Technische Universiteit **Eindhoven** University of Technology

Lattice algorithms for the closest vector problem with preprocessing

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Lattices

Basics

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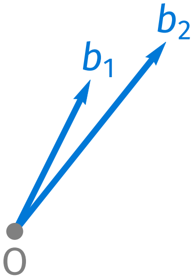

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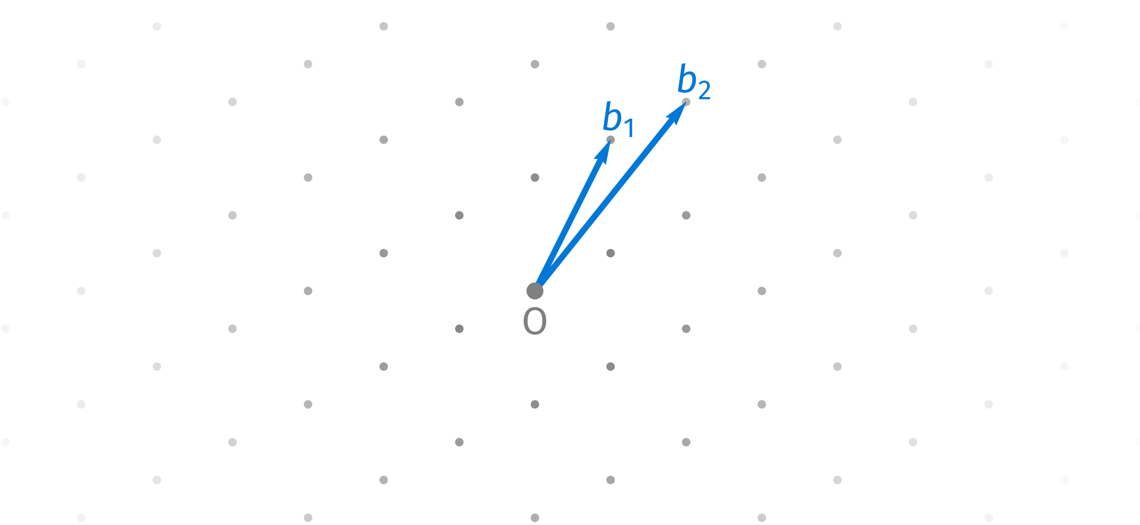
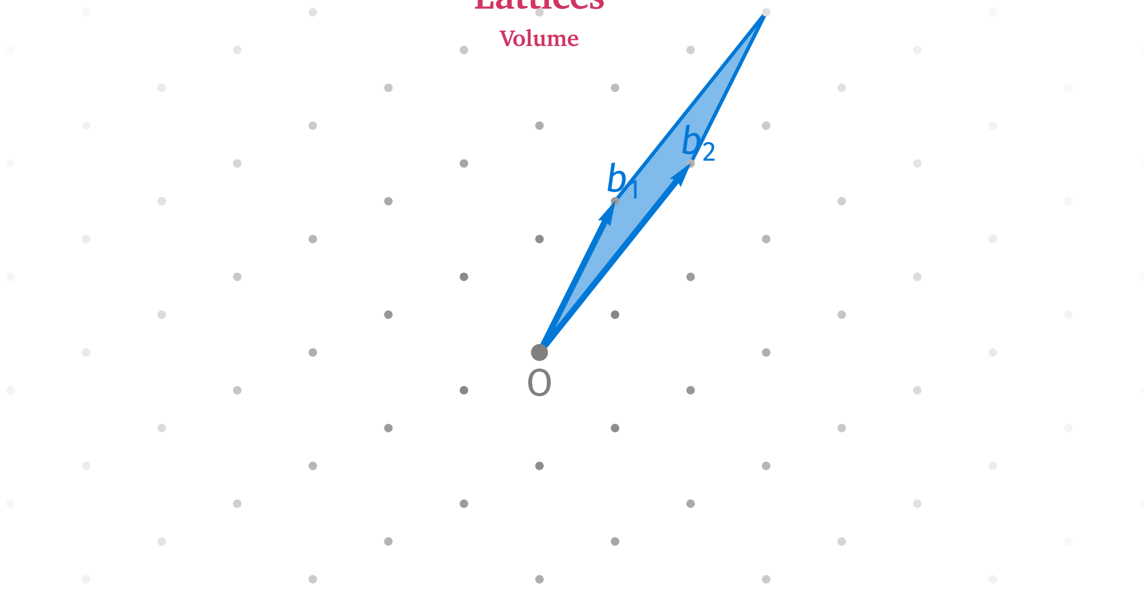


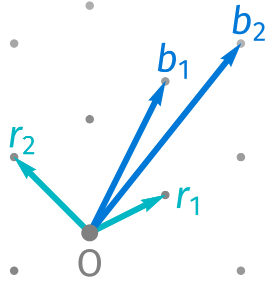
Lattices

Volume



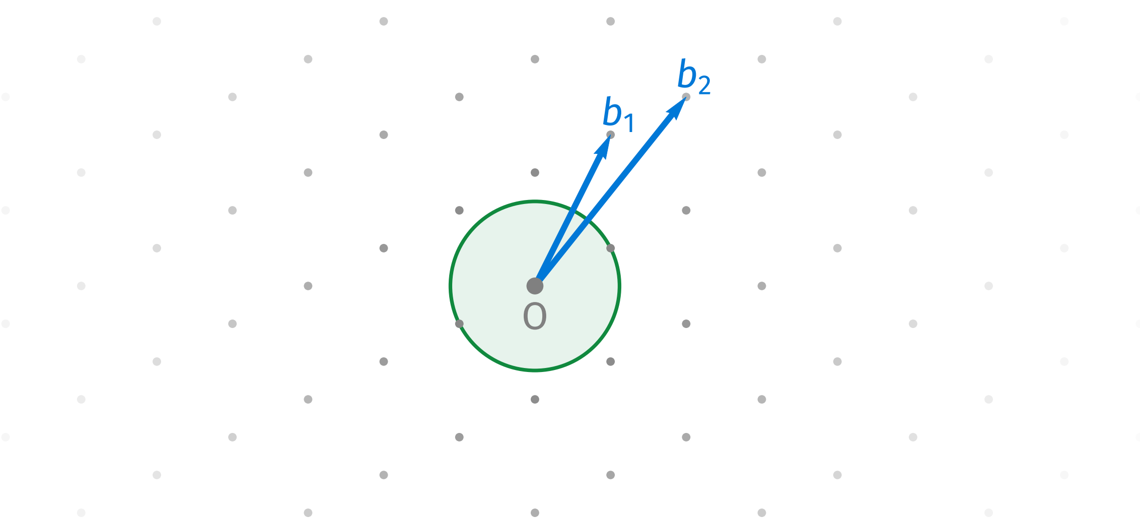
Lattices

Lattice basis reduction



Lattice problems

Shortest Vector Problem (SVP)



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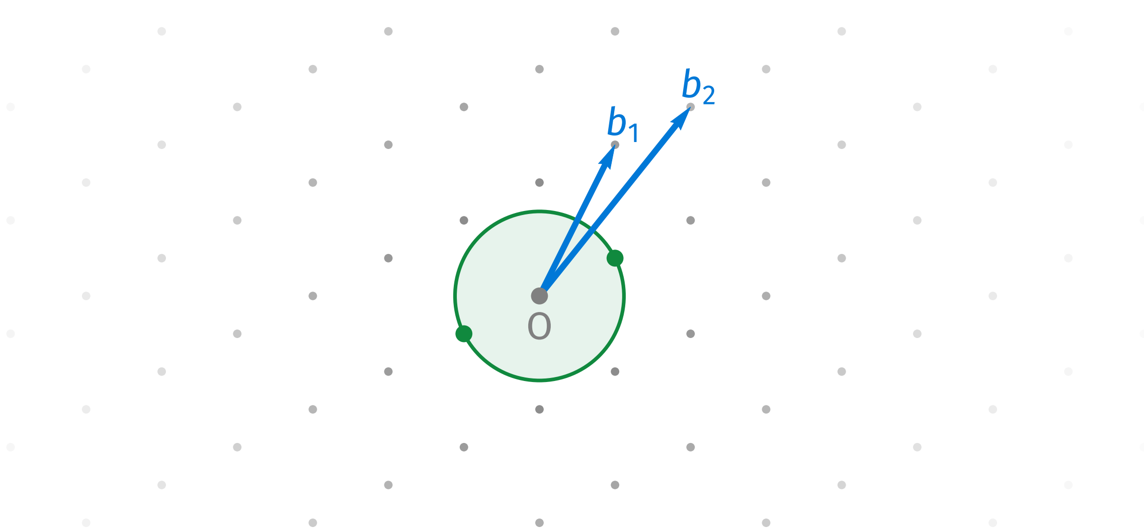

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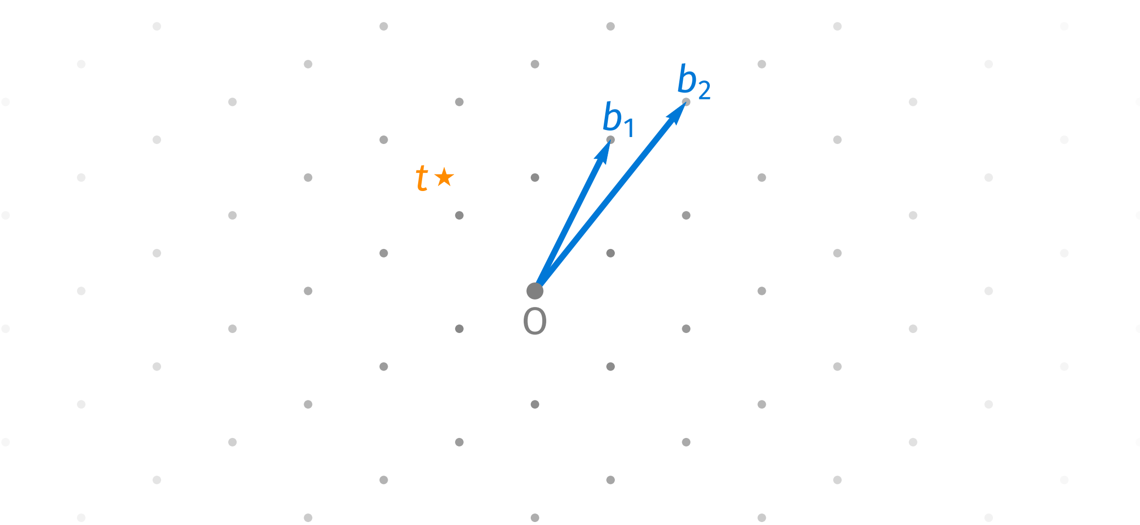

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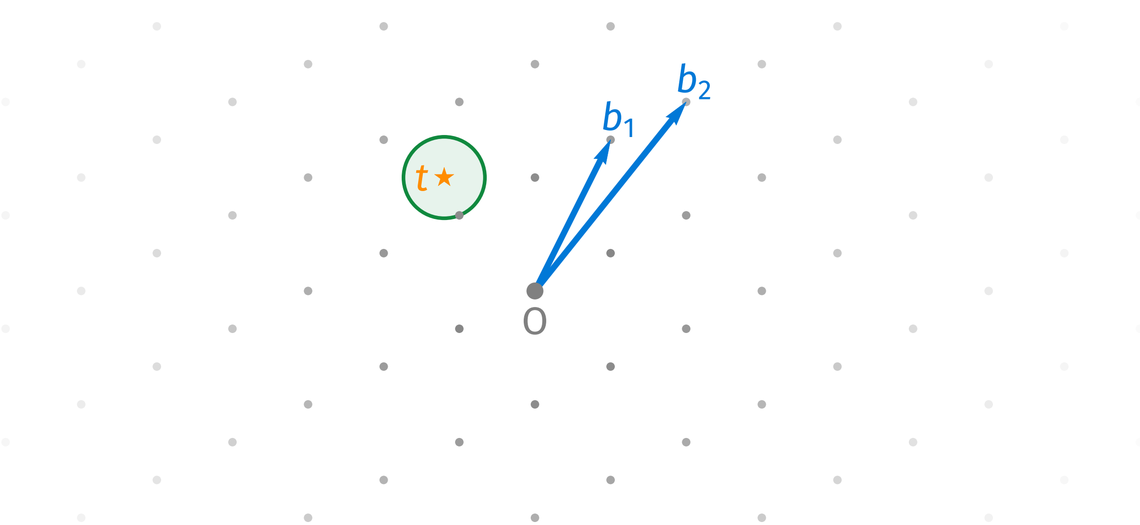

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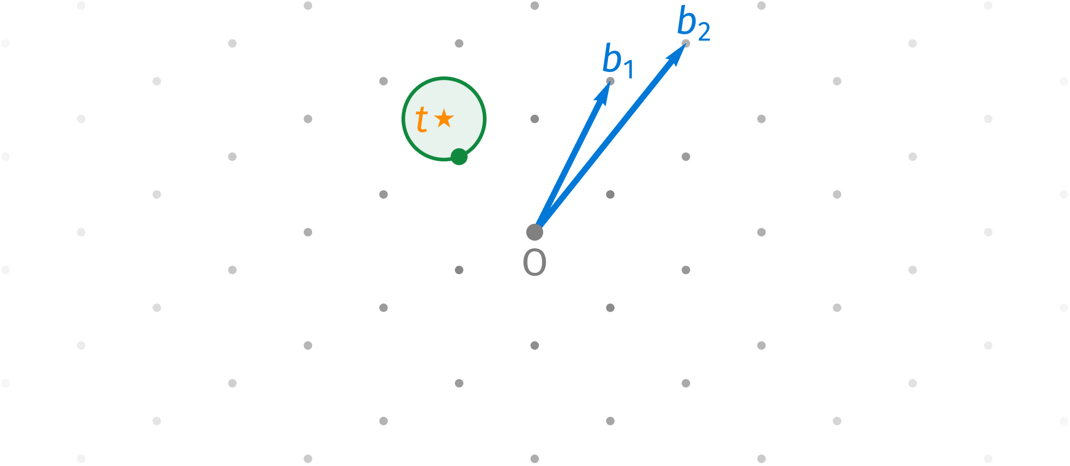


