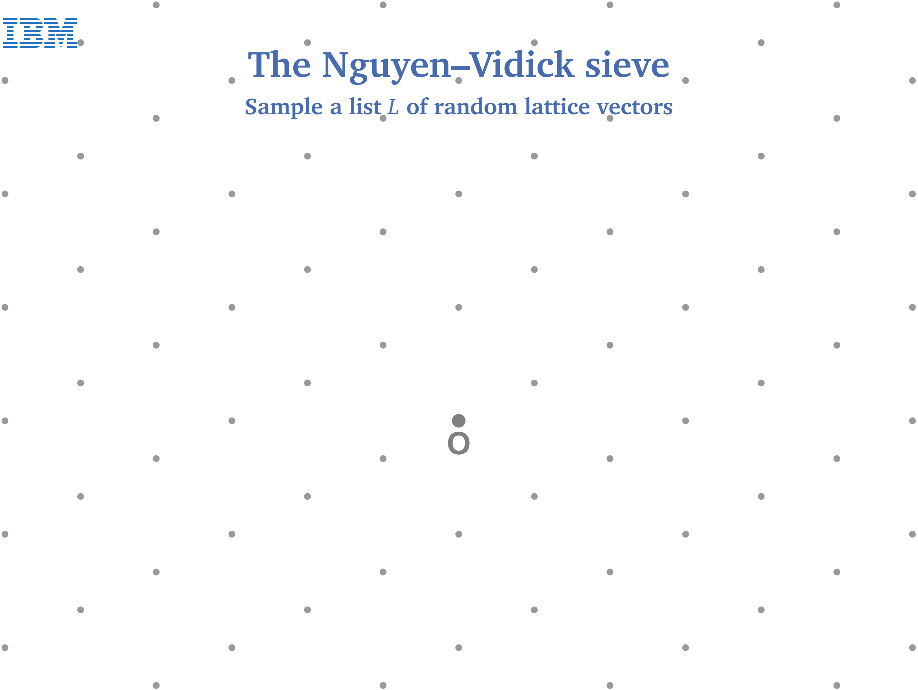
Conclusion

The Nguyen–Vidick sieve



The Nguyen–Vidick sieve

Sample a list L of random lattice vectors



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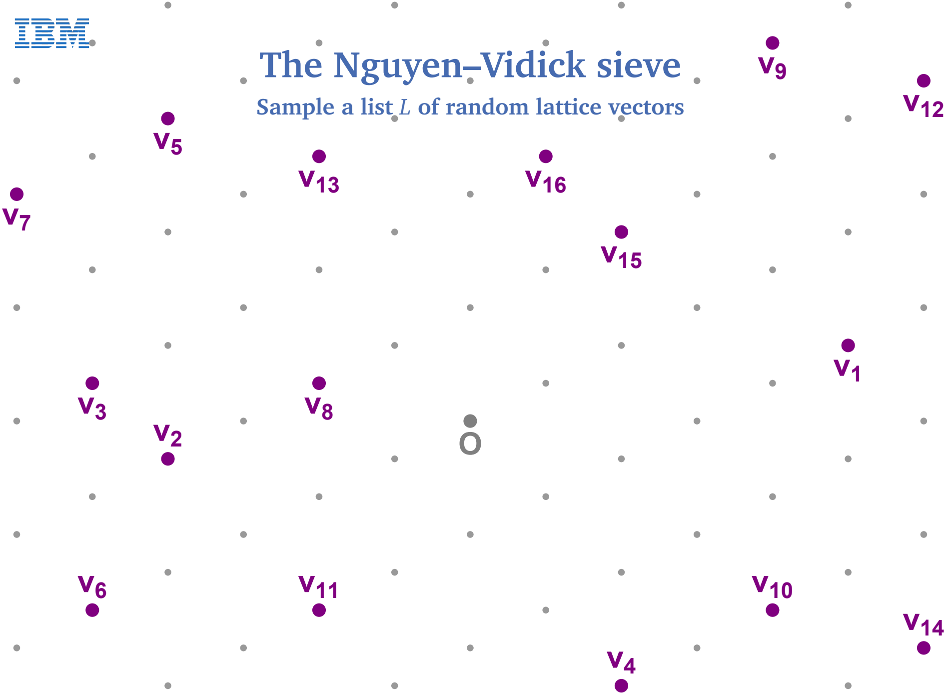
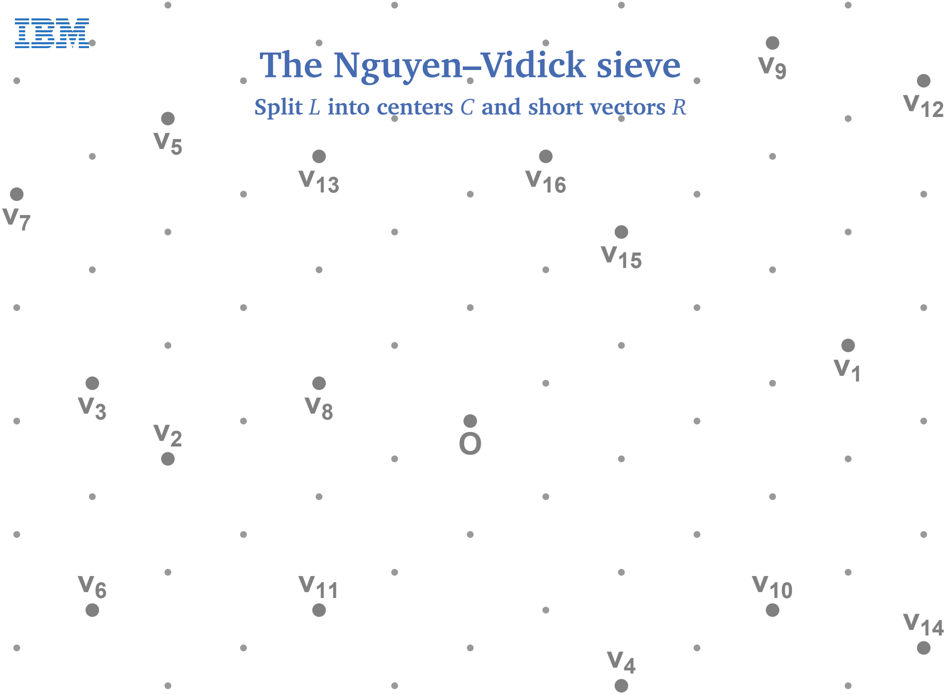


The Nguyen–Vidick sieve

Split L into centers C and short vectors R



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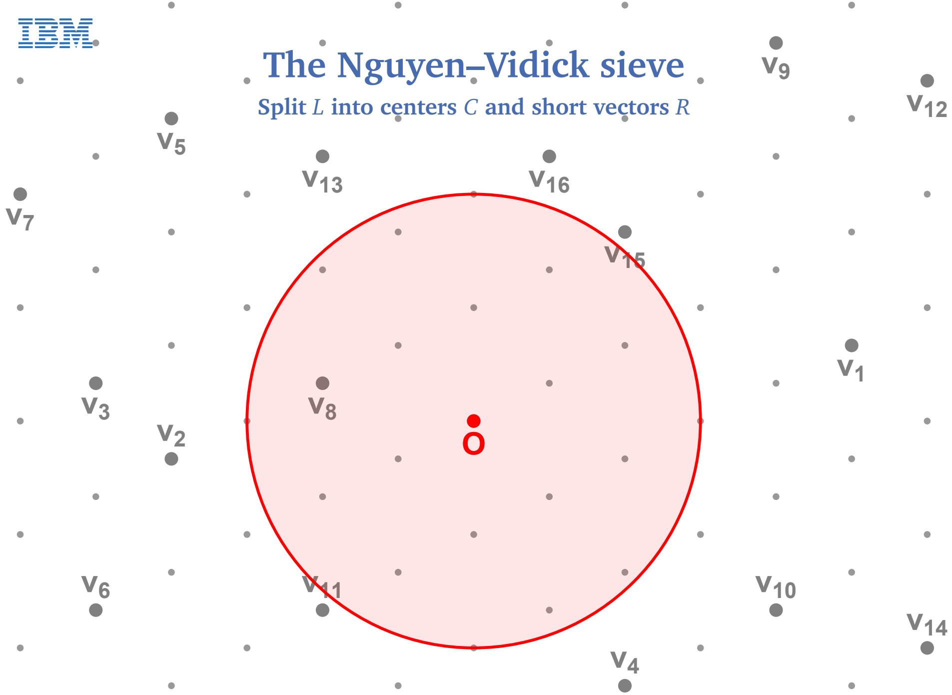

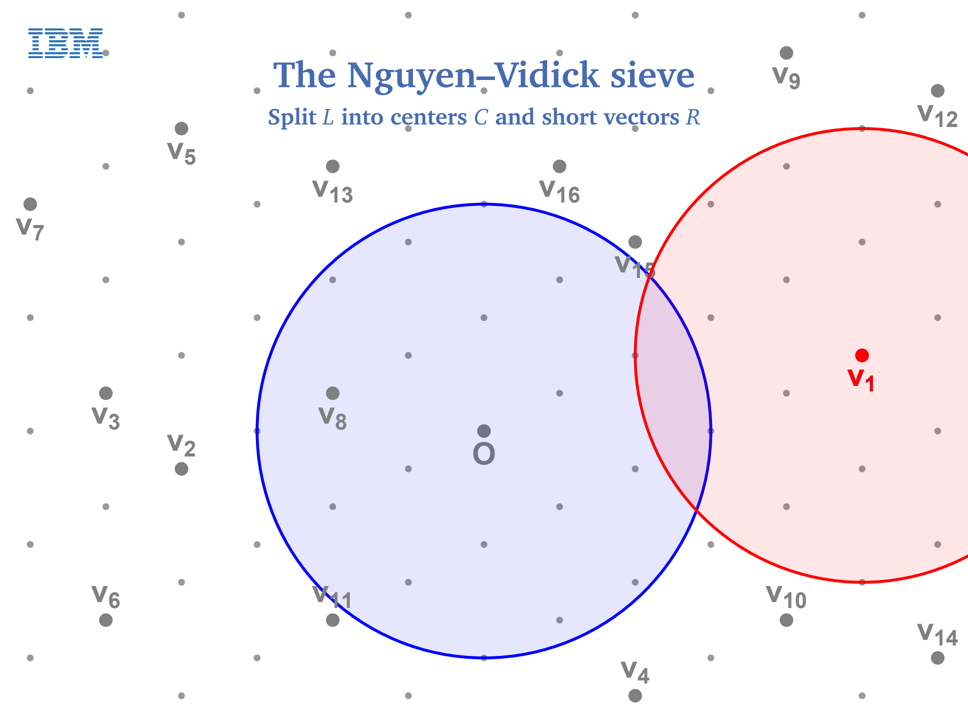
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The Nguyen–Vidick sieve
Split L into centers C and short vectors R
v9
v12
v5
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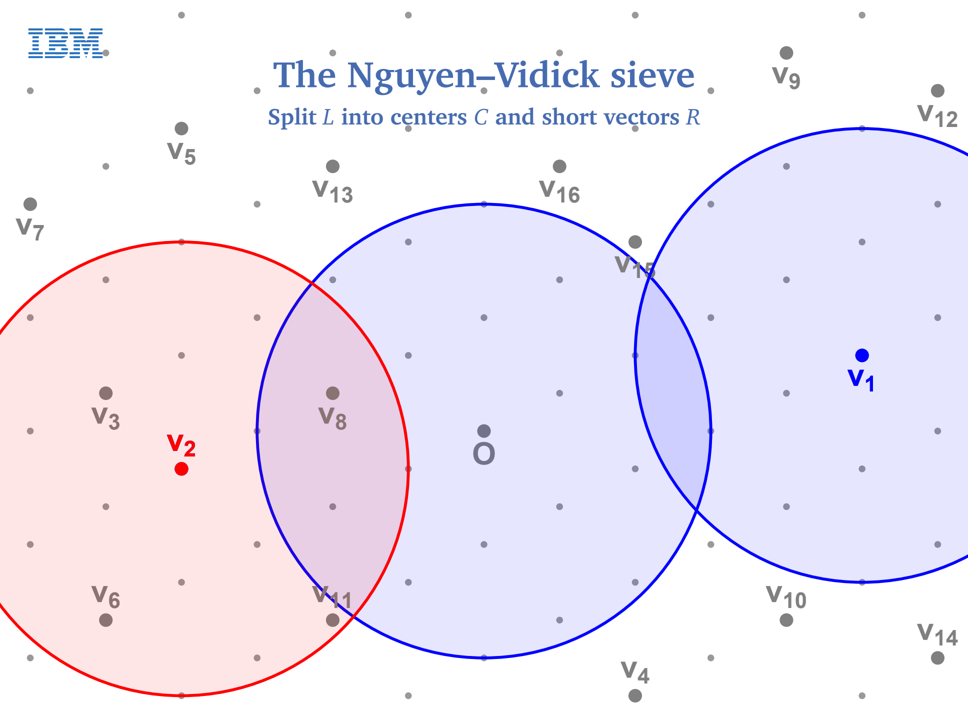
v11
v10
v14
v4


The Nguyen–Vidick sieve
Split L into centers C and short vectors R
v_1
v_2
v_3
v_4
v_5
v_6
v_7
v_8
v_9
v_{10}
v_{11}
v_{12}
v_{13}
v_{14}
v_{15}
v_{16}
O

The Nguyen–Vidick sieve

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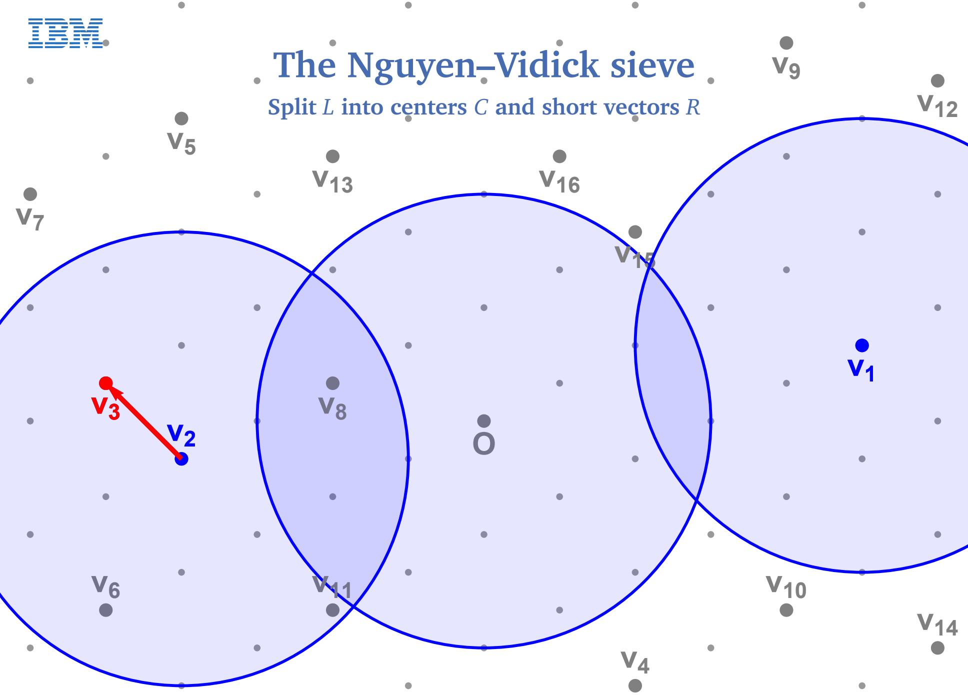

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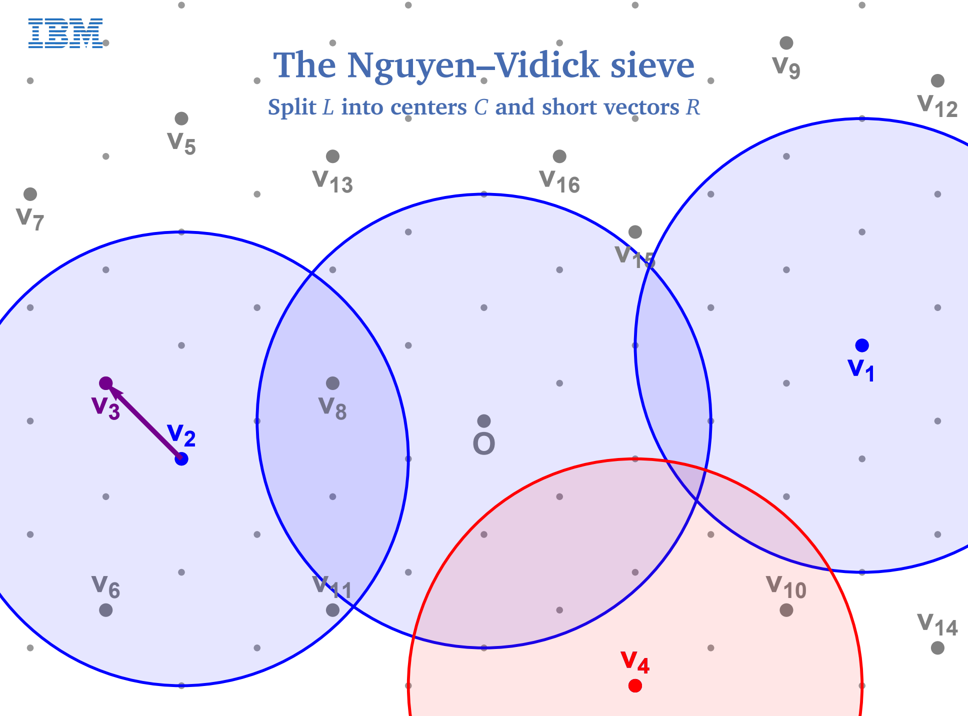

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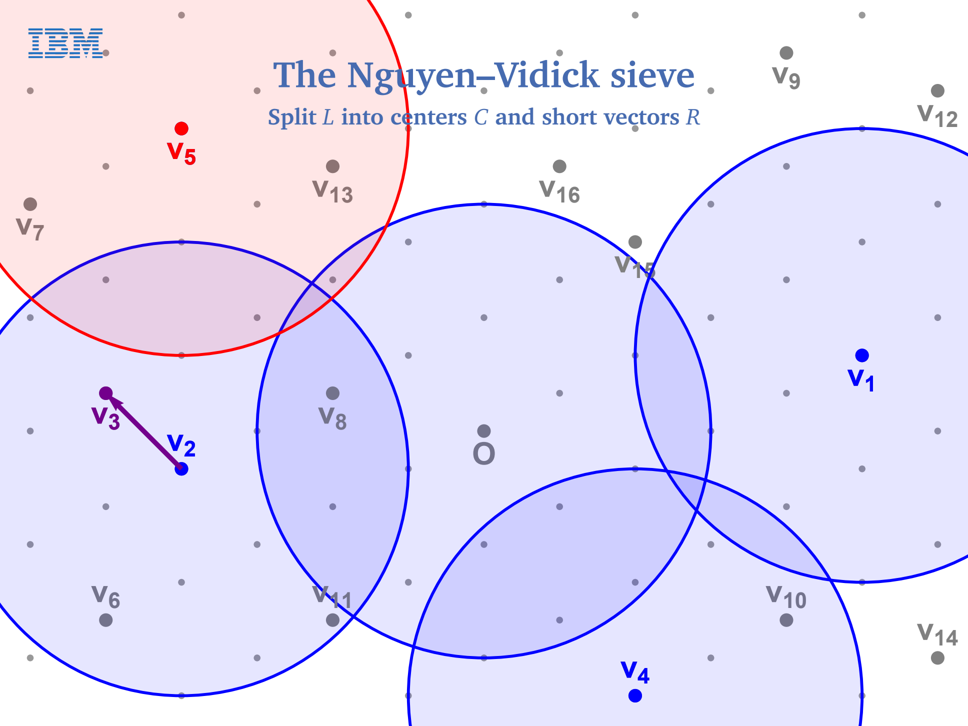

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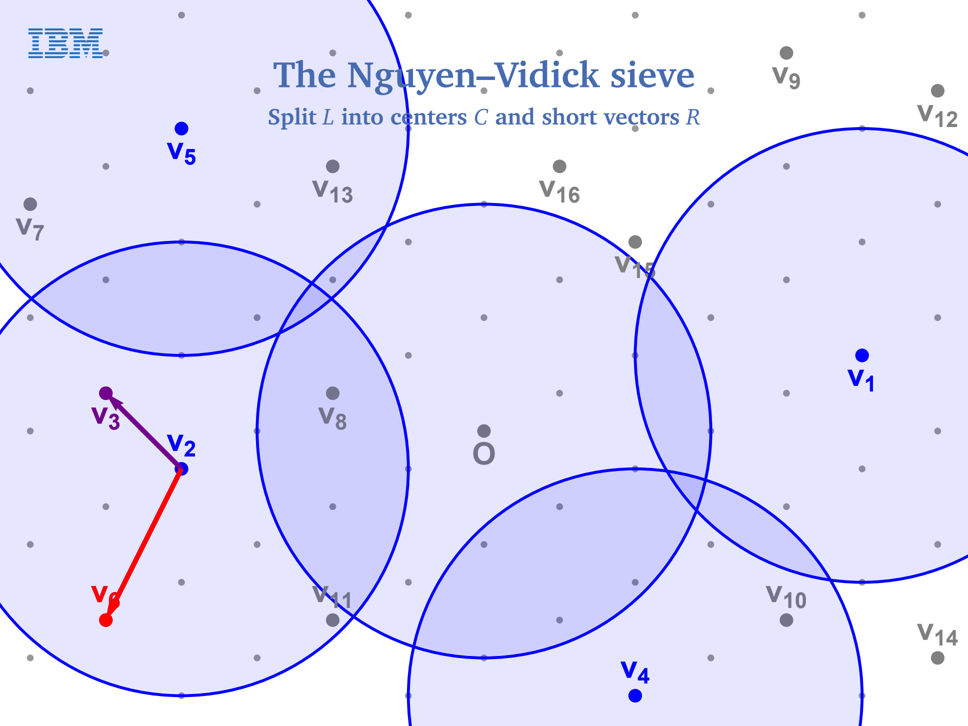

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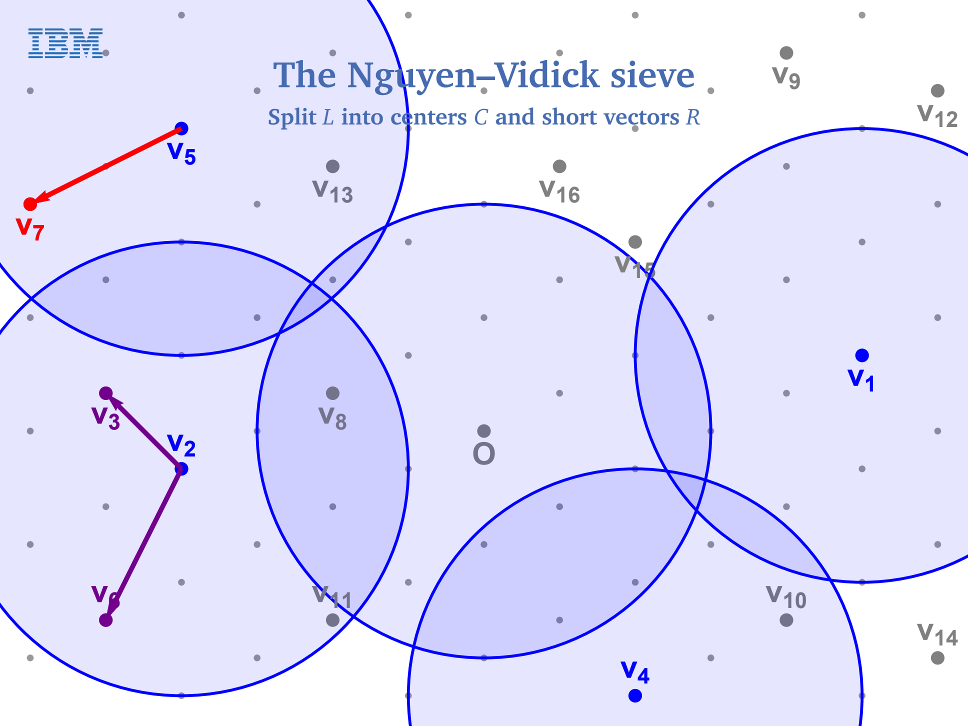

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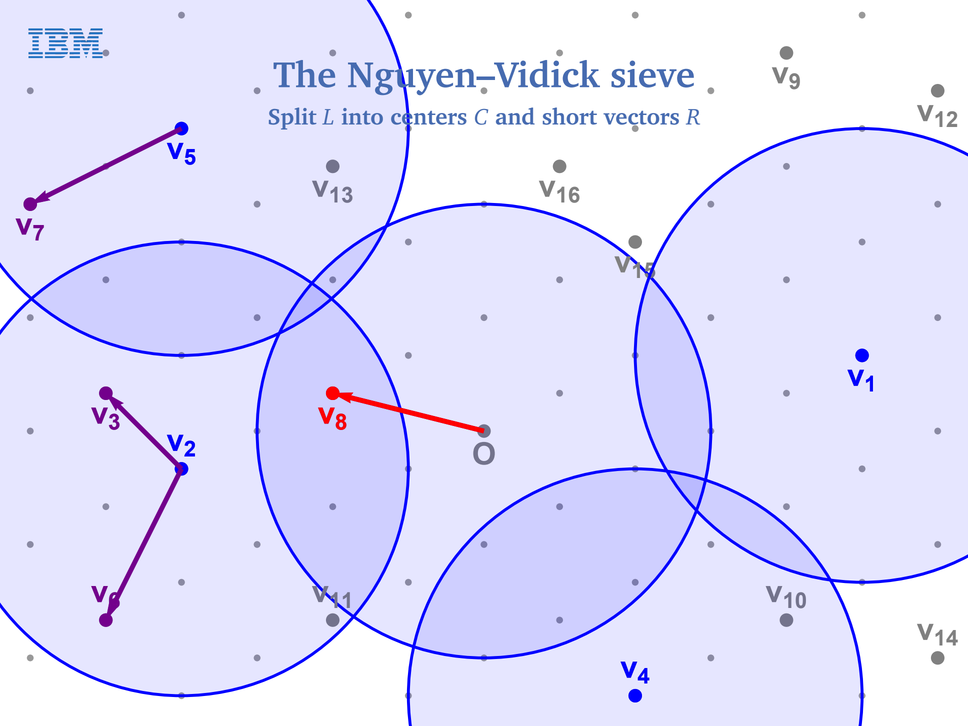

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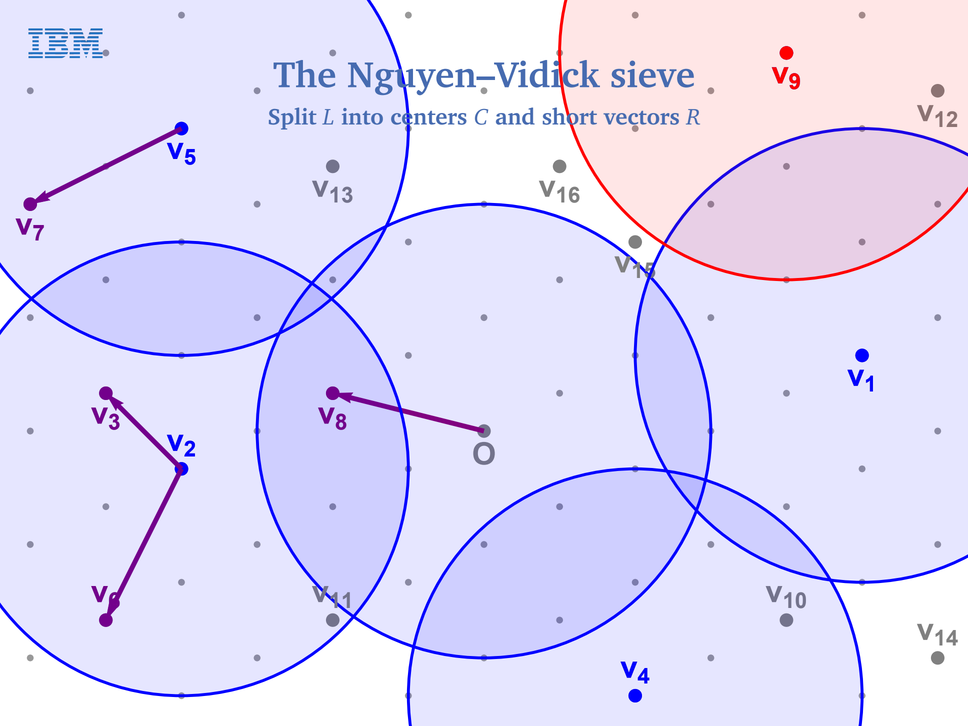

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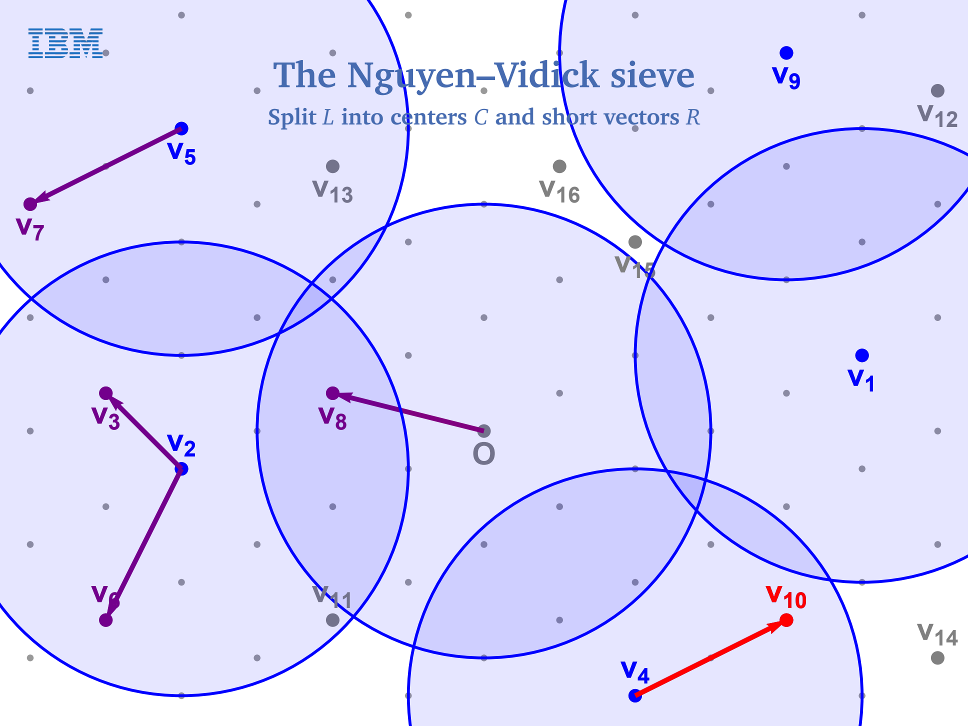

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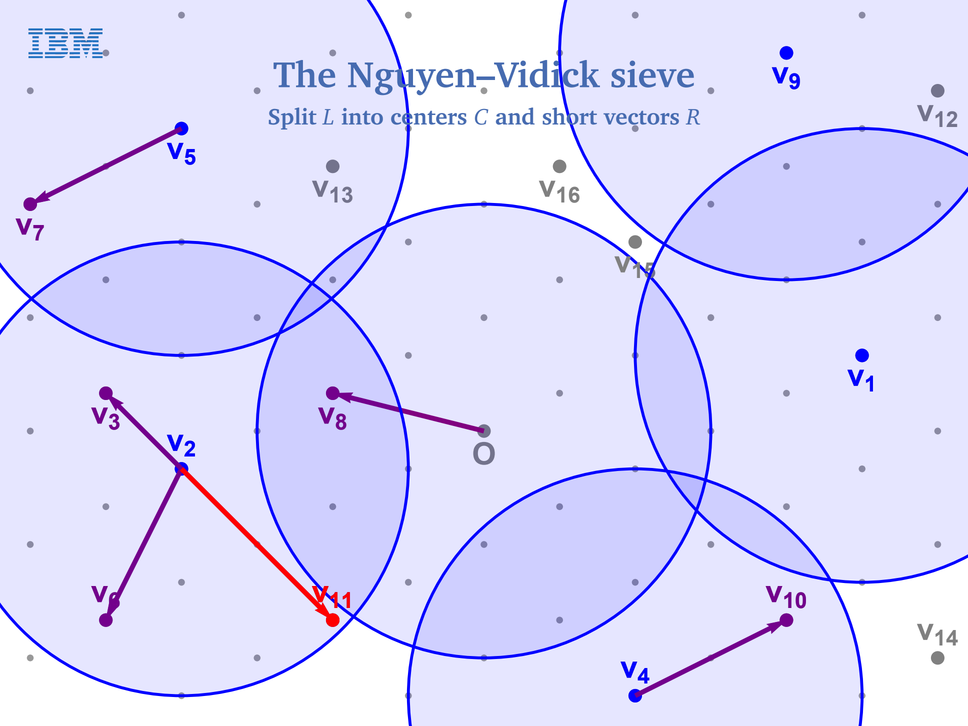

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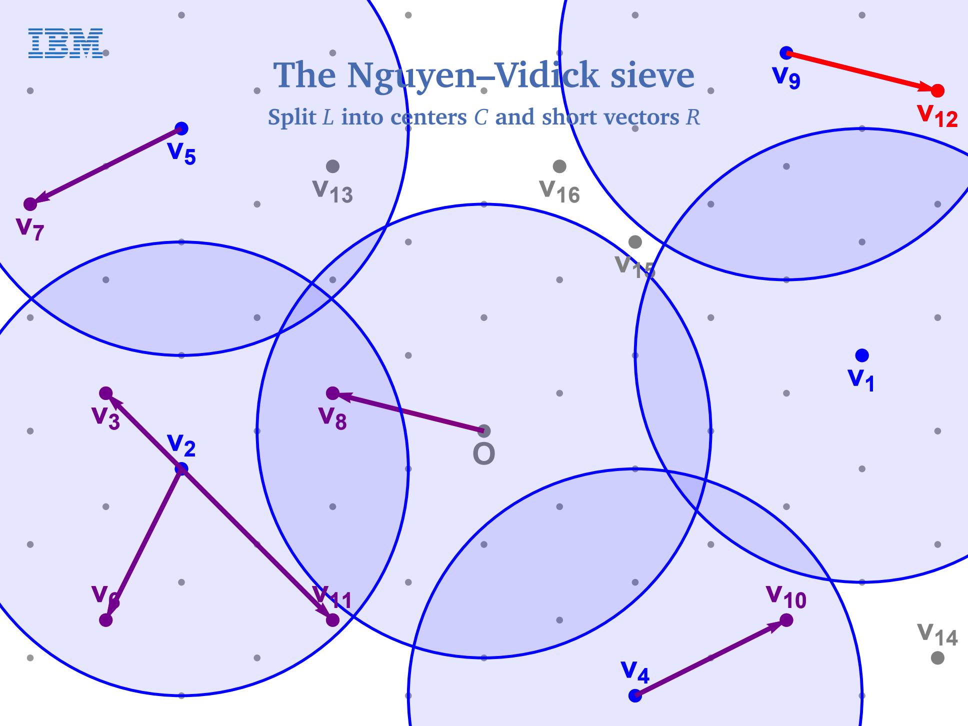

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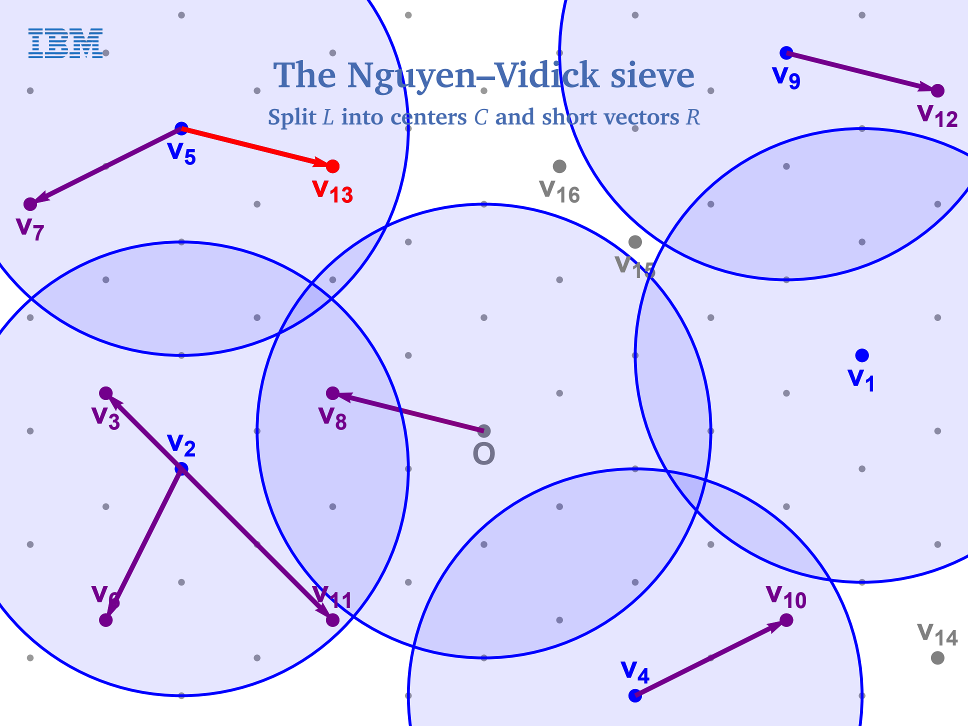

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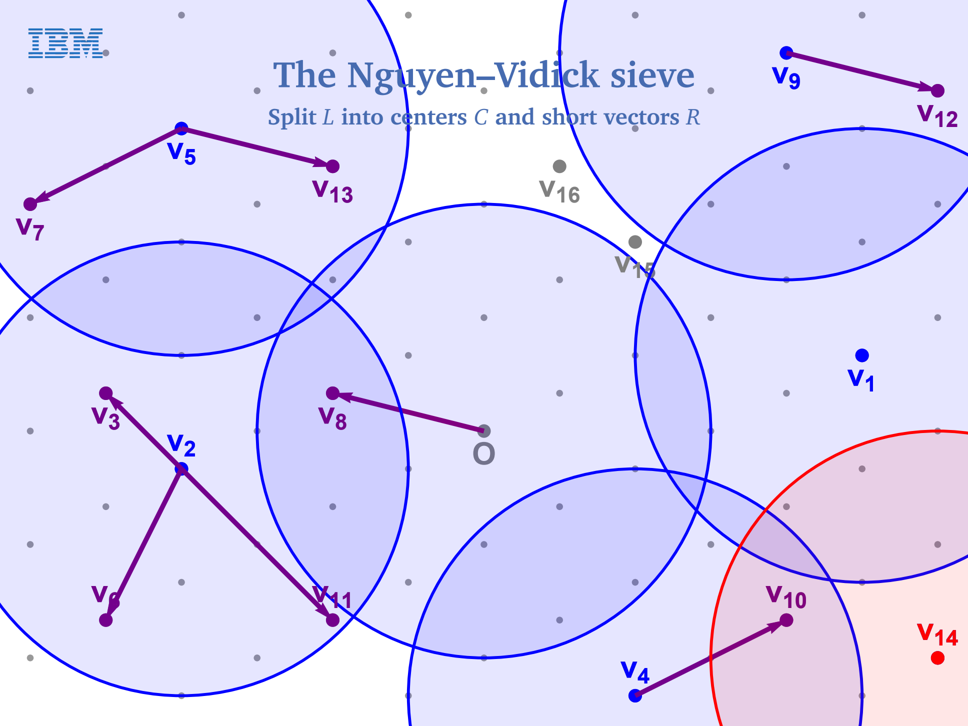

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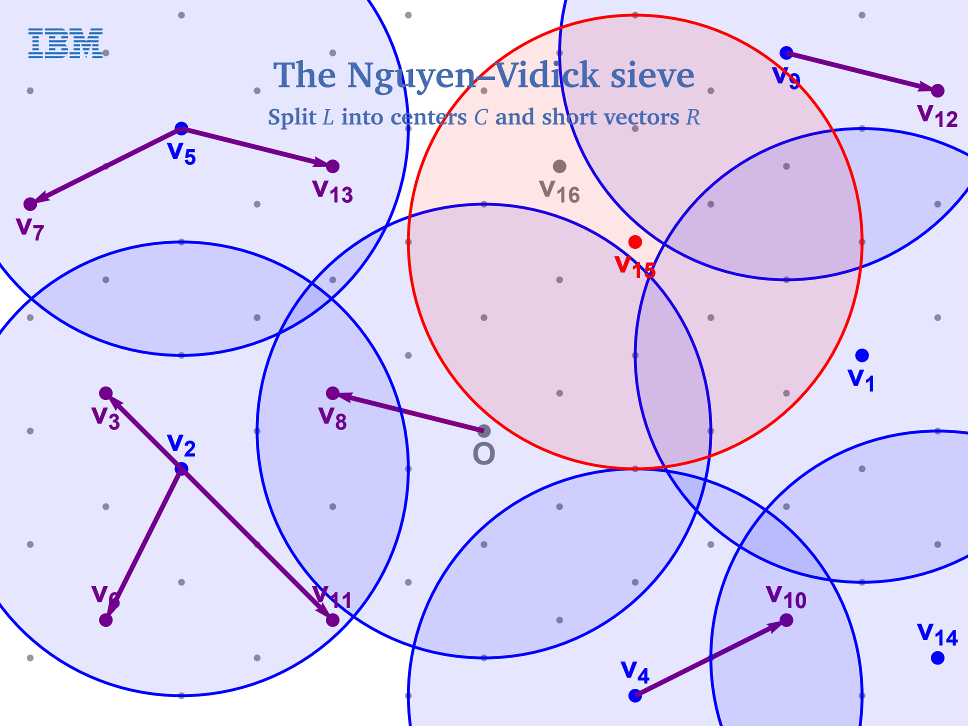