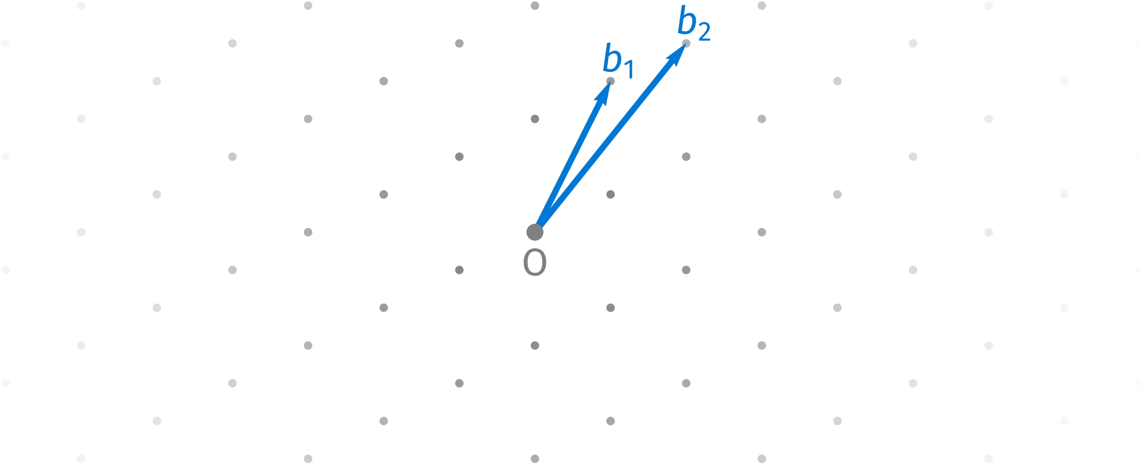
Lattice problems

SVP/CVP asymptotics

	Algorithm	$\log_2(\text{Time})$	$\log_2(\text{Space})$	Experiments
Worst-case SVP	Enumeration [Poh81, Kan83, ..., MW15, AN17]	$O(n \log n)$	$O(\log n)$	**152**
	AKS-sieve [AKS01, NV08, MV10, HPS11]	$3.398n$	$1.985n$	–
	Birthday sieves [PS09, HPS11]	$2.465n$	$1.233n$	–
	Enumeration/DGS hybrid [CCL17]	$2.048n$	$0.500n$	–
	Voronoi cell algorithm [AEVZ02, MV10b, BD15]	$2.000n$	$1.000n$	40
	Quantum sieve [LMP13, LMP15]	$1.799n$	$1.286n$	–
	Quantum enum/DGS [CCL17]	$1.256n$	**0.500n**	–
	Discrete Gaussian sampling [ADRS15, ADS15, AS18]	**1.000n**	$1.000n$	–
Average-case SVP	The Nguyen–Vidick sieve [NV08]	$0.415n$	$0.208n$	50
	GaussSieve [MV10, ..., IKMT14, BNvdP16, YKYC17]	$0.415n$	$0.208n$	130*
	Triple sieve [BLS16, HK17]	$0.396n$	$0.189n$	80
	Kleinjung sieve [Kle14]	$0.379n$	$0.189n$	116
	Leveled sieving [WLTB11, ZPH13]	$0.378n$	$0.283n$	–
	Overlattice sieve [BGJ14]	$0.377n$	$0.293n$	90
	Triple sieve with NNS [HK17, HKL18]	$0.359n$	**0.189n**	76
	Single filters [DL17, ADH+19]	$0.349n$	$0.246n$	**155**
	Hyperplane LSH [Cha02, FBB+14, Laa15, ..., LM18]	$0.337n$	$0.337n$	107
	Hypercube LSH [TT07, Laa17]	$0.322n$	$0.322n$	–
	May–Ozerov NNS [MO15, BGJ15]	$0.311n$	$0.311n$	–
	Quantum sieve [LMP13]	$0.311n$	$0.208n$	–
	Spherical LSH [AINR14, LdW15]	$0.297n$	$0.297n$	–
	Cross-polytope LSH [TT07, AILRS15, BL16, KW17]	$0.297n$	$0.297n$	80
	Spherical LSF [BDGL16, MLB17, ALRW17, Chr17]	**0.292n**	$0.292n$	92
	Quantum NNS sieve [LMP15, Laa16]	**0.265n**	$0.265n$	–

Lattice problems

Closest Vector Problem with Preprocessing (CVPP)



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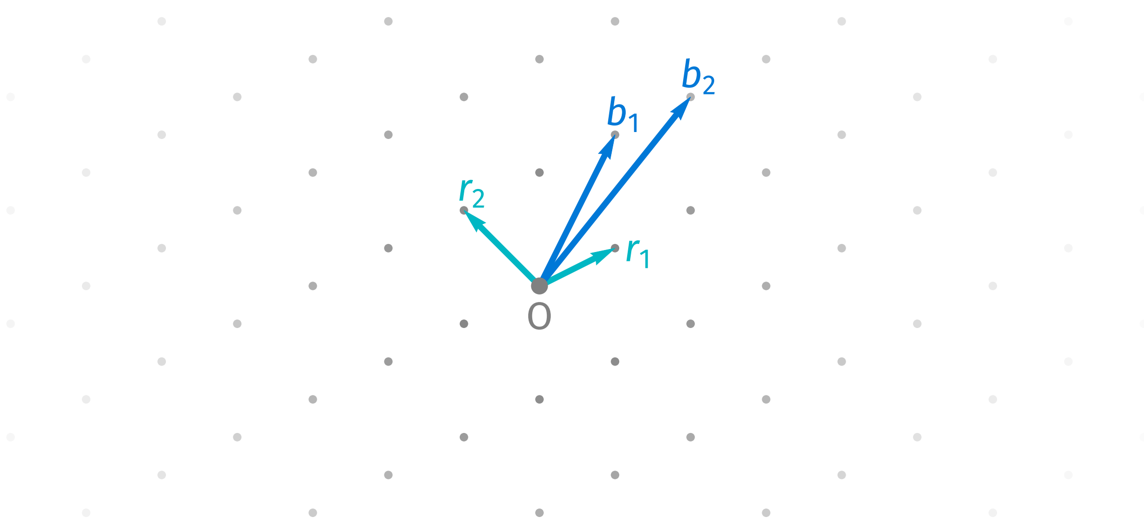

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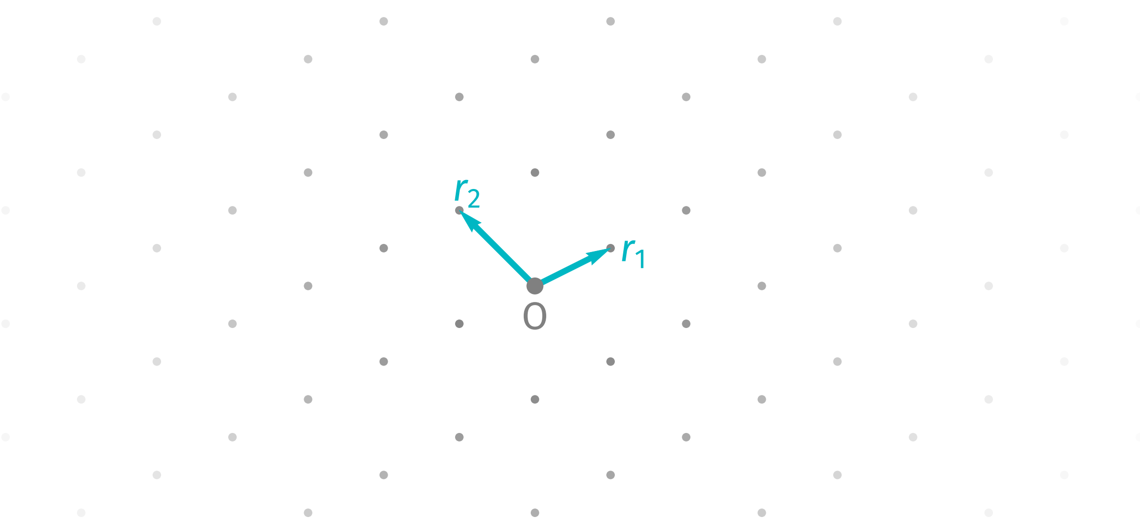

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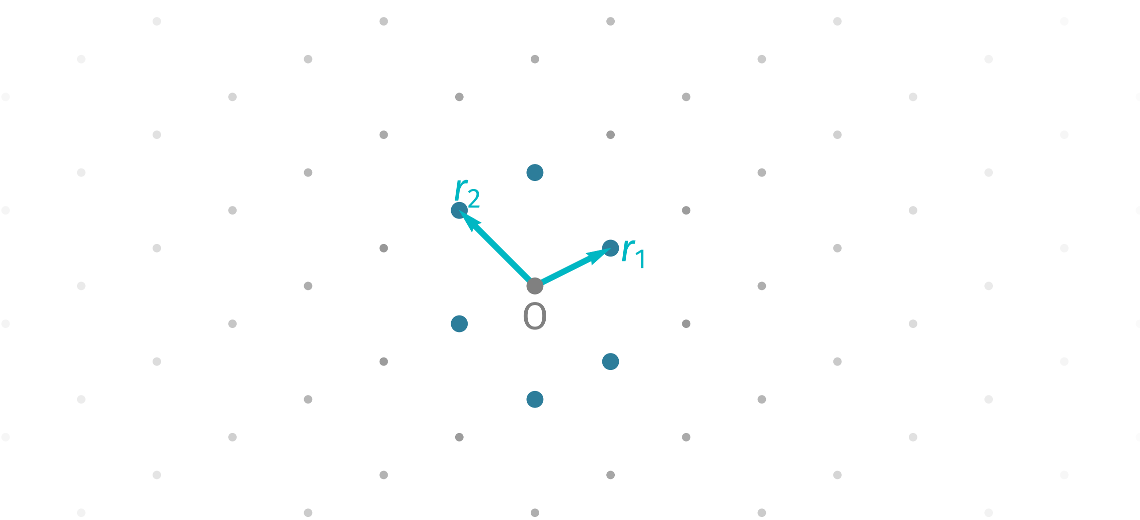

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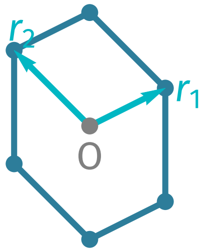

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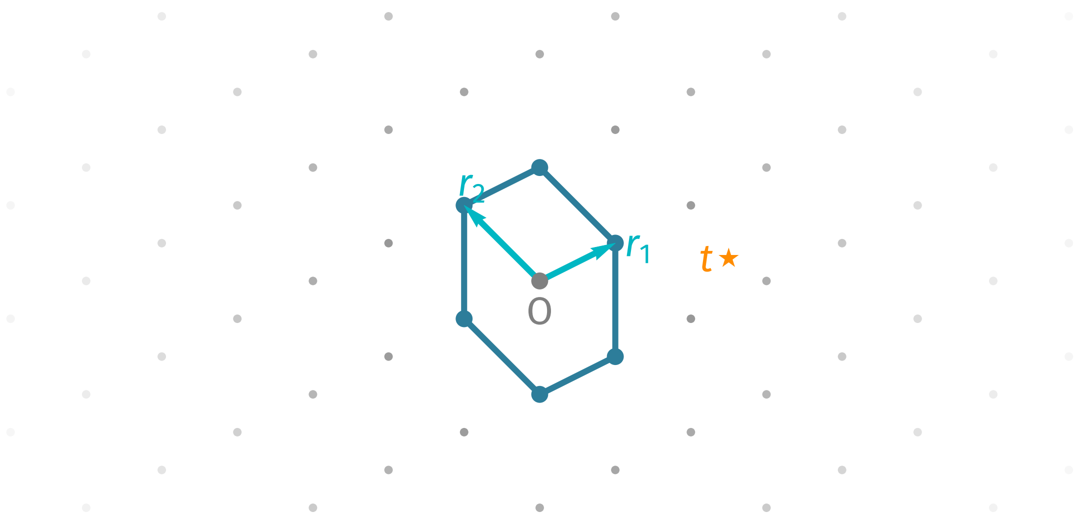

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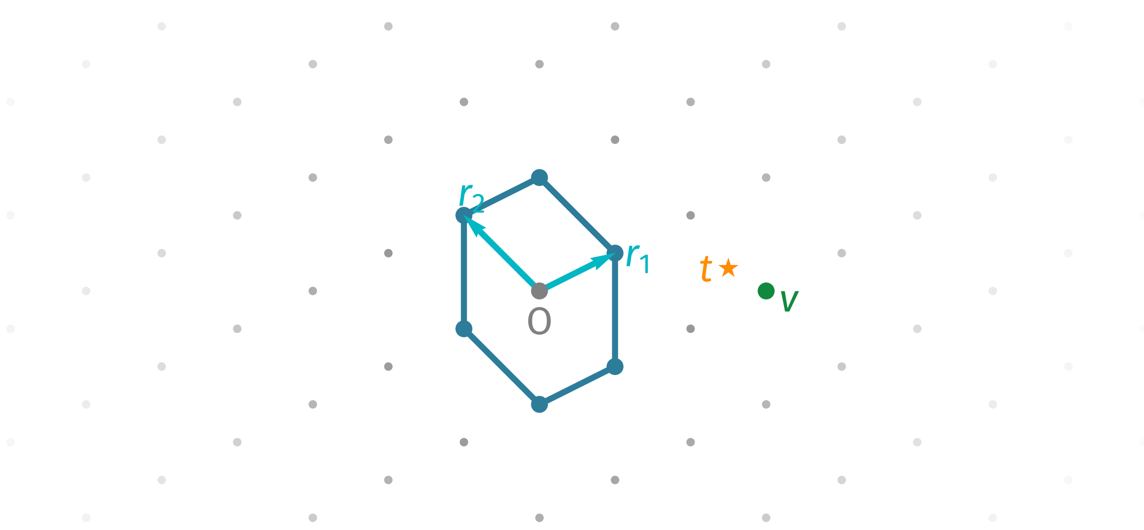

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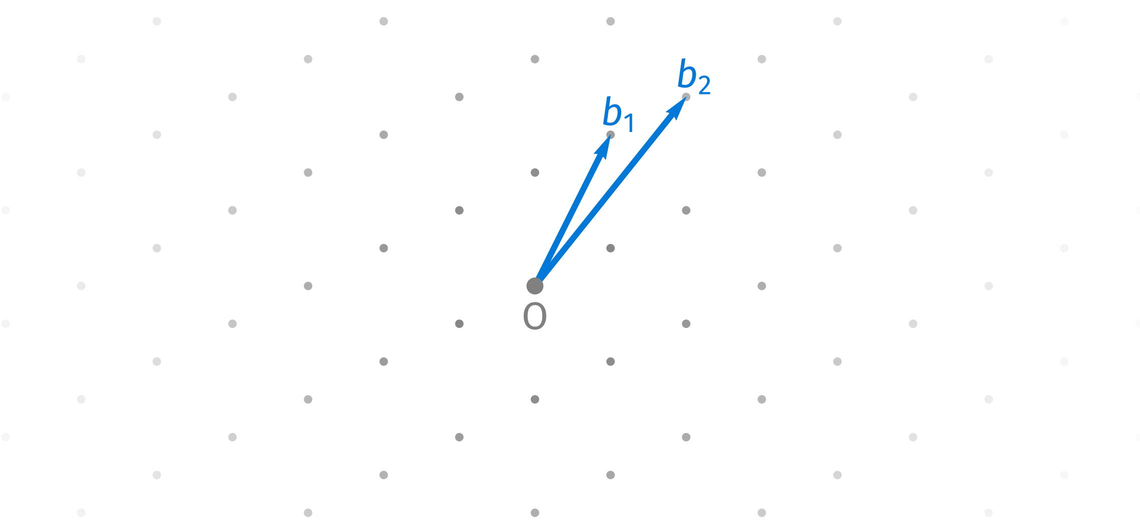

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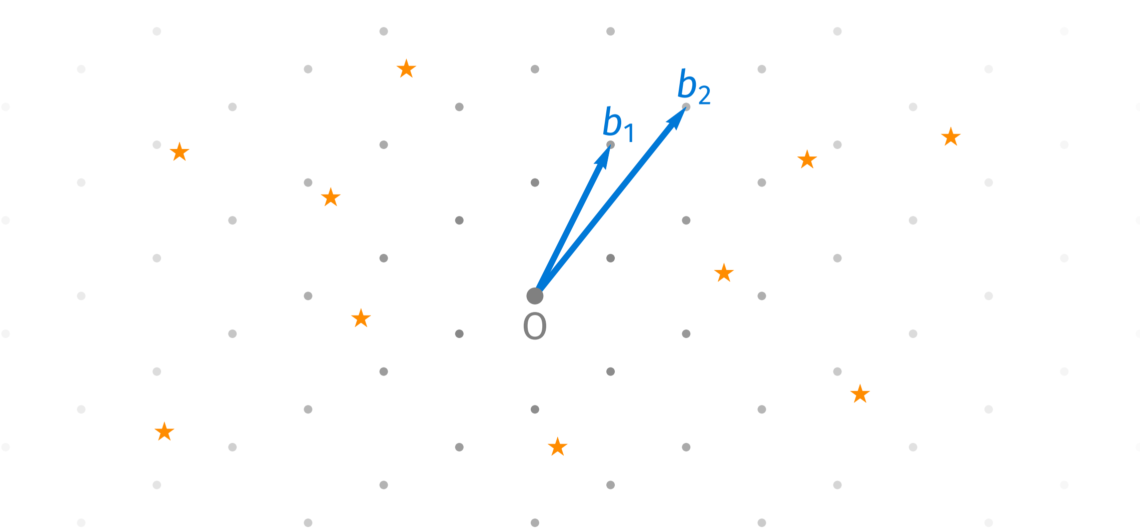

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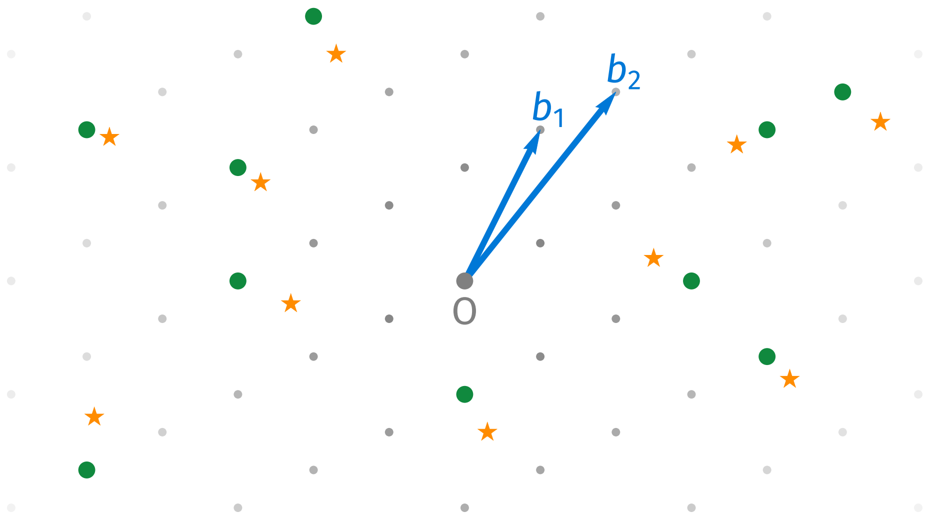

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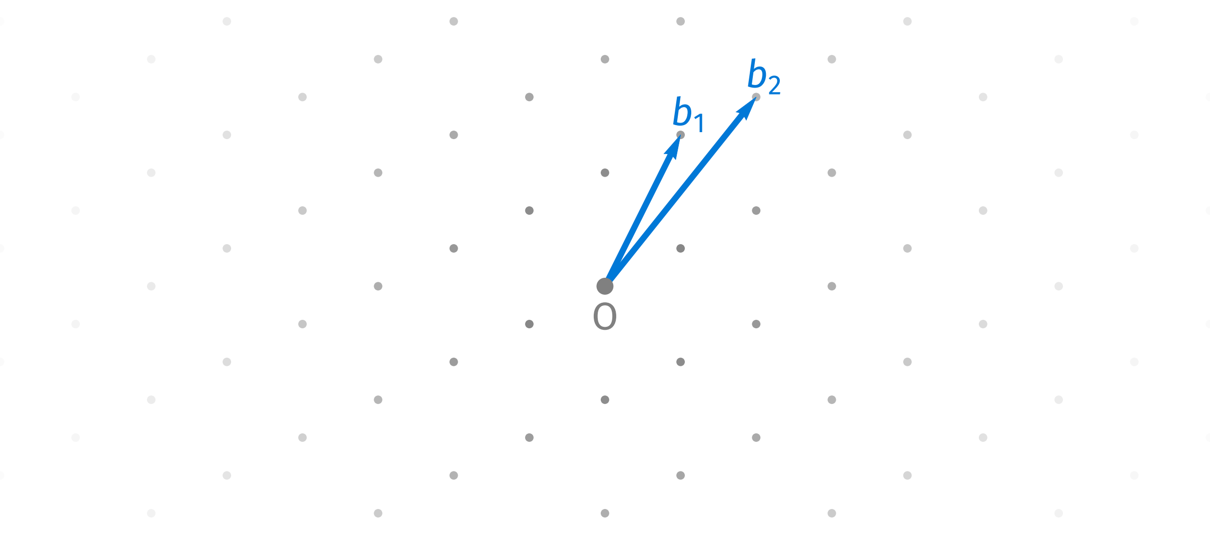

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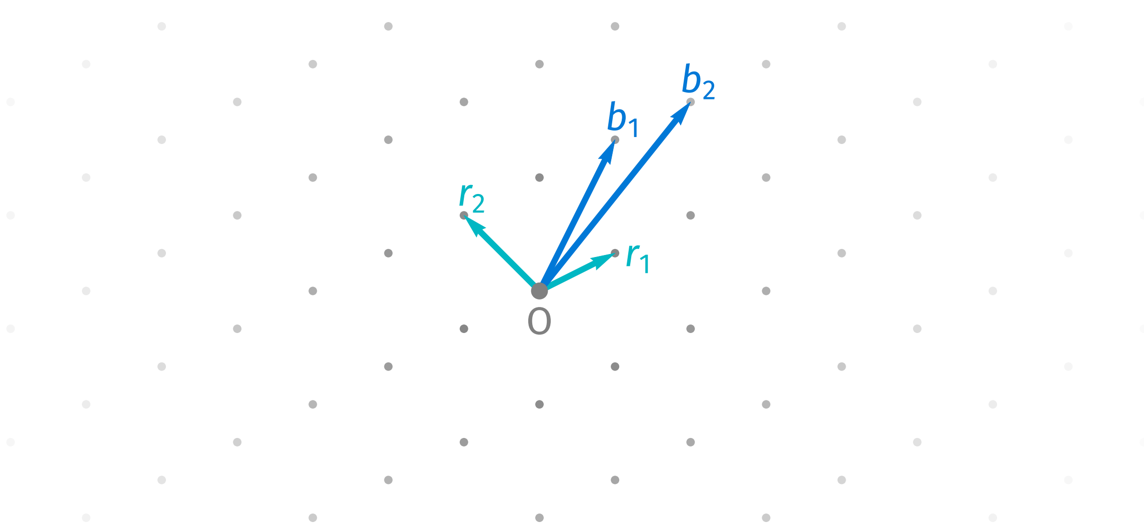

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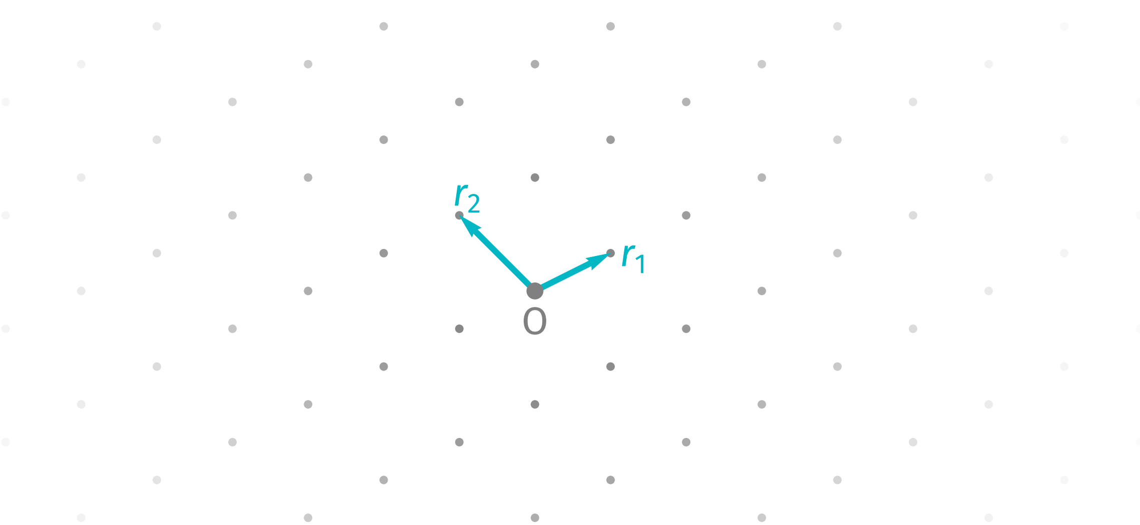

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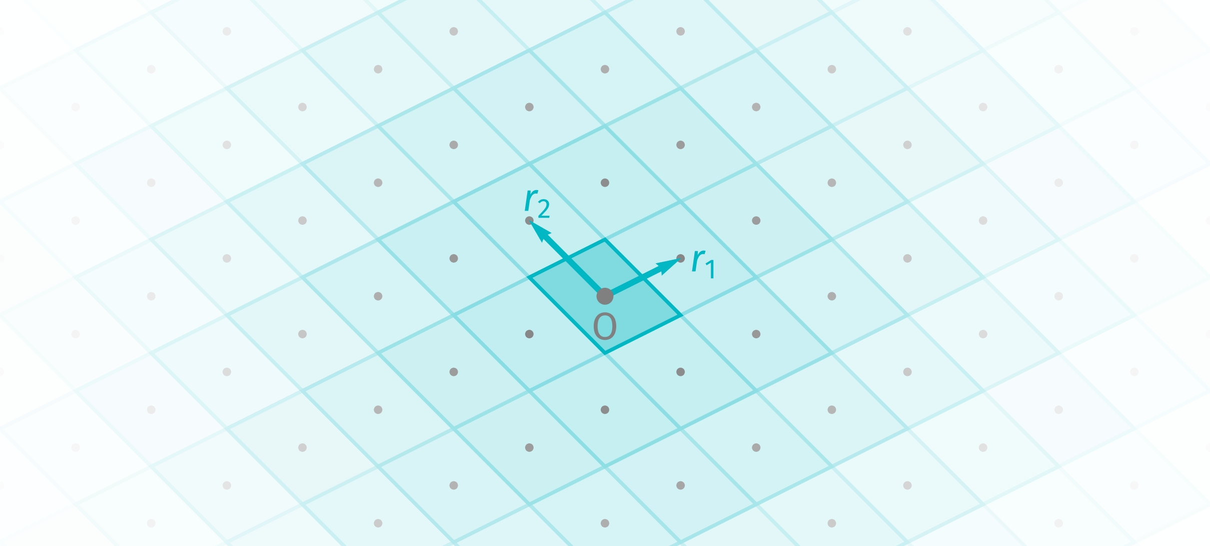

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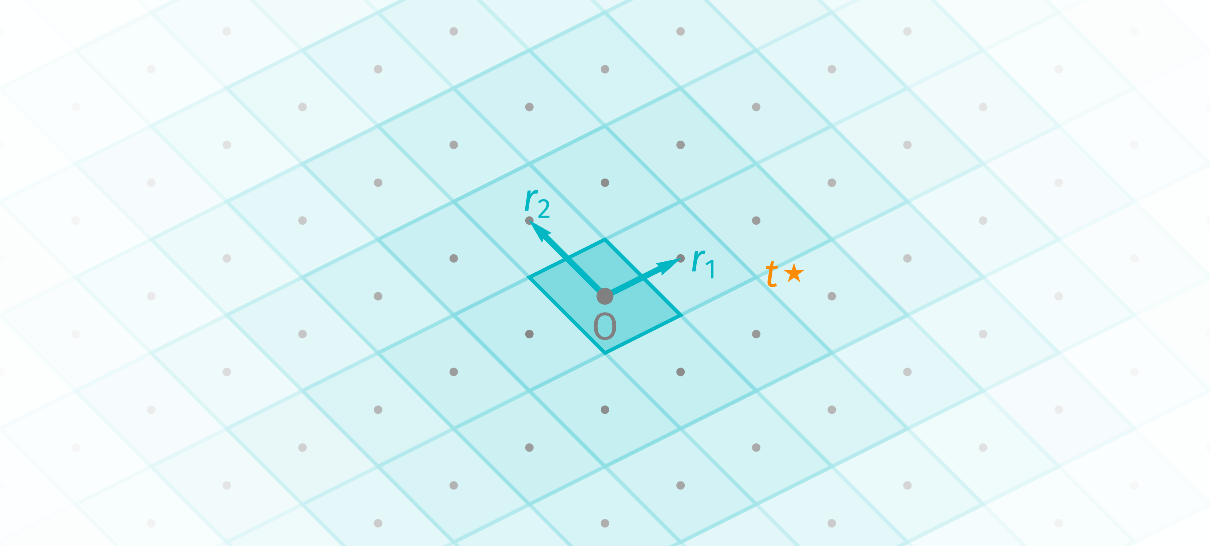

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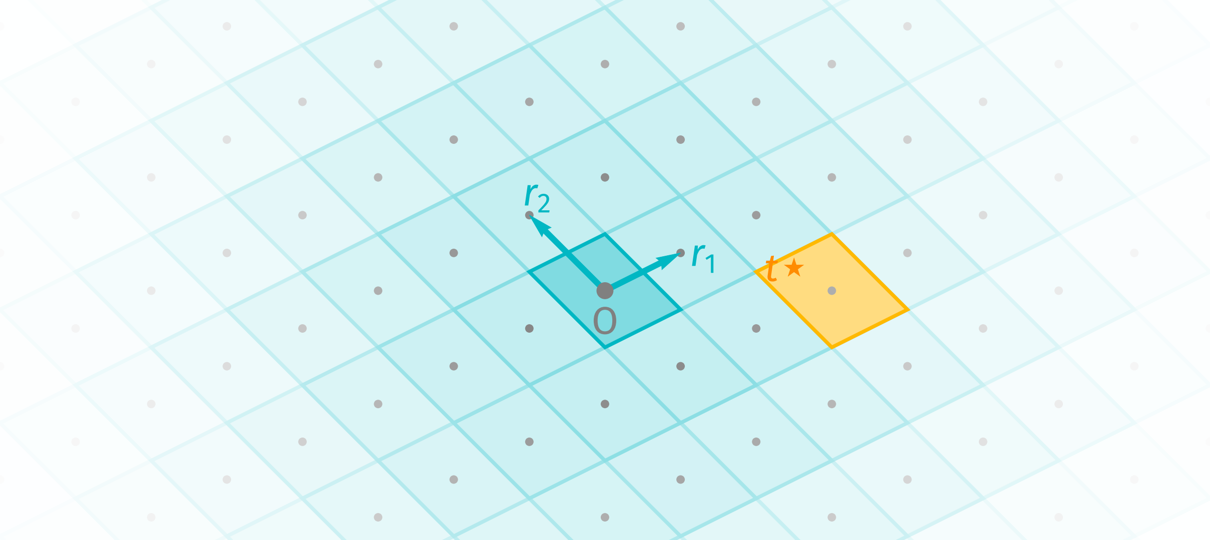