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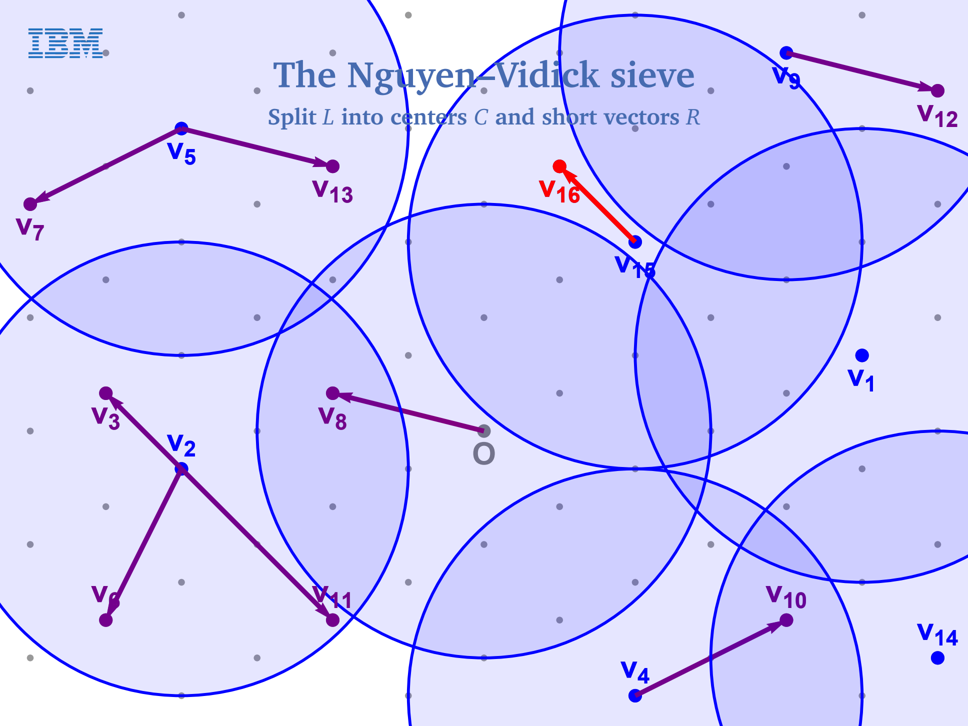

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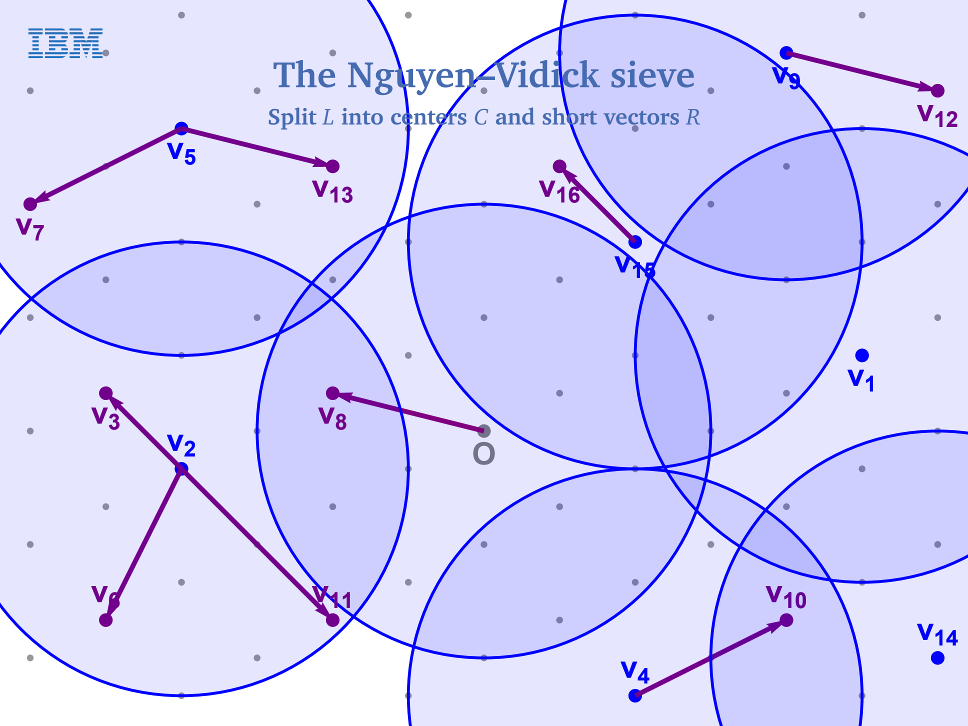
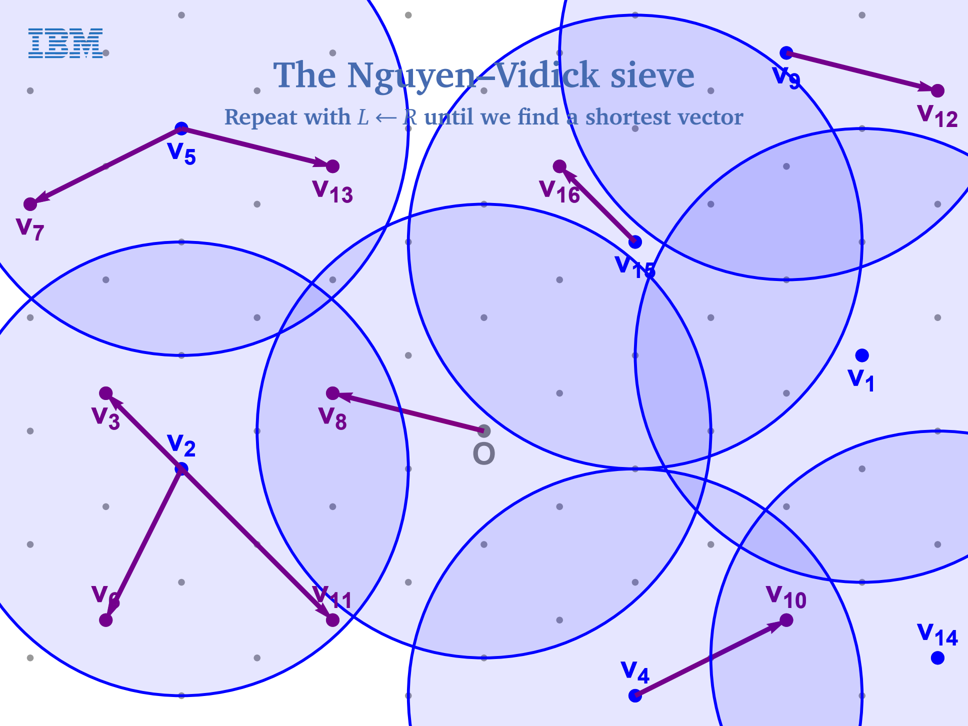


The Nguyen–Vidick sieve

Repeat with $L \leftarrow R$ until we find a shortest vector



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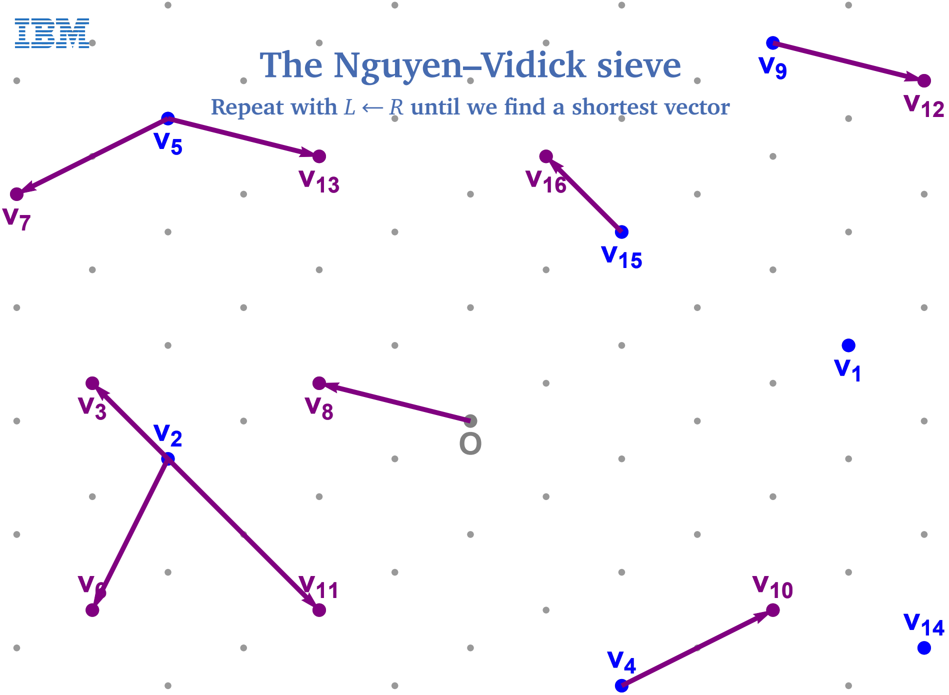

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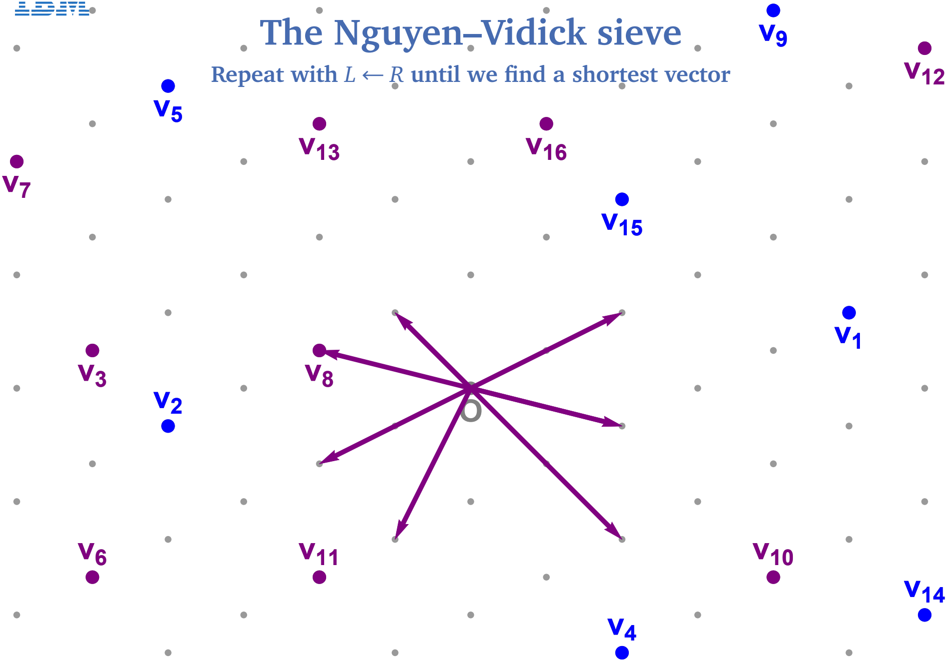

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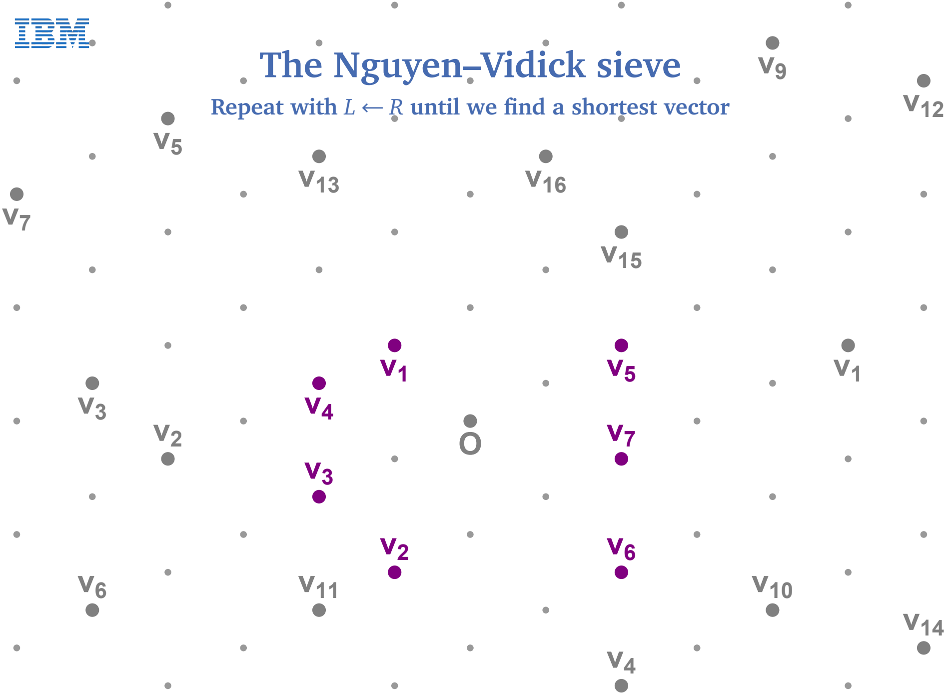
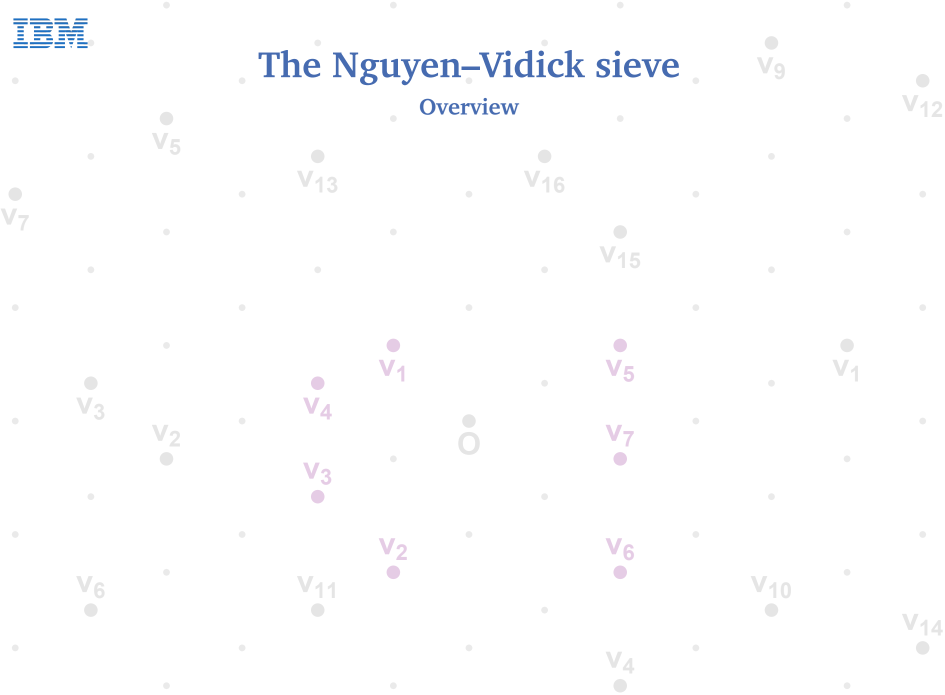


The Nguyen–Vidick sieve

Overview



The Nguyen–Vidick sieve

Overview

- Space complexity: $\sqrt{4/3}^n \approx 2^{0.21n+o(n)}$ vectors
  - Need $\sqrt{4/3}^n$ vectors to cover all corners of $\mathbb{R}^n$

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- Time complexity: $(4/3)^n \approx 2^{0.42n+o(n)}$
  - Comparing a target vector to all centers: $2^{0.21n+o(n)}$
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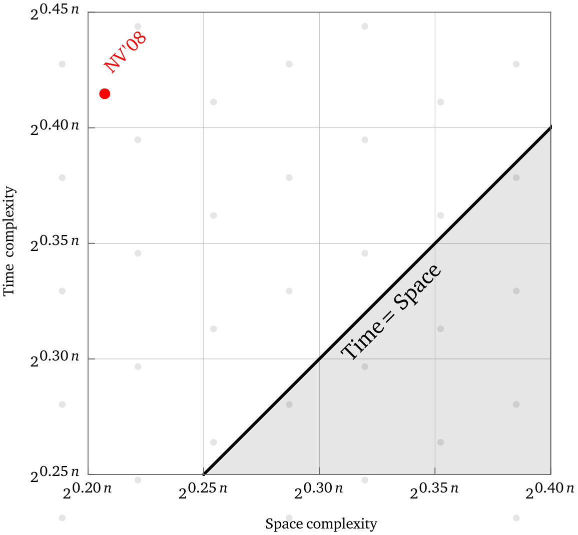
Heuristic result (Nguyen–Vidick, J. Math. Crypt. '08)

The NV-sieve runs in time $2^{0.42n+o(n)}$ and space $2^{0.21n+o(n)}$.

The Nguyen–Vidick sieve

Space/time trade-off



The GaussSieve

Generate random lattice vectors

The GaussSieve

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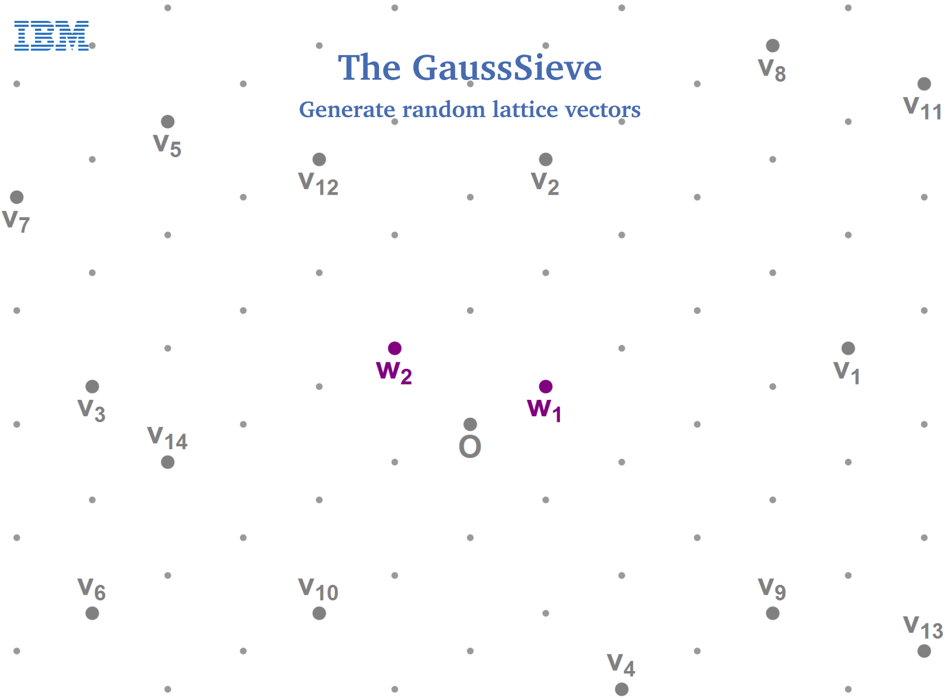
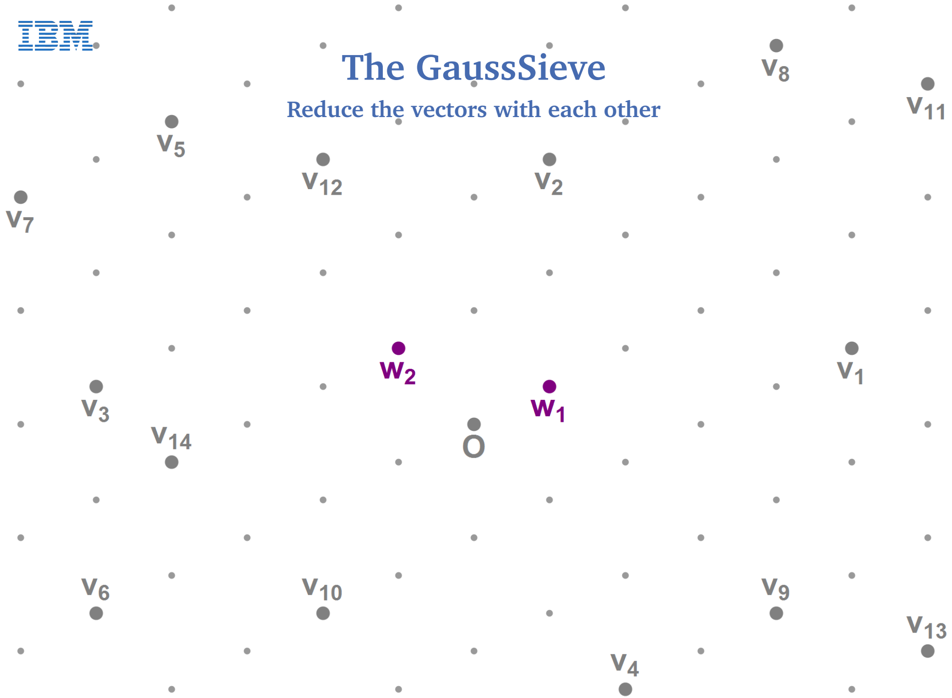


The GaussSieve

Reduce the vectors with each other



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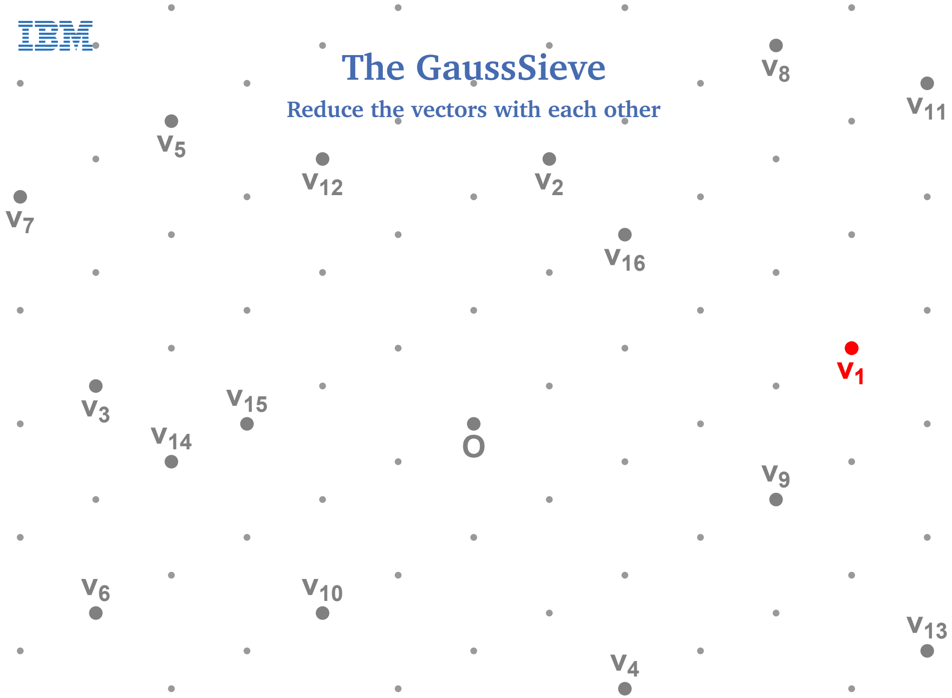

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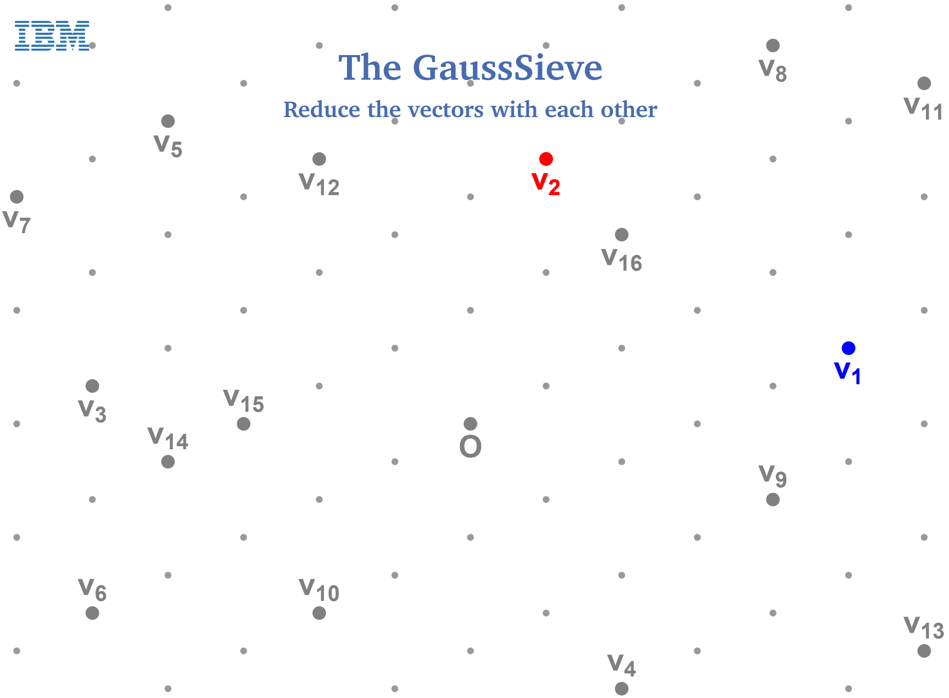

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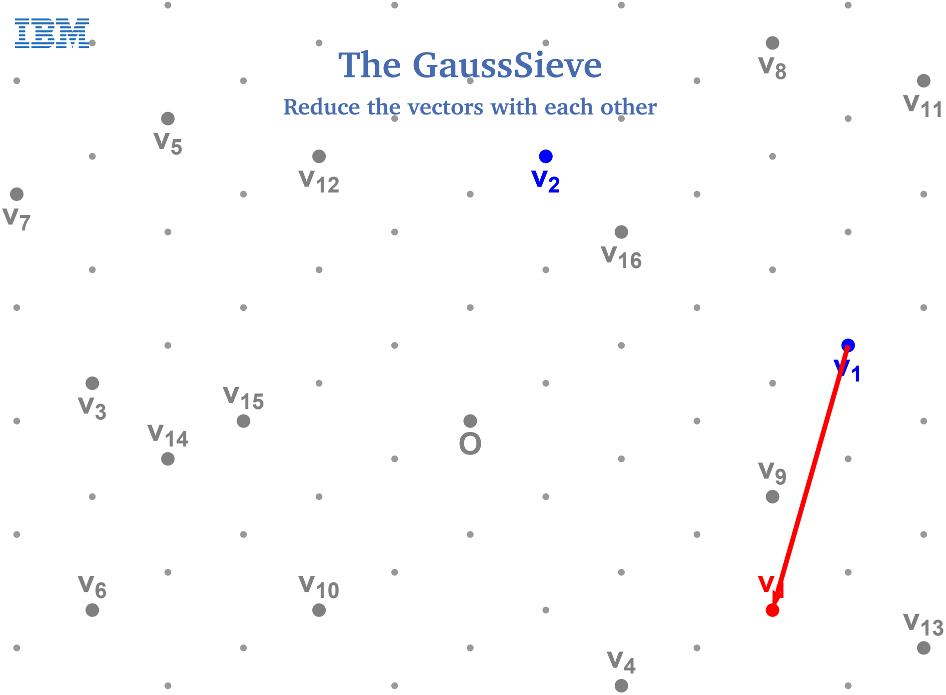

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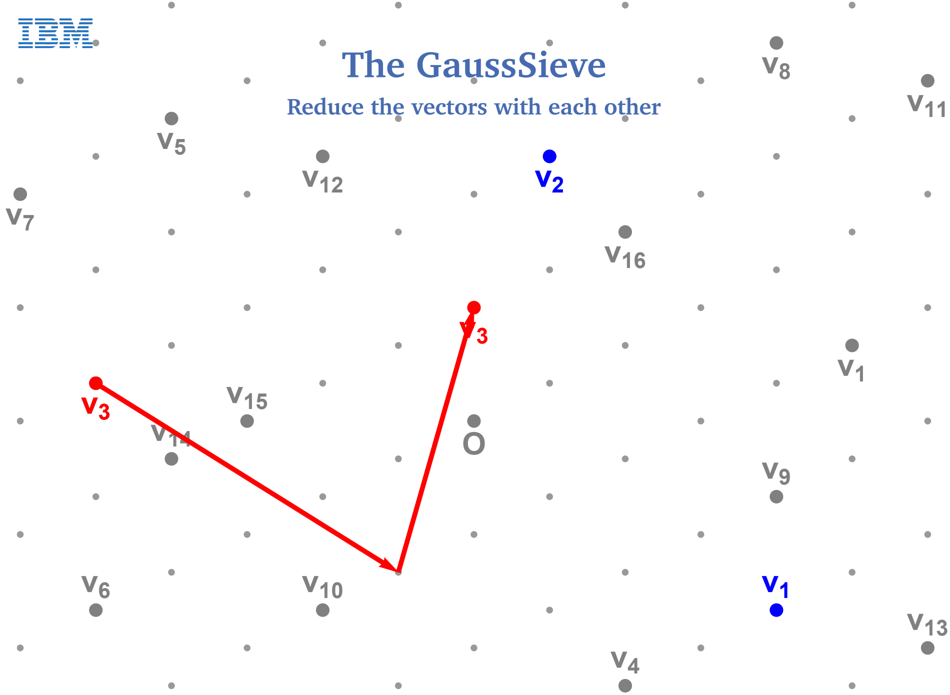

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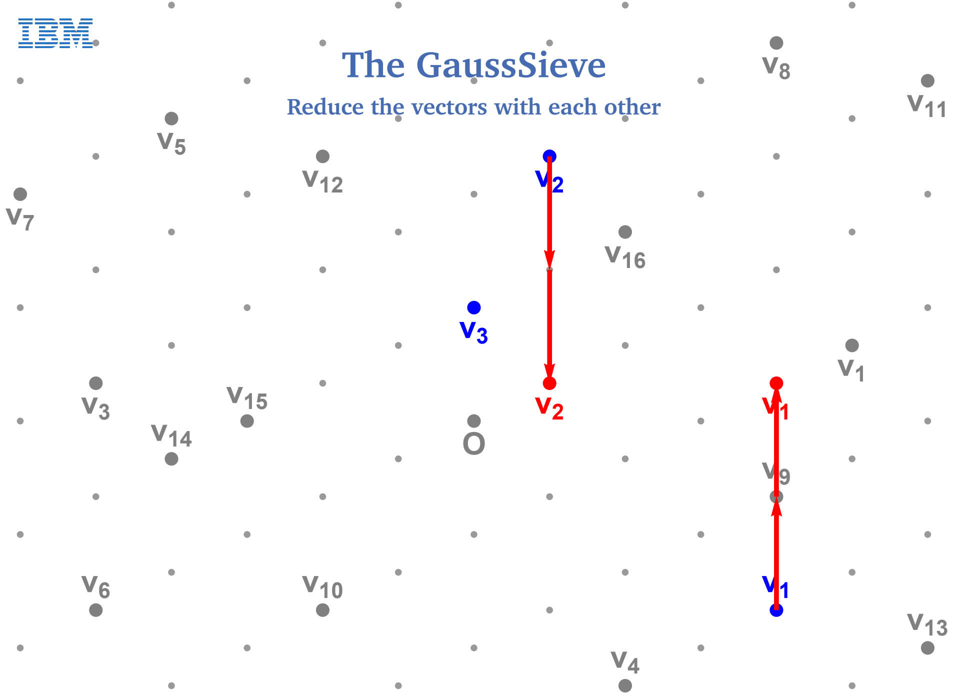

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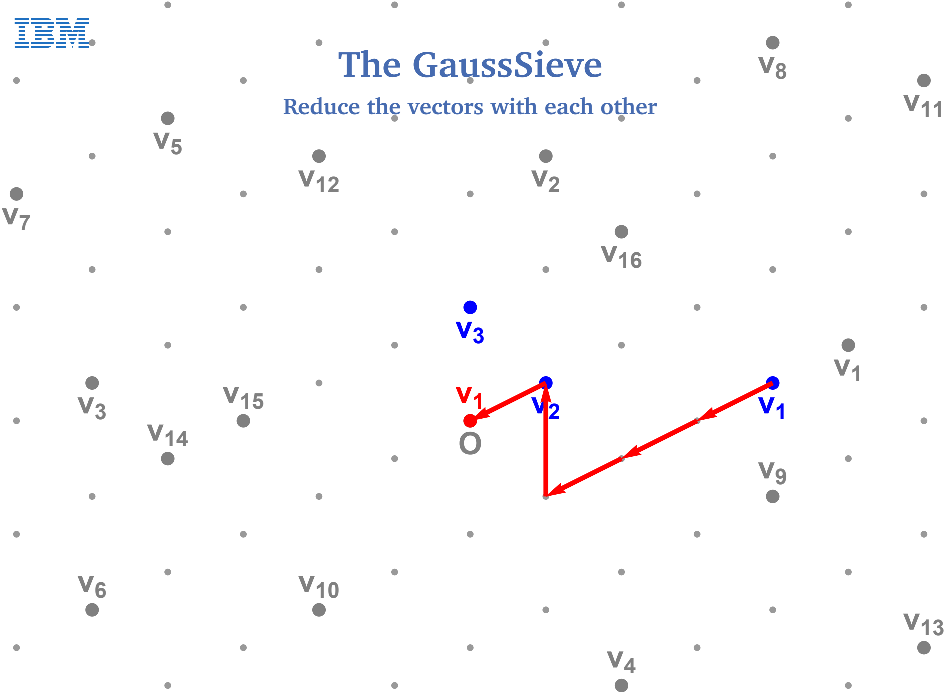

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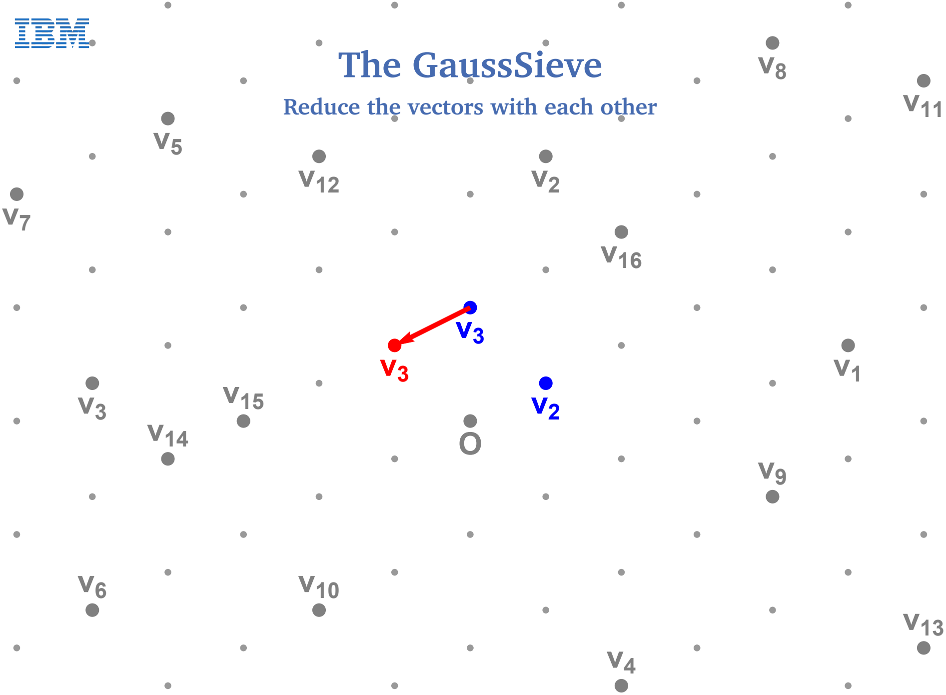

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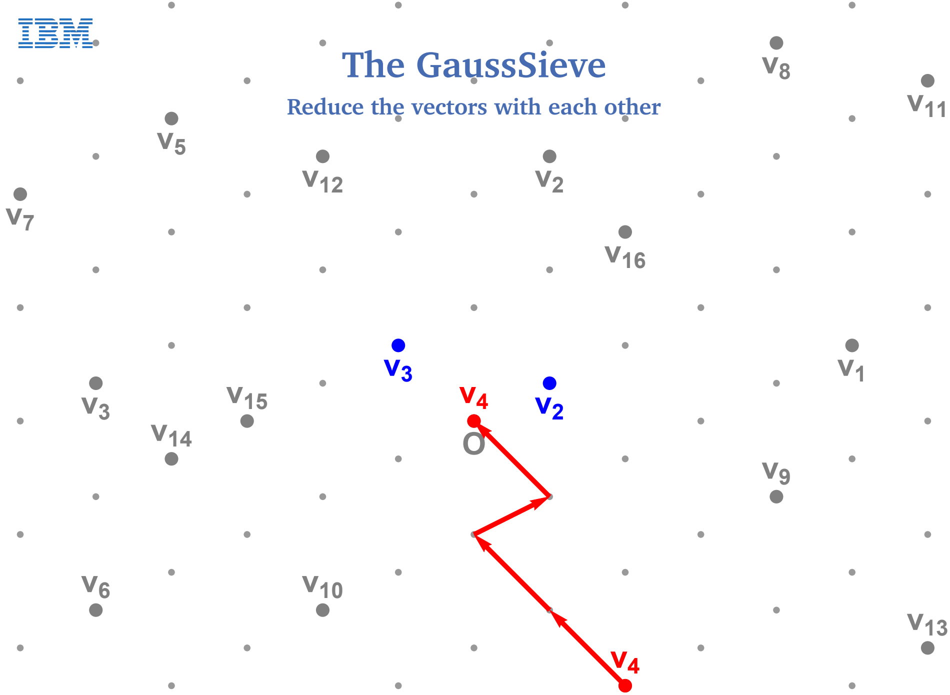

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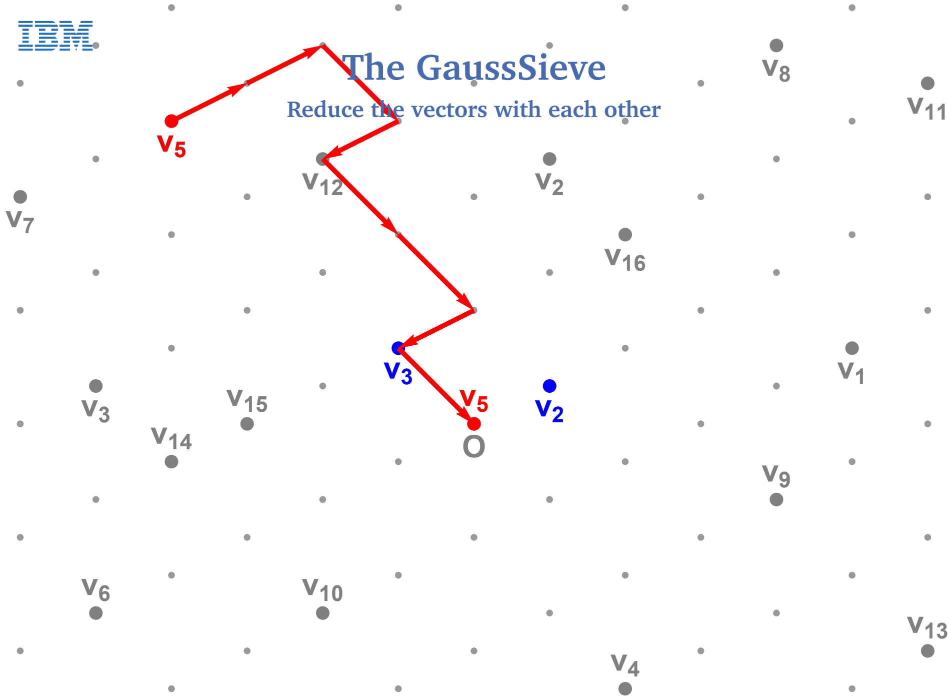

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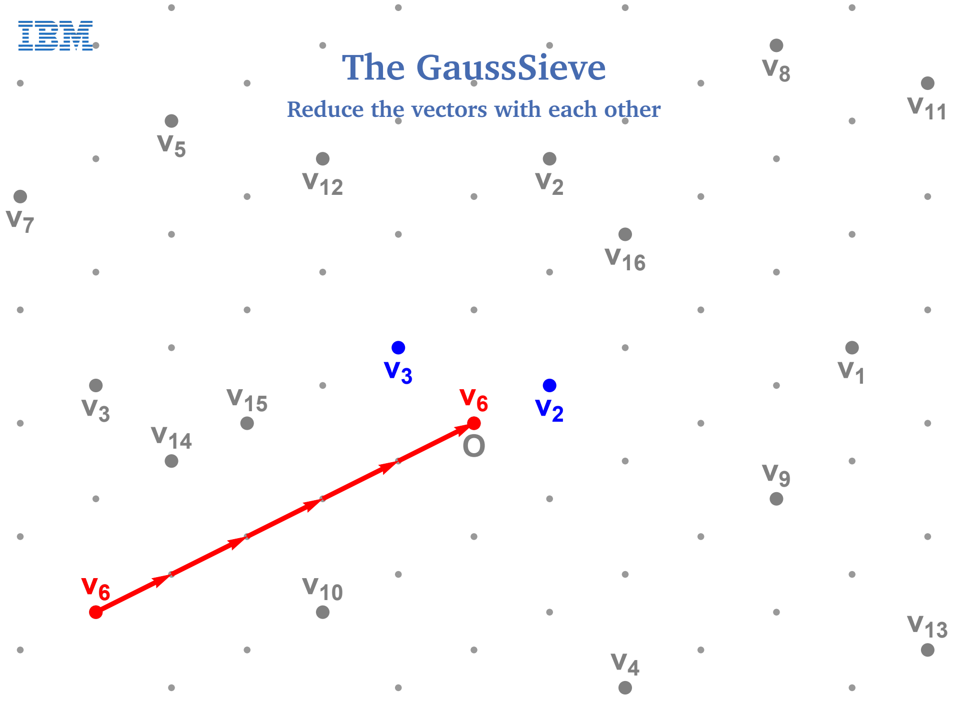

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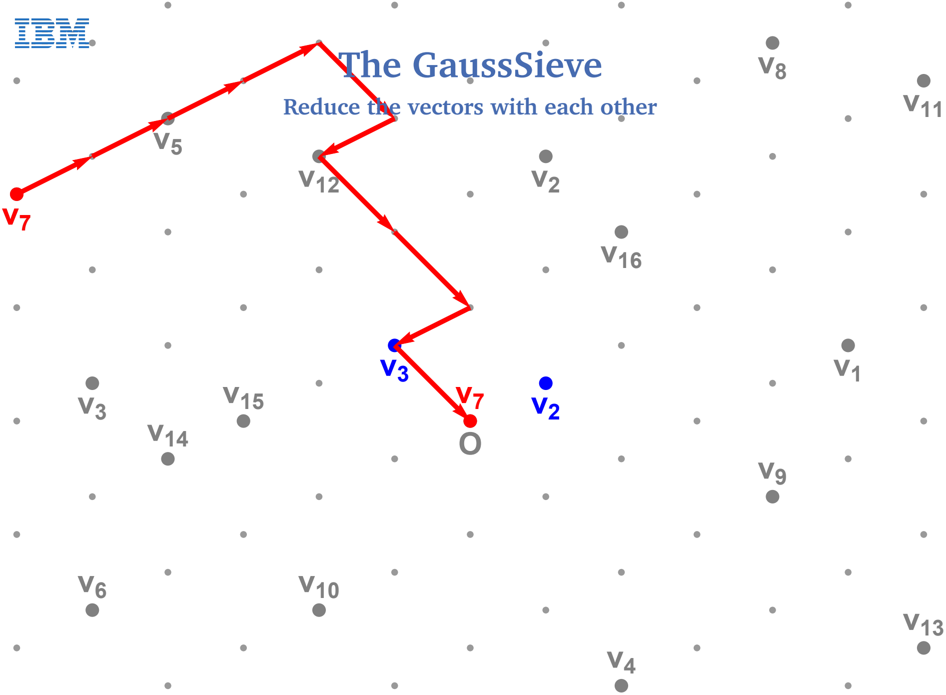

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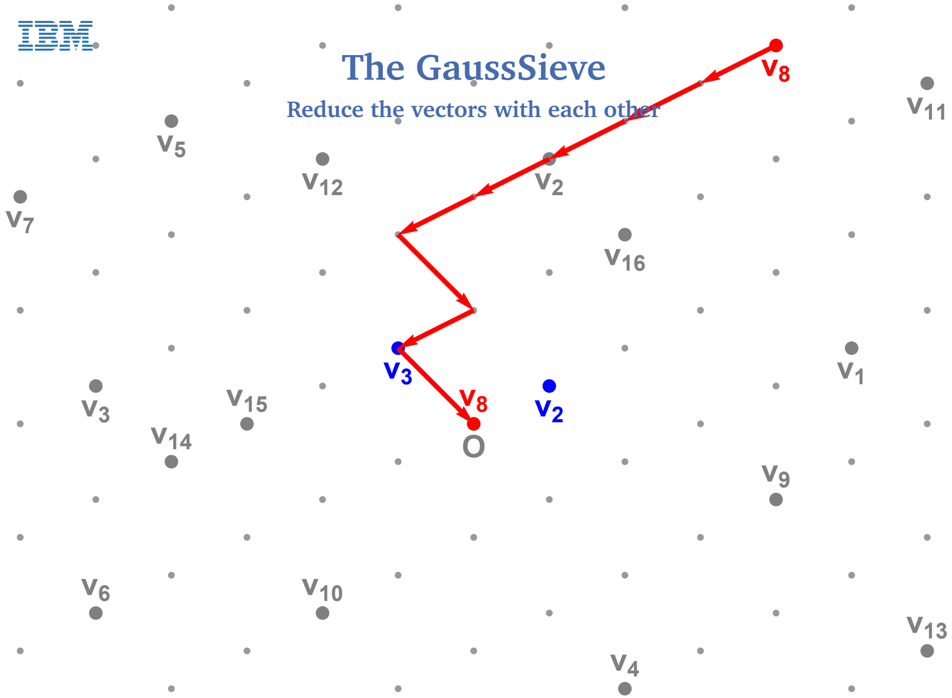

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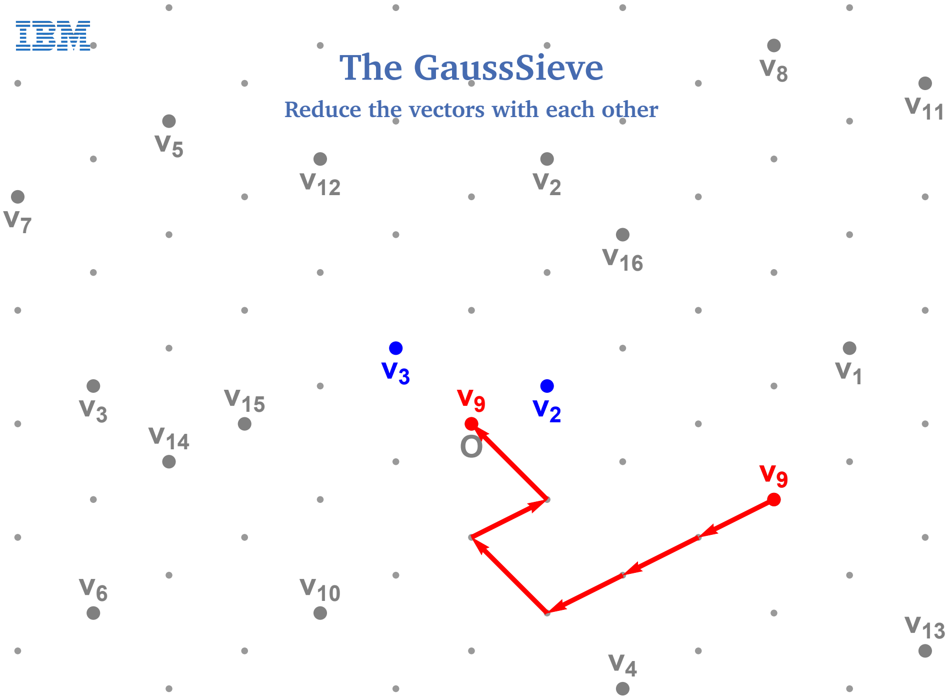

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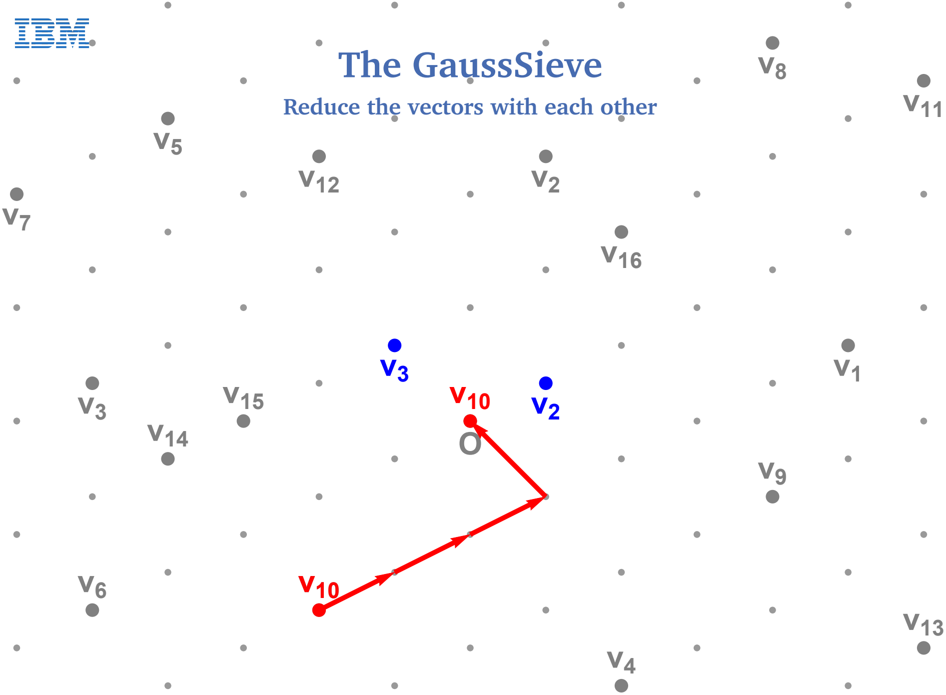

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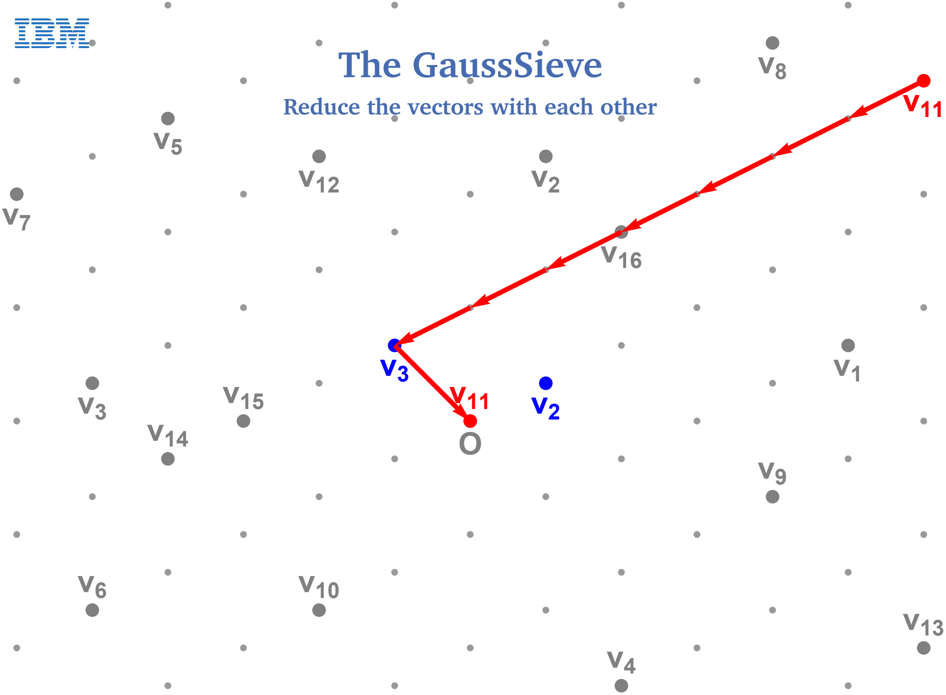
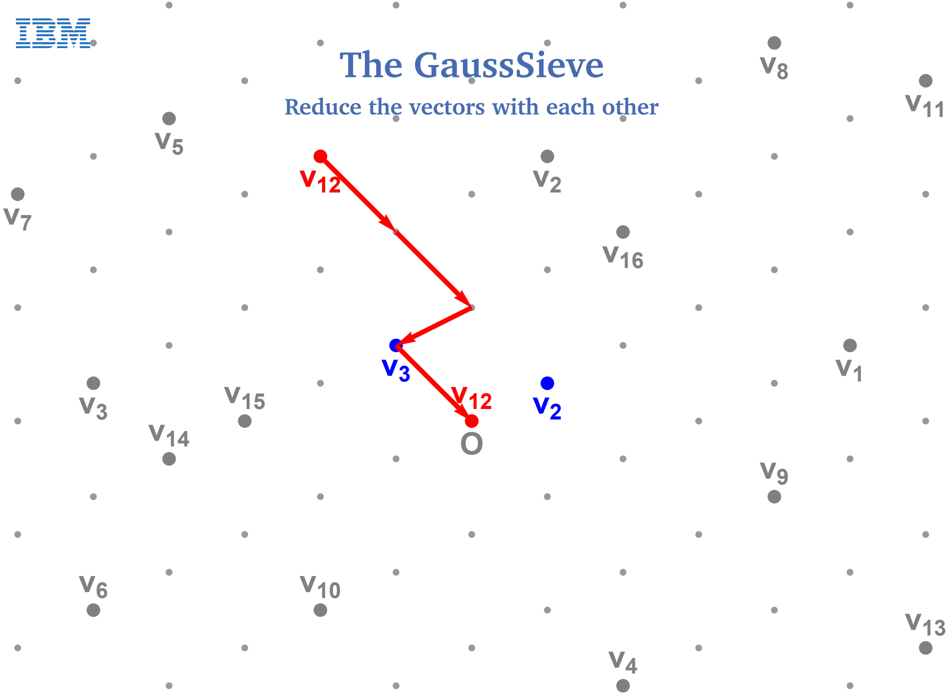


The GaussSieve

Reduce the vectors with each other



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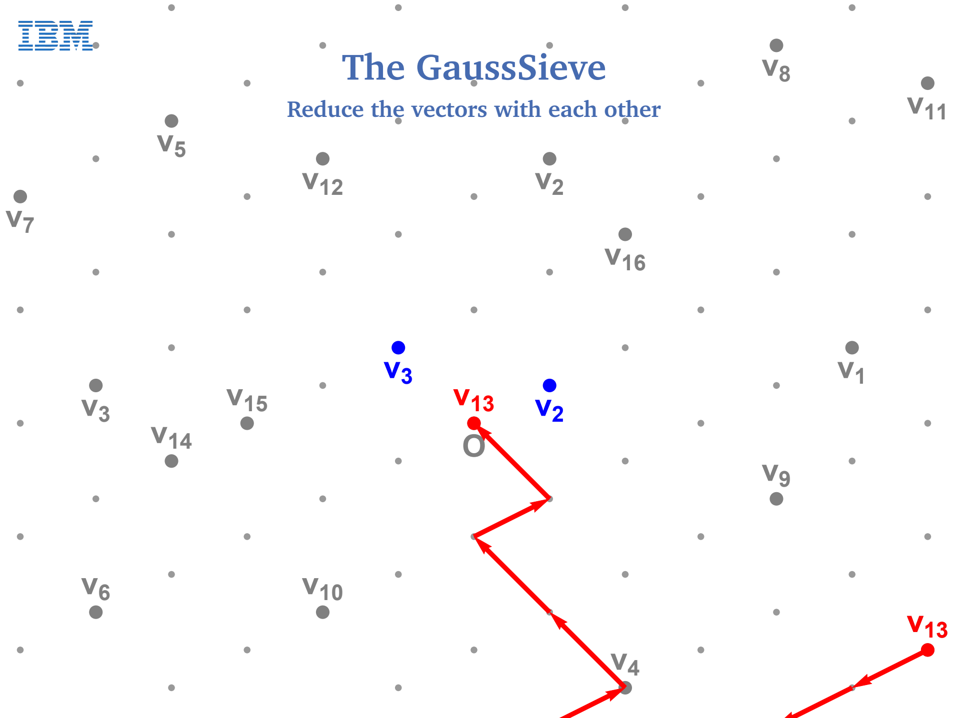

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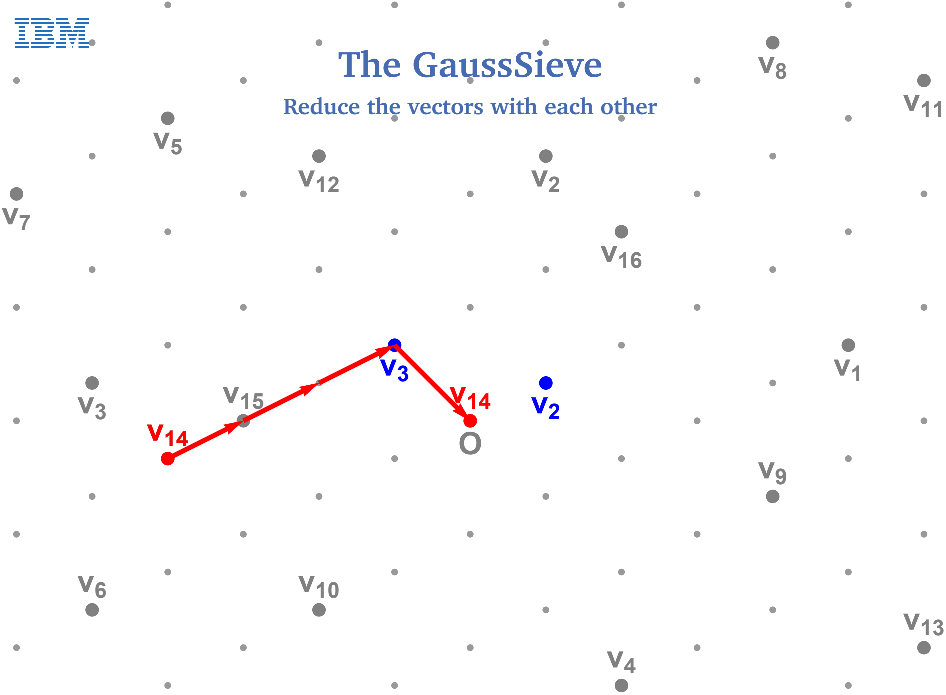

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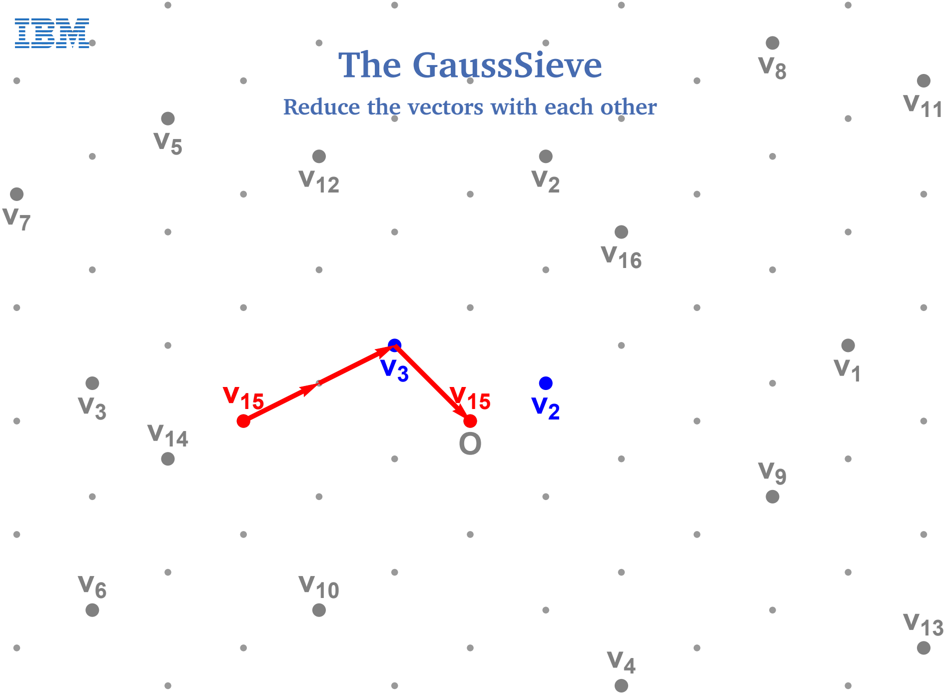

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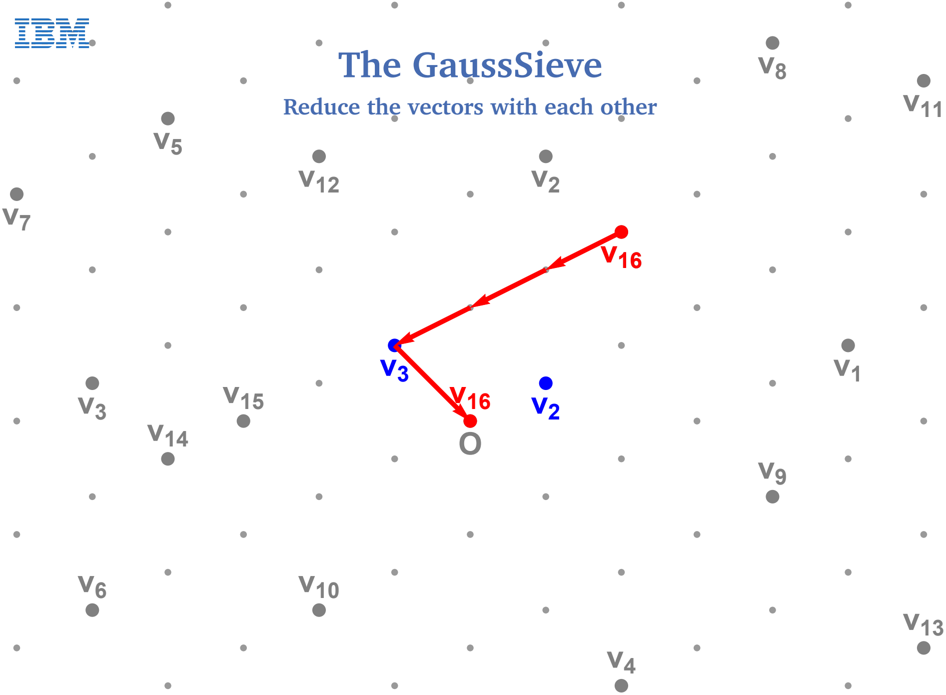

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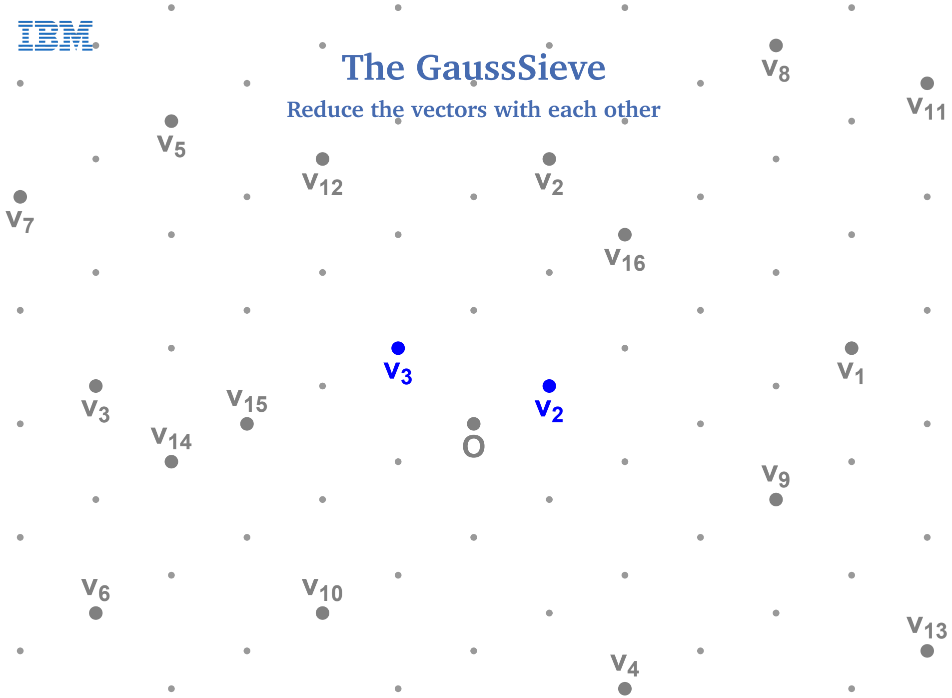
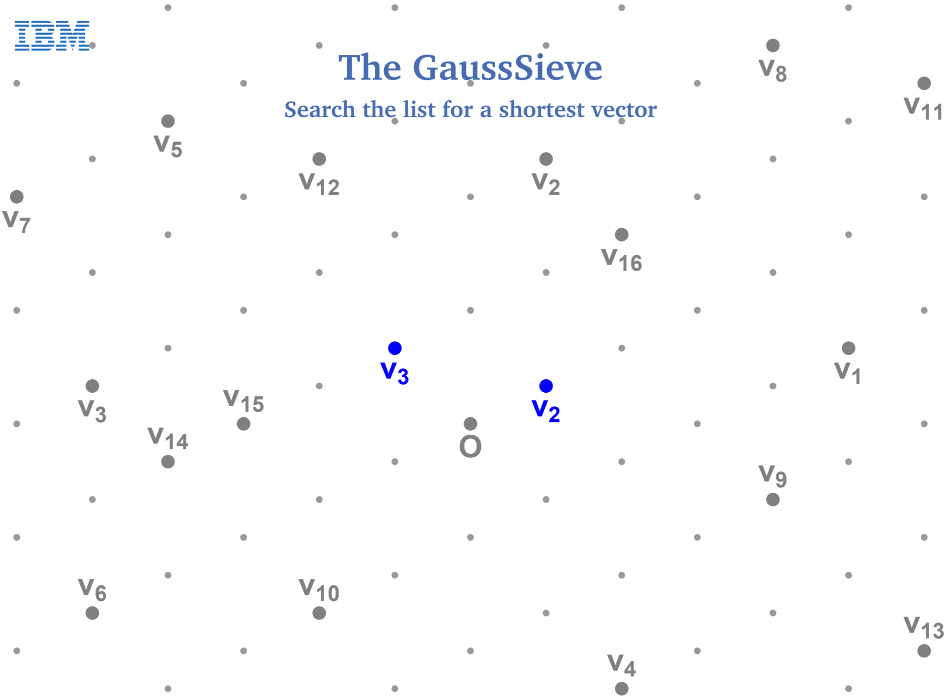


The GaussSieve

Search the list for a shortest vector

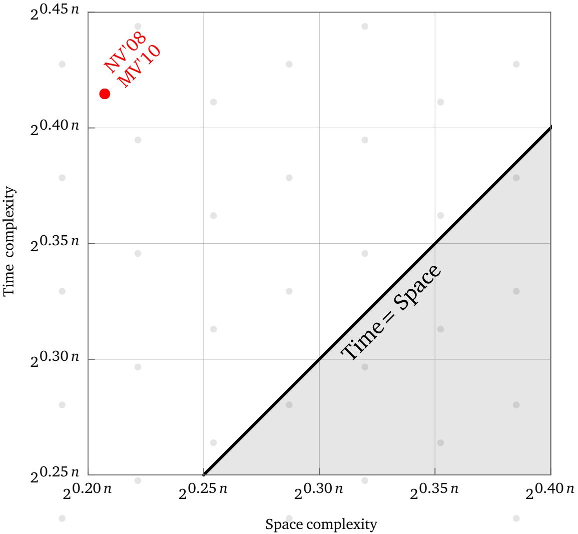


The GaussSieve

Search the list for a shortest vector

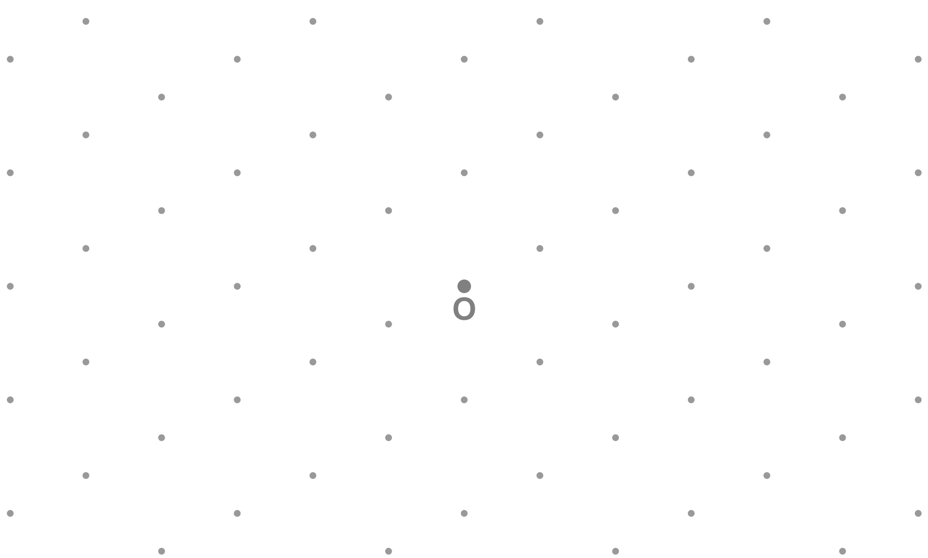
The GaussSieve

Space/time trade-off



Two-level sieve

Sample a list L of random lattice vectors



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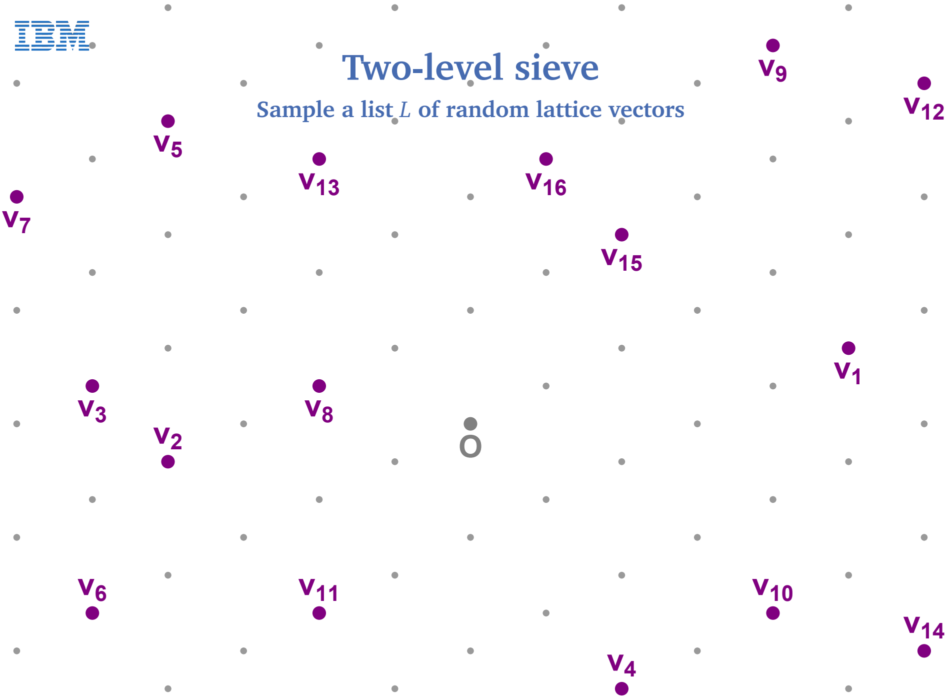
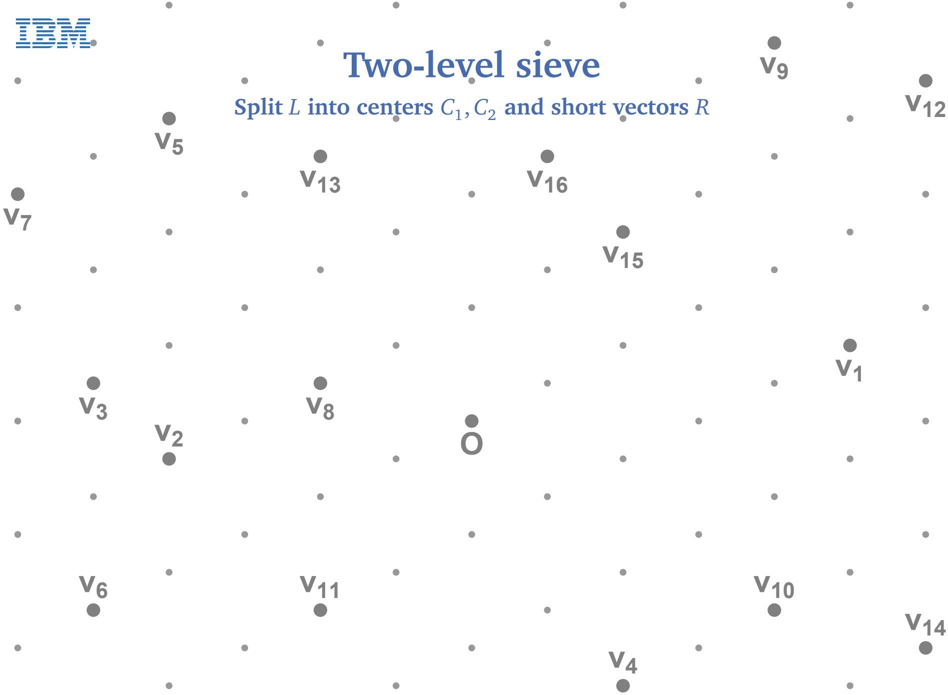


Two-level sieve

Split L into centers C_1, C_2 and short vectors R



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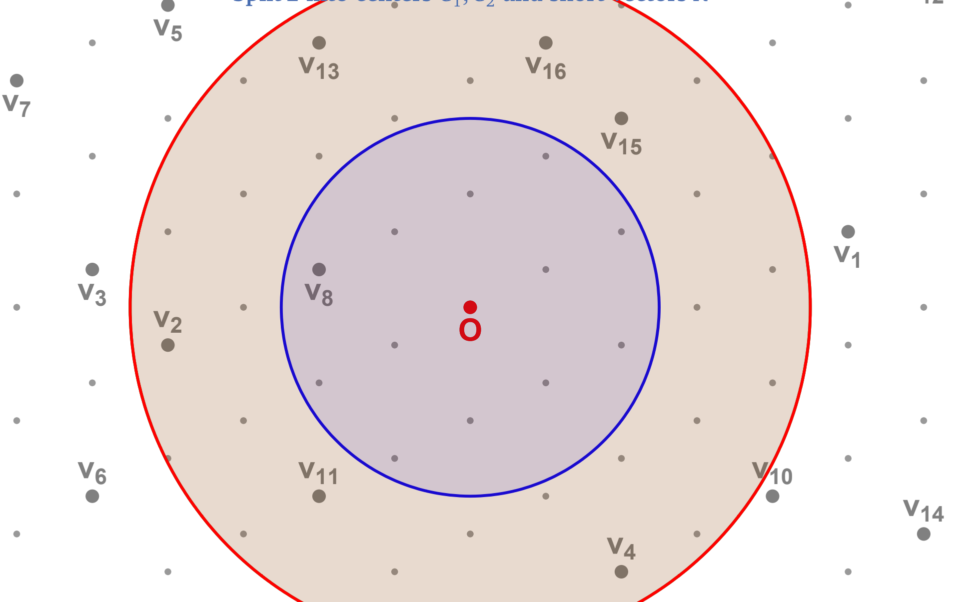

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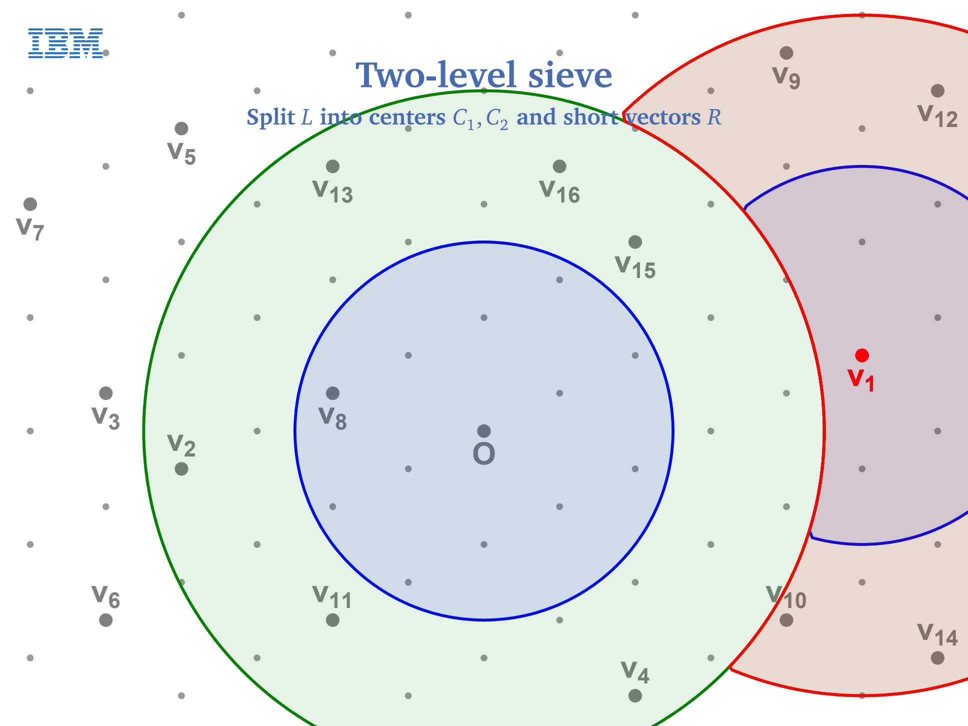

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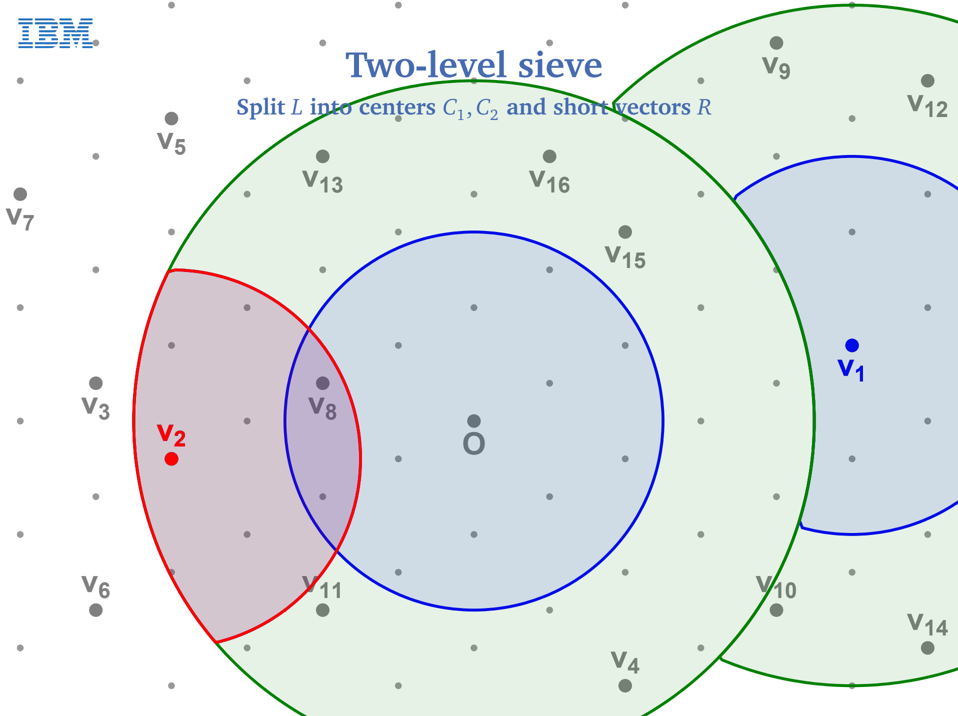

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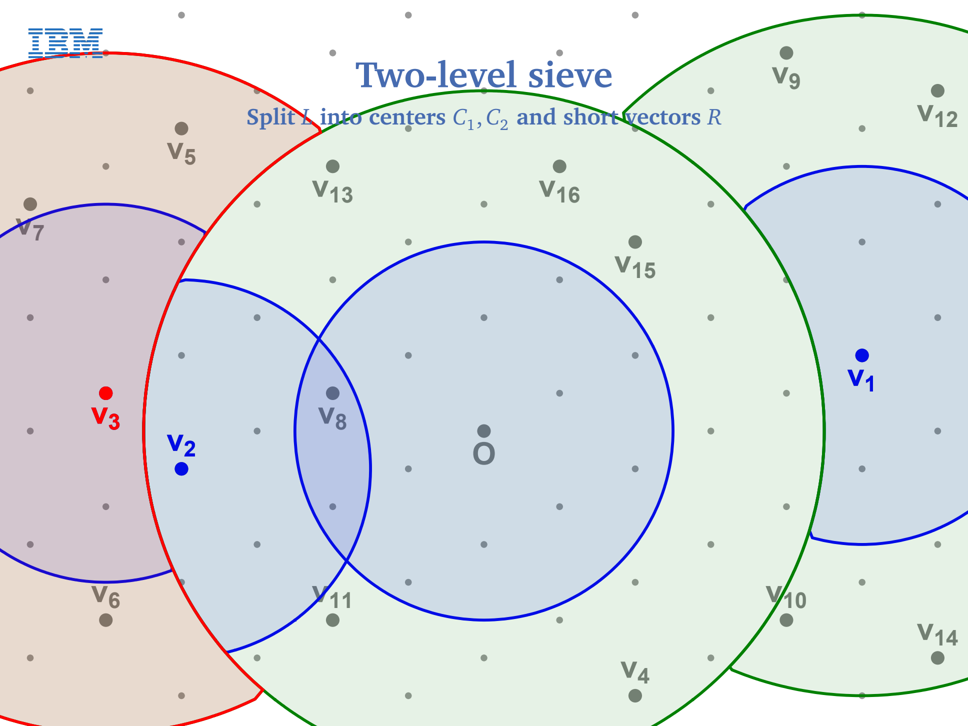