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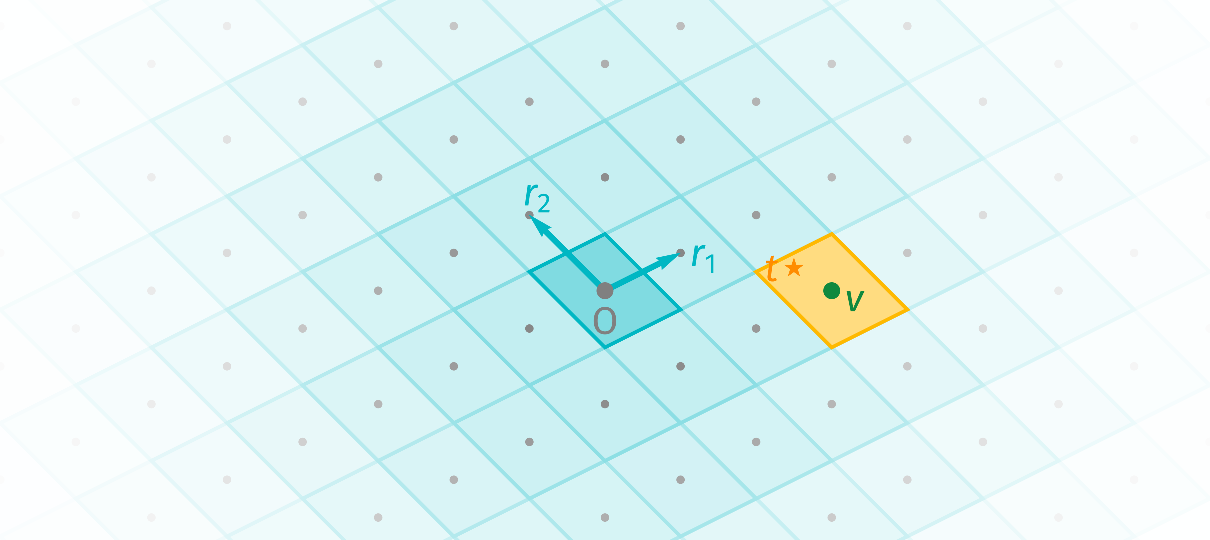

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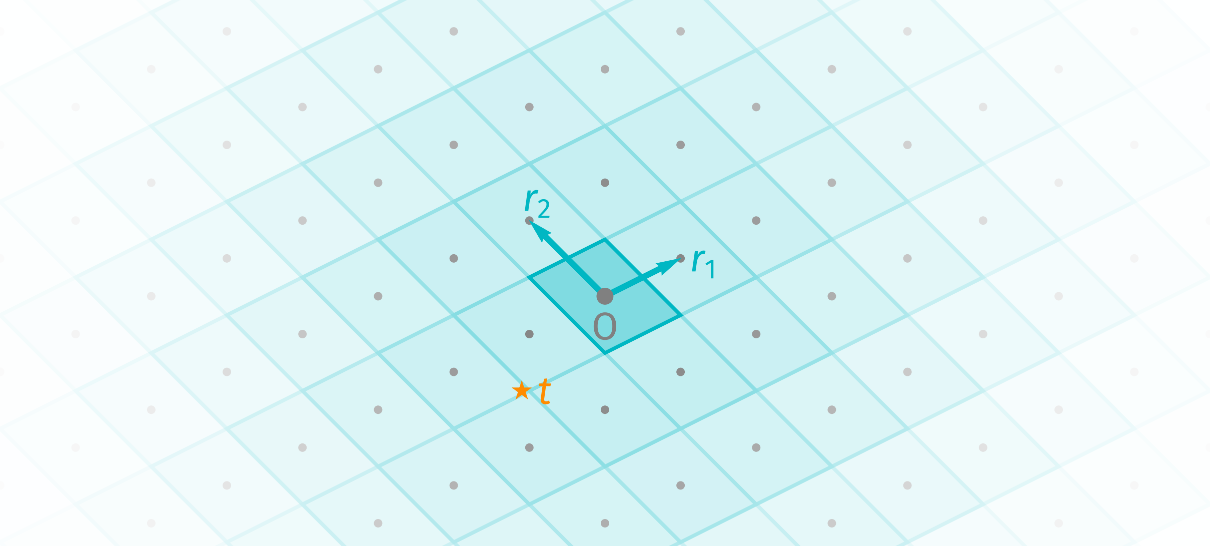

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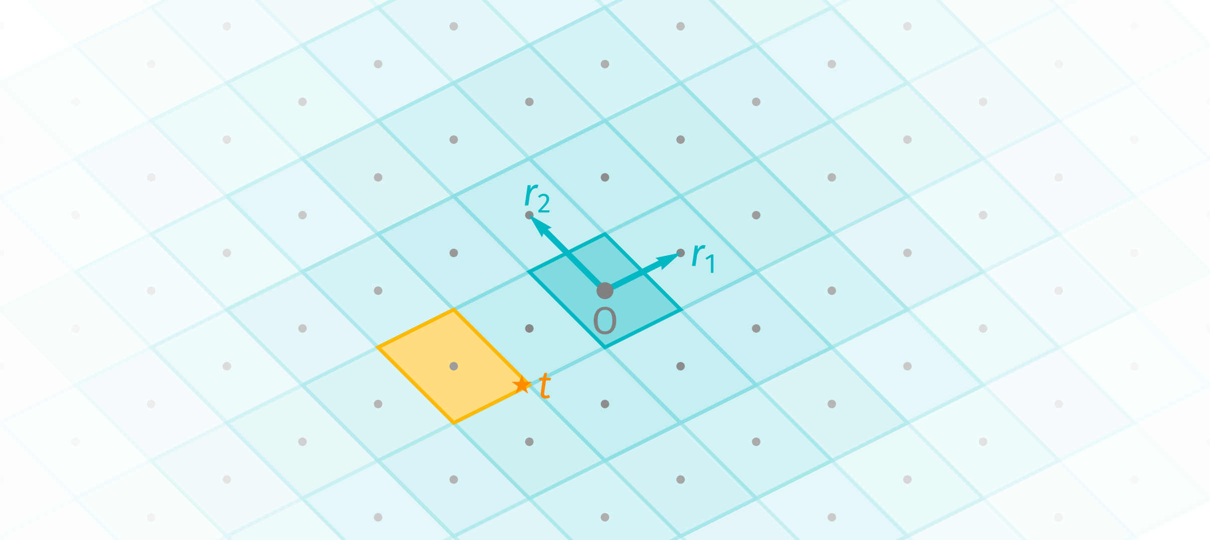

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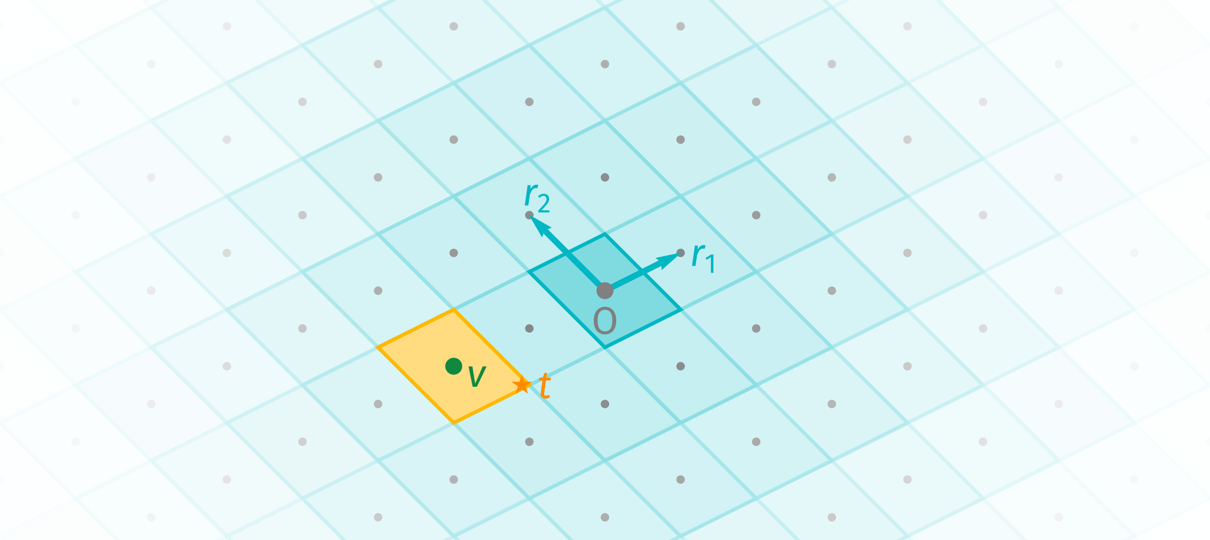

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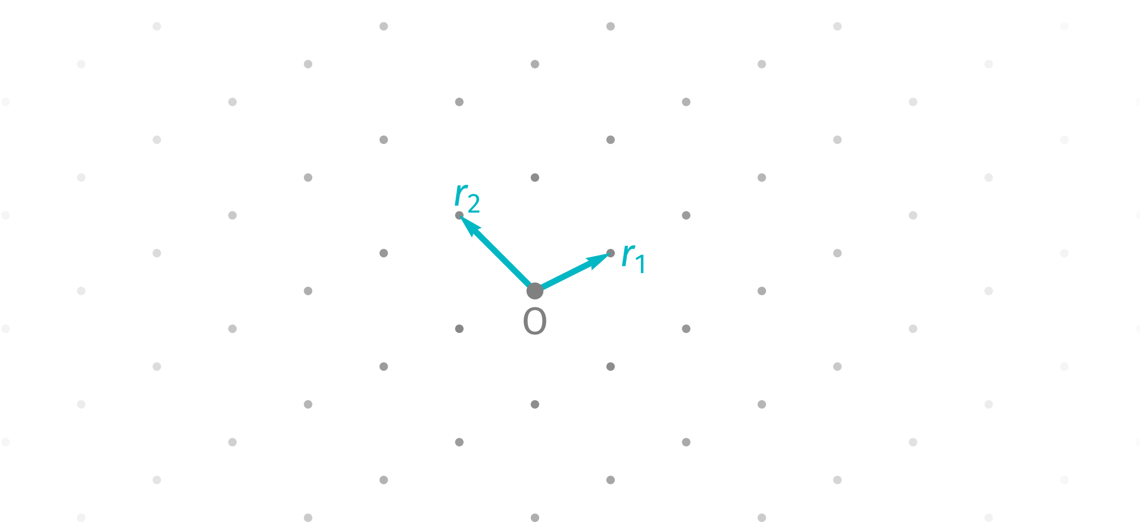

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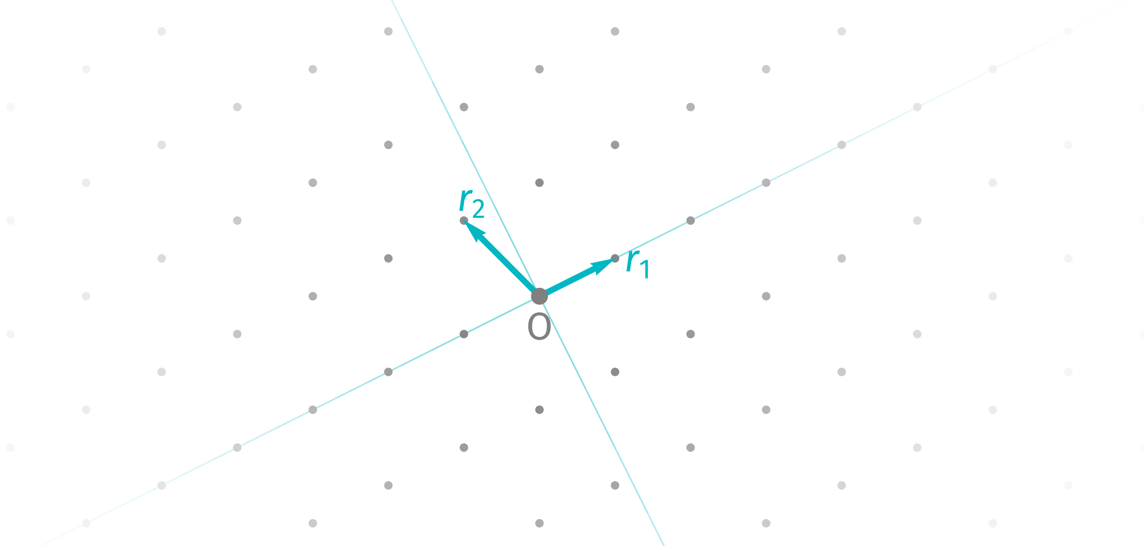

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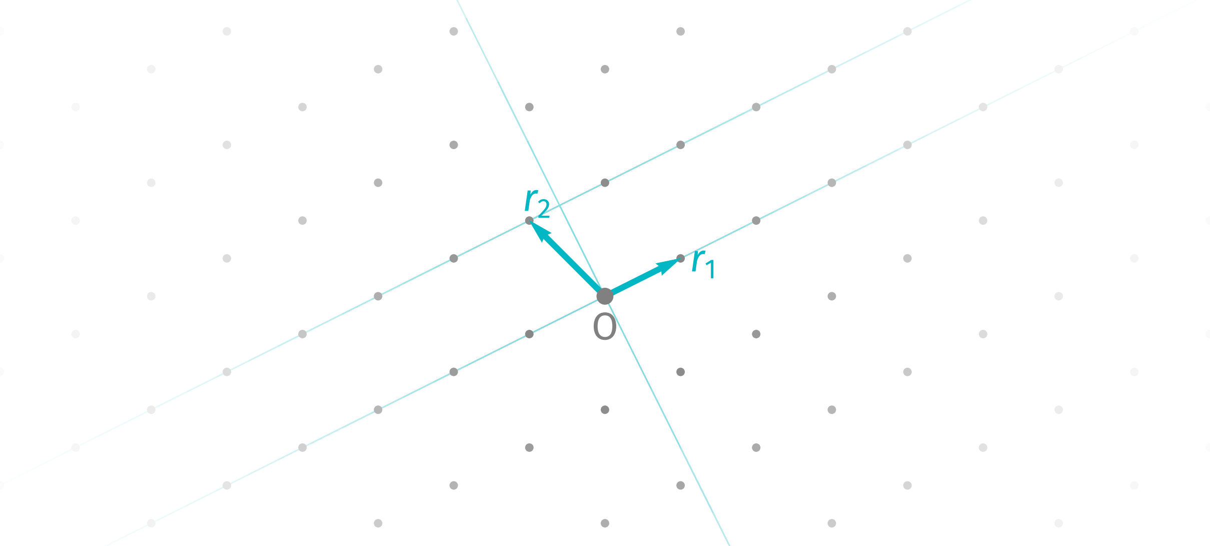

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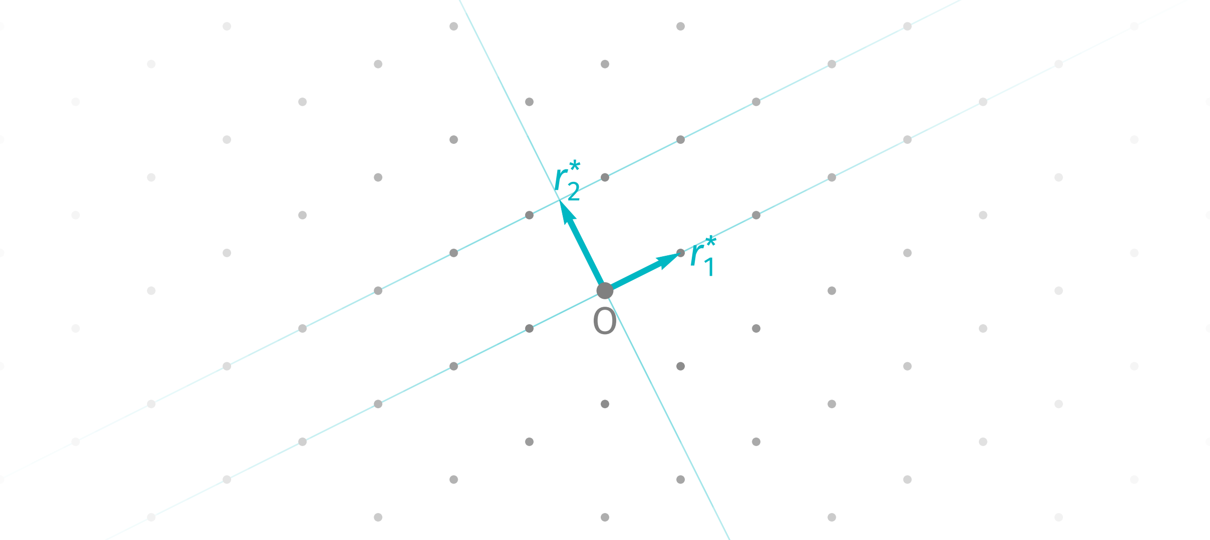

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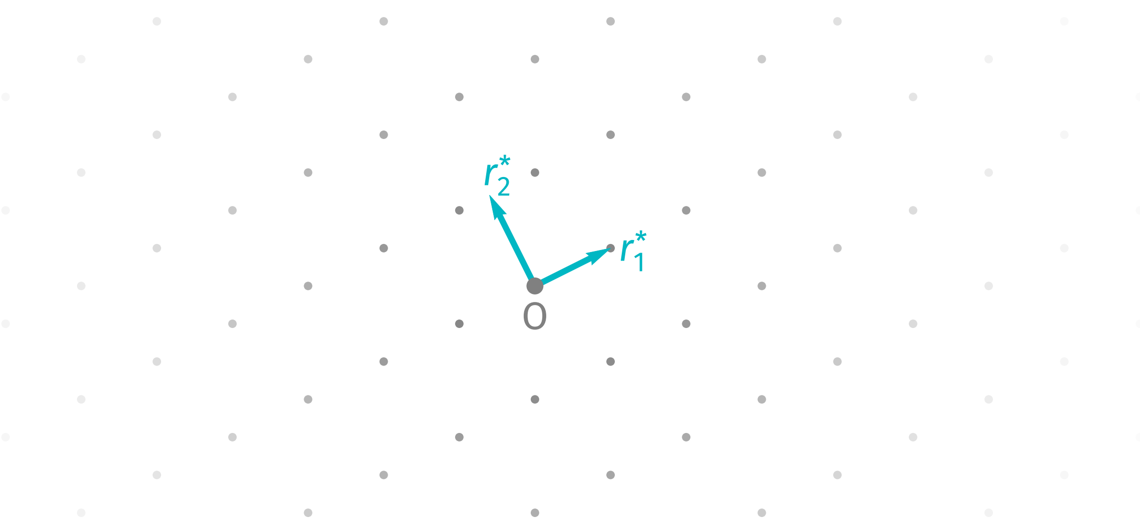

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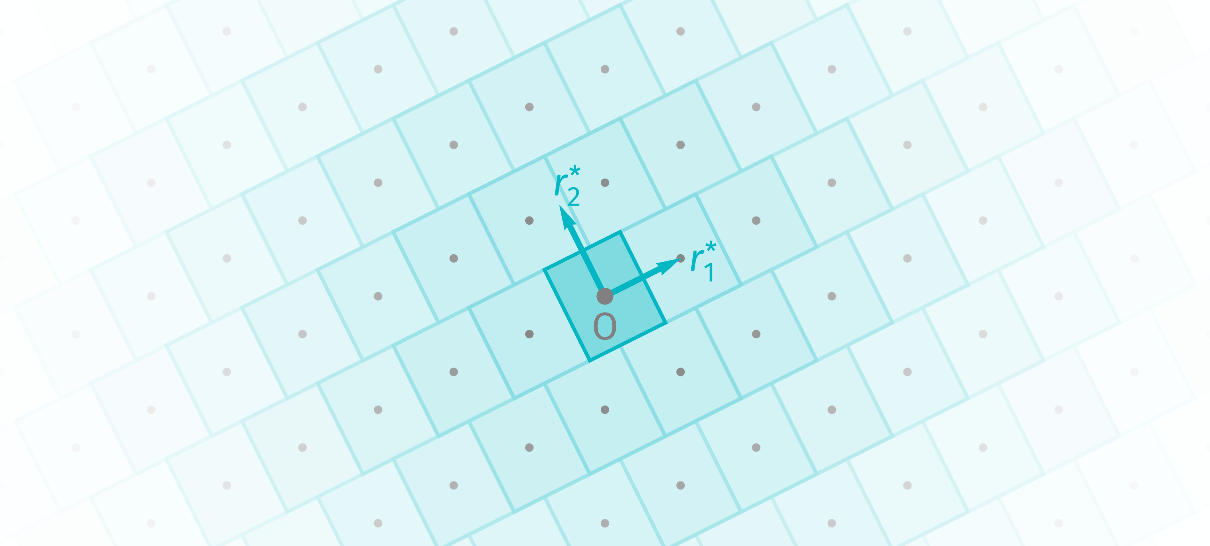

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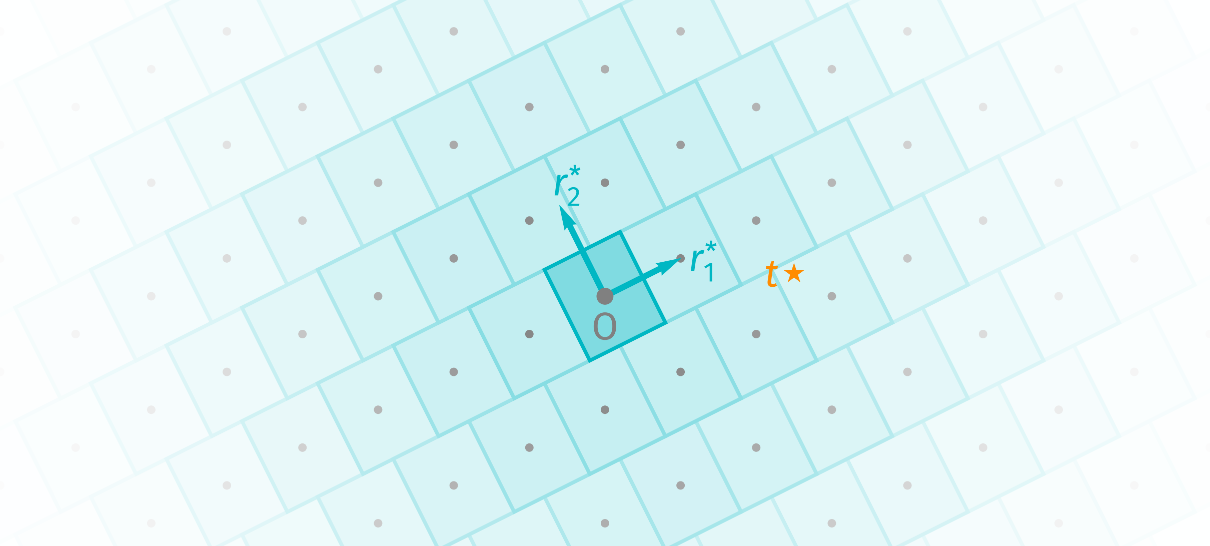

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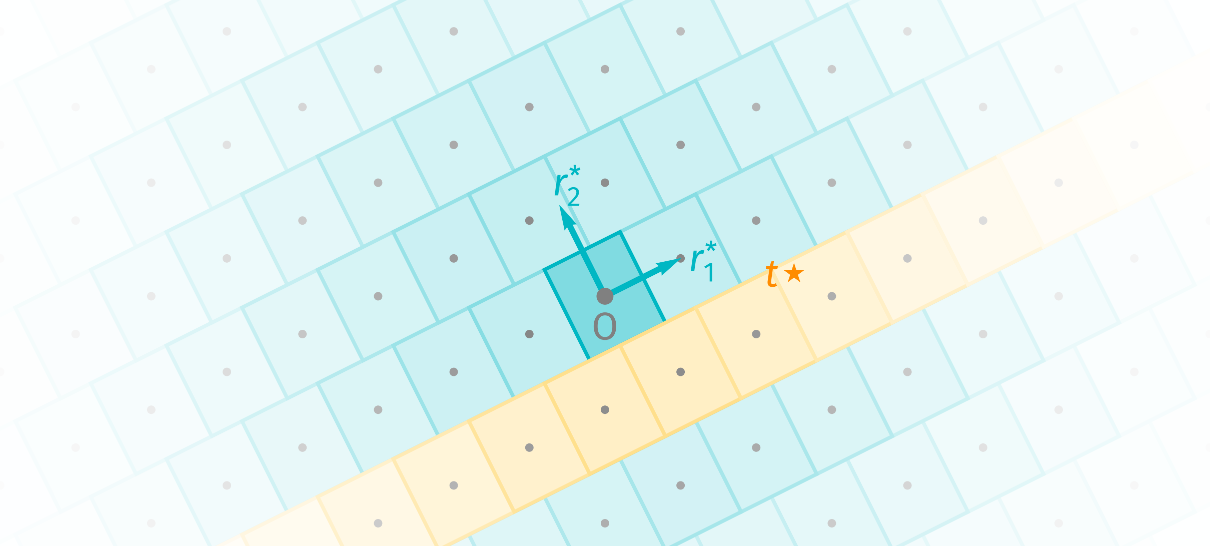

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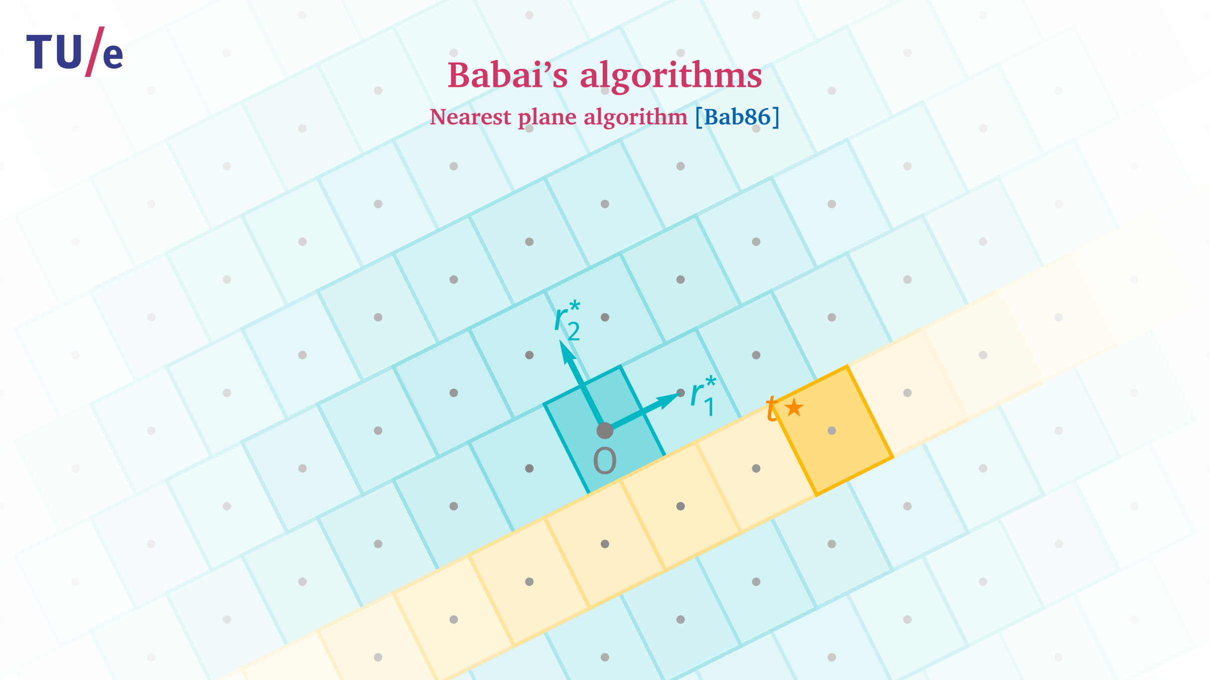

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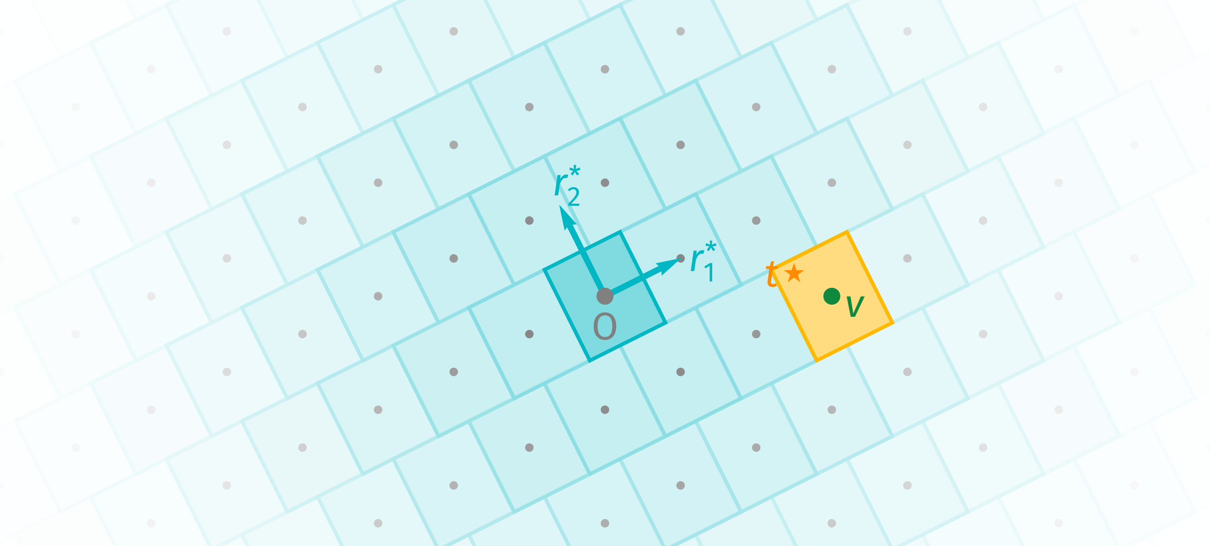