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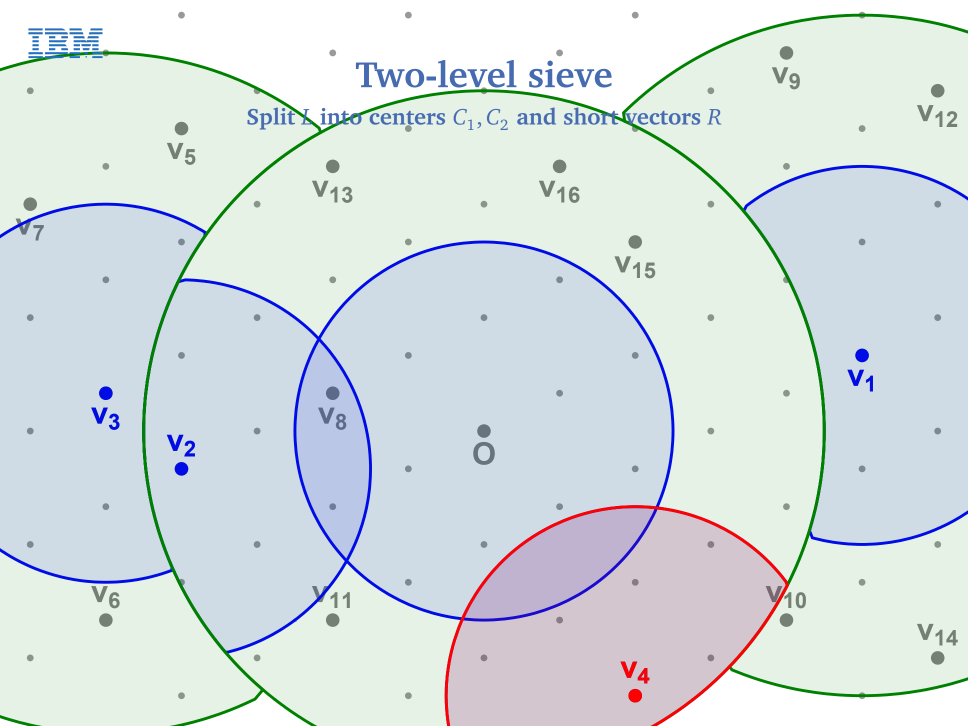

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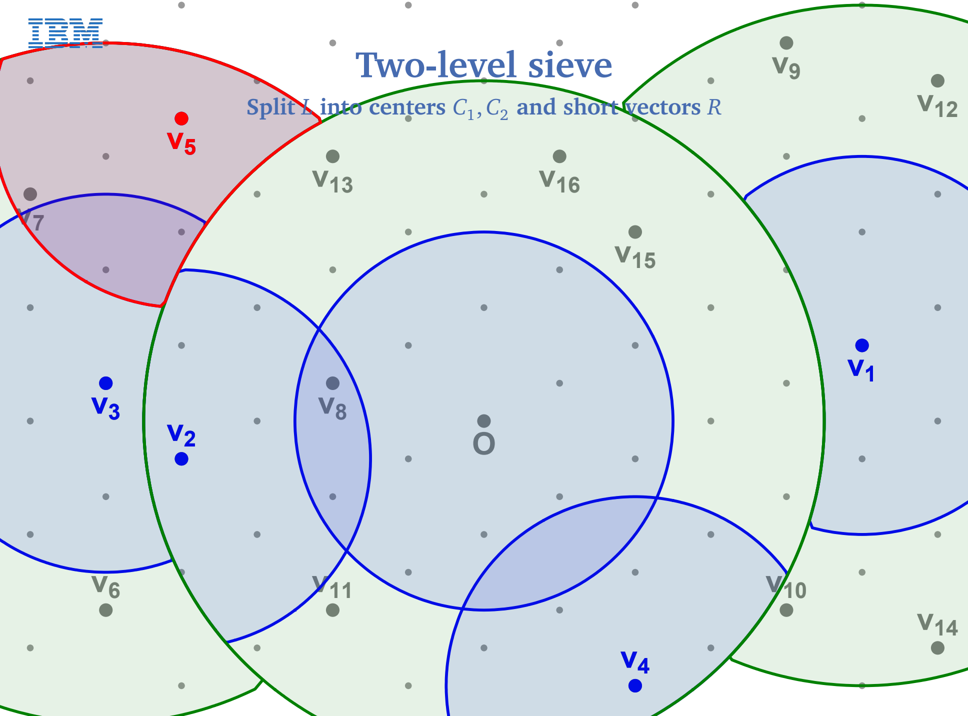

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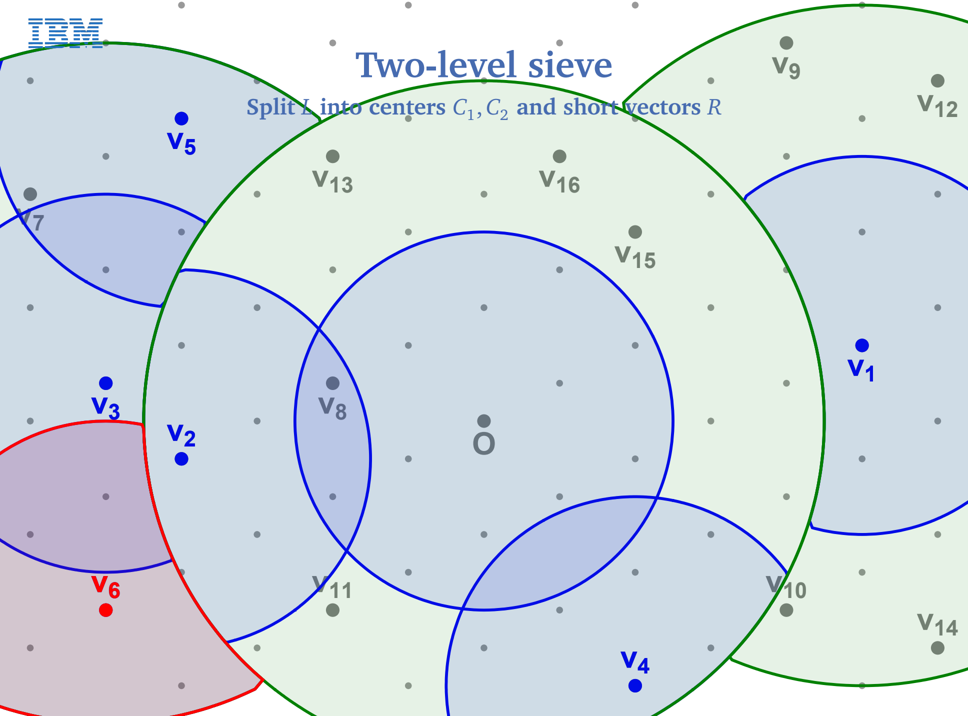

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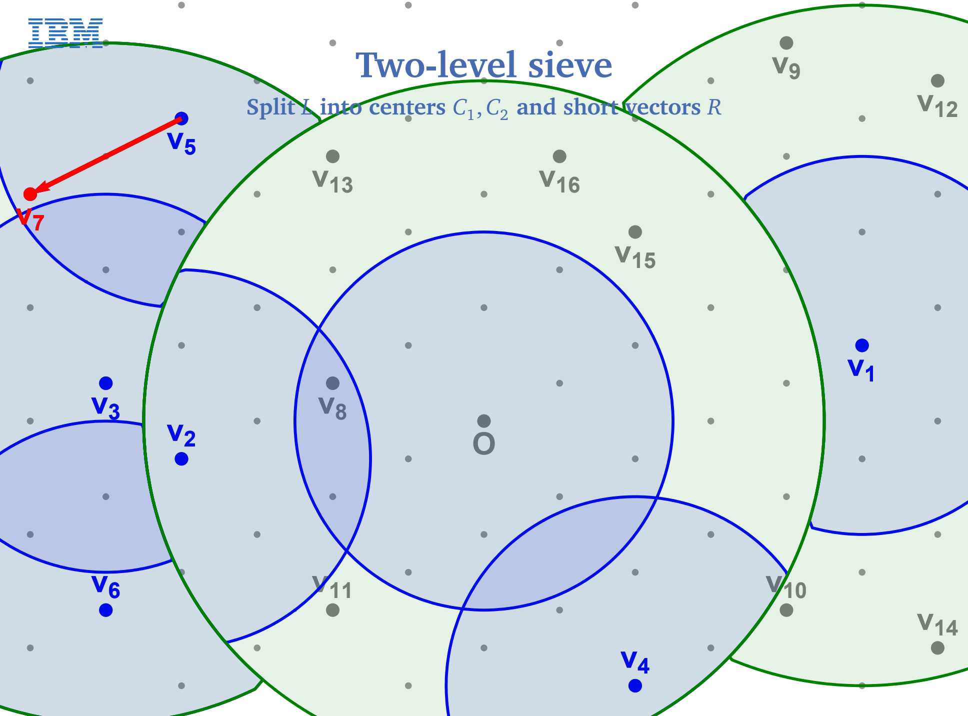

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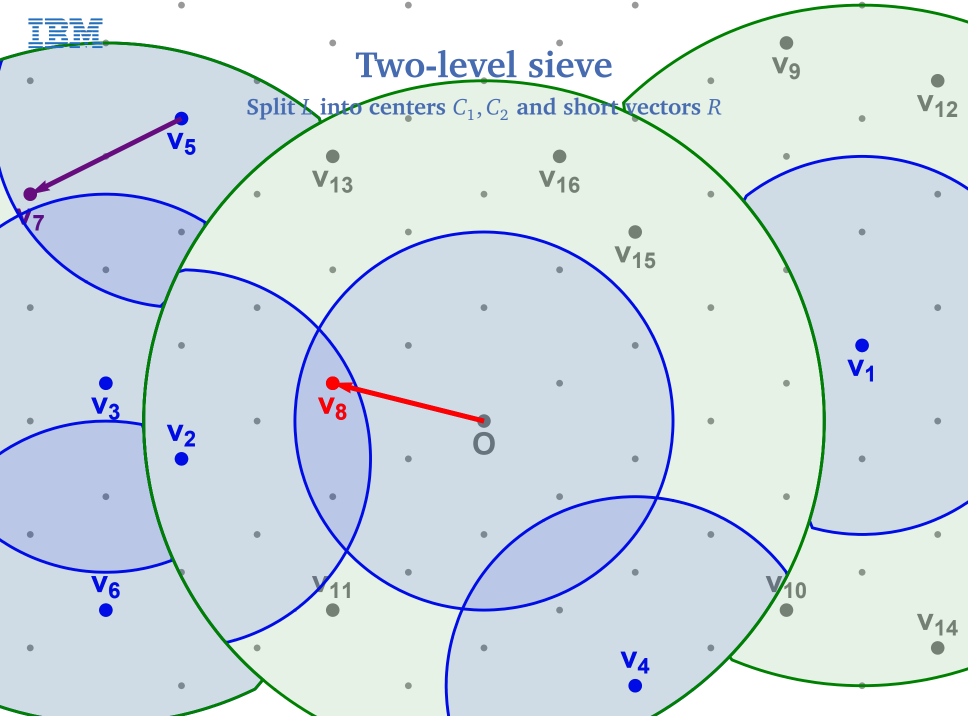

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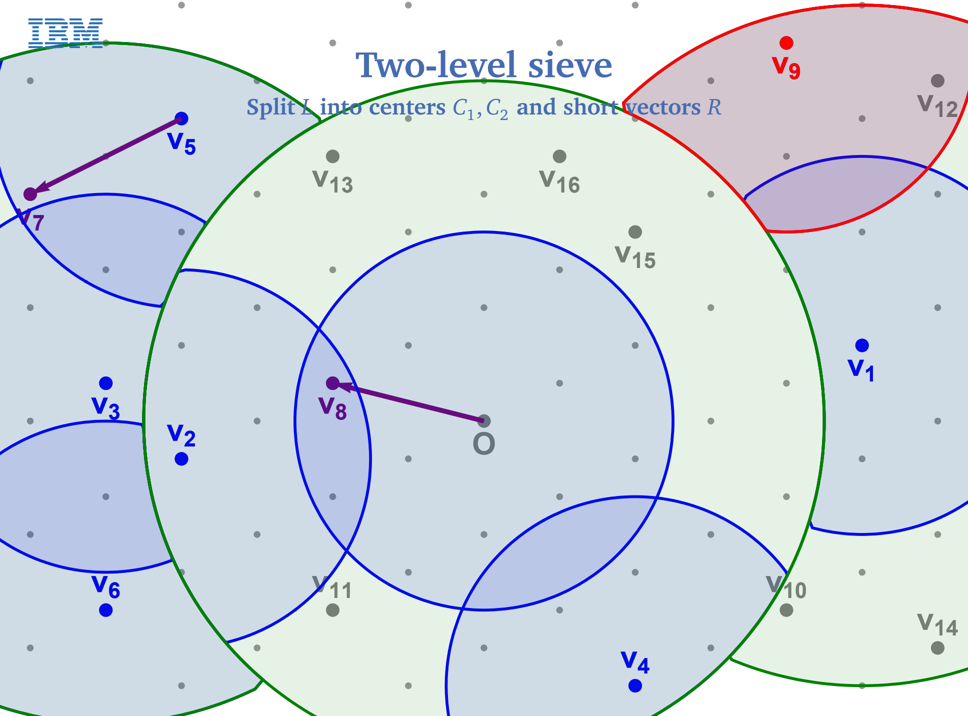

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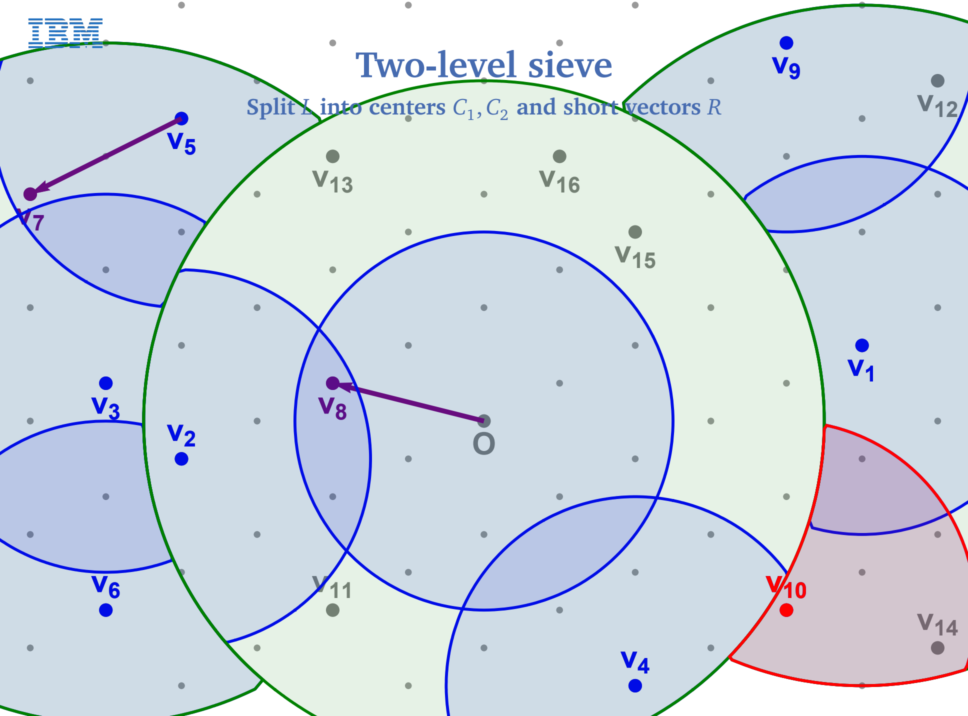

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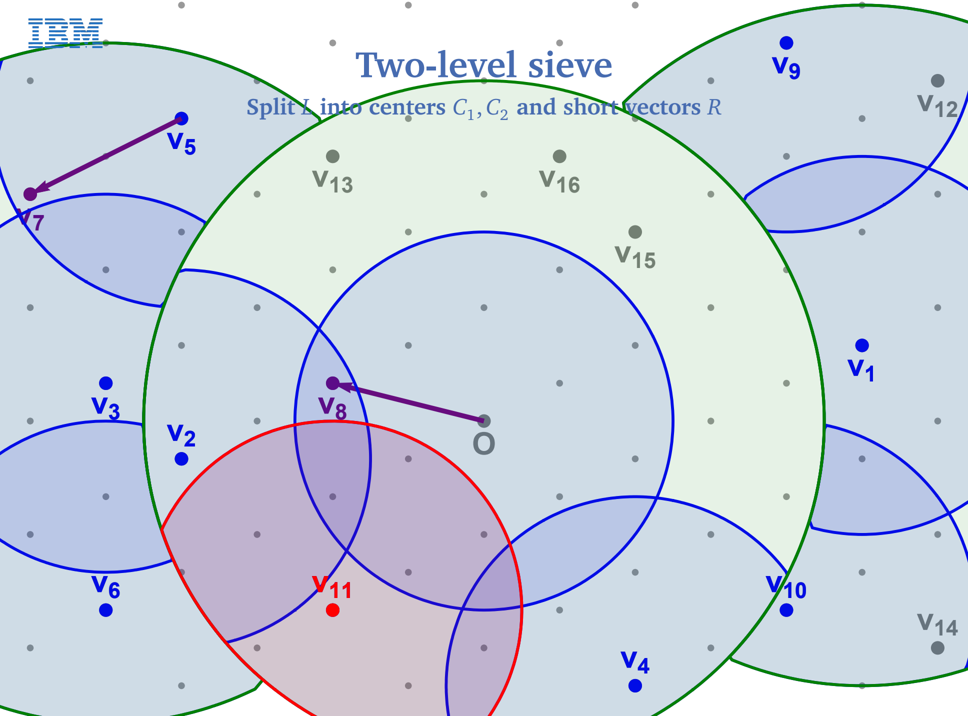

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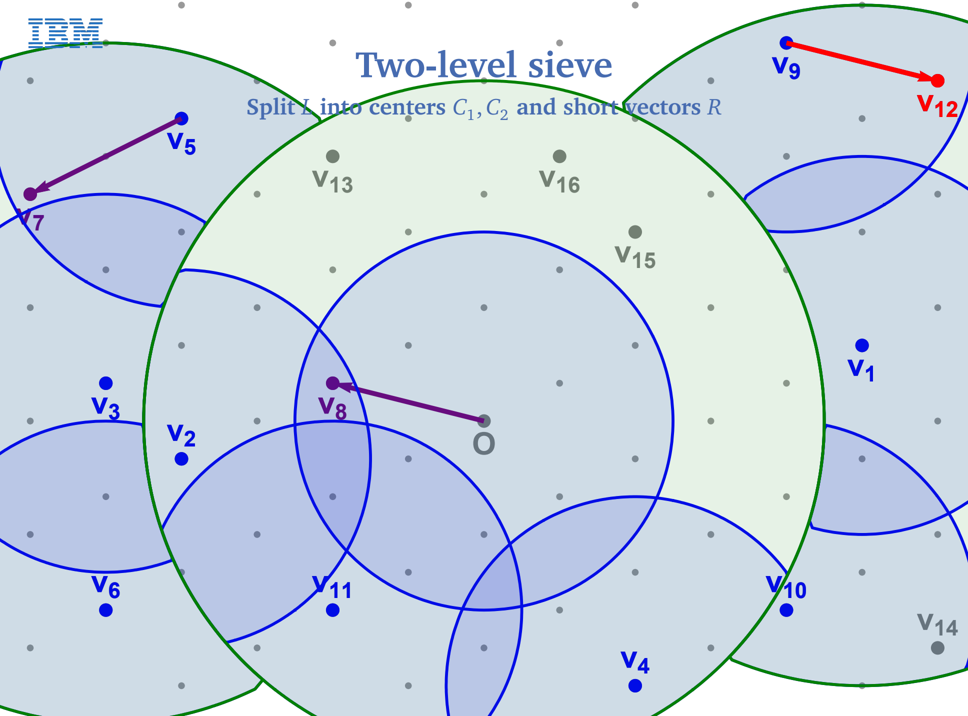

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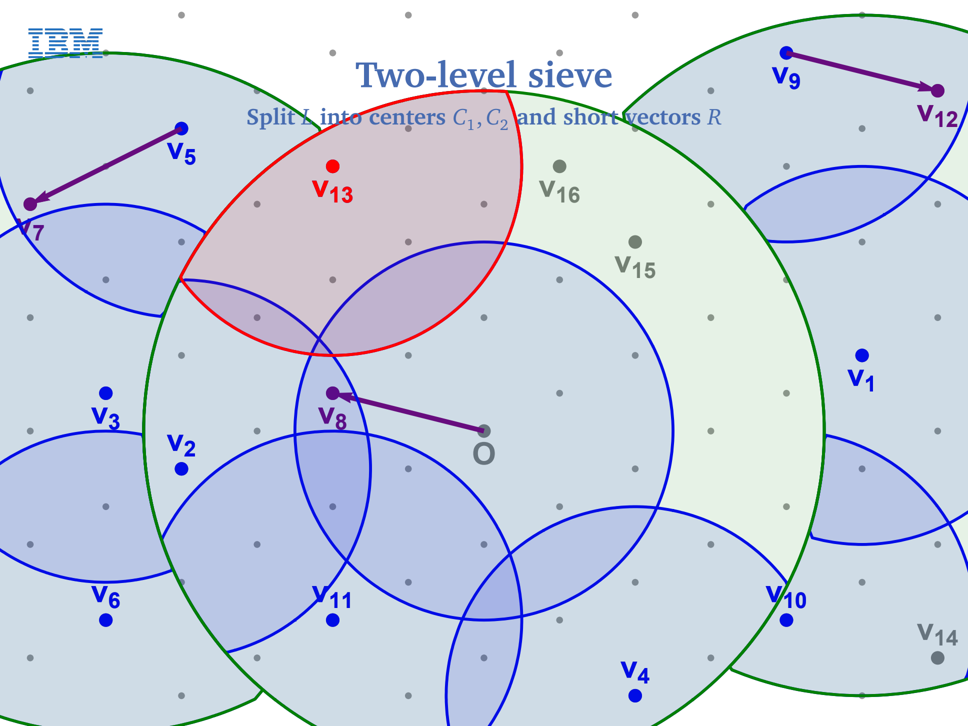

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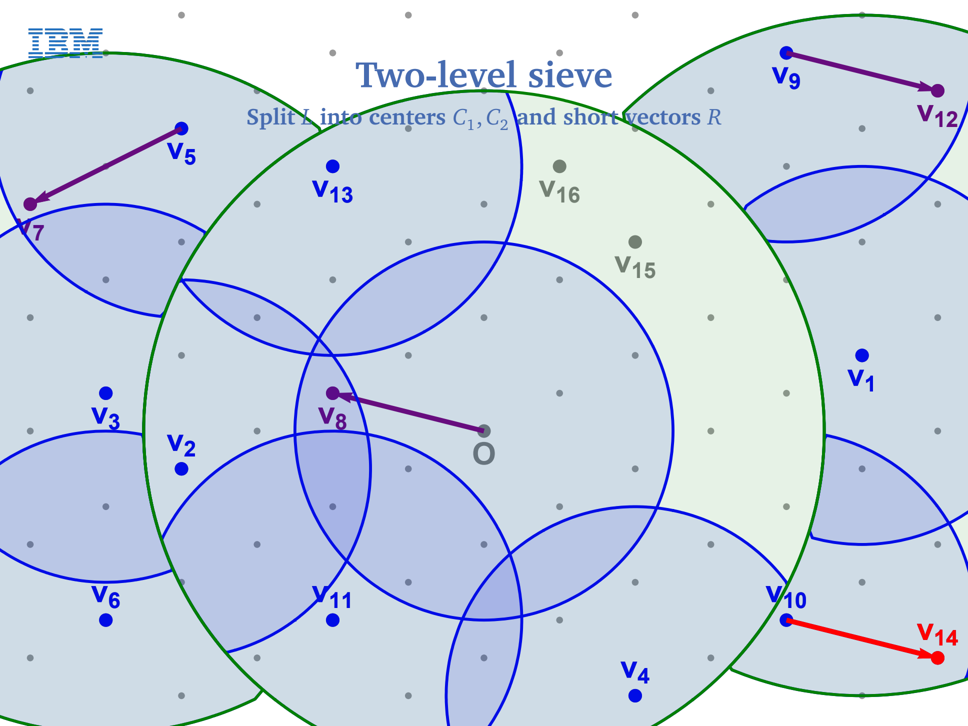

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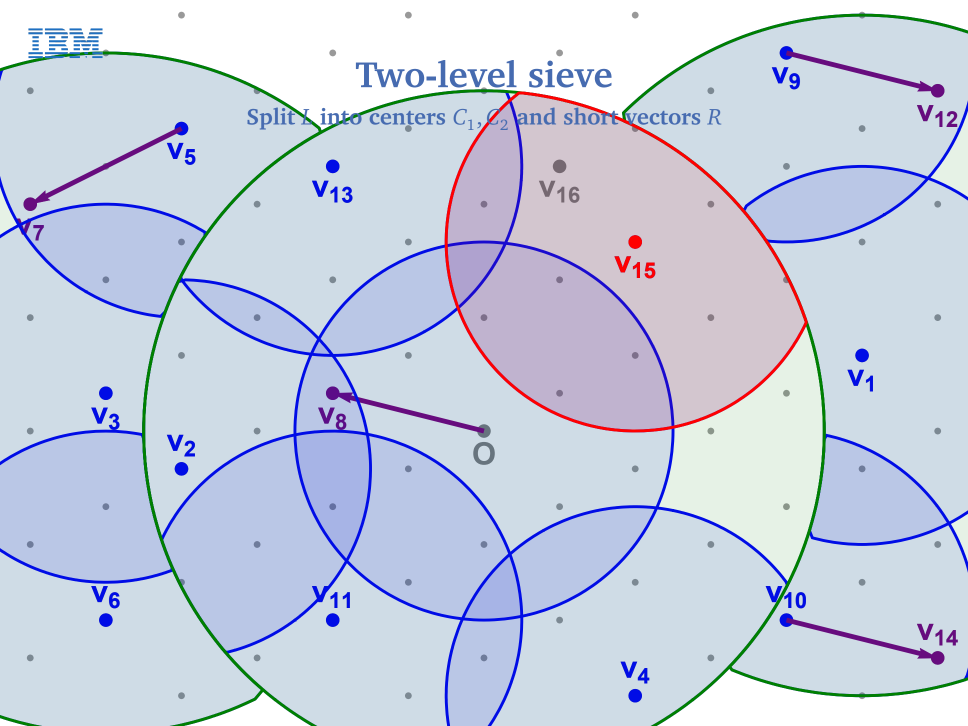

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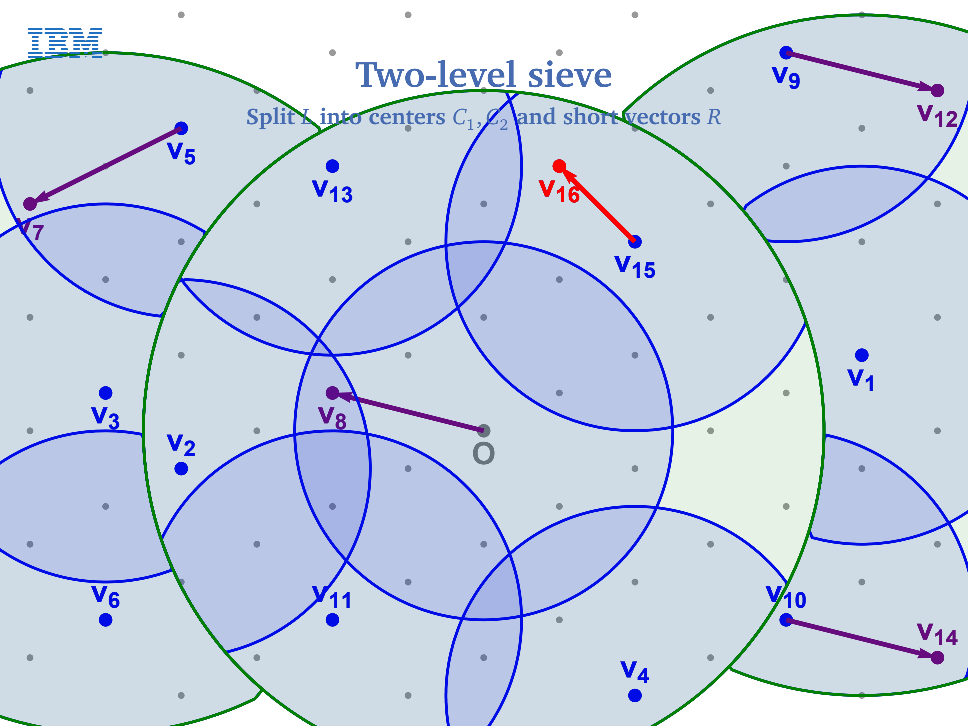

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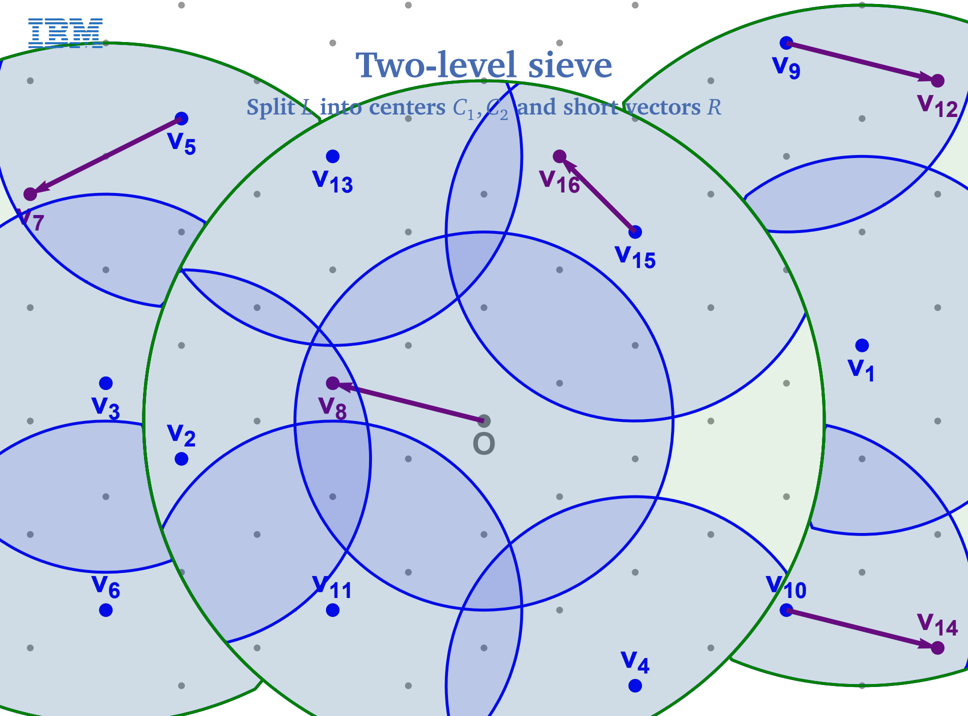

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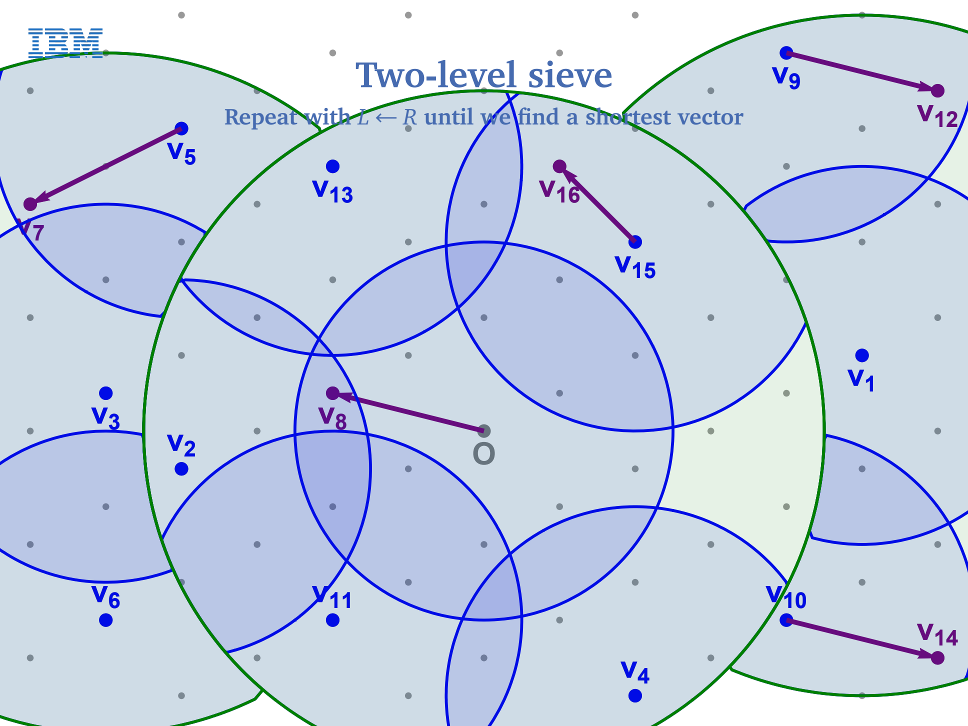

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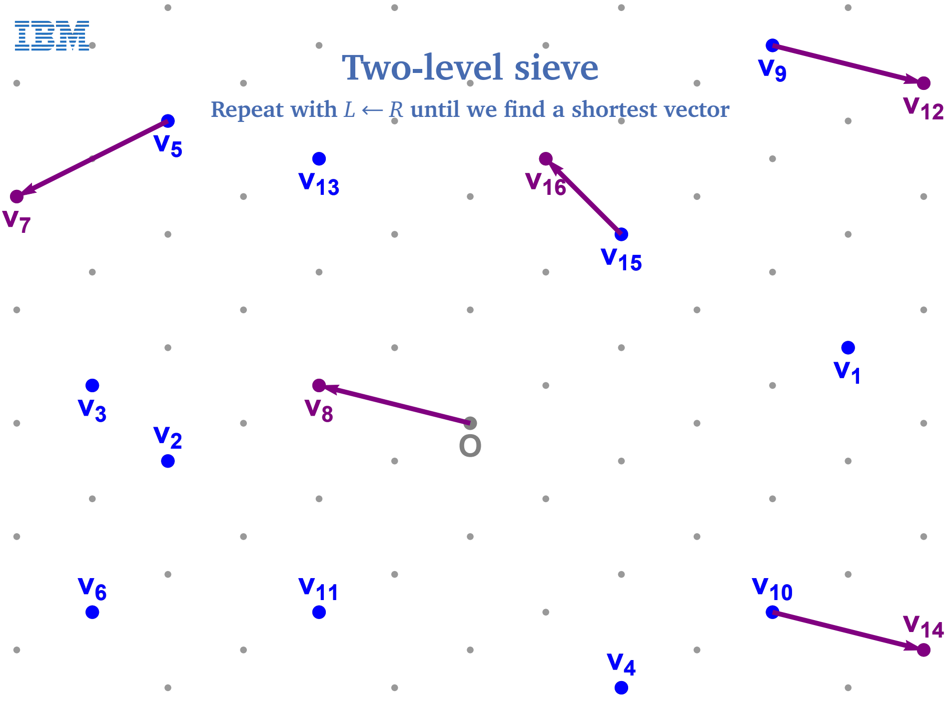

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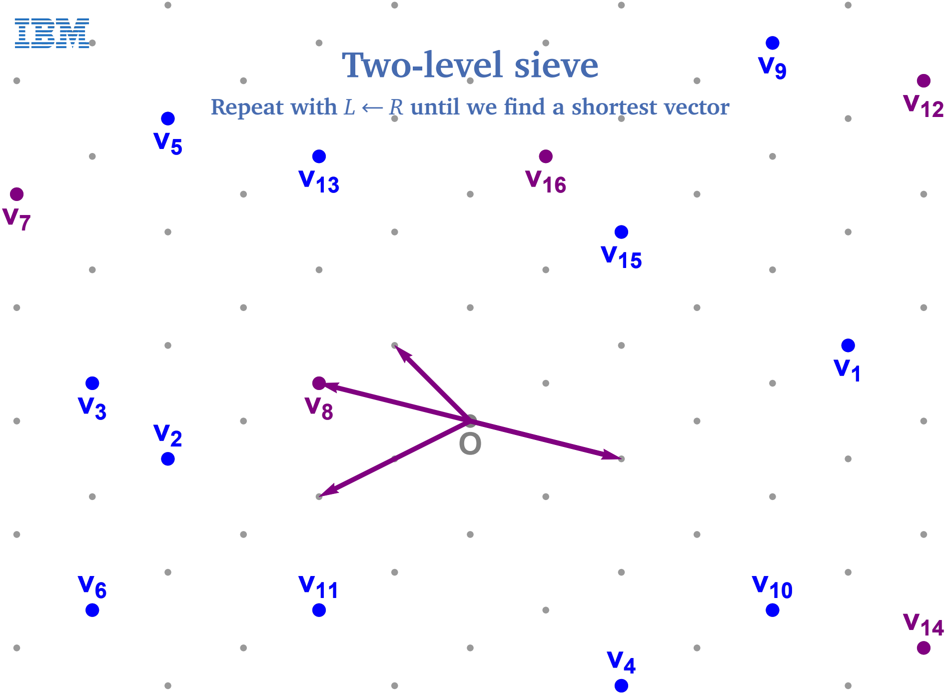

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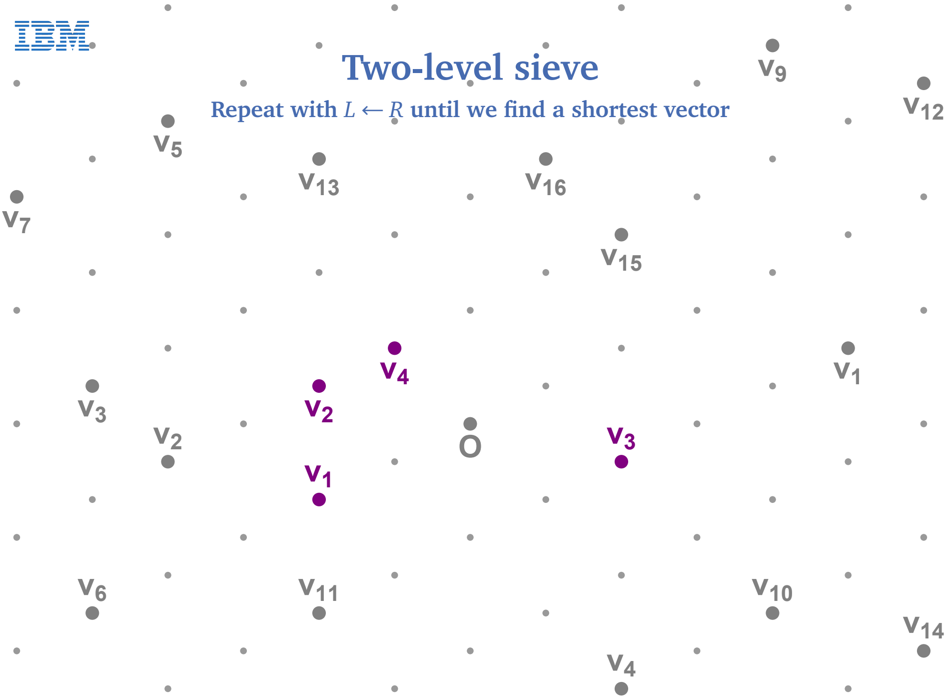
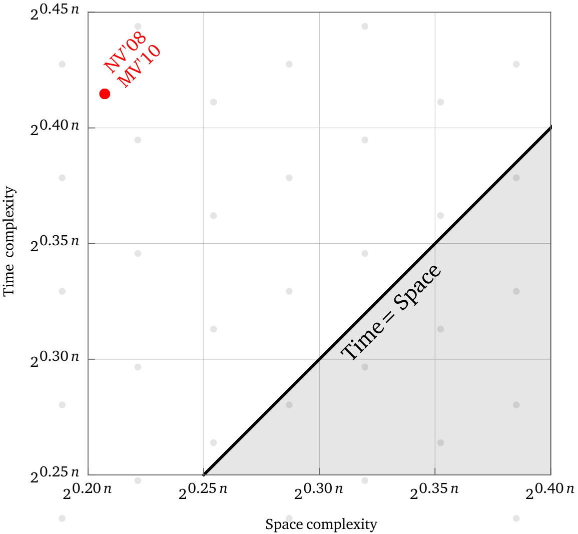


Two-level sieve

Space/time trade-off



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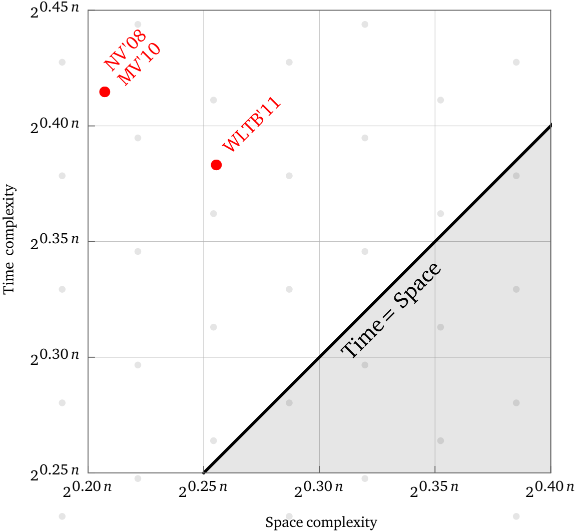

Three-level sieve

Overview

Heuristic result (Nguyen–Vidick, J. Math. Crypt. '08)

The one-level sieve runs in time $2^{0.4150n}$ *and space* $2^{0.2075n}$.