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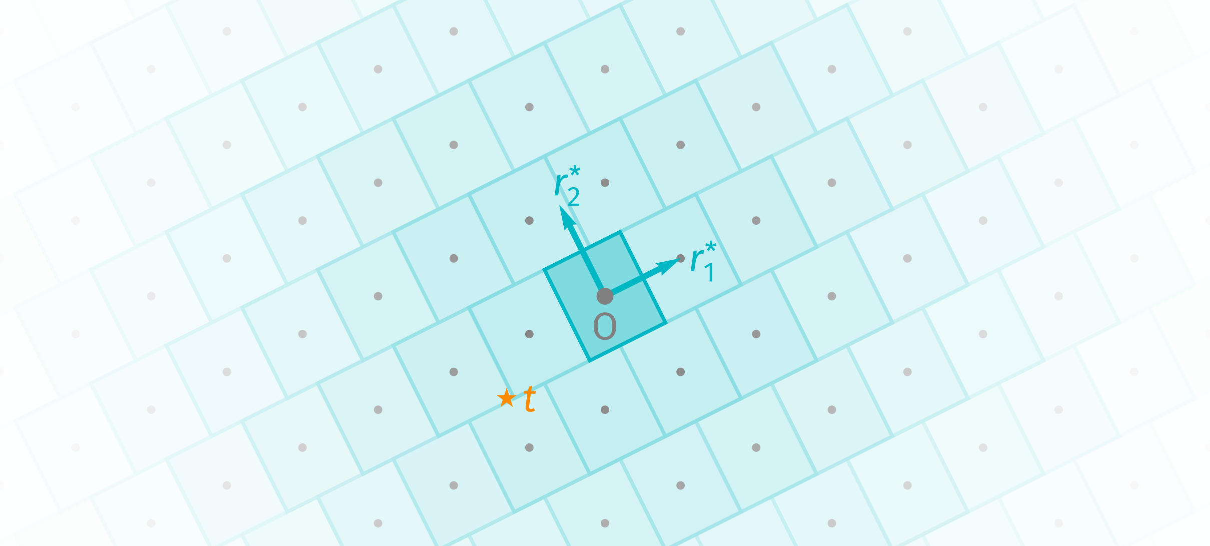

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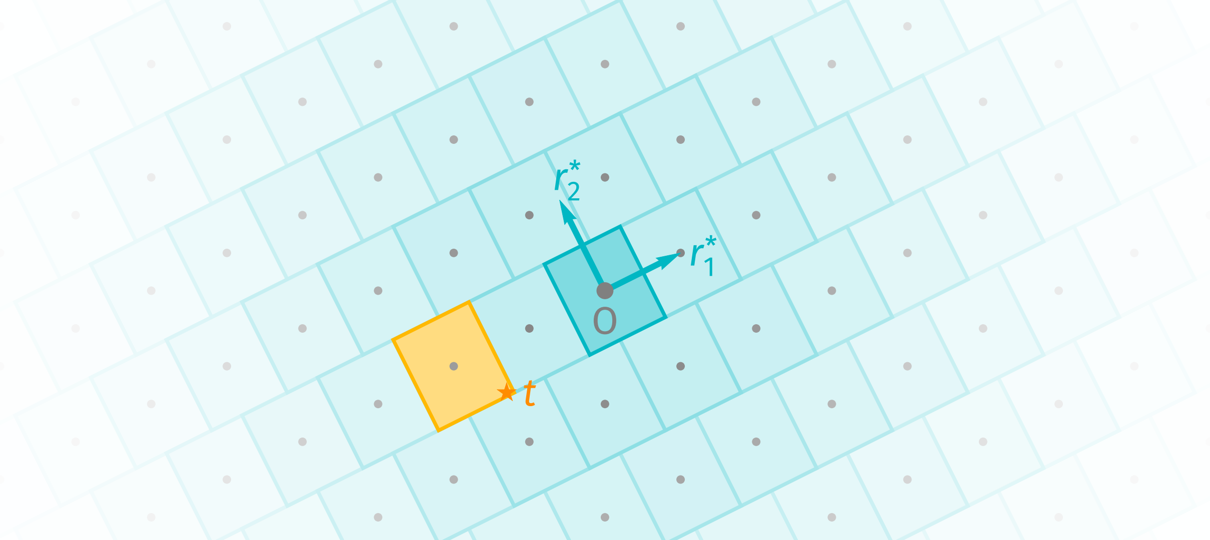

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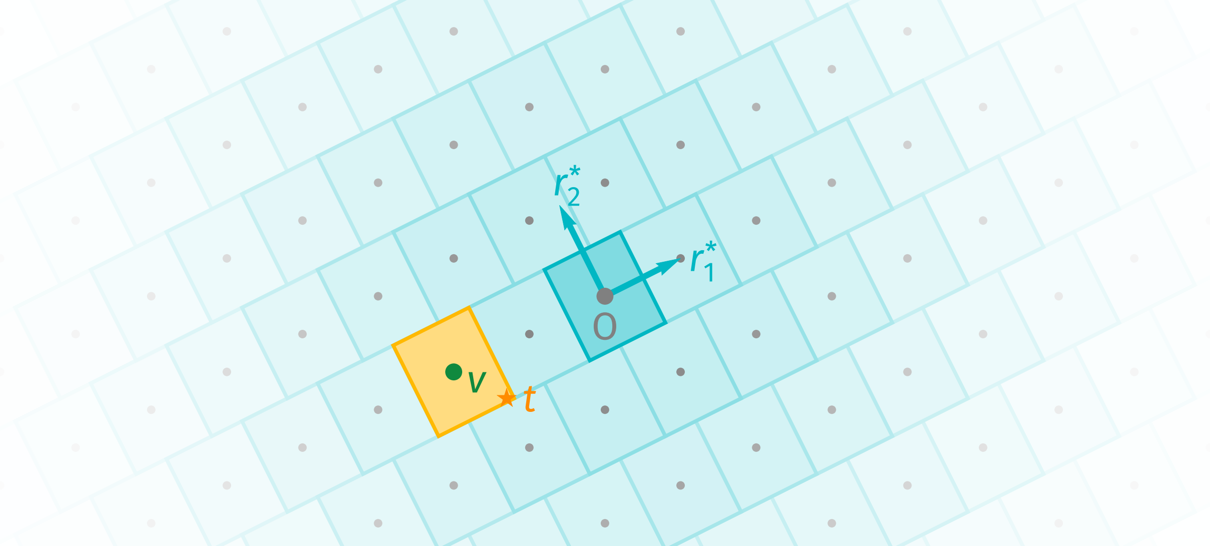

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Overview

- *Preprocessing*: find a short basis ($2^{O(n)}$ time, poly(n) space)

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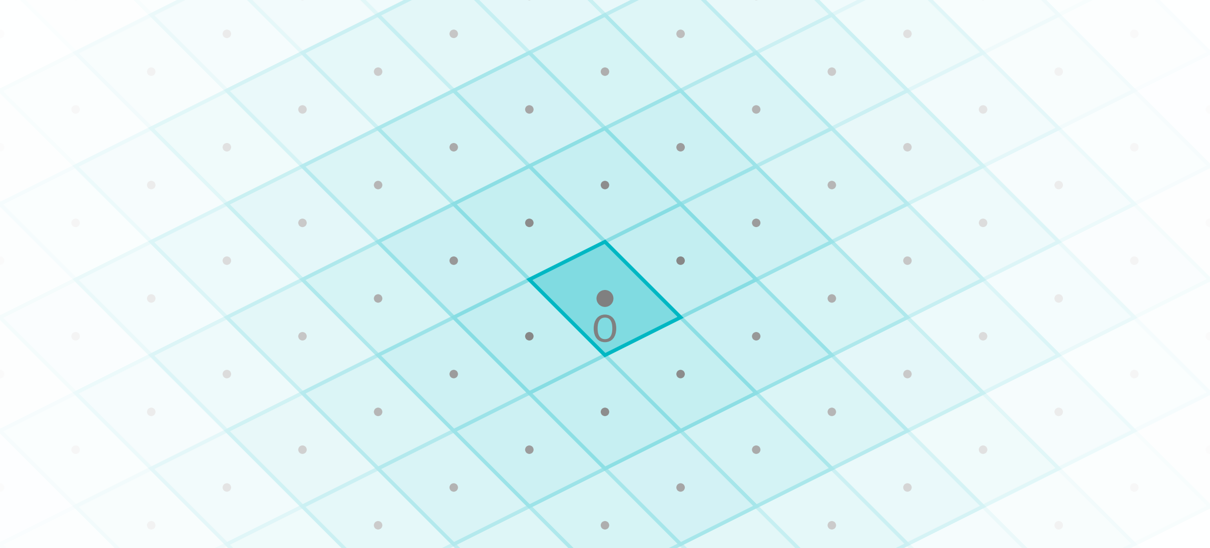
- *Preprocessing*: find a short basis ($2^{O(n)}$ time, poly(n) space)
- *Query*: round-off or nearest-planes (poly(n) time)
- *Strengths*: fast and simple algorithms
- *Limitations*: does not always solve CVPP

Voronoi cells

Round-off tiling



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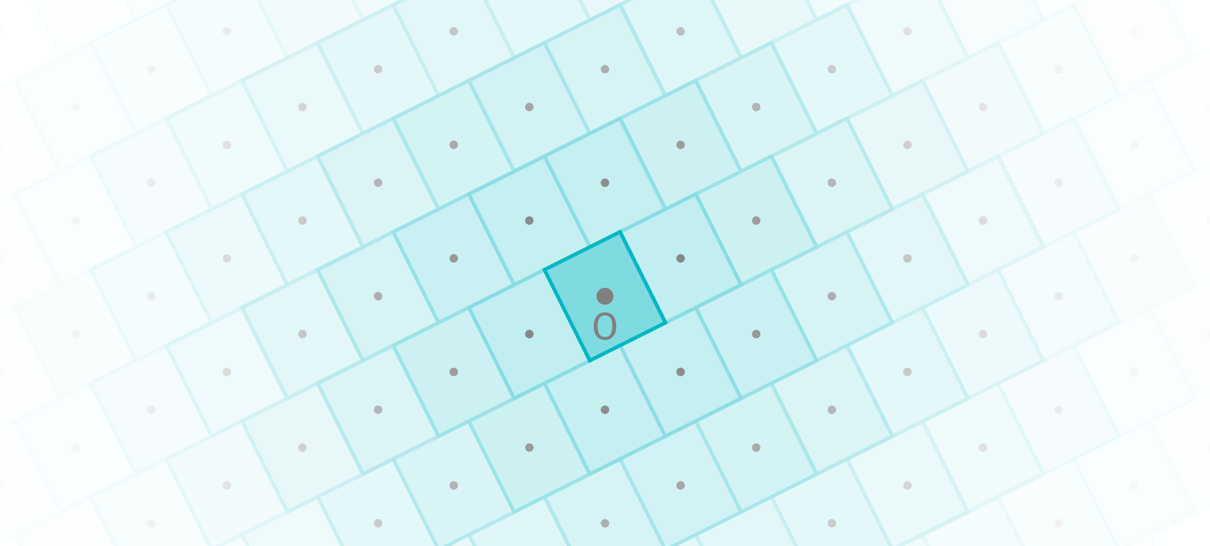
Nearest-plane tiling



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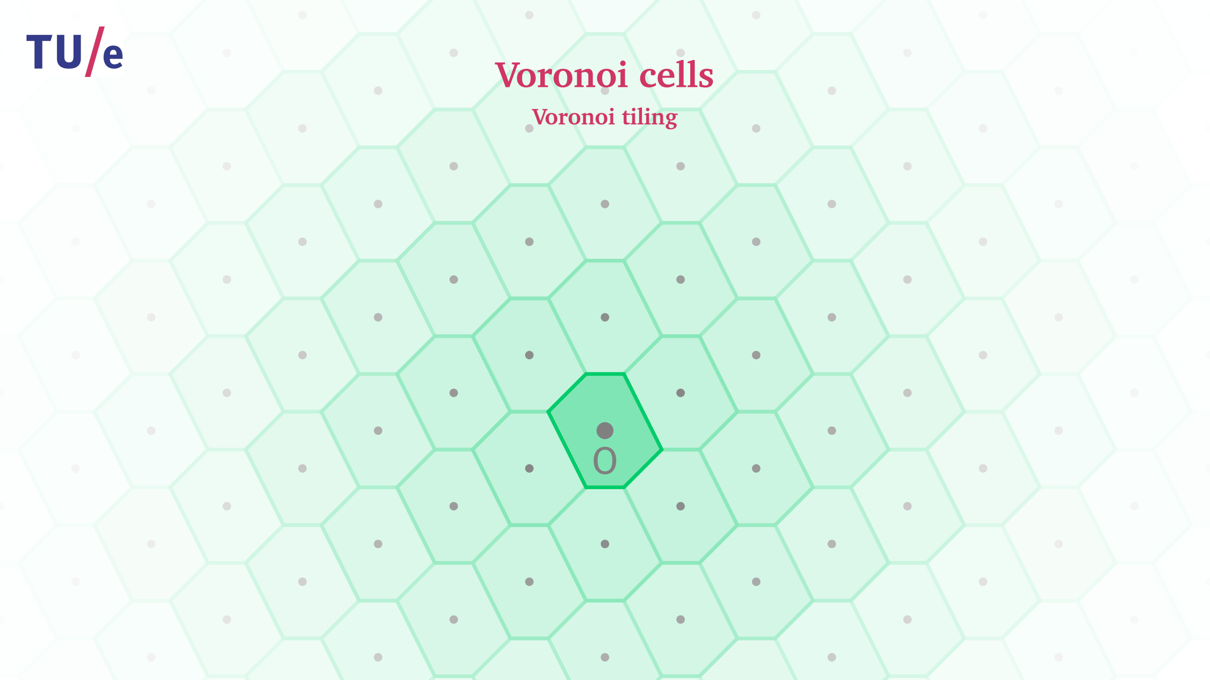

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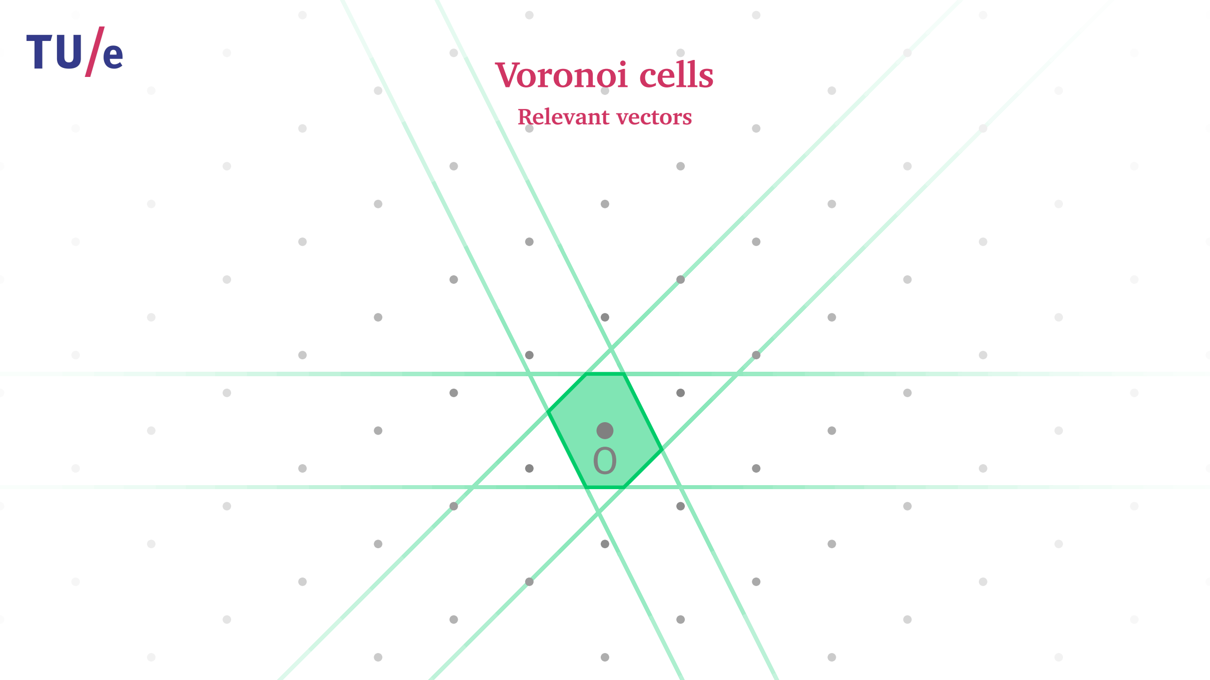
Relevant vectors



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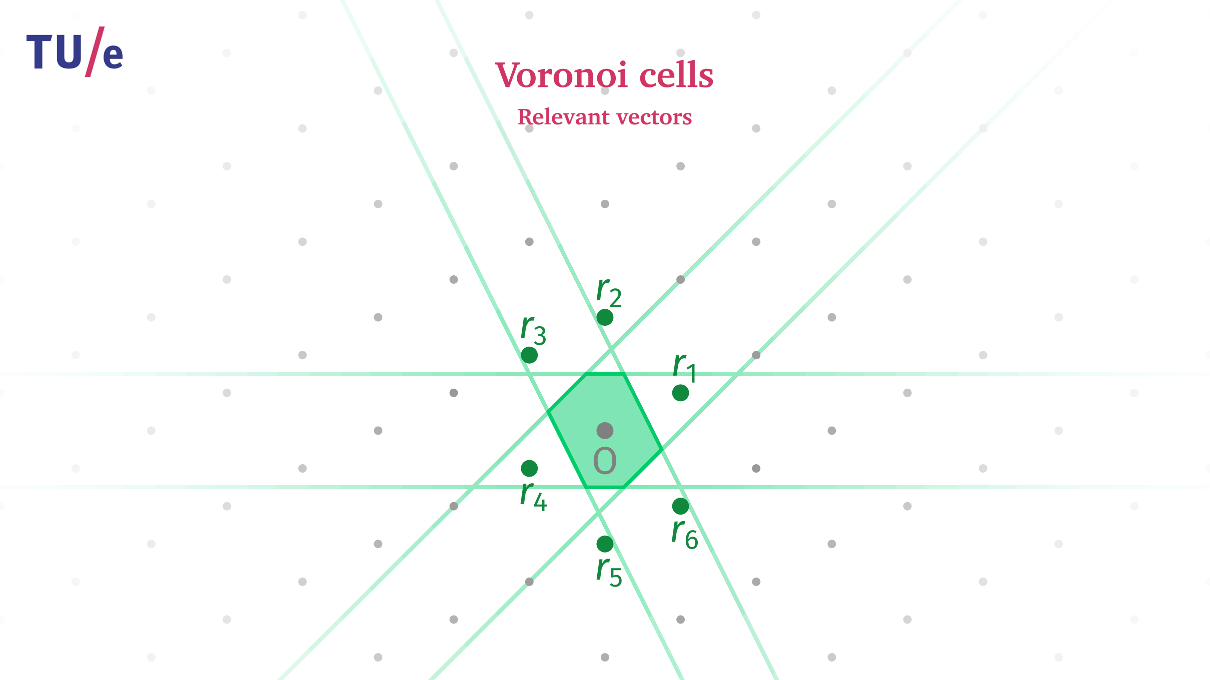

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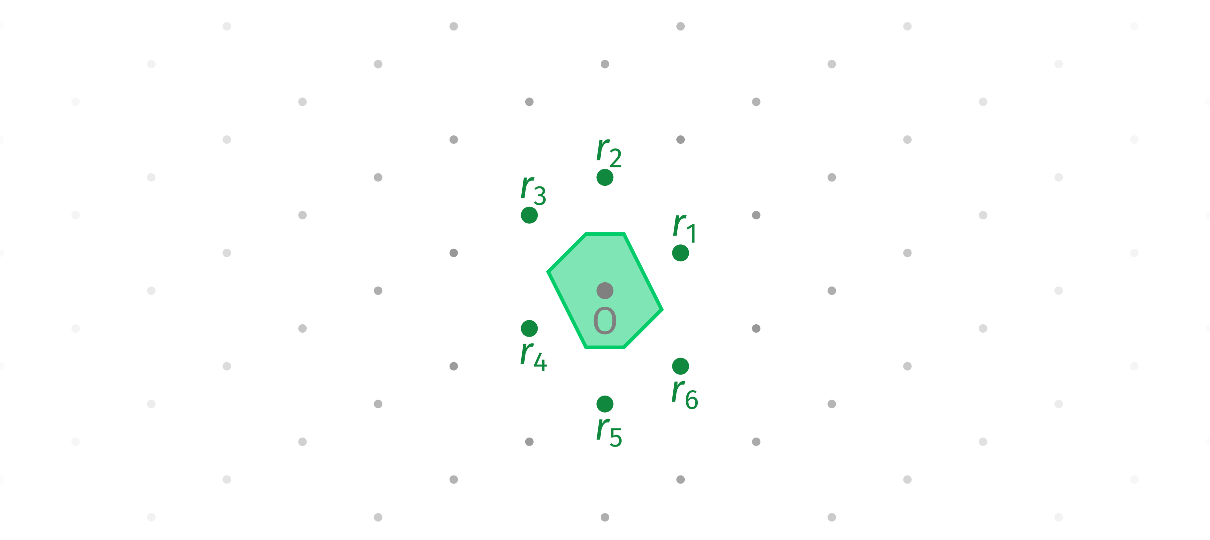

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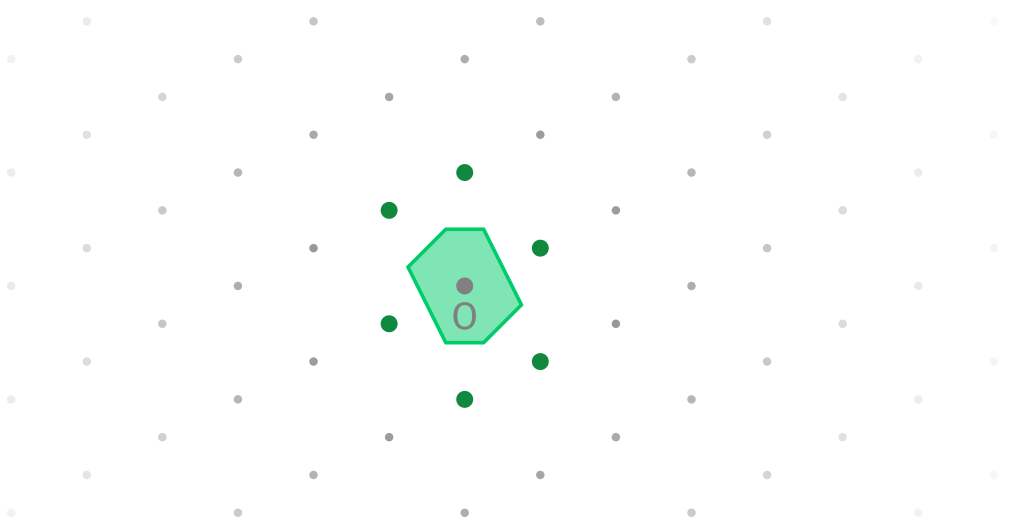

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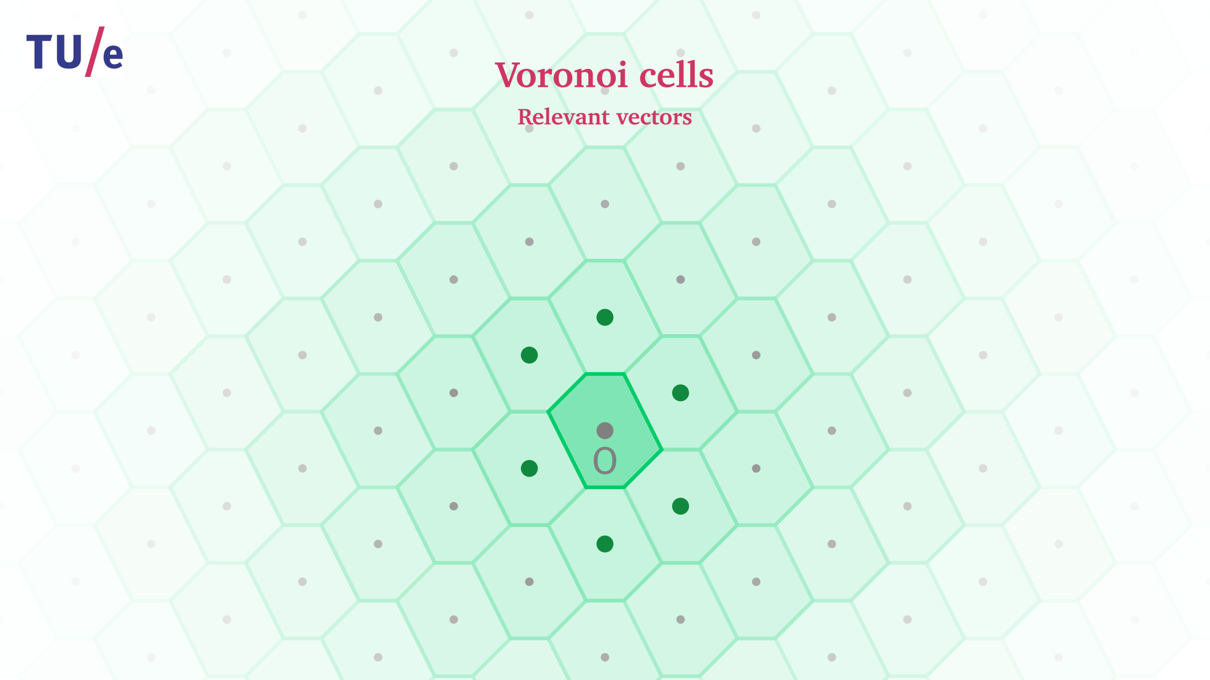

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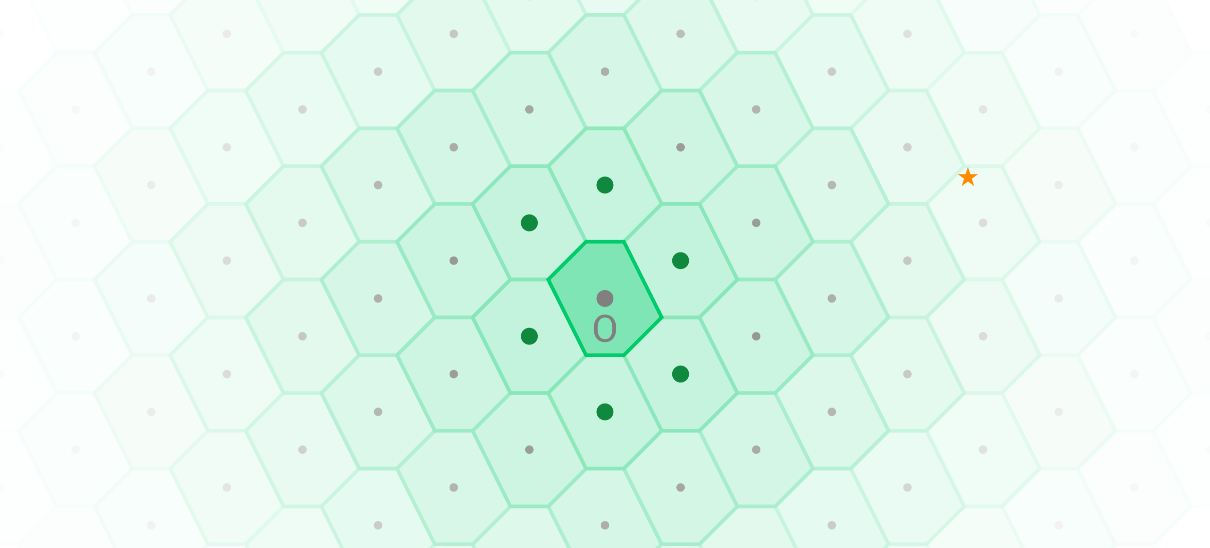
Iterative slicer [SFS09]



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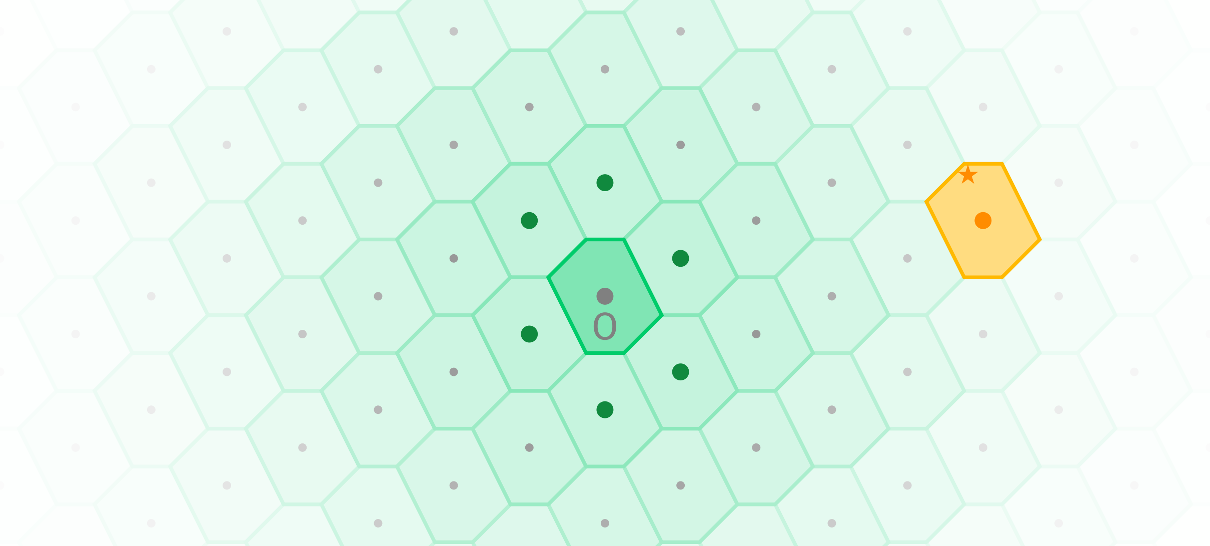