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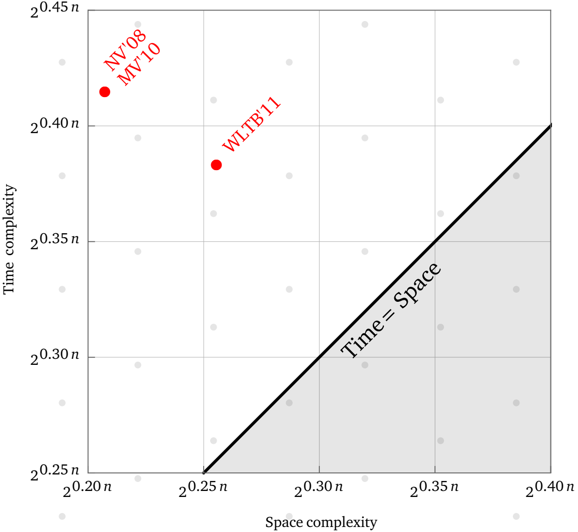
Conjecture

The four-level sieve runs in time $2^{0.3774n}$ *and space* $2^{0.2925n}$*, and higher-level sieves are not faster than this.*

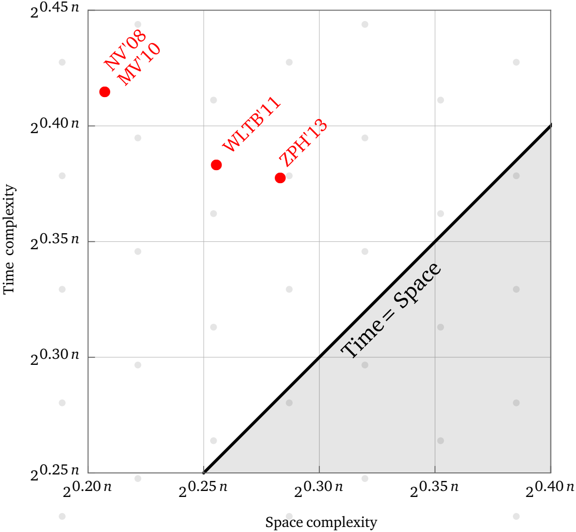
Three-level sieve

Space/time trade-off



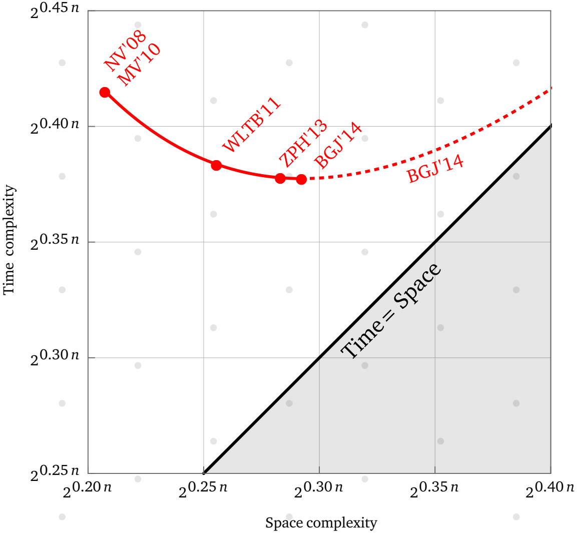
Three-level sieve

Space/time trade-off



Decomposition approach

Space/time trade-off



Locality-sensitive hashing

Introduction

Problem: Given a high-dimensional data set $D \subset \mathbb{R}^n$, preprocess it such that when later given a target $\boldsymbol{t} \in \mathbb{R}^n$, we can quickly find a nearby vector to $\boldsymbol{t}$ in D.

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> *"The key idea is to use hash functions such that the probability of collision is much higher for objects that are close to each other than for those that are far apart."*
>
> — Indyk–Motwani, *STOC'98*

Hyperplane LSH

Sample a list L of random lattice vectors

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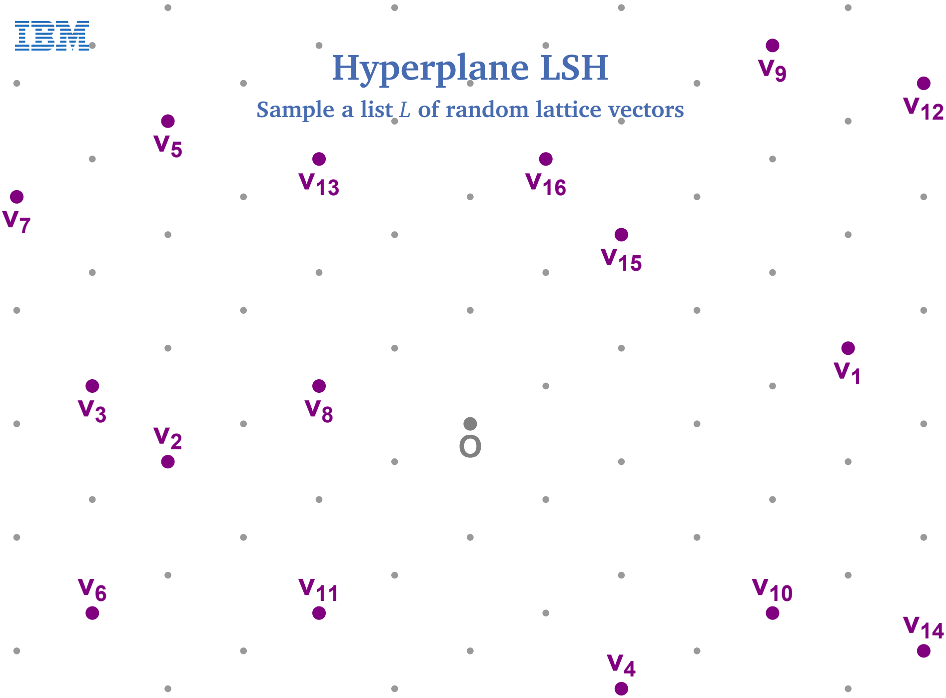
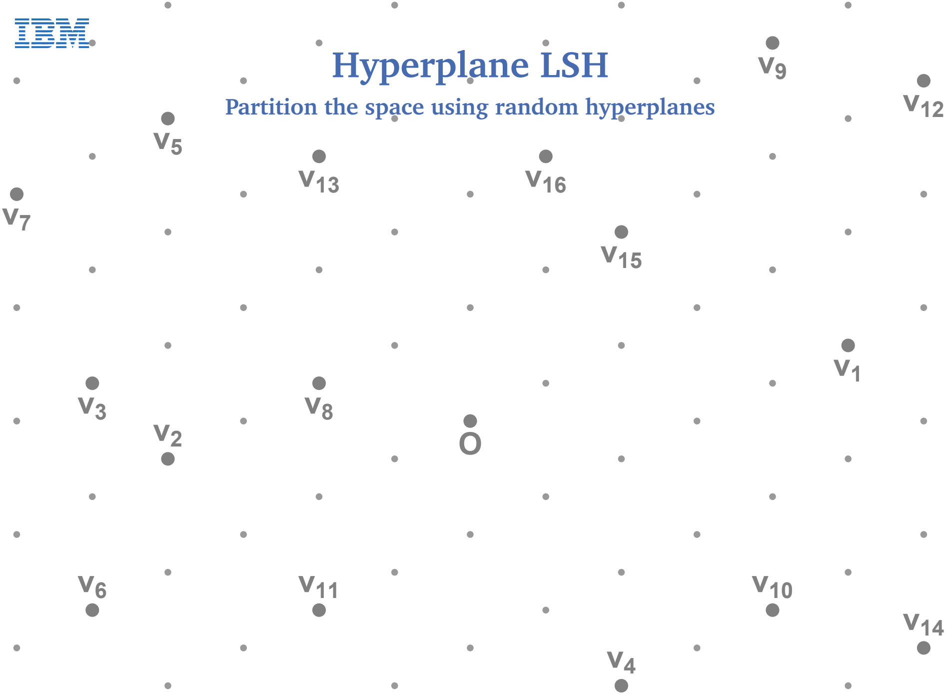


Hyperplane LSH

Partition the space using random hyperplanes



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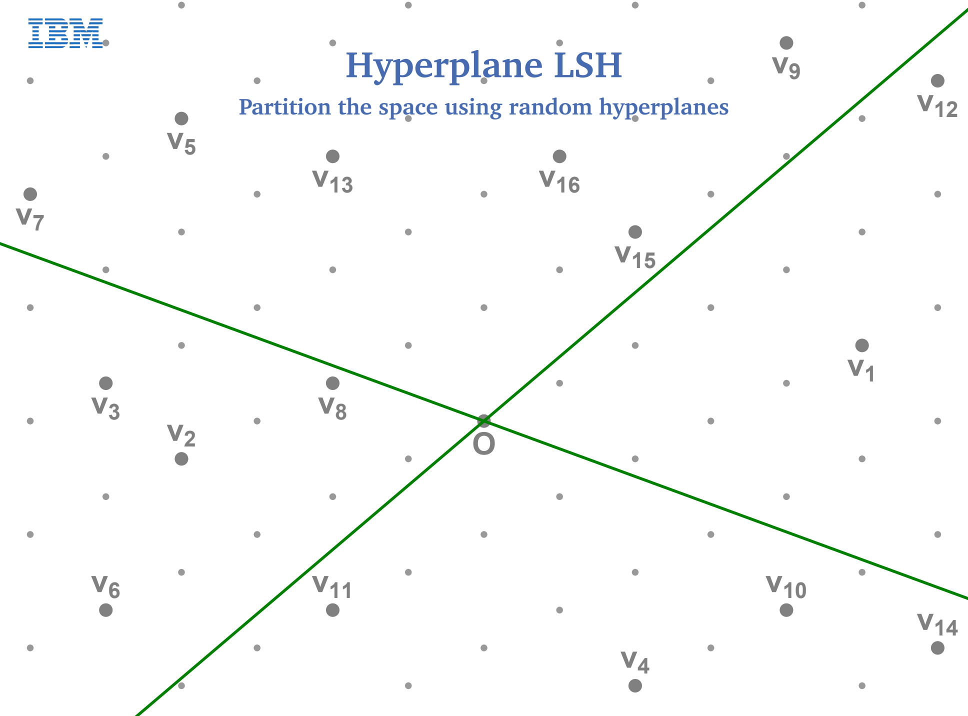
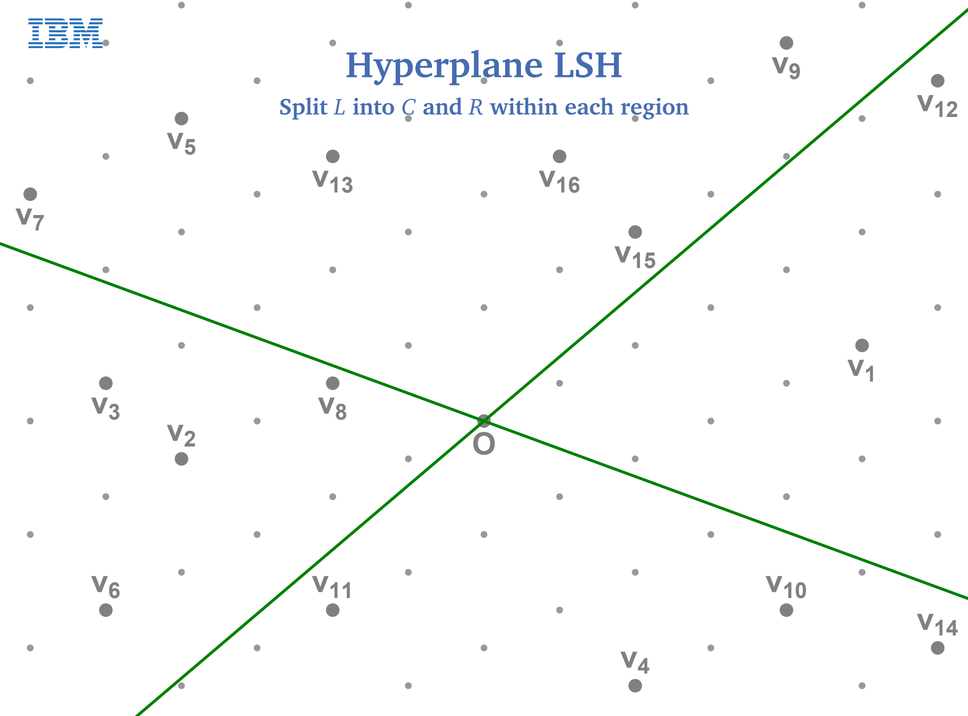


IBM

Hyperplane LSH

Split L into C and R within each region



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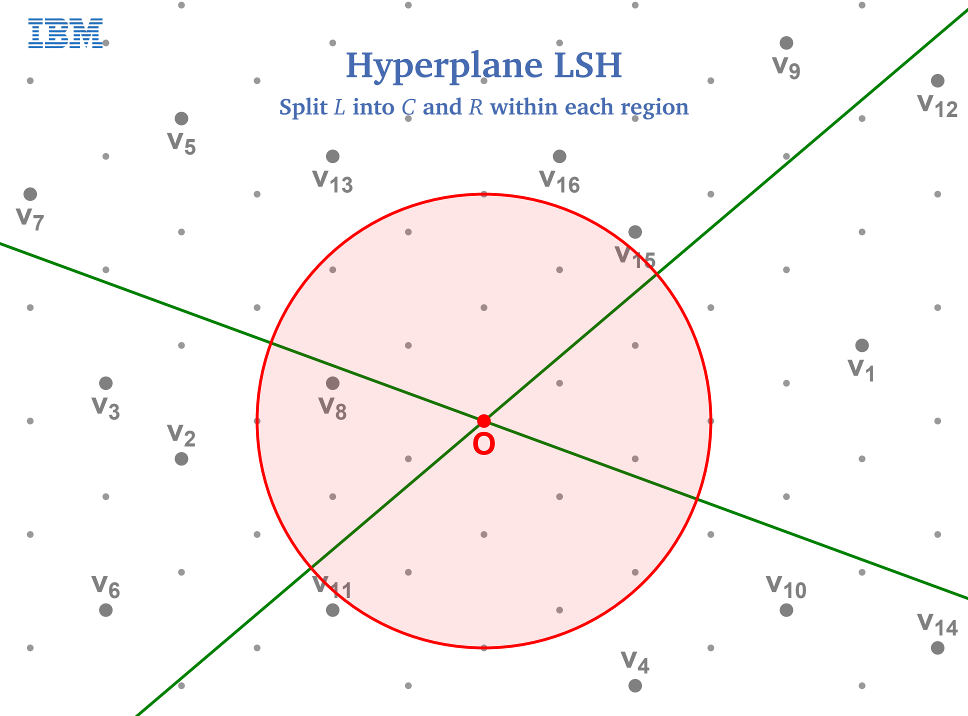

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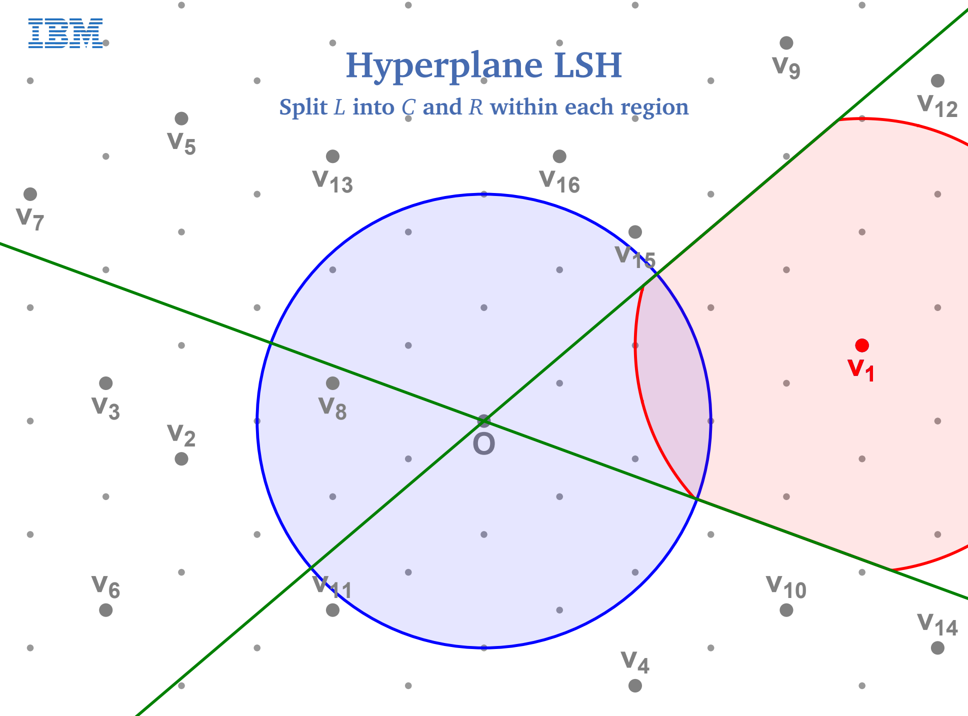
Split *L* into *C* and *R* within each region



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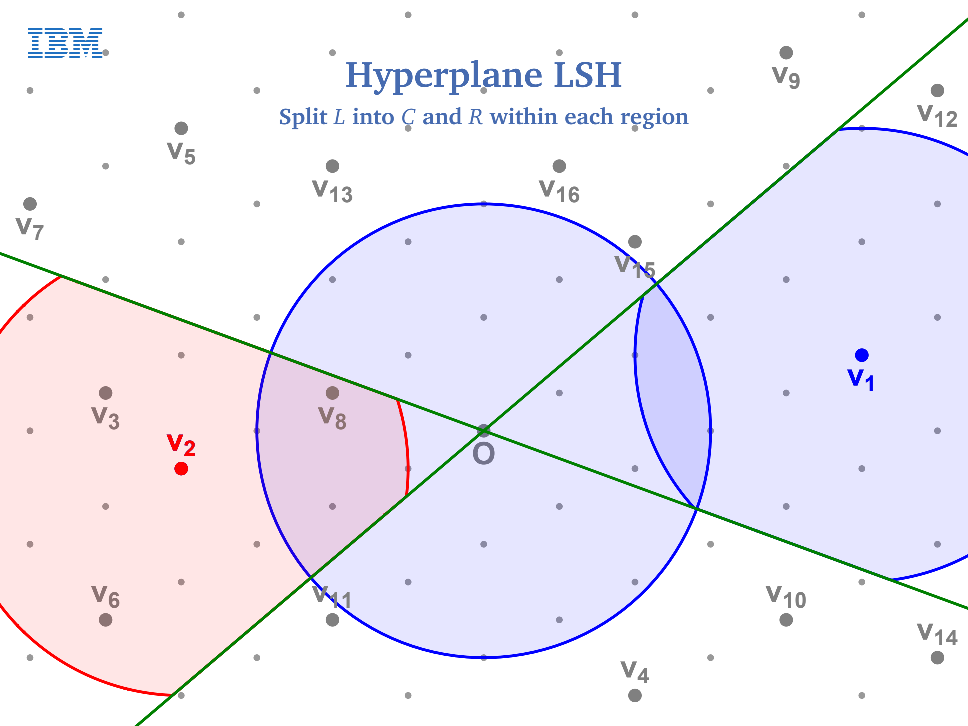

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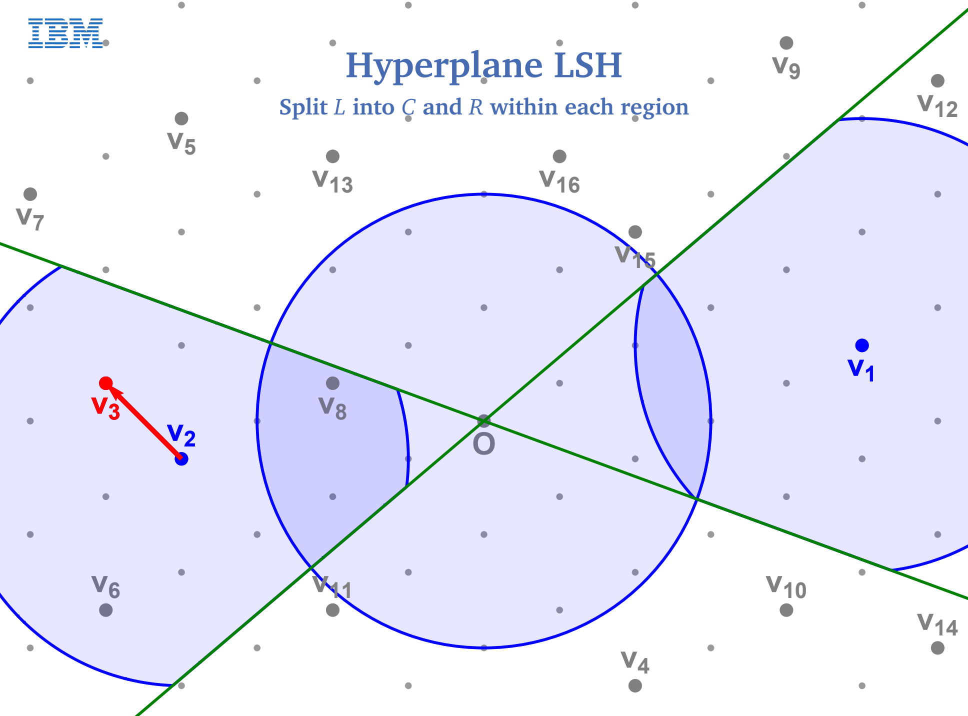

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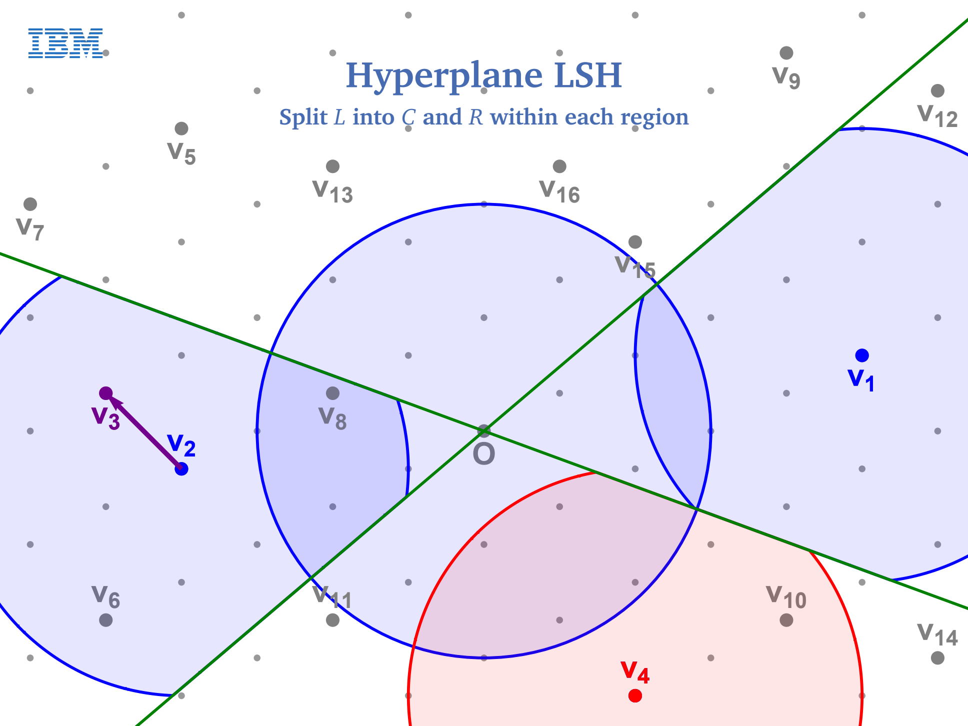

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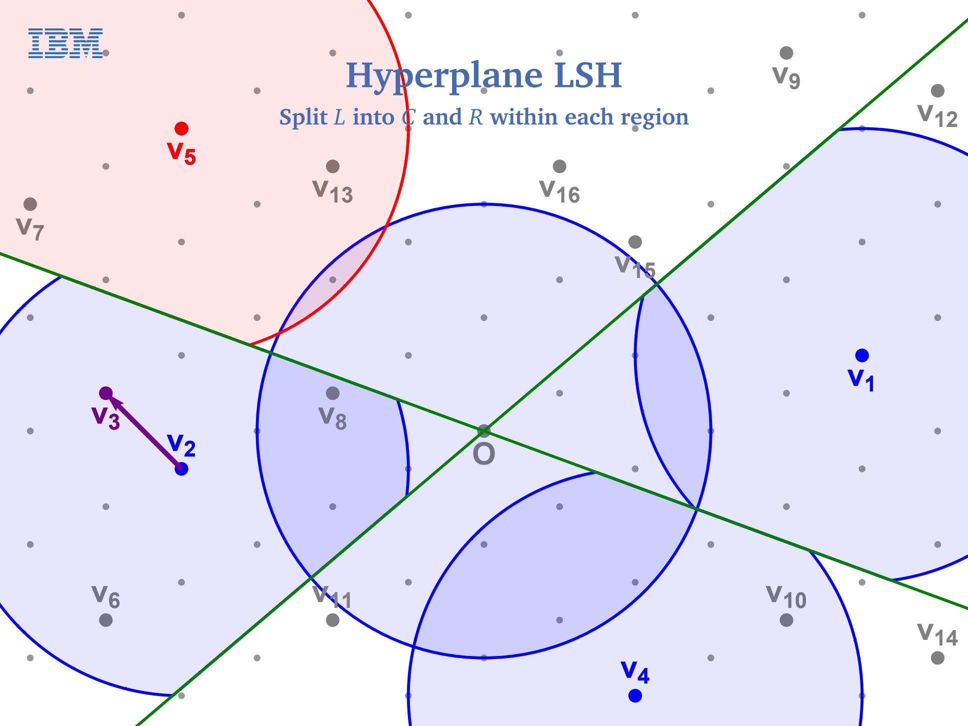

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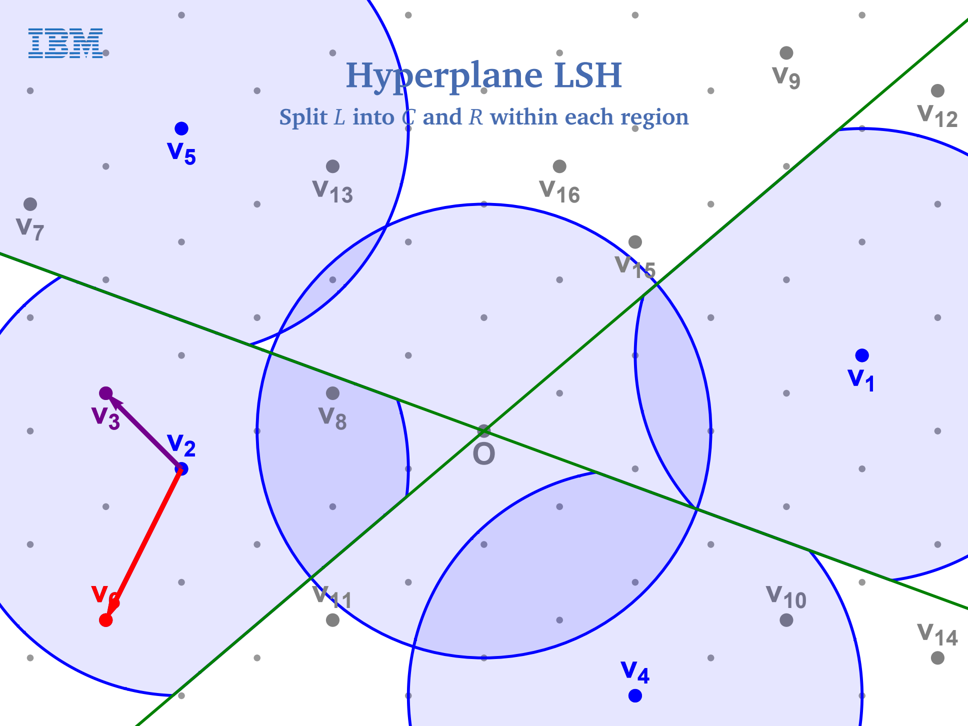

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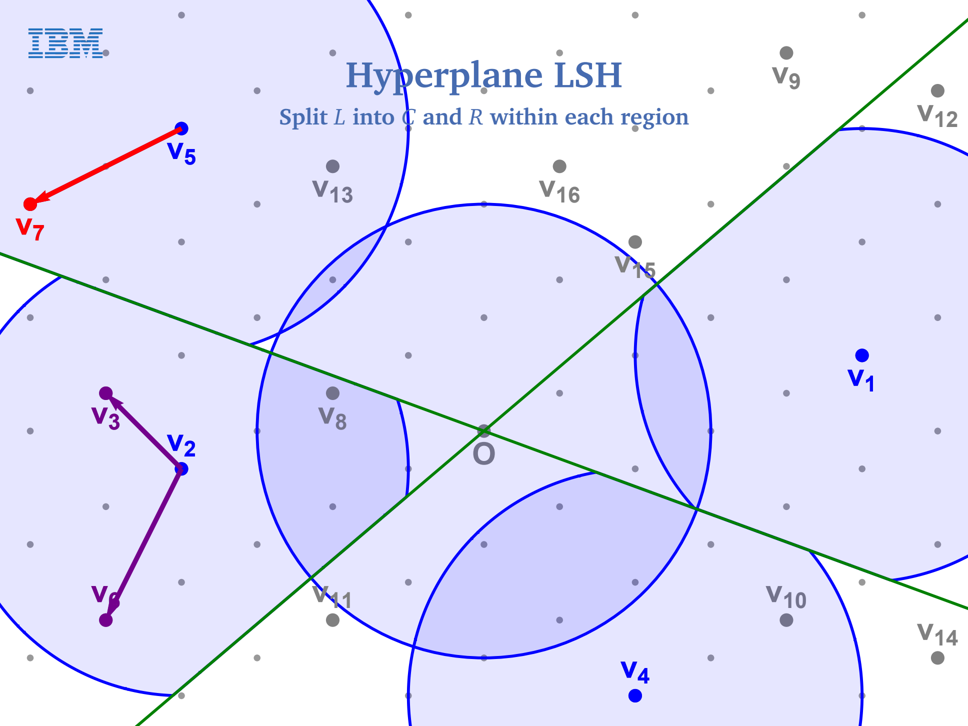

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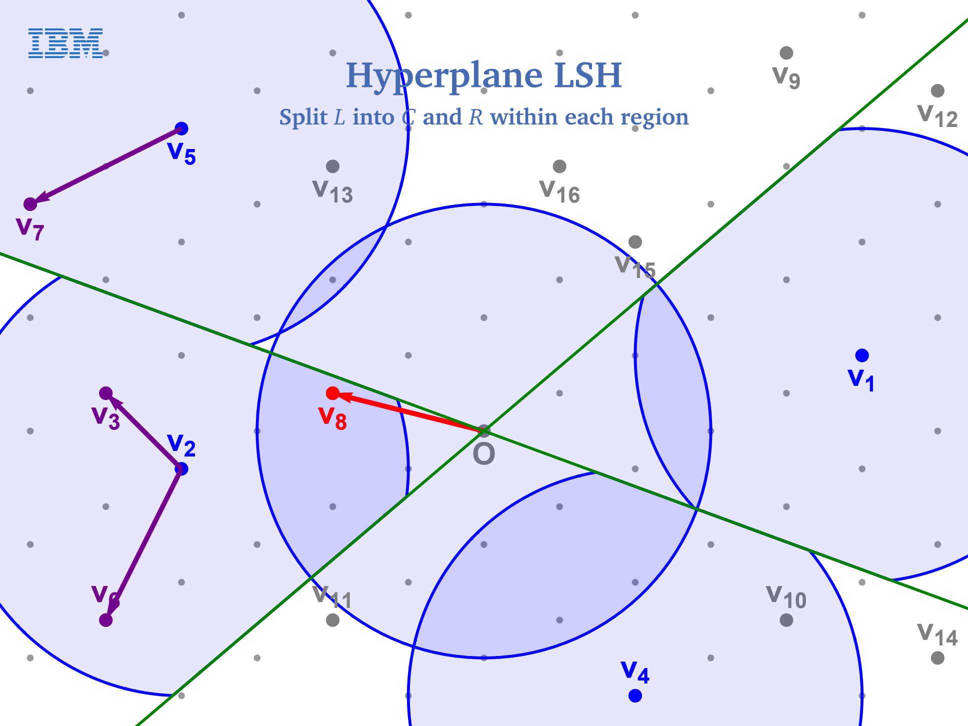

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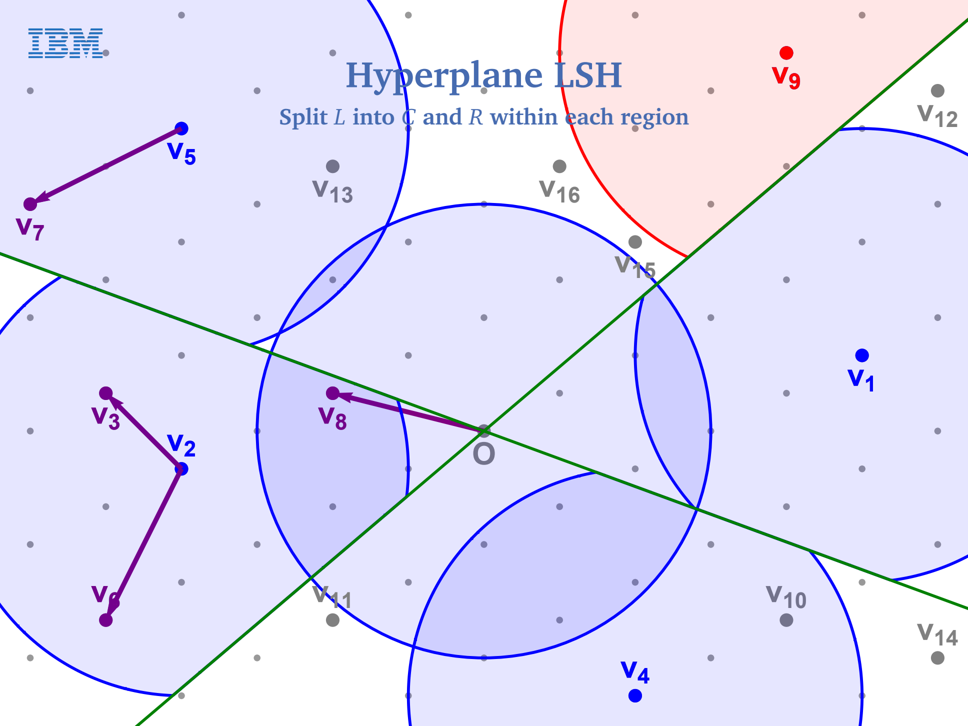

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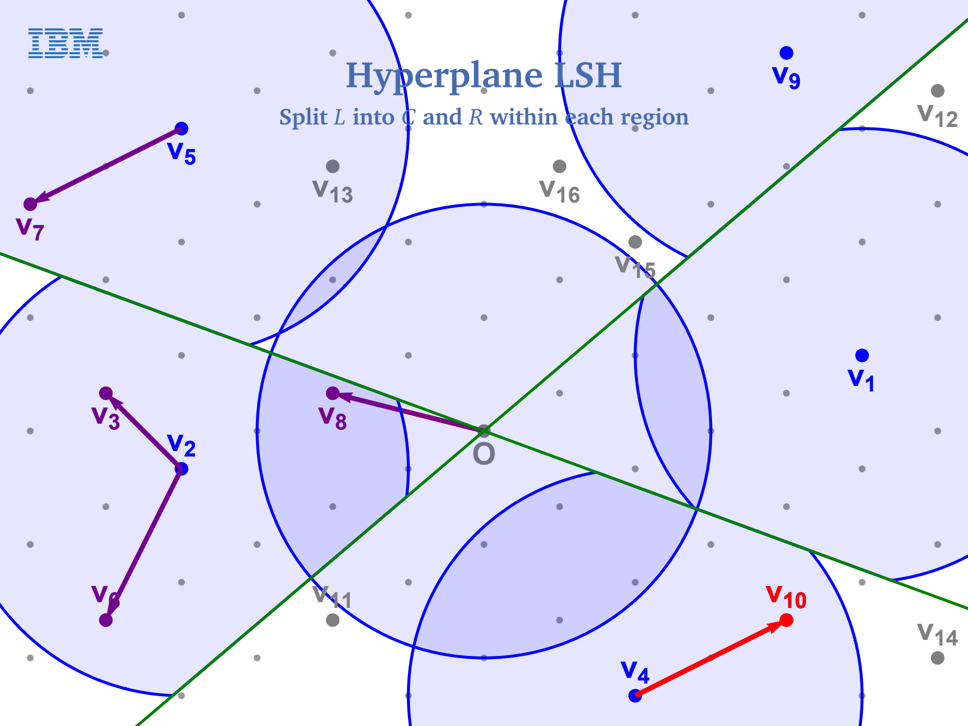

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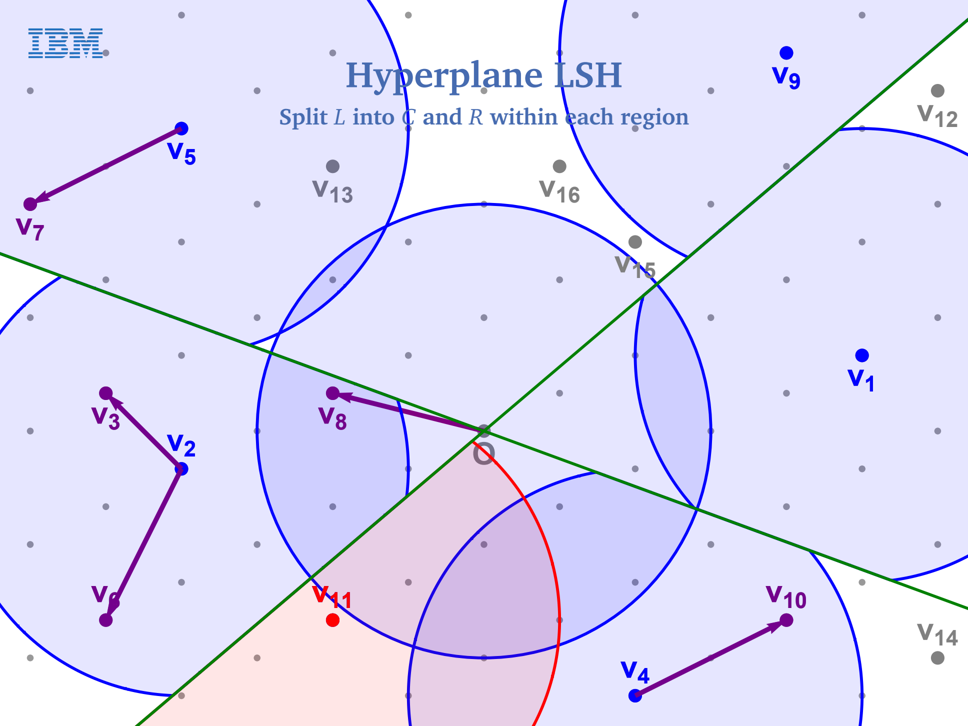

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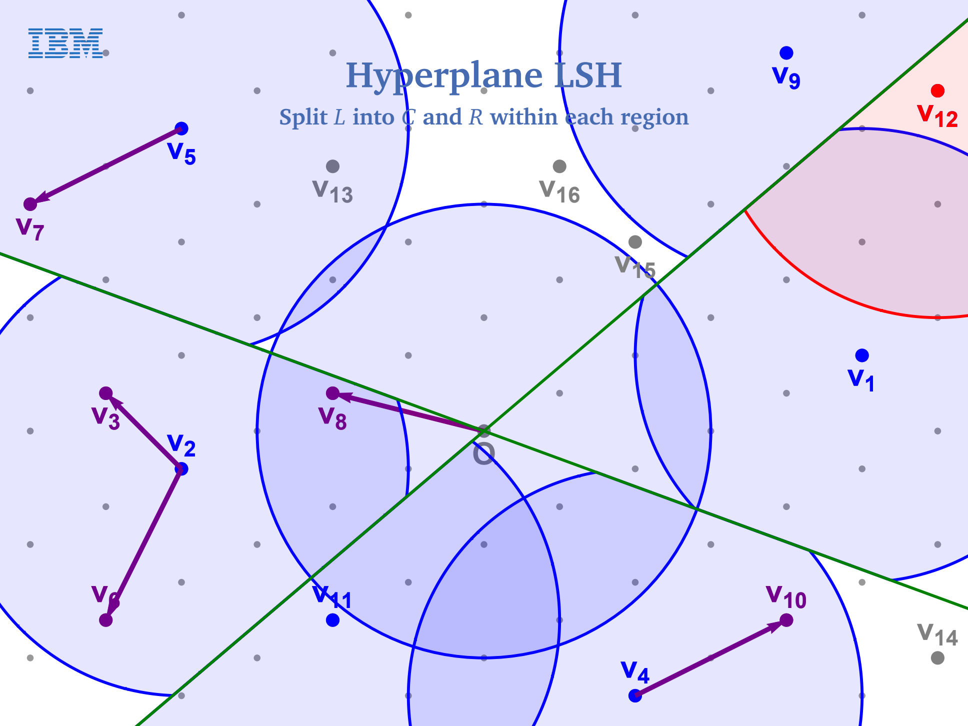

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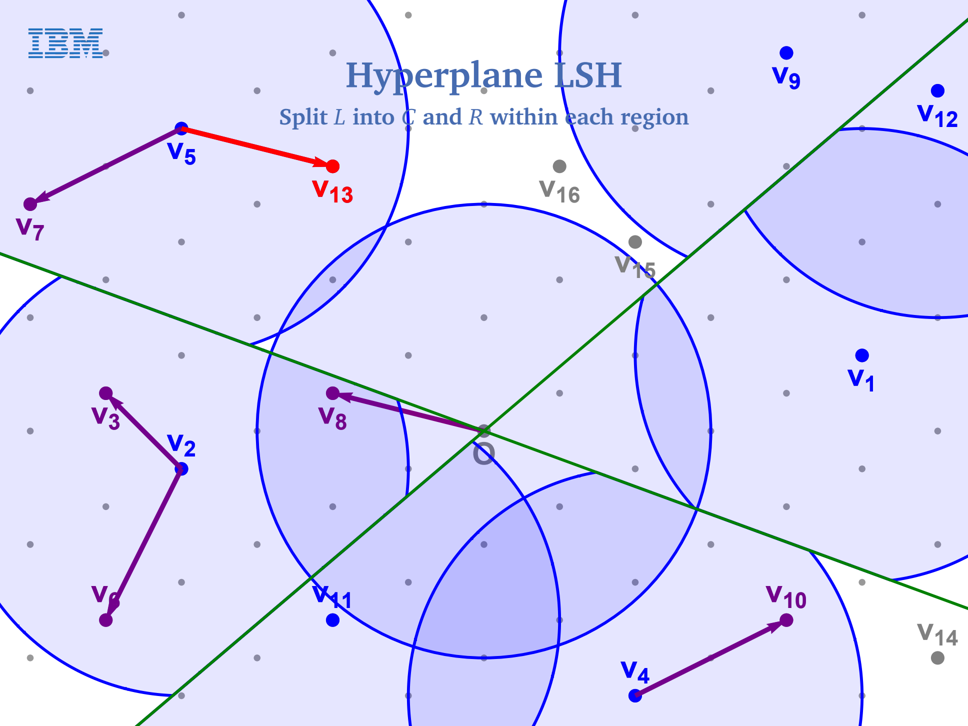

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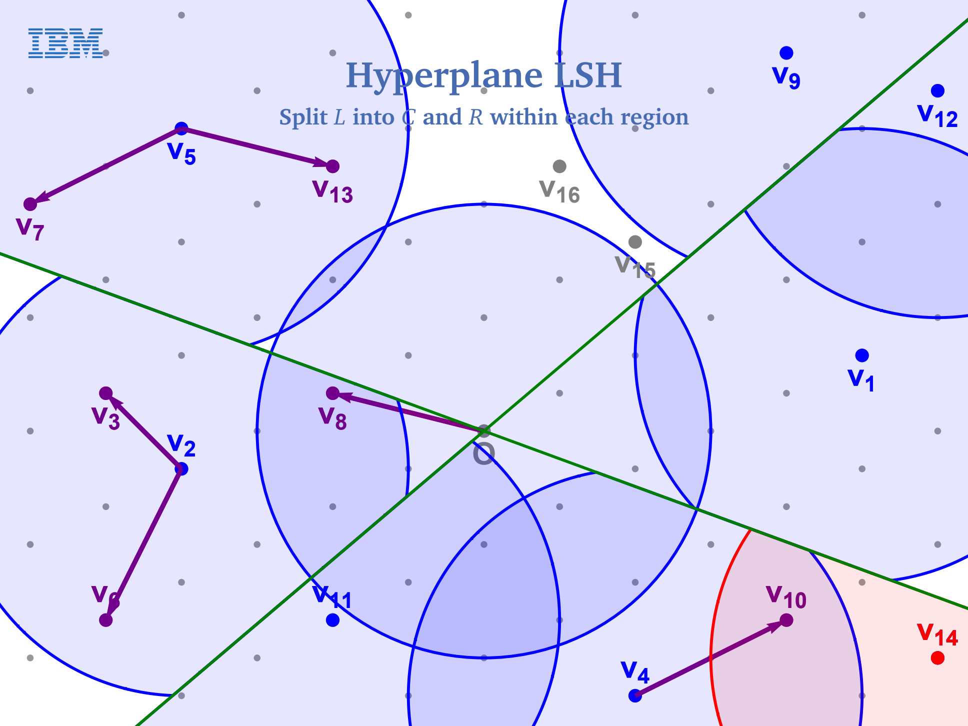

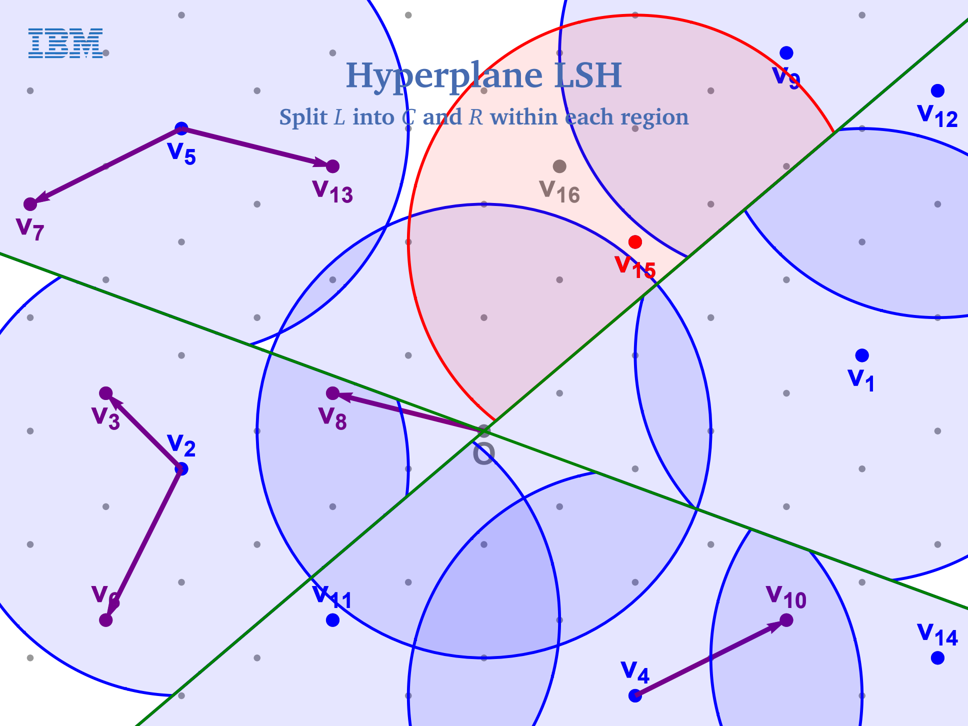
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v_9
v_{12}
v_5
v_{13}
v_{16}
v_7
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v_1
v_3
v_8
v_2
O
v_{11}
v_{10}
v_{14}
v_4

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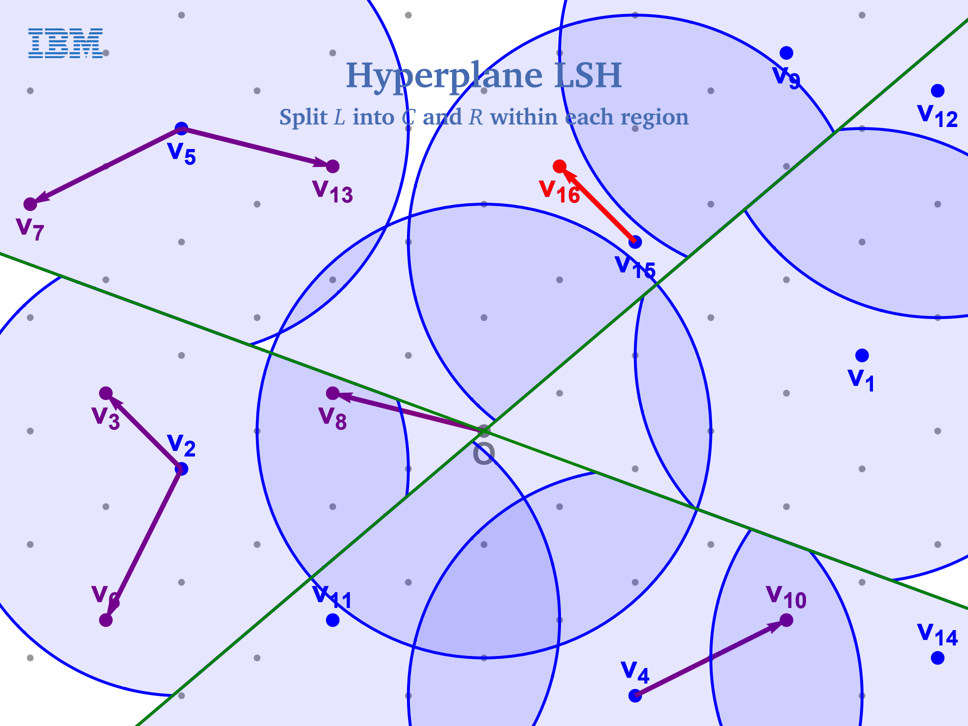

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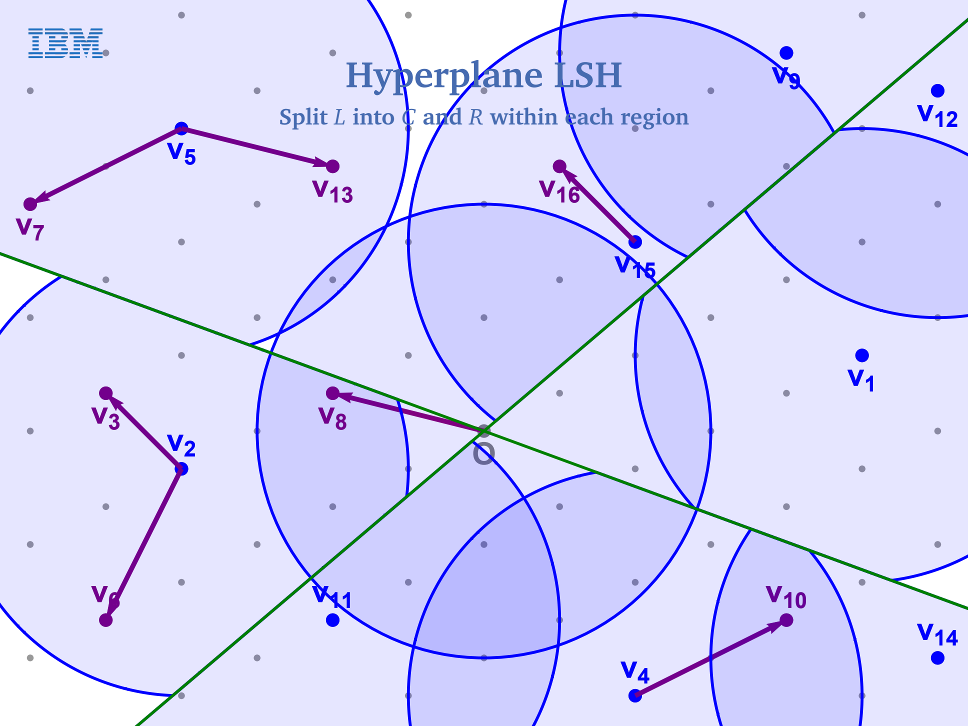
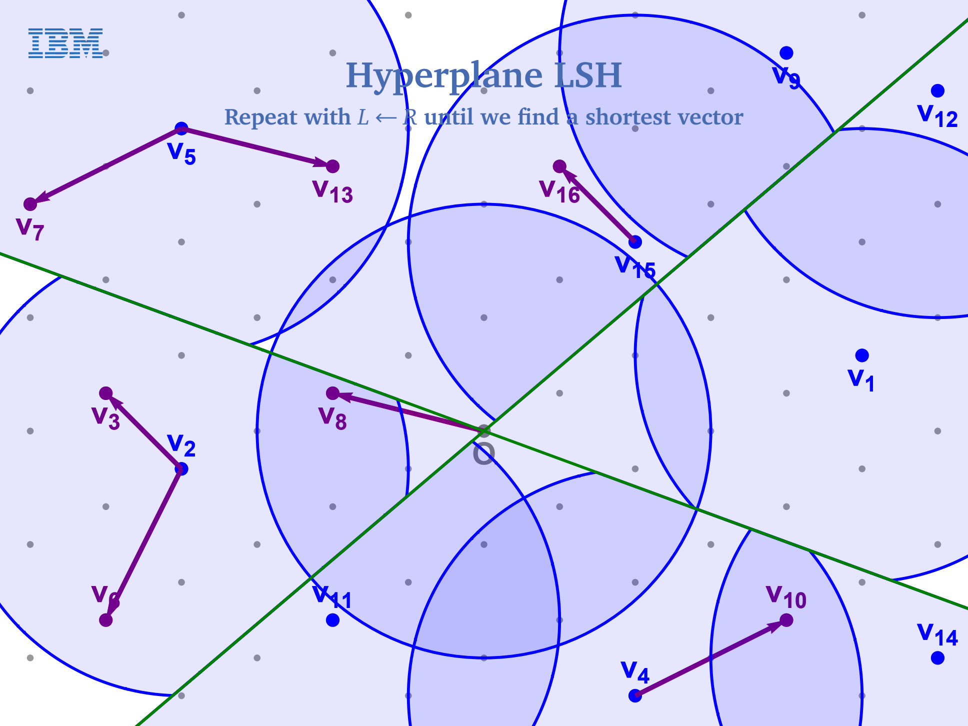


IBM

Hyperplane LSH

Repeat with $L \leftarrow R$ until we find a shortest vector



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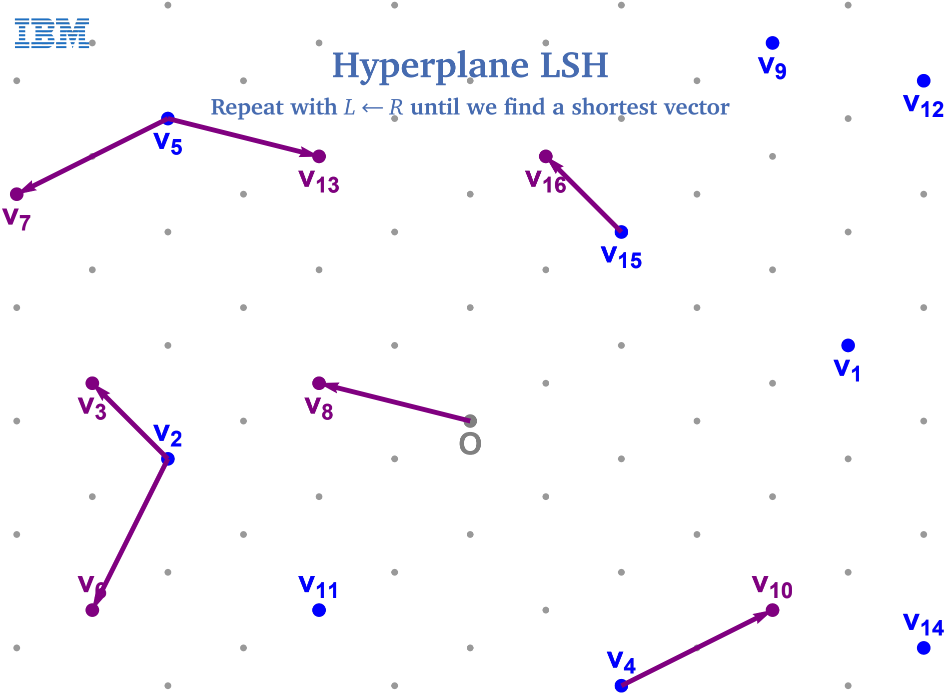

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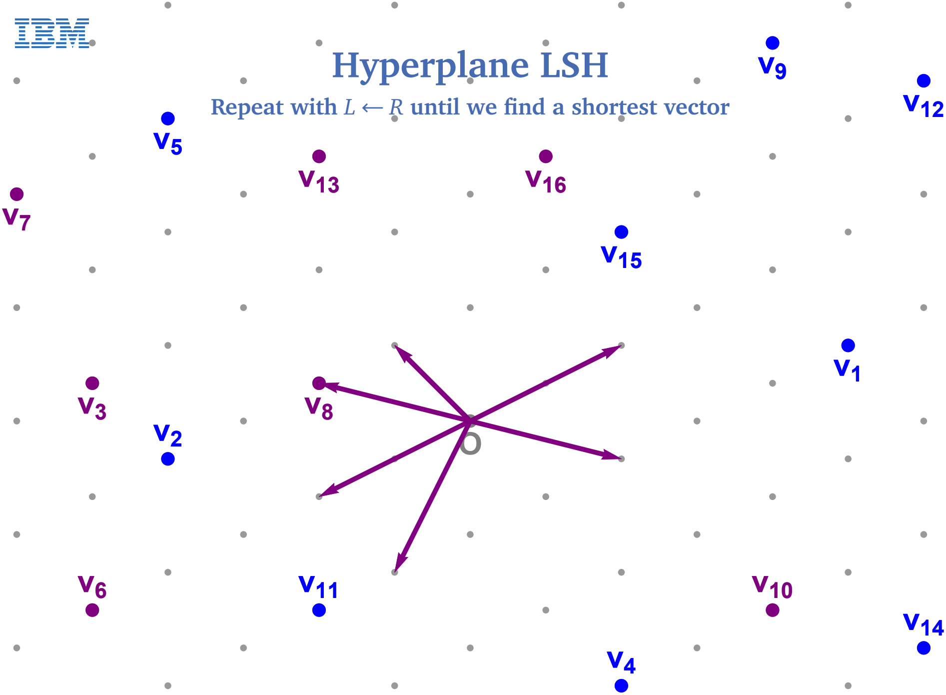

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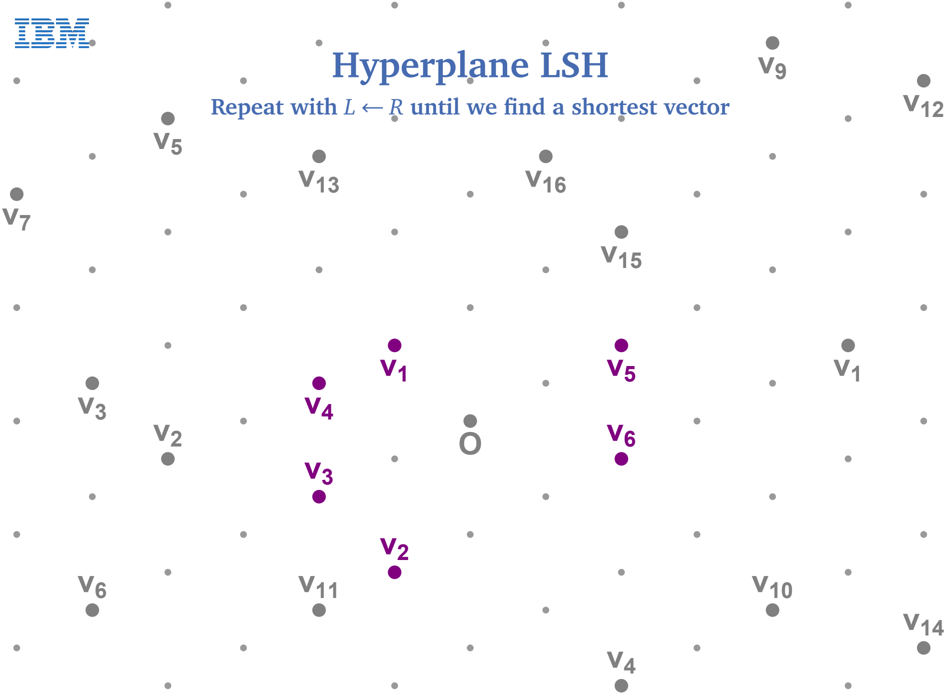
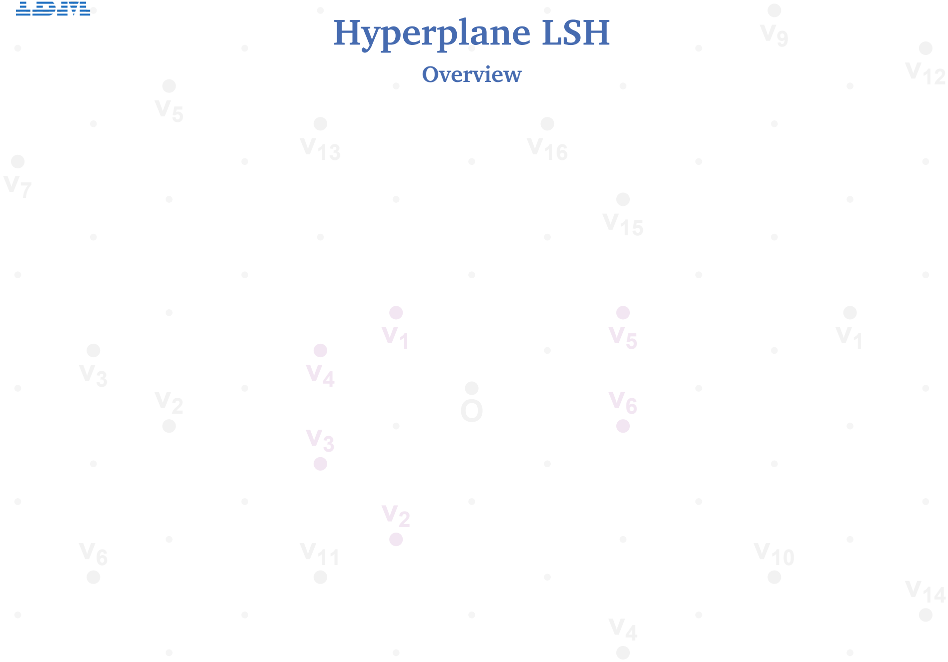


Hyperplane LSH

Overview



Hyperplane LSH

Overview

- Two parameters to tune
  - $k = O(n)$: Number of hyperplanes, leading to 2^k regions
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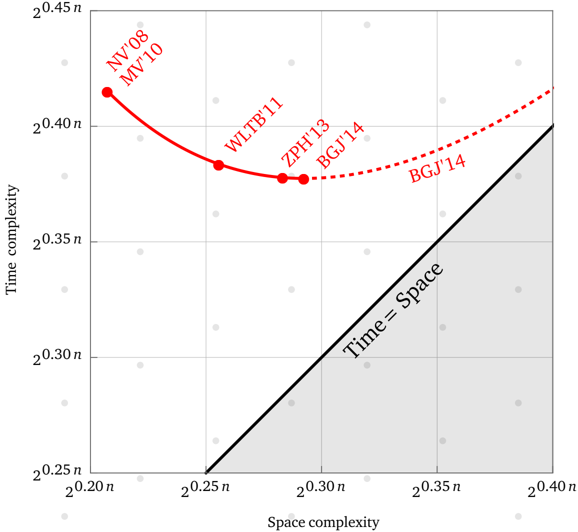
Heuristic result (Laarhoven, CRYPTO'15)

Sieving with hyperplane LSH solves SVP in time $2^{0.337n+o(n)}$.

Hyperplane LSH

Space/time trade-off



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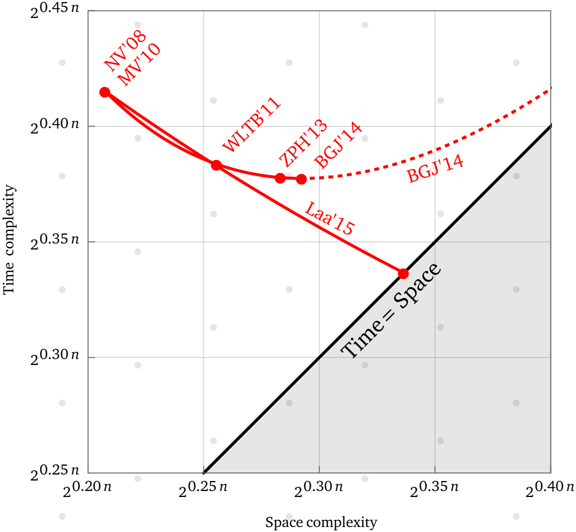

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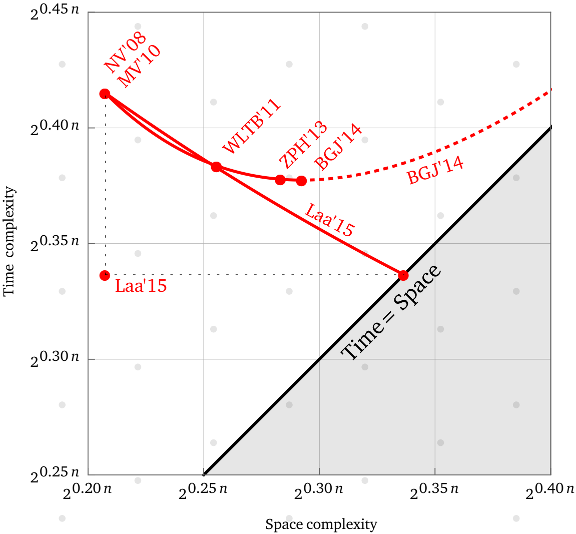

Spherical LSH

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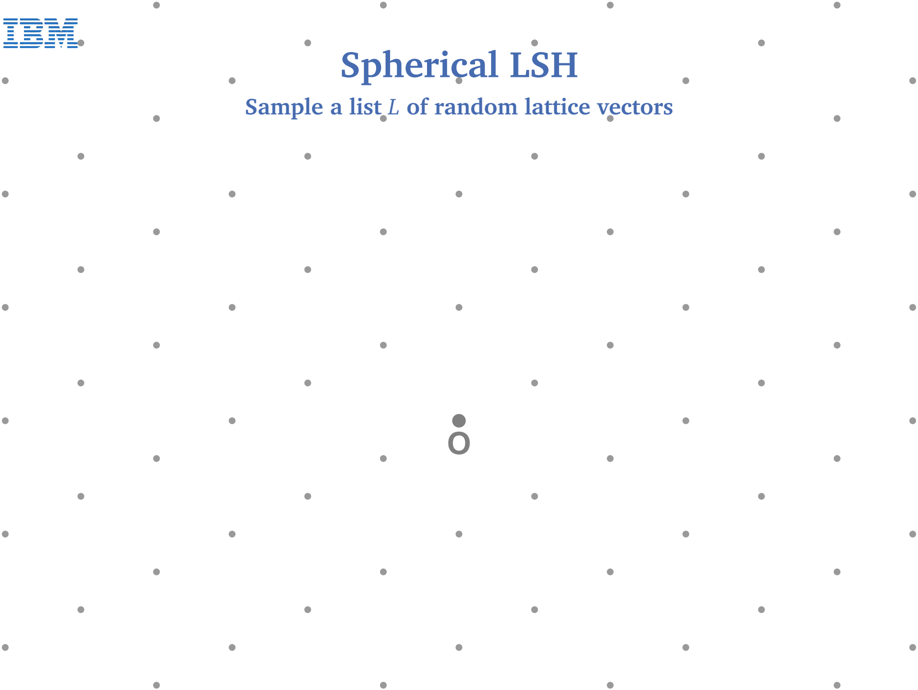

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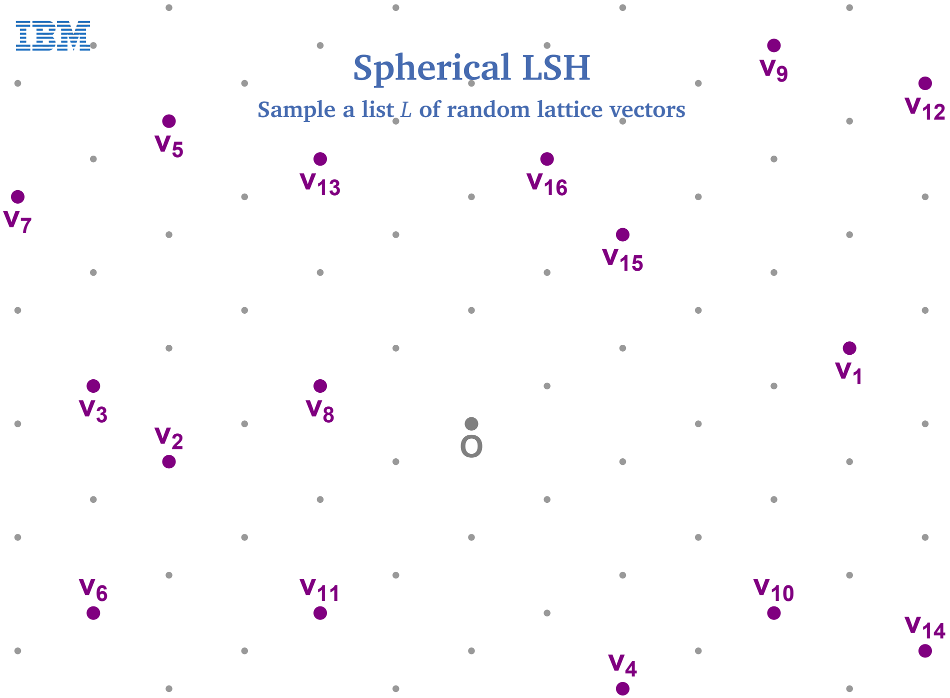
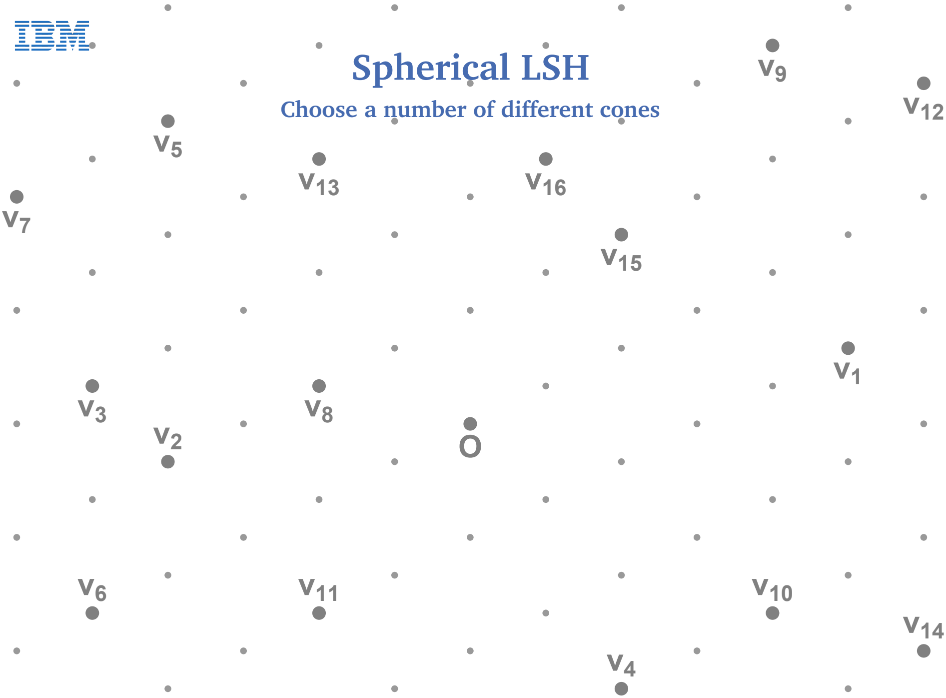


Spherical LSH

Choose a number of different cones



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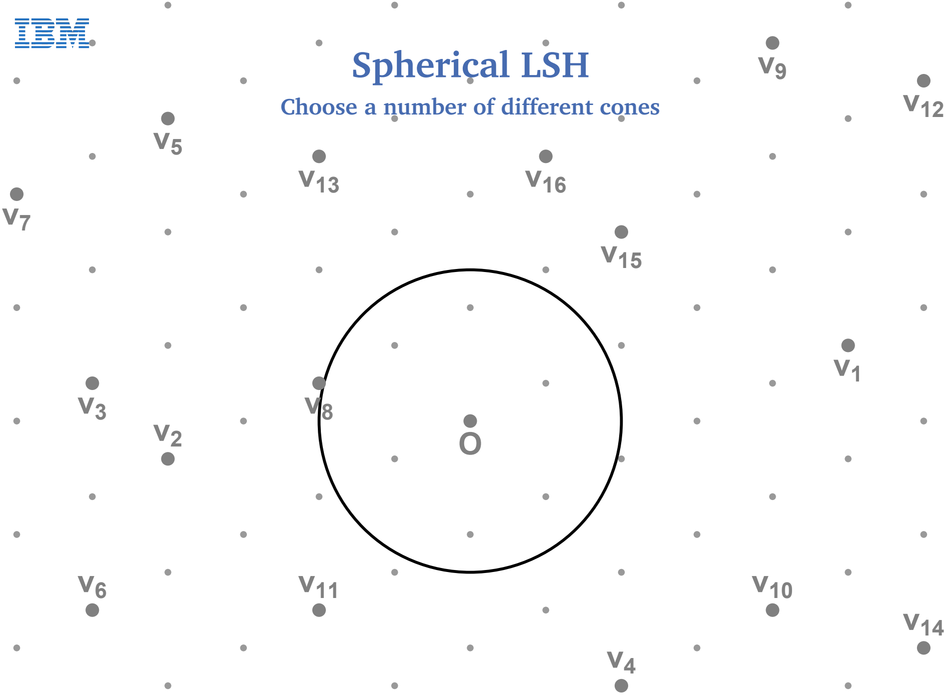

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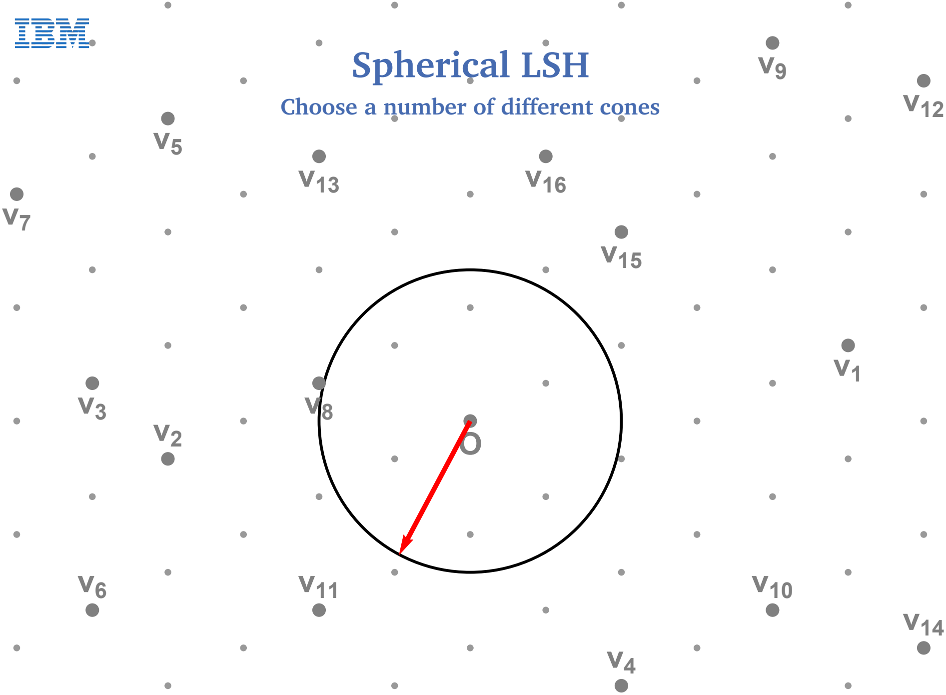

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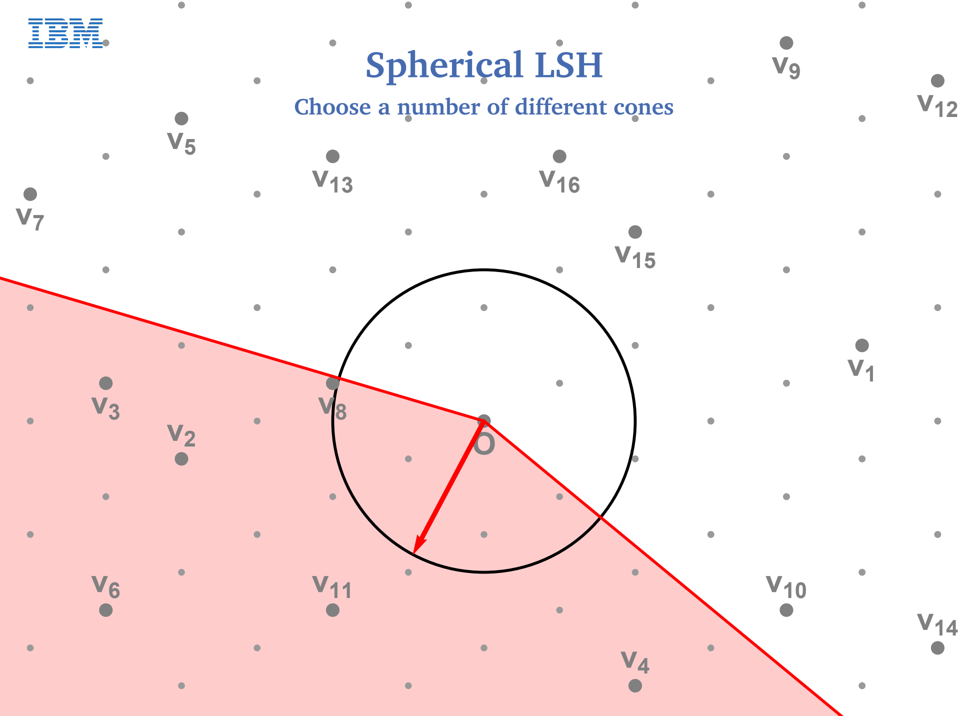

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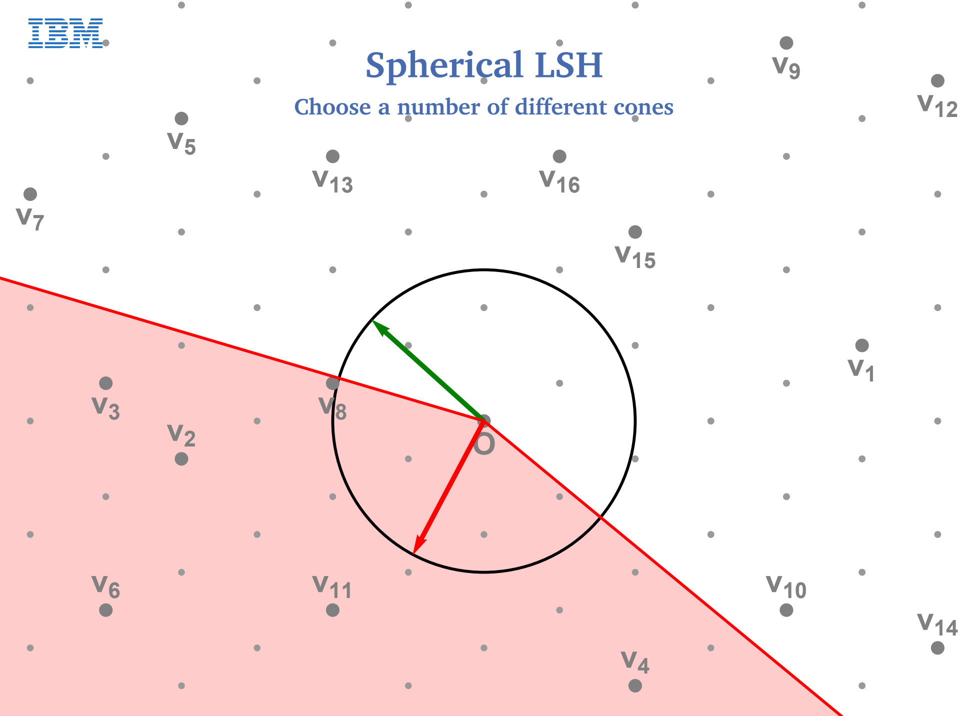

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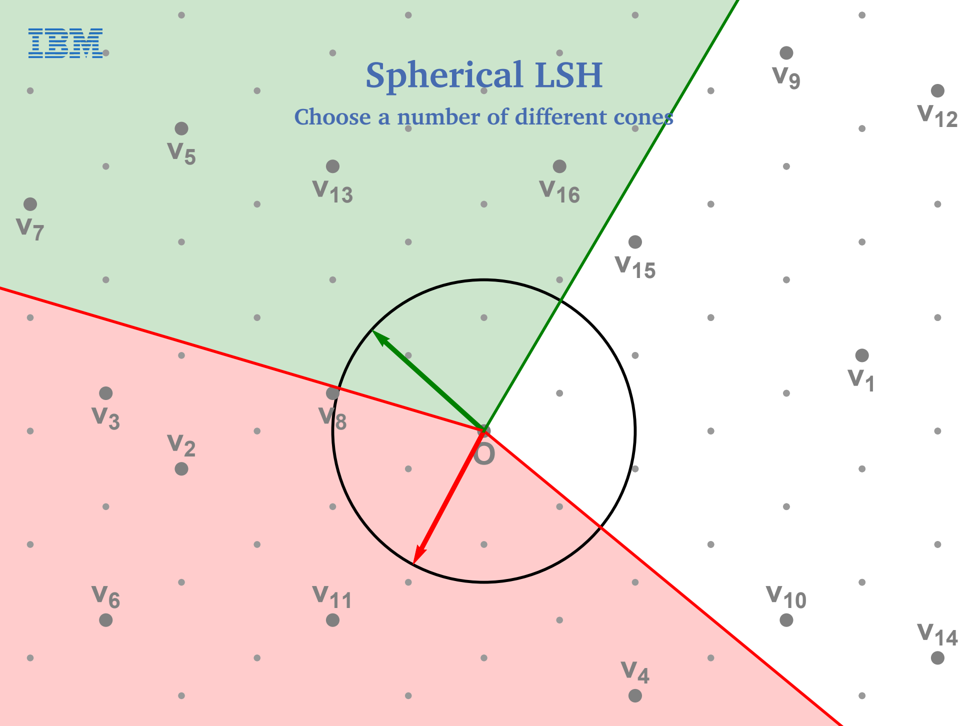

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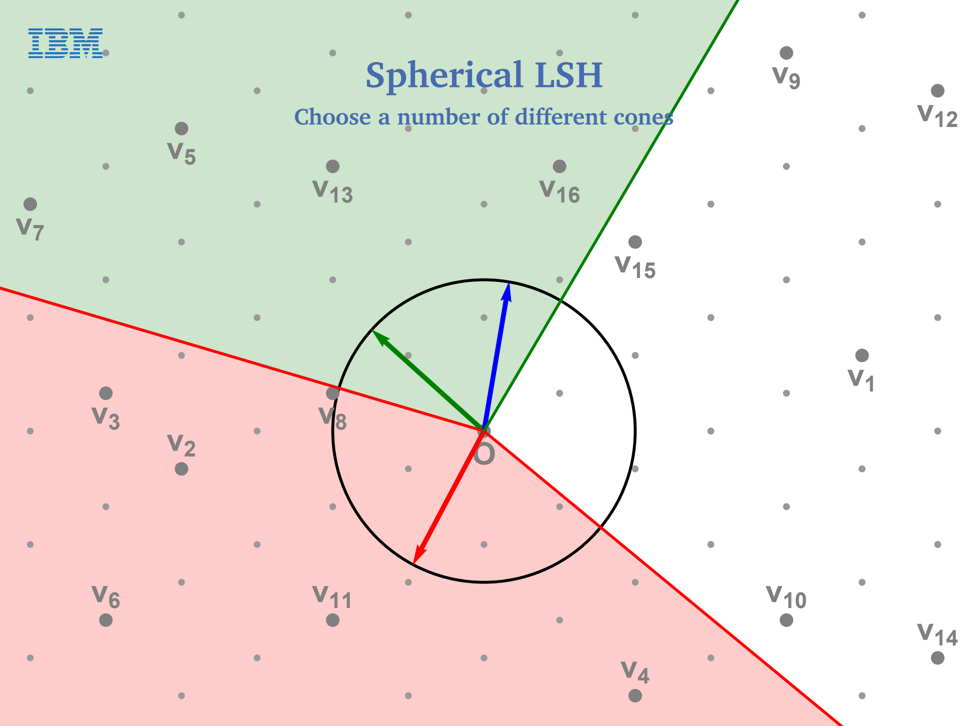

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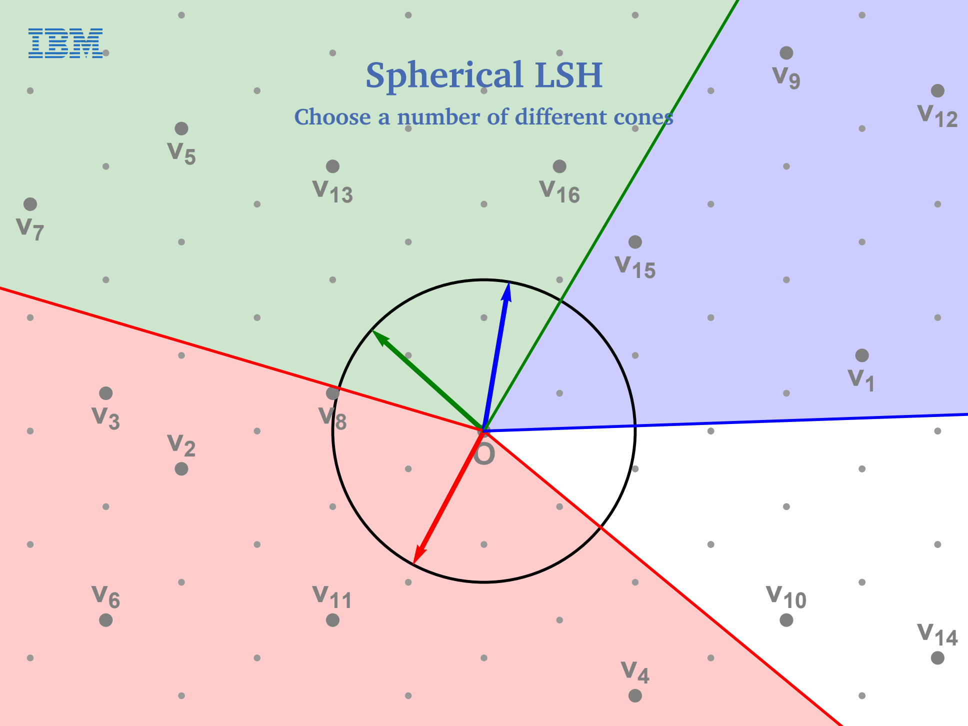

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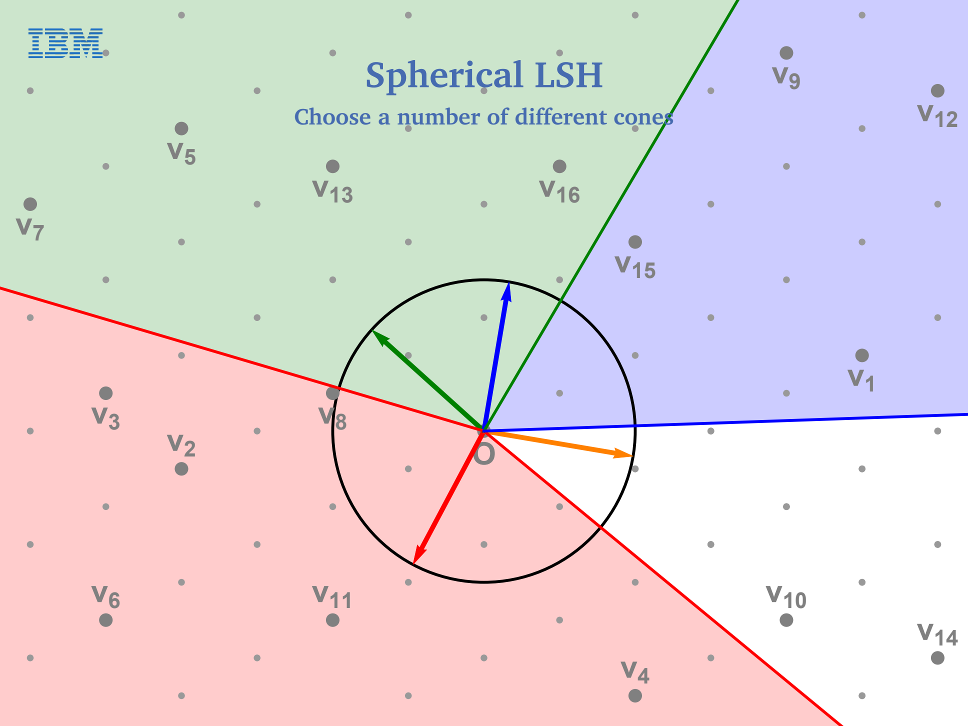

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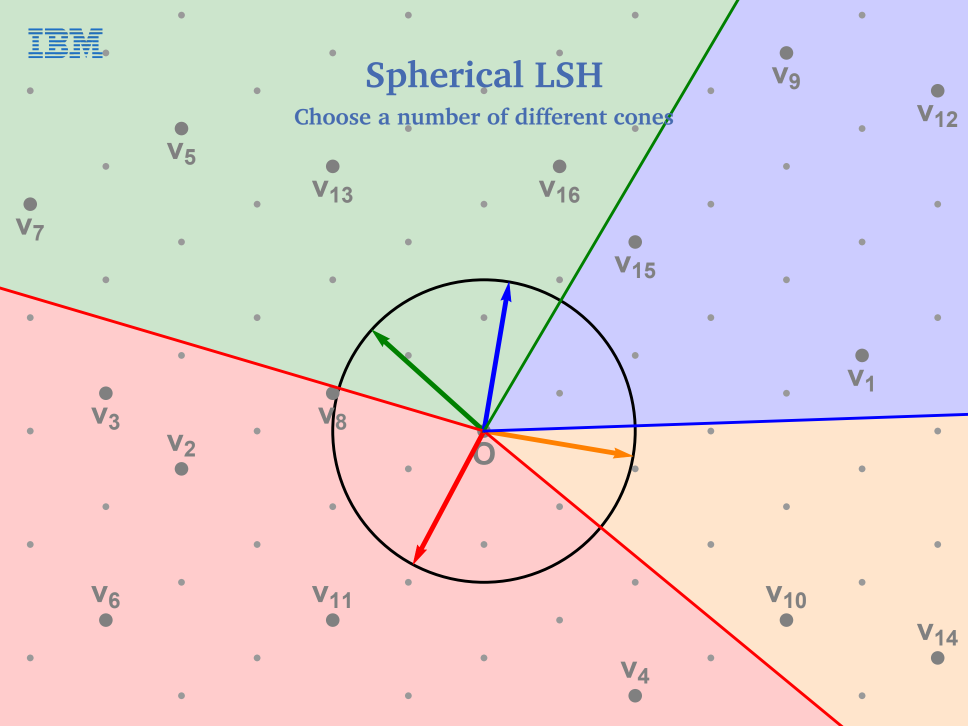

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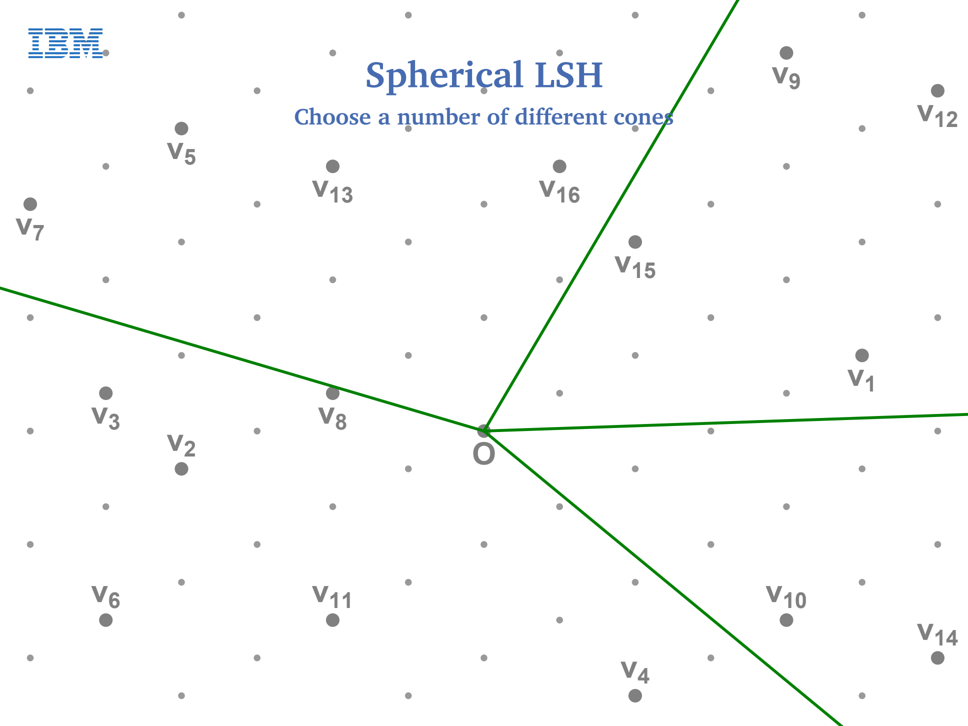

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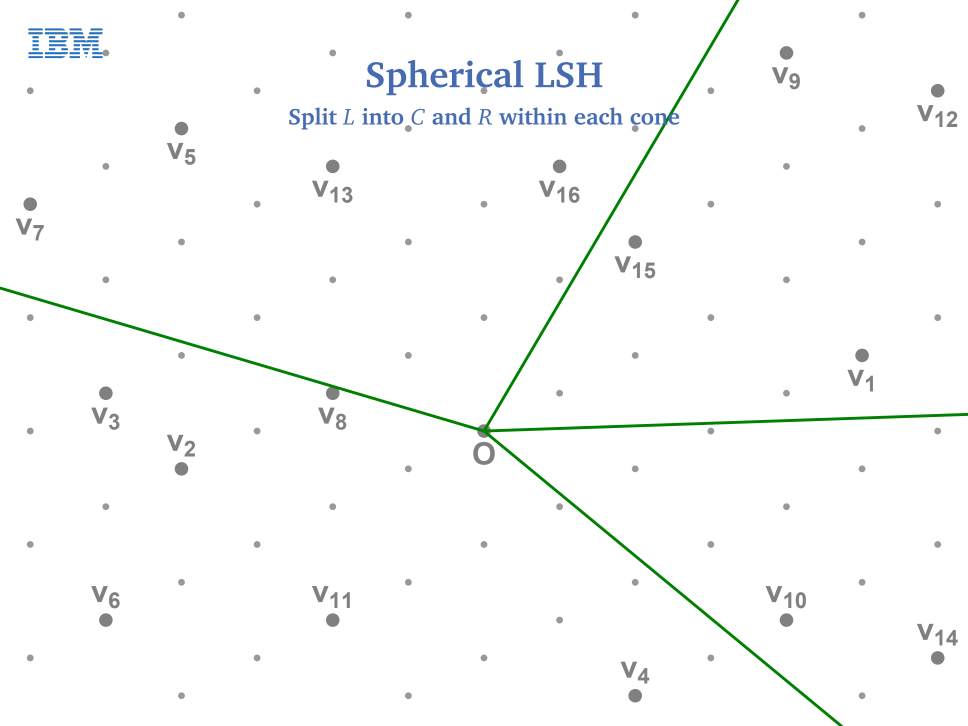

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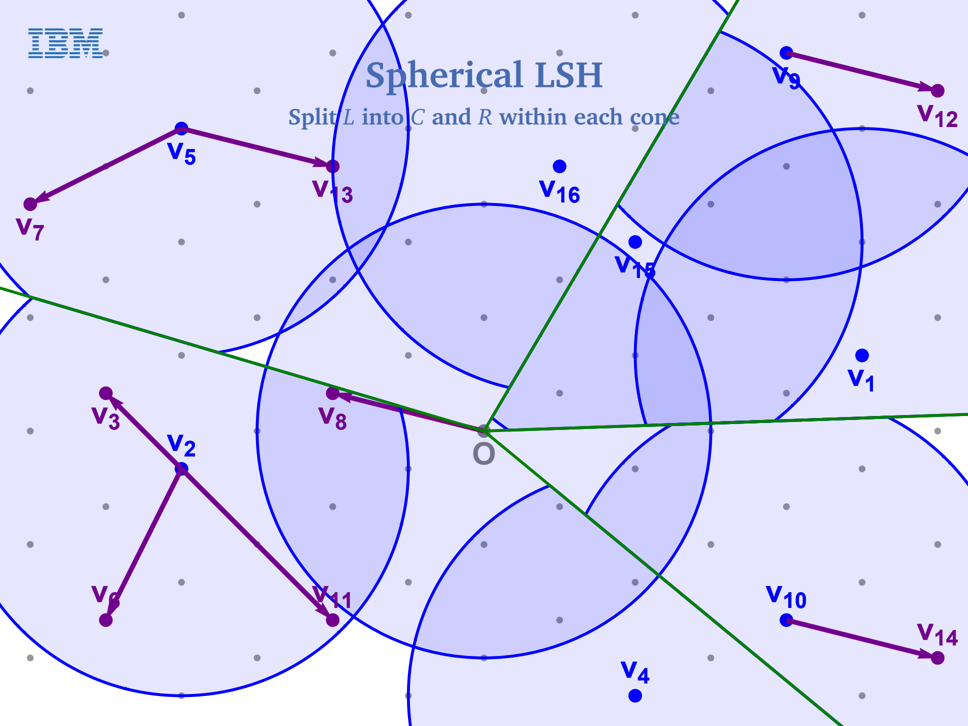

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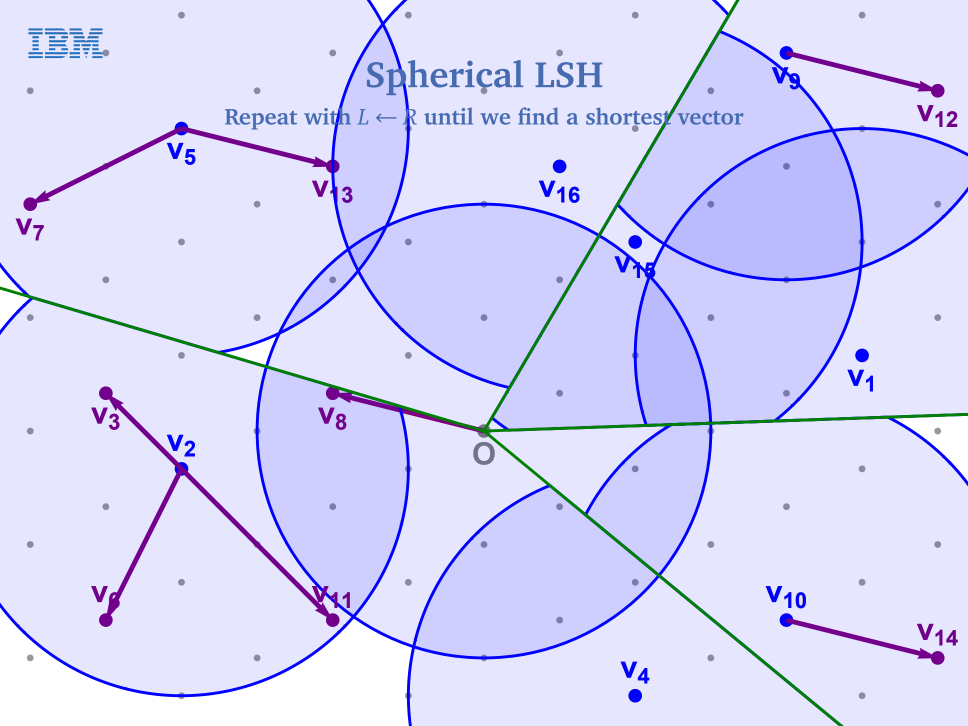

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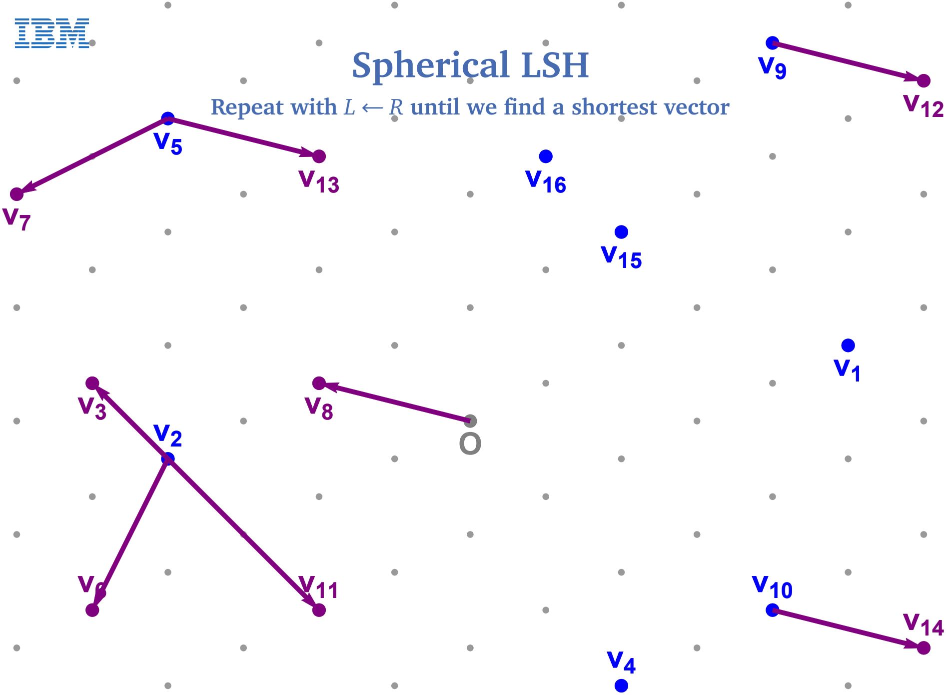

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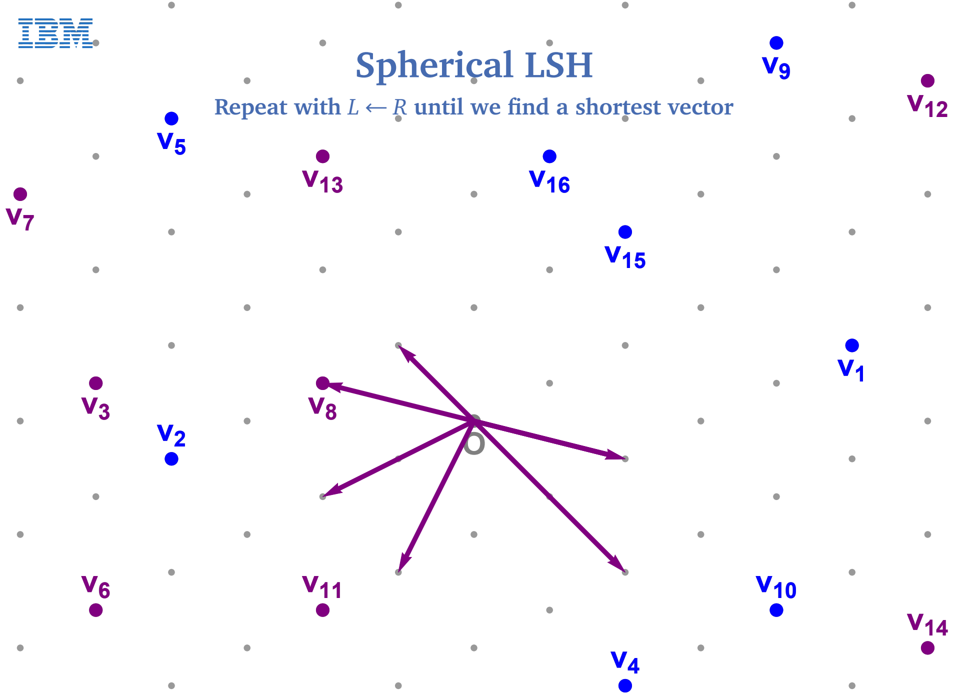

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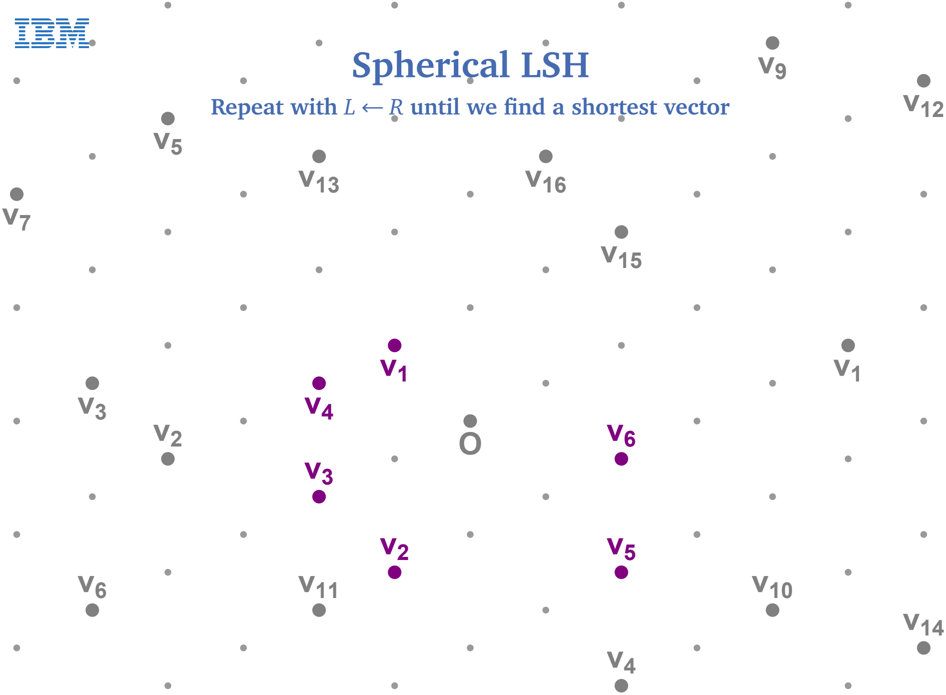
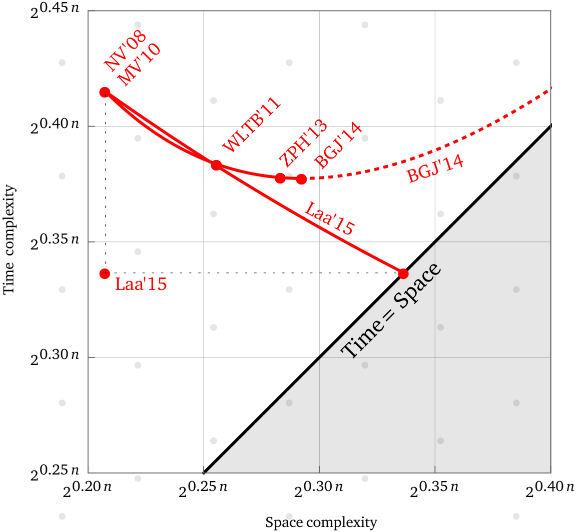


Spherical LSH

Space/time trade-off



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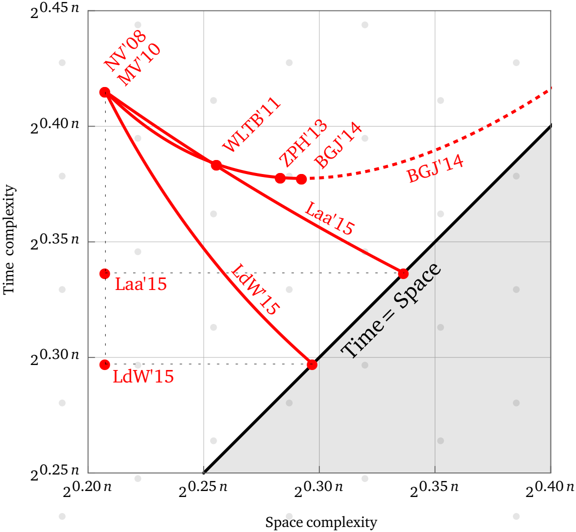
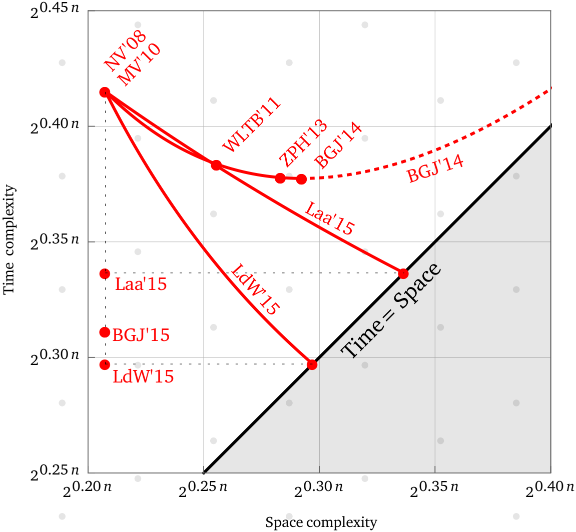


May and Ozerov's NNS method

Space/time trade-off



Cross-Polytope LSH

Sample a list L of random lattice vectors

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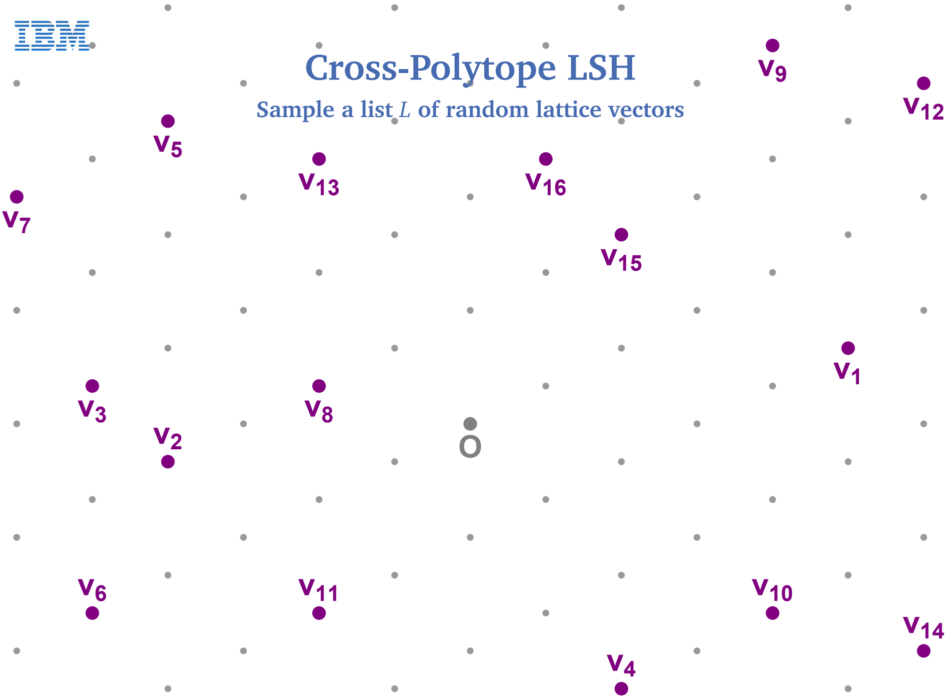
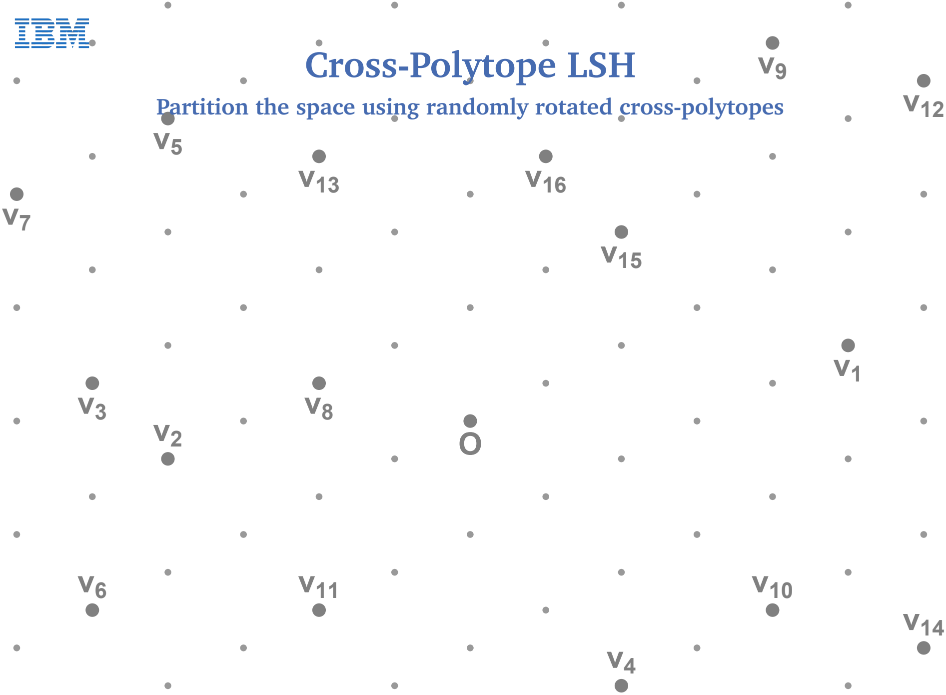


Cross-Polytope LSH

Partition the space using randomly rotated cross-polytopes



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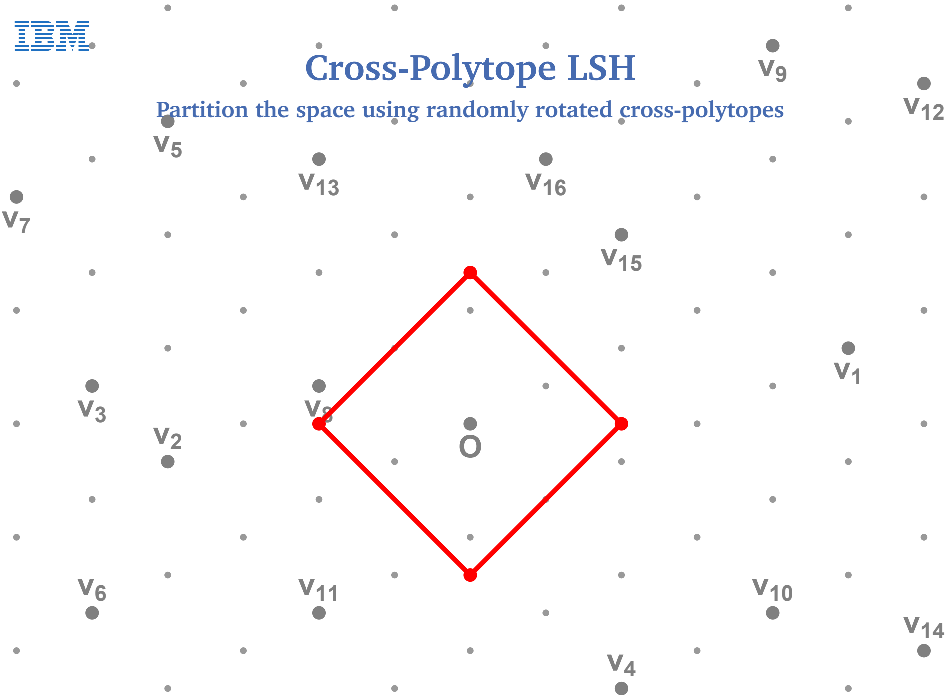

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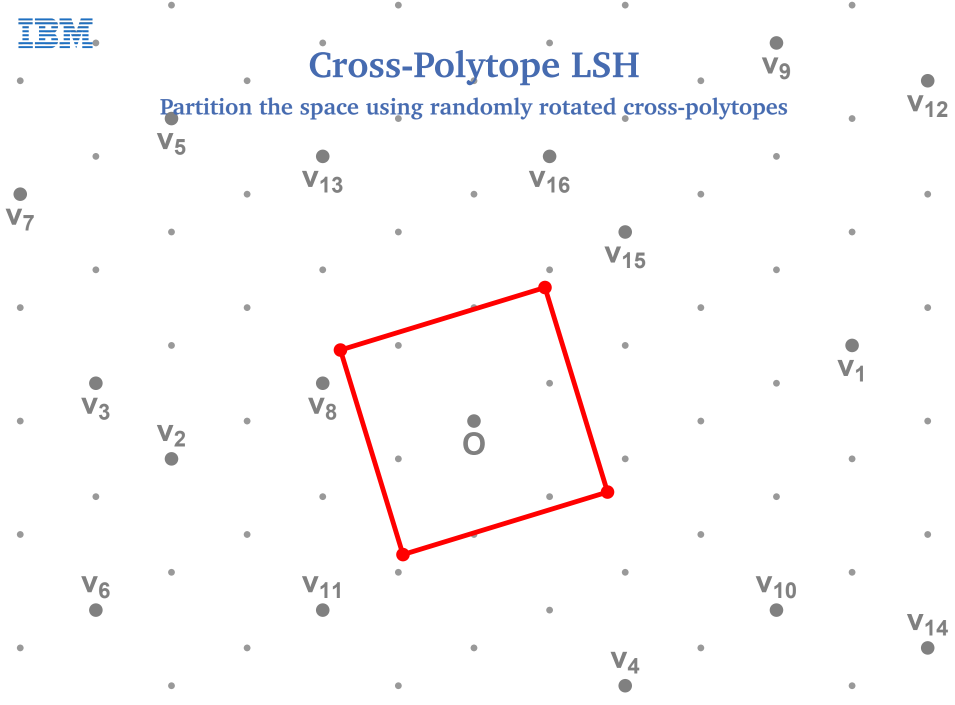

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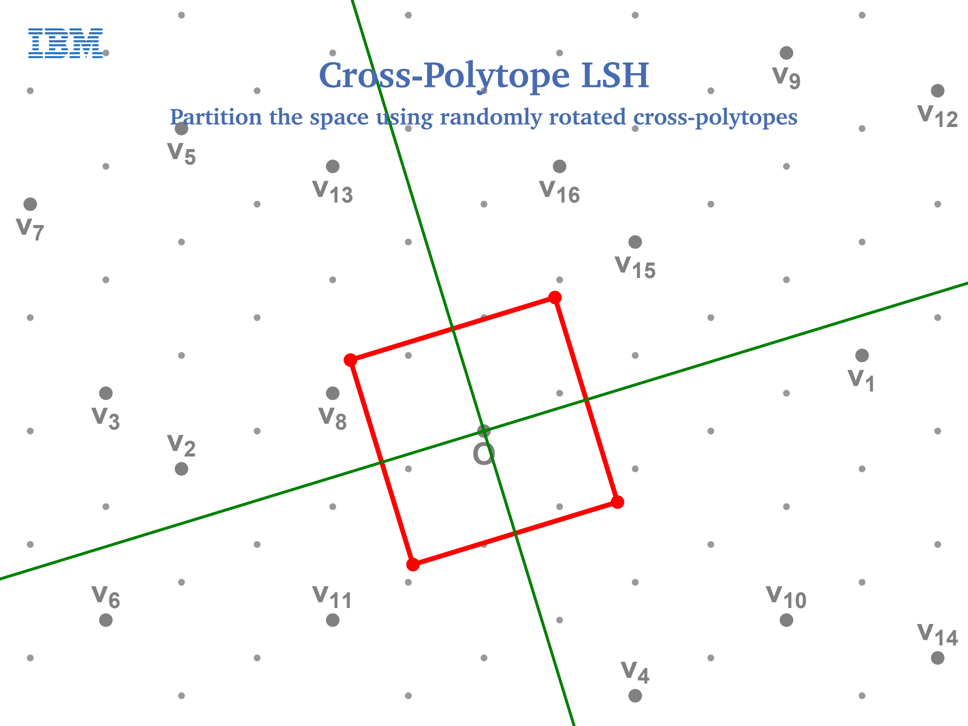

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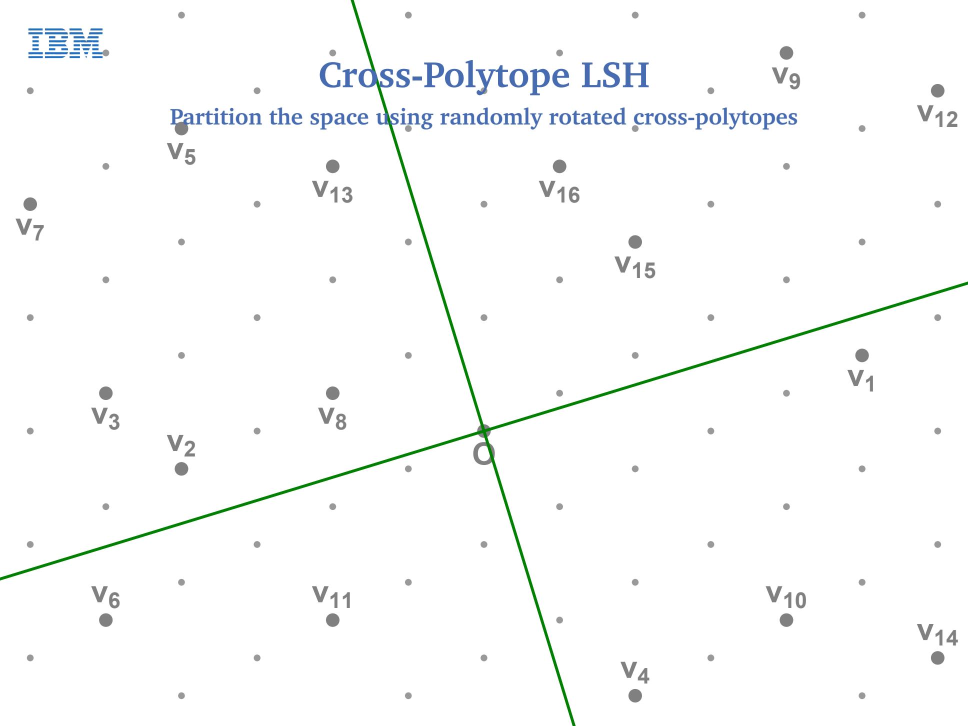

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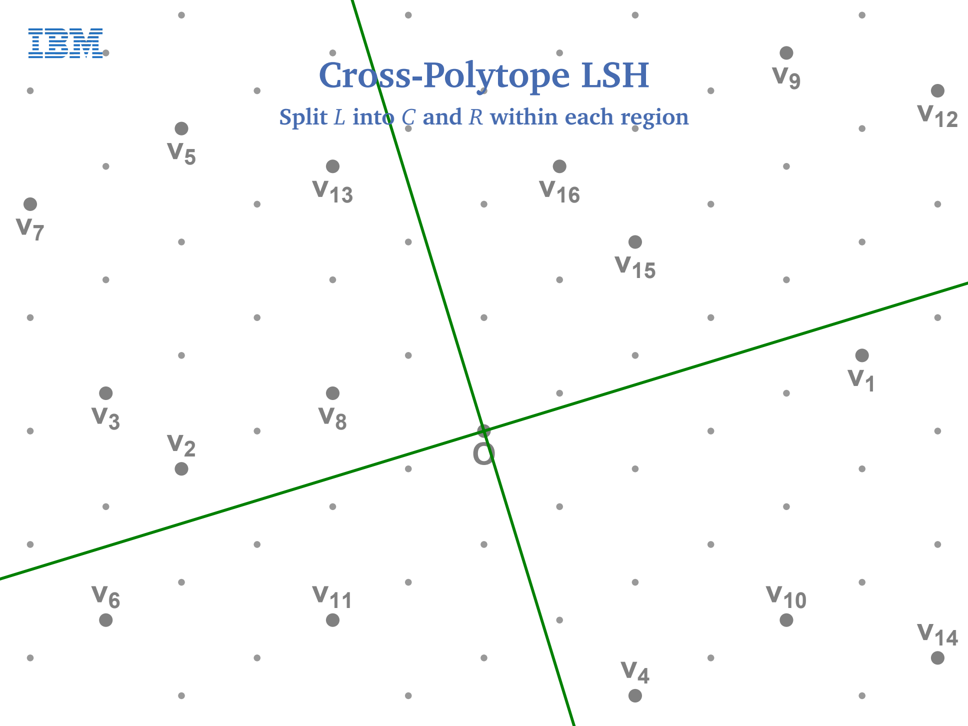

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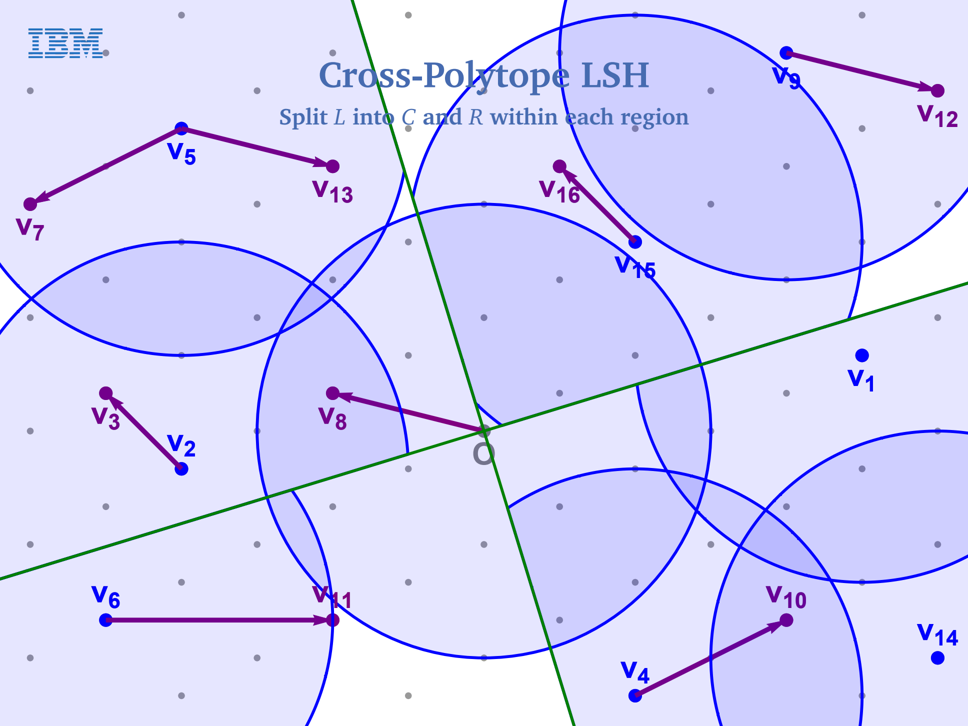

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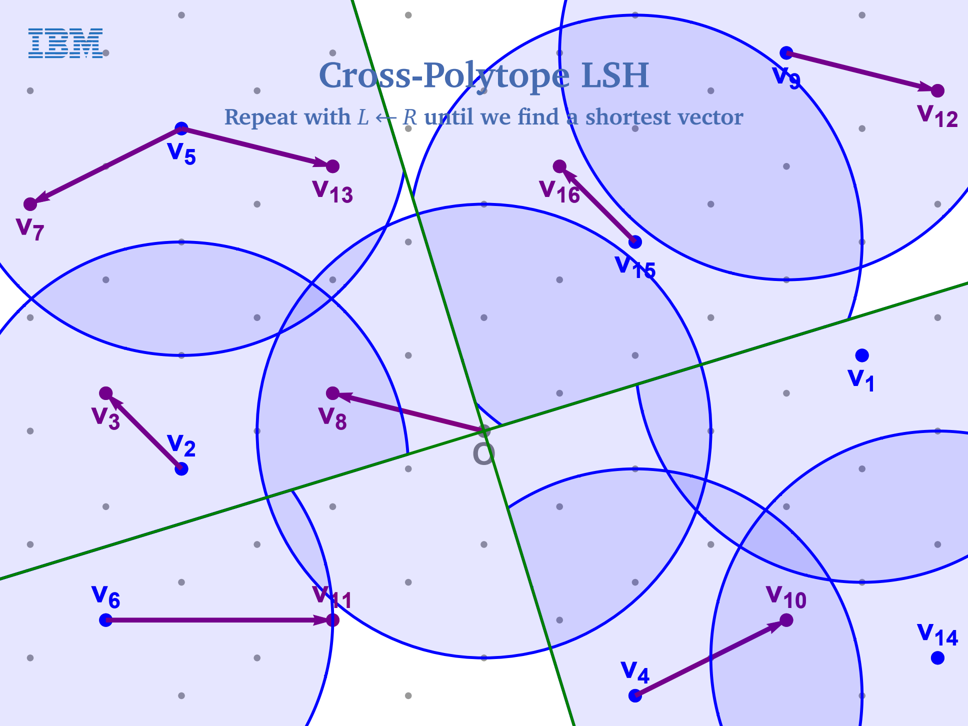

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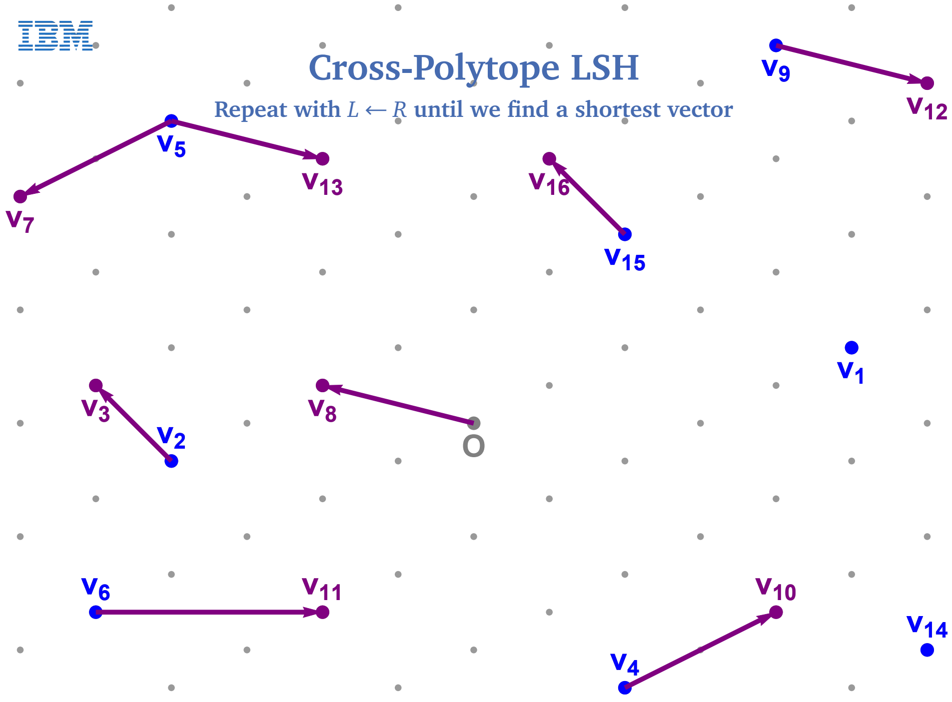

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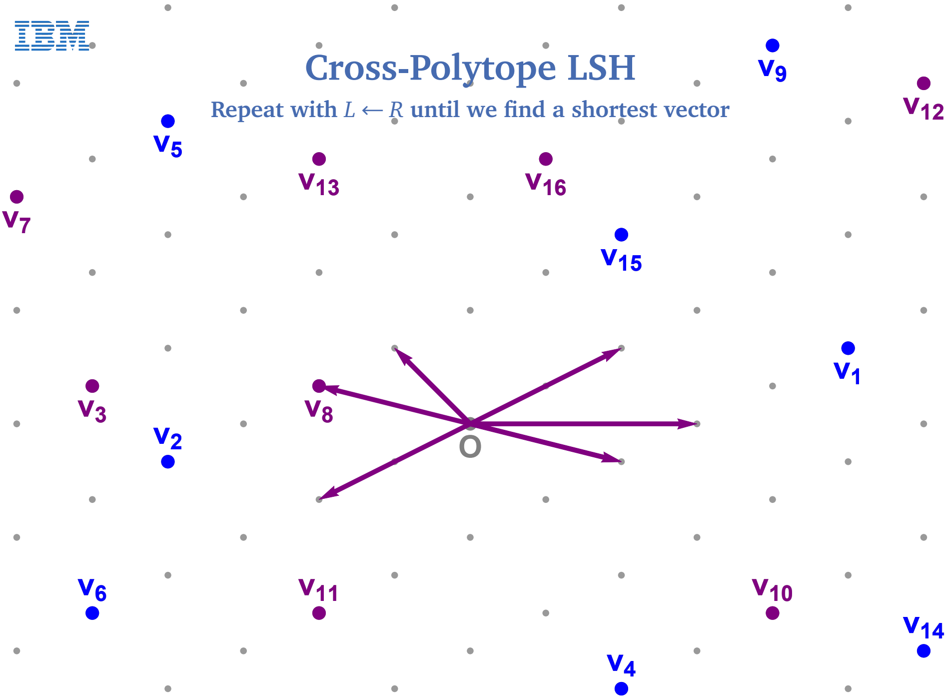

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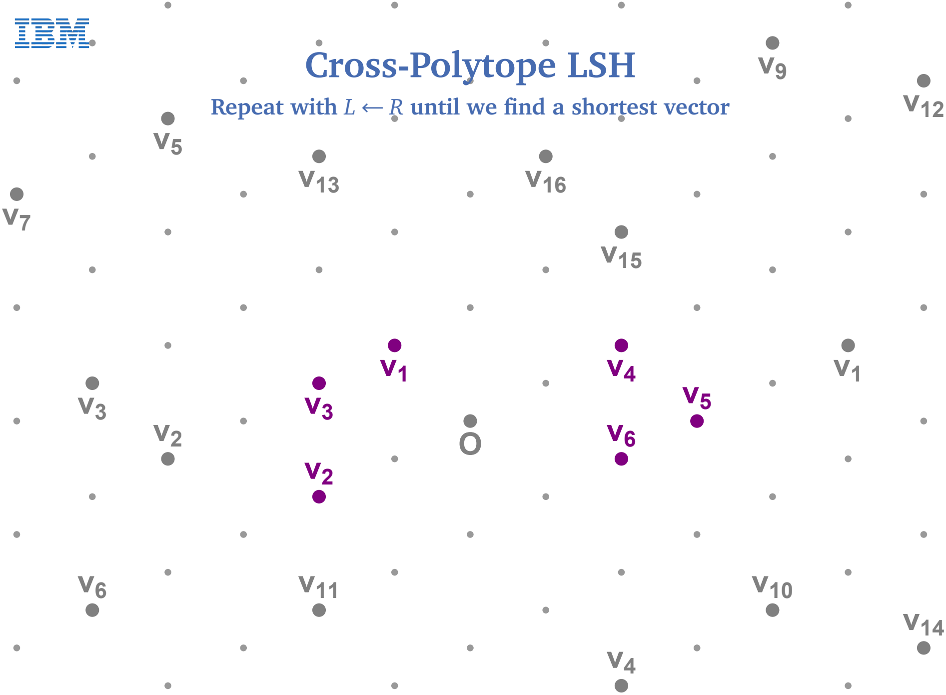

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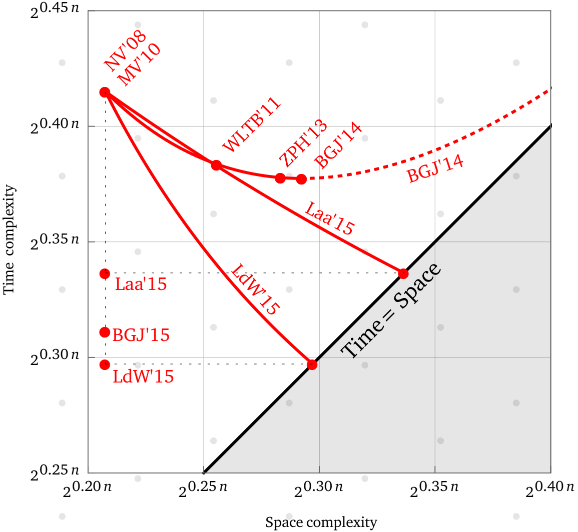

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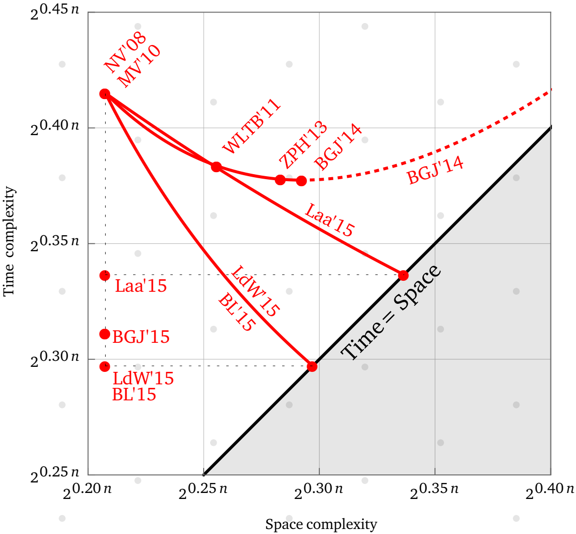

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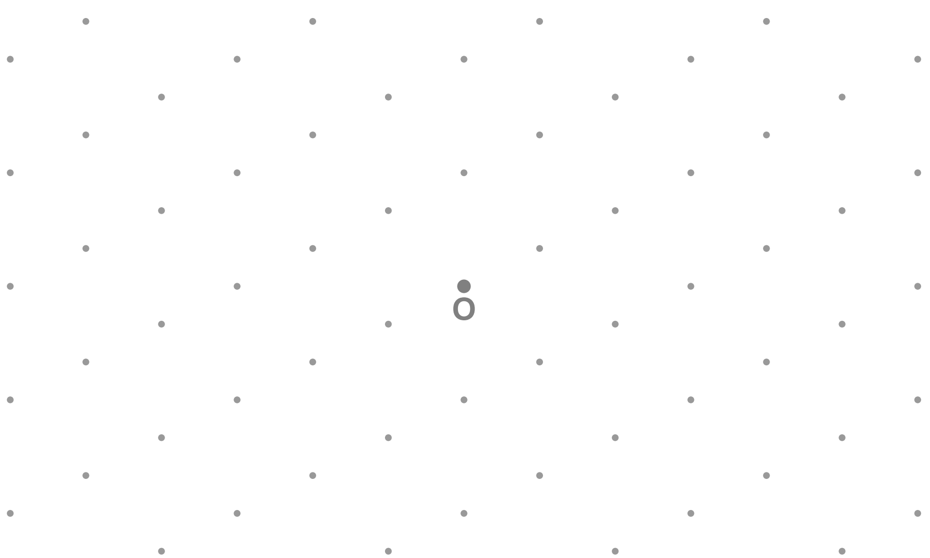

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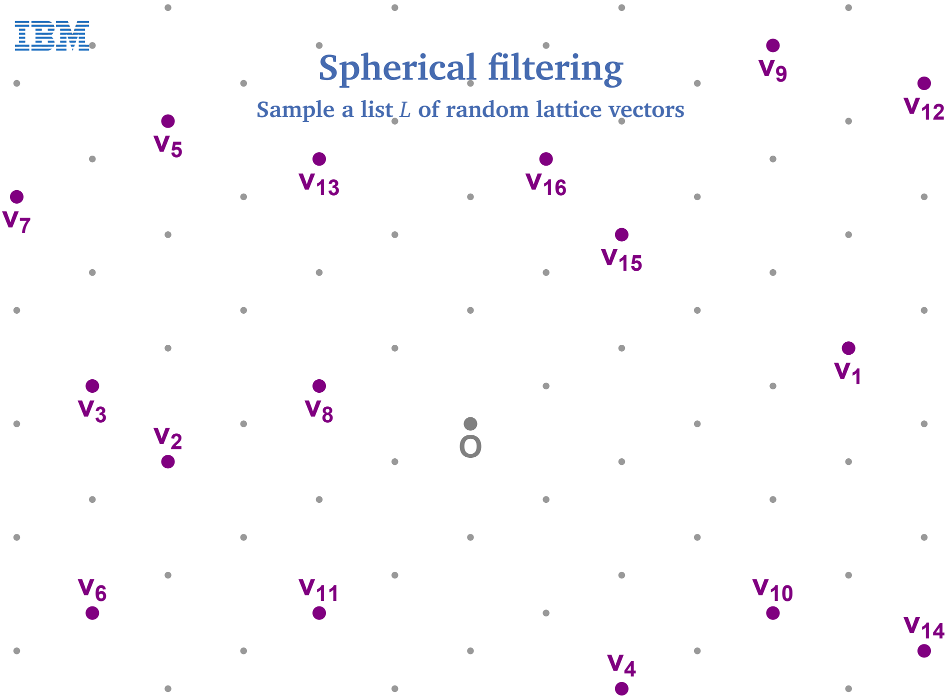
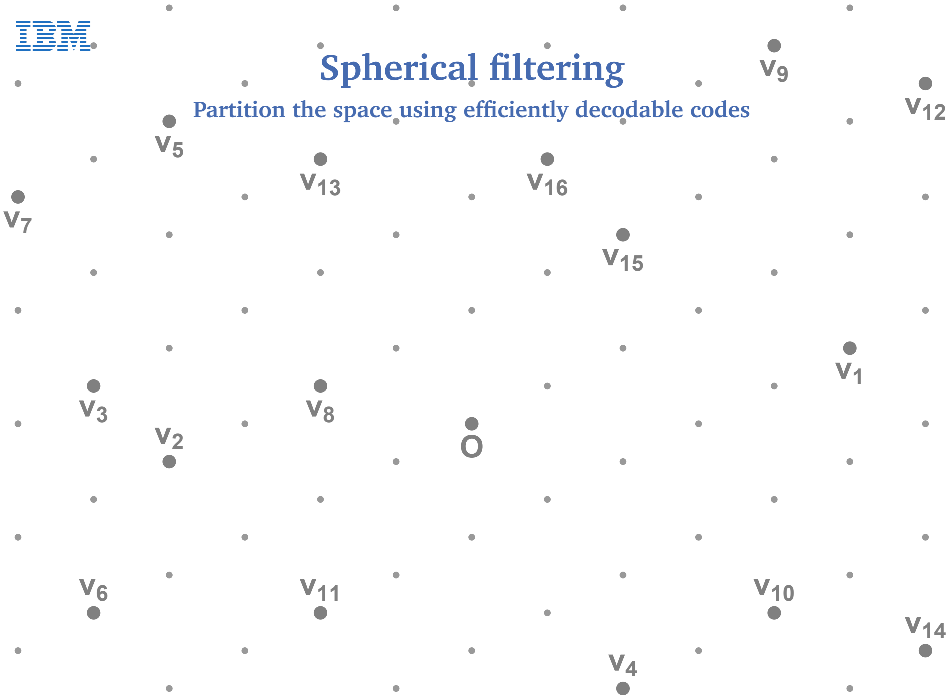


Spherical filtering

Partition the space using efficiently decodable codes



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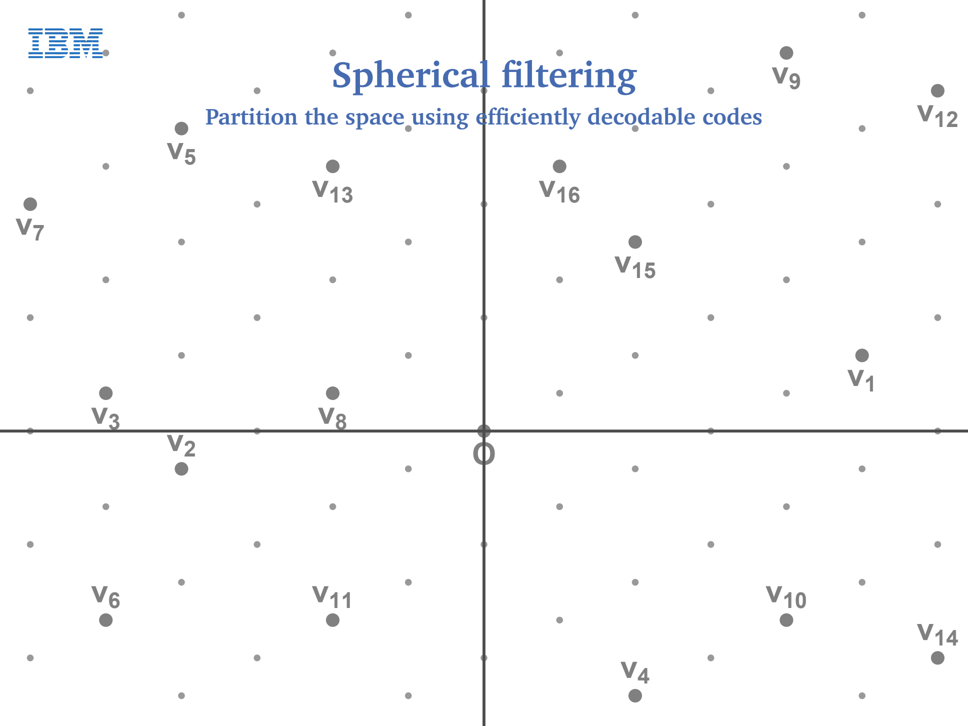

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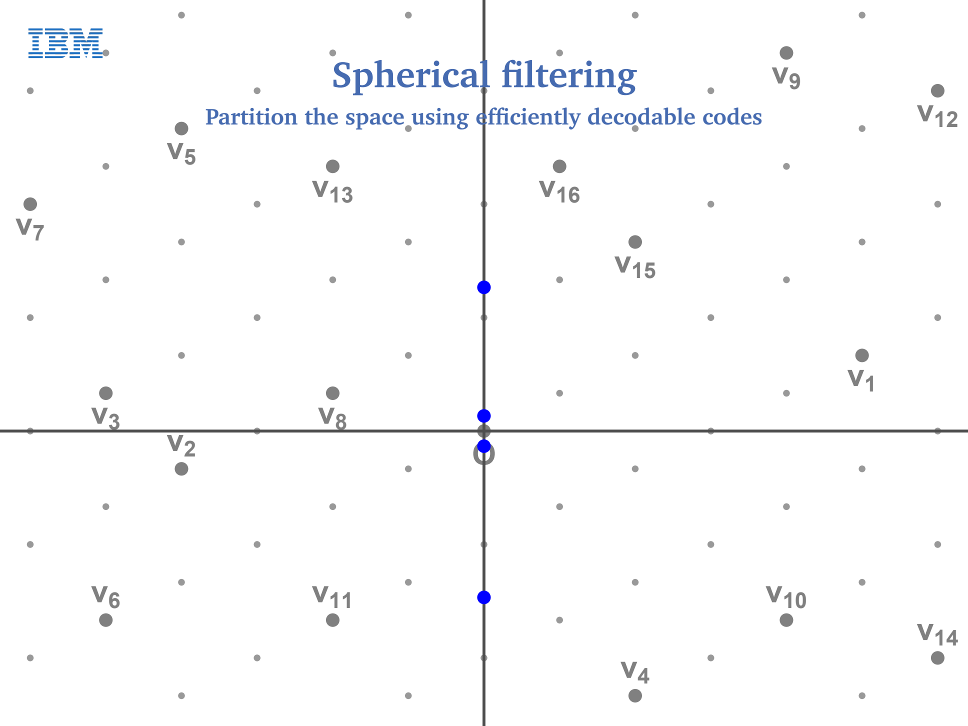

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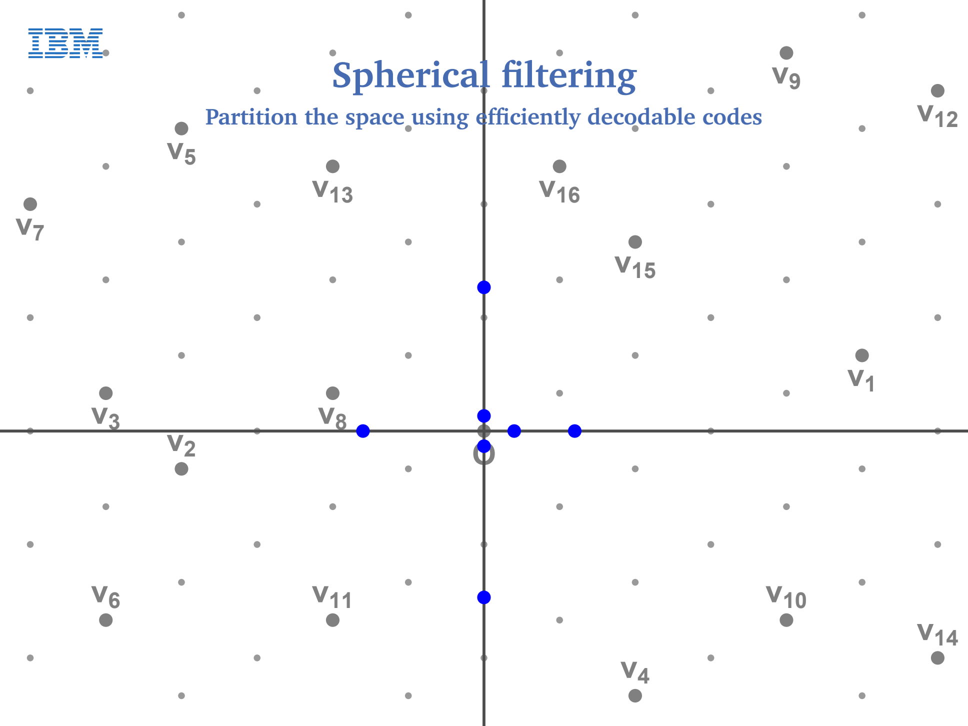

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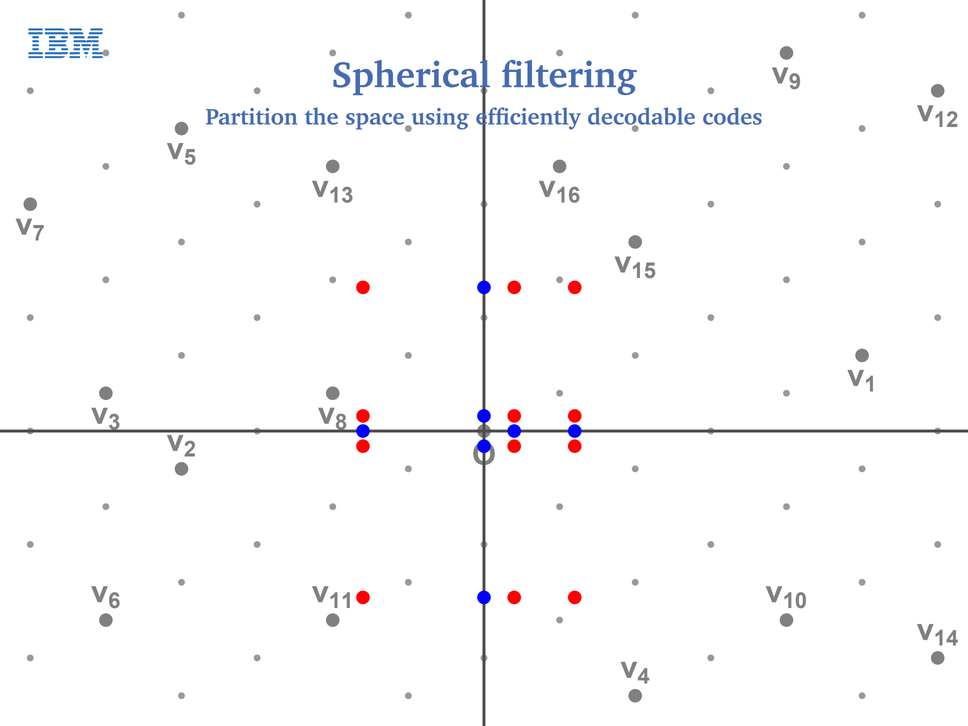

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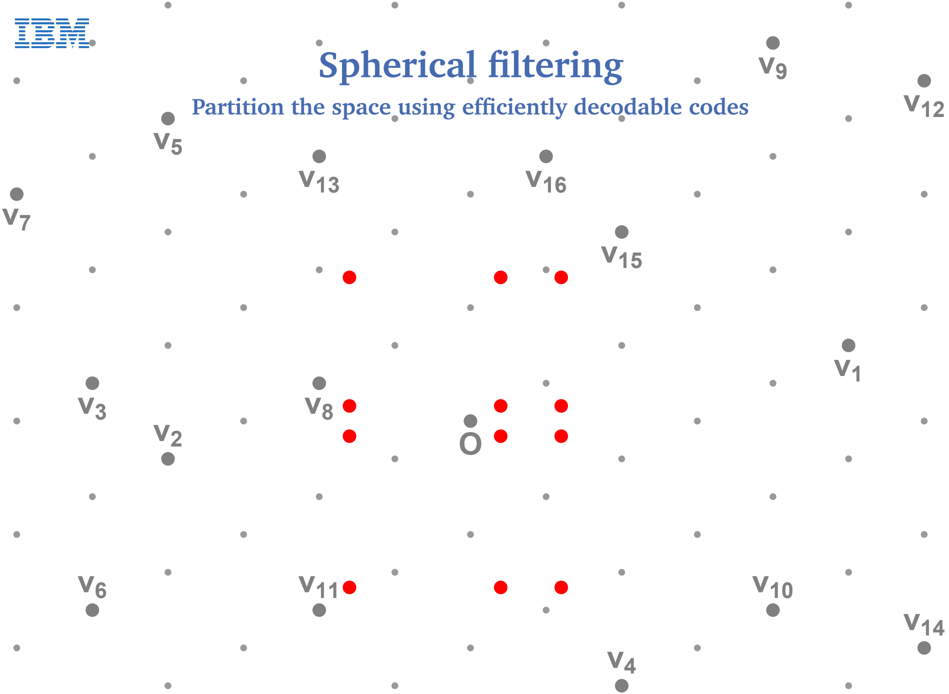

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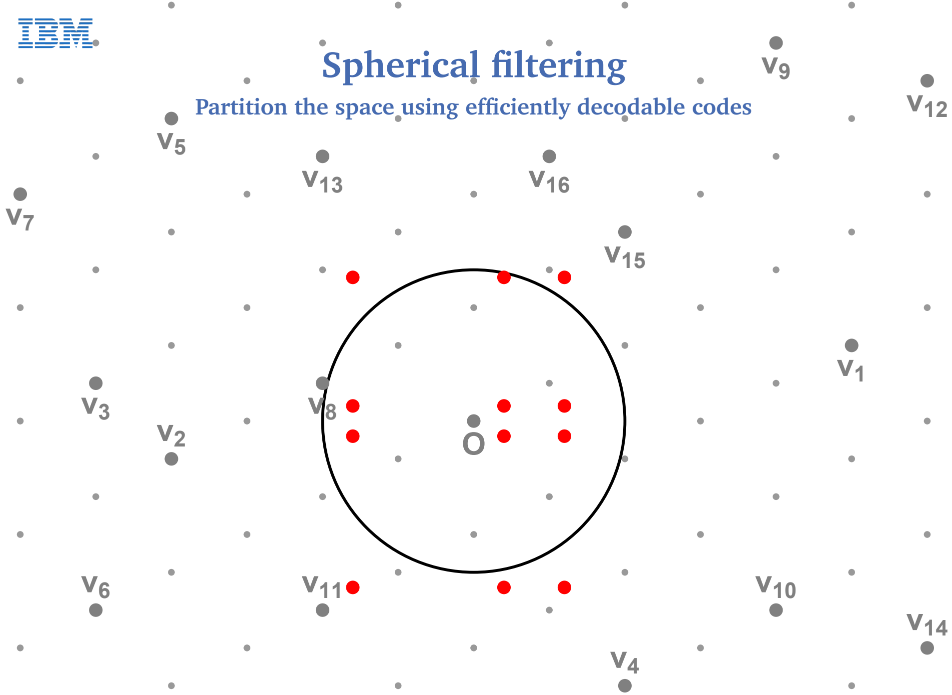

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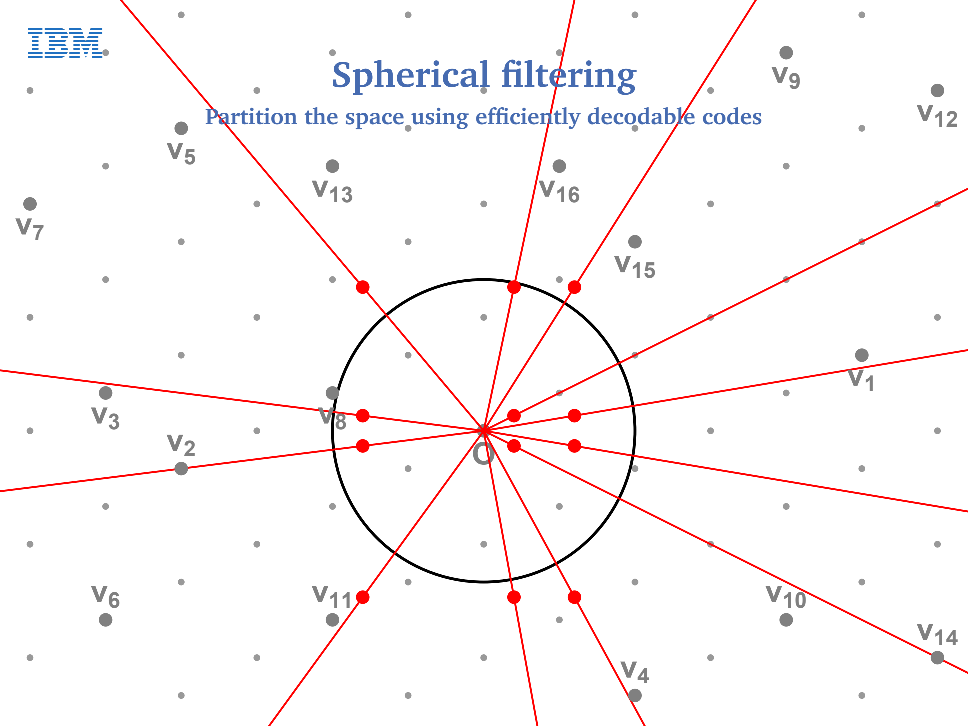

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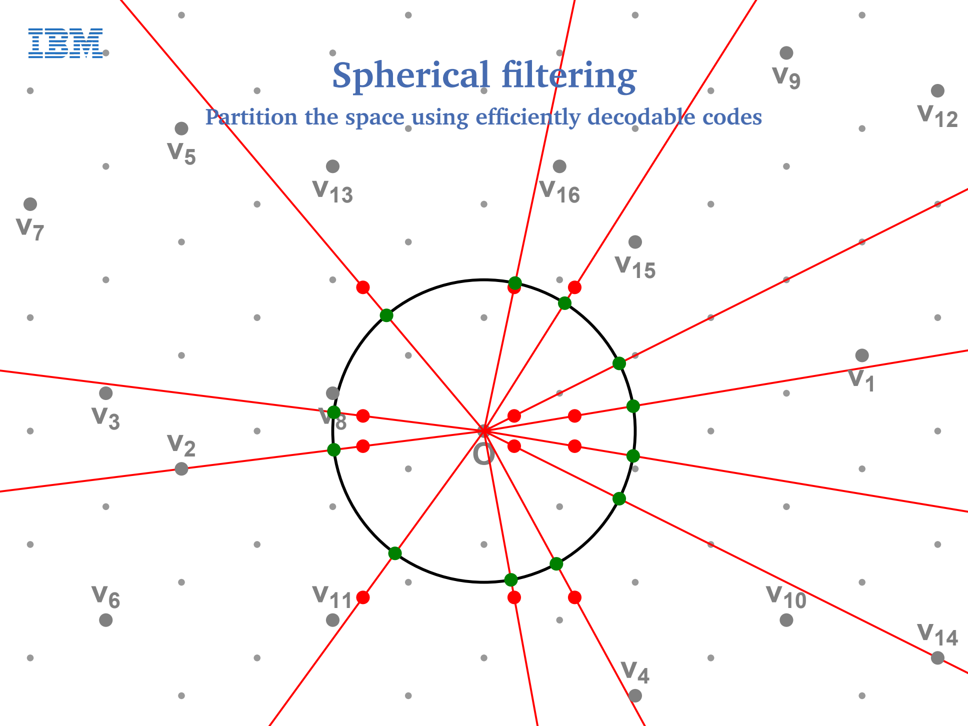

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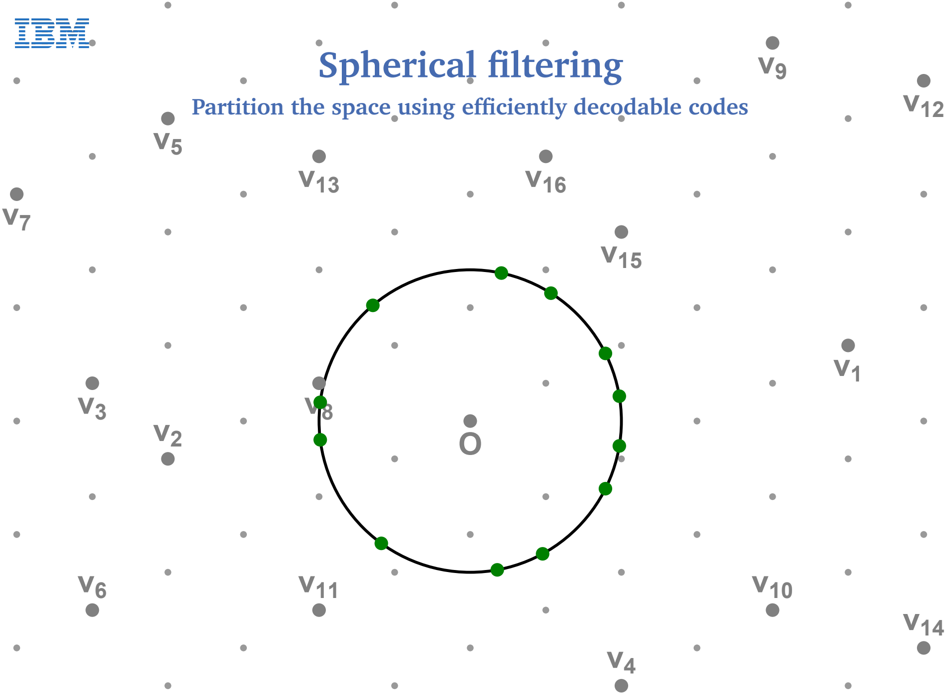

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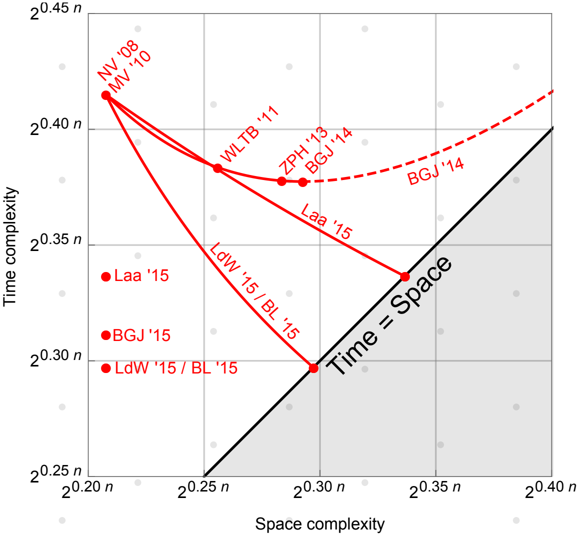

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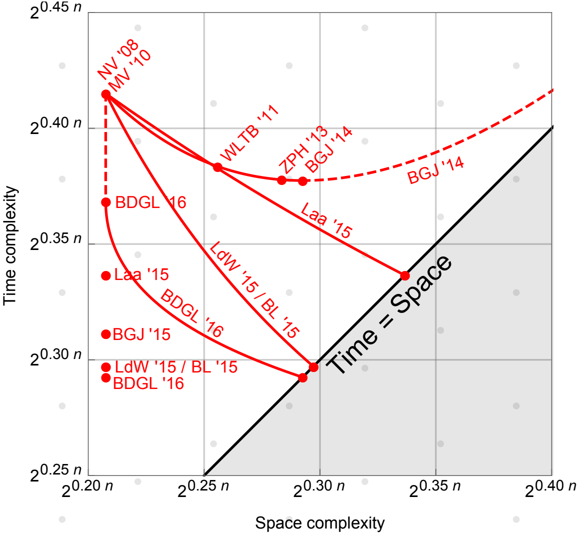

SVP in practice

"We expect our [enumeration] algorithm to be more efficient than lattice sieving up to dimension $n = 1895$."

— Micciancio–Walter, *SODA'15*

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— Bernstein, *Google groups '16*

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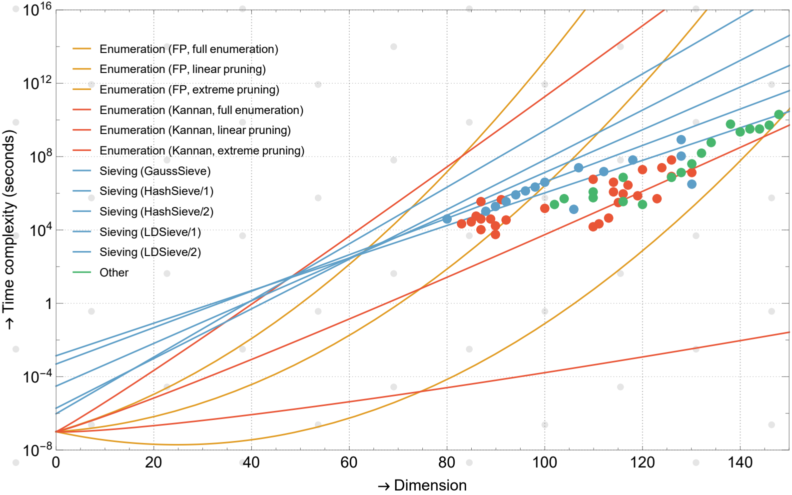
— Bernstein, *Google groups '16*

"I compute a cross-over point [between enumeration and the HashSieve] at dimension $b = 217$."

— Ducas, *Google groups '16*

IBM

SVP in practice



Open problems

- Exponential time, polynomial space for SVP(?)
- Effects of pruning on Kannan enumeration
- Close gap between provable and heuristic sieving
- Close gap between provable and heuristic enumeration
- Mixing/interpolating between different methods
- Sieving as BKZ subroutine
- Lower bounds on SVP complexity

Questions?