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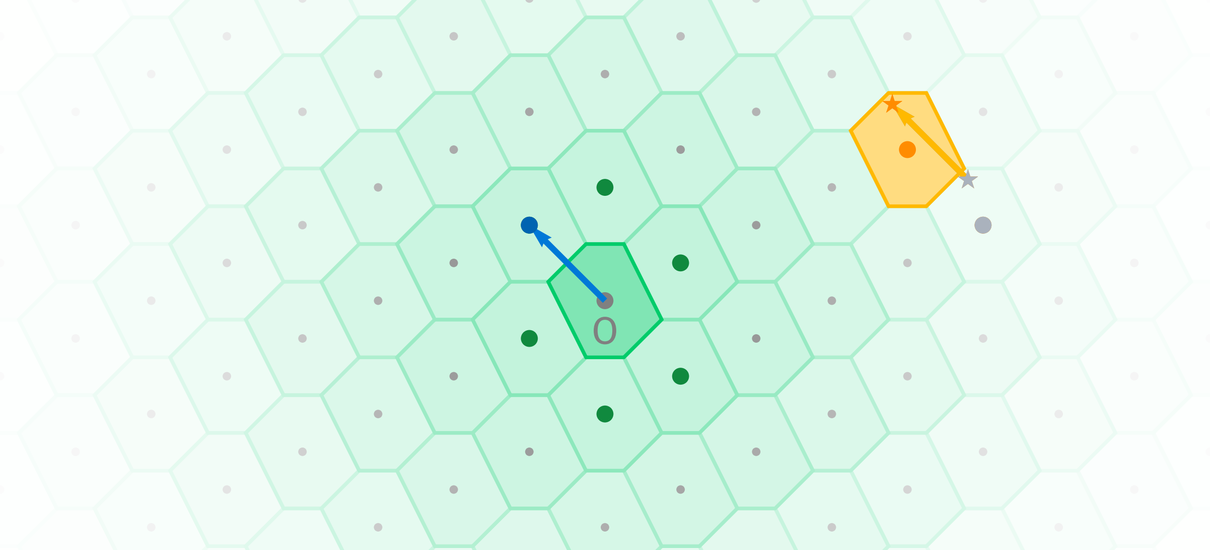

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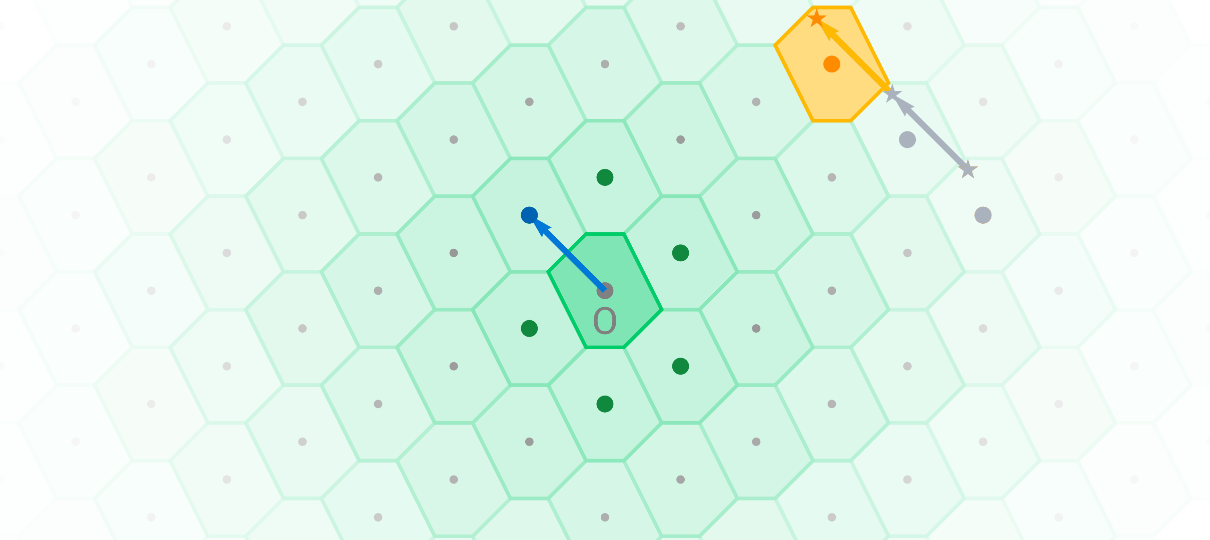

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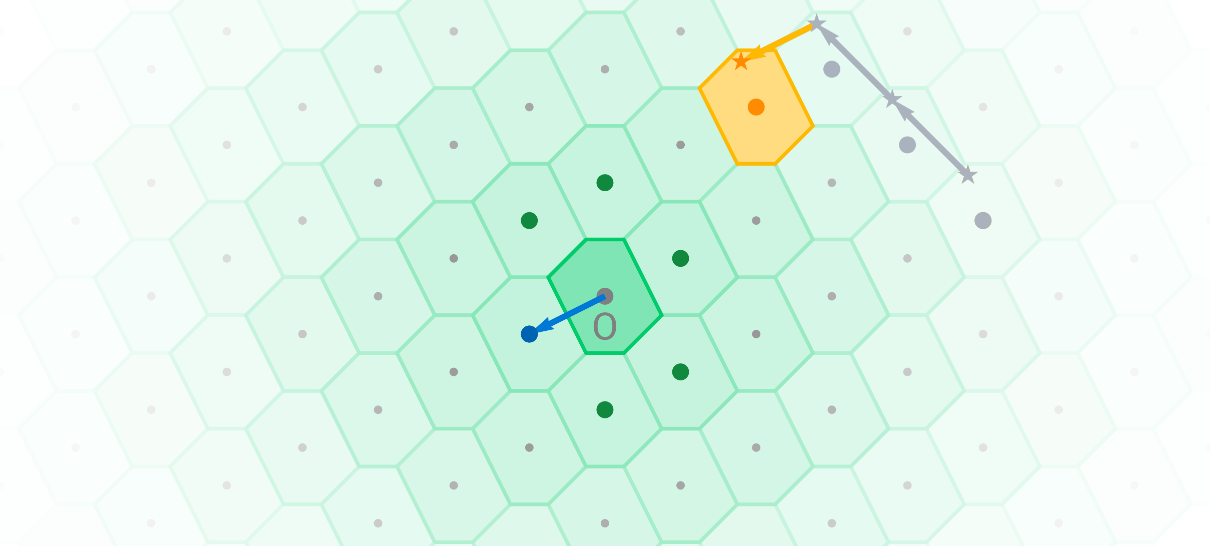

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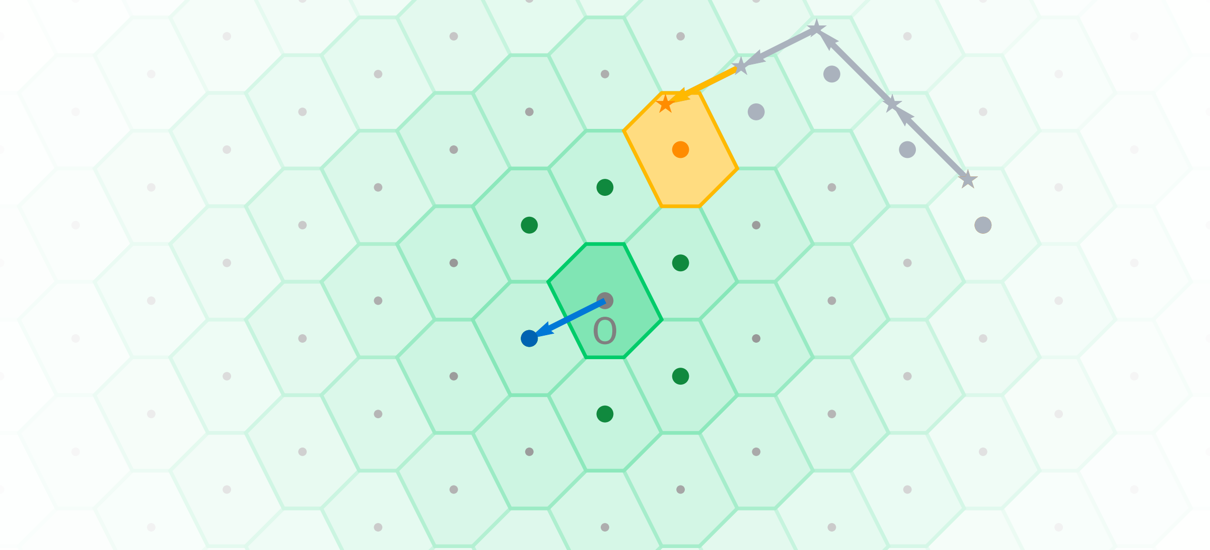

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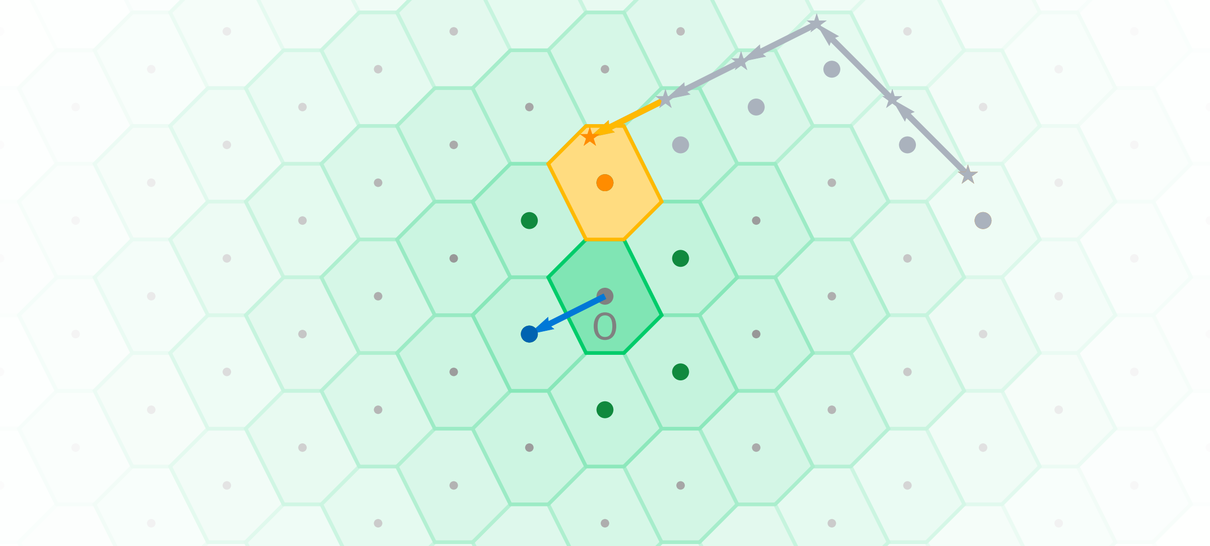

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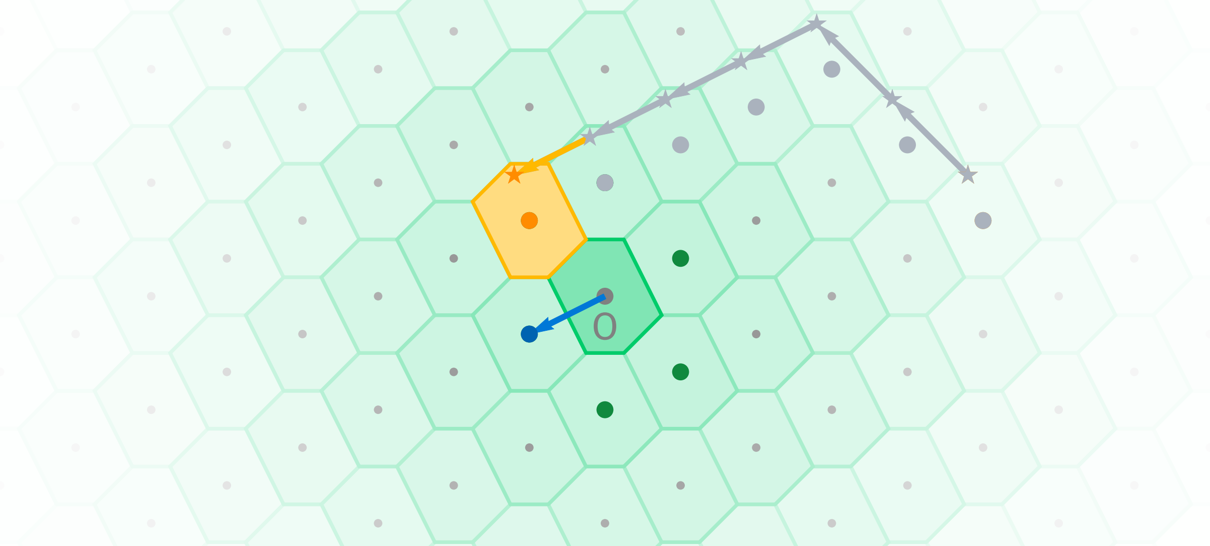

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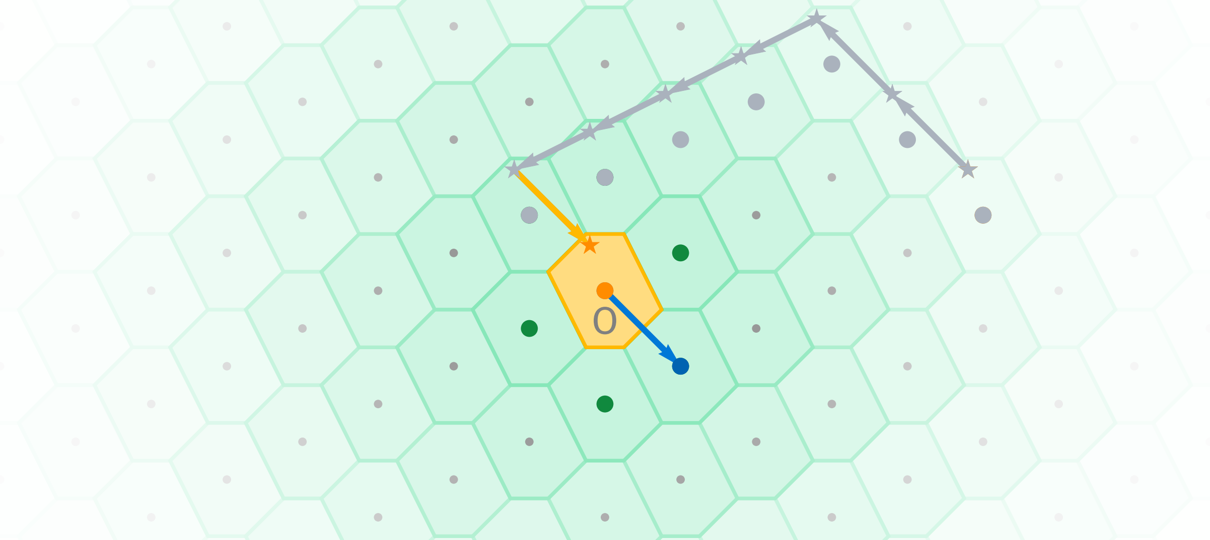

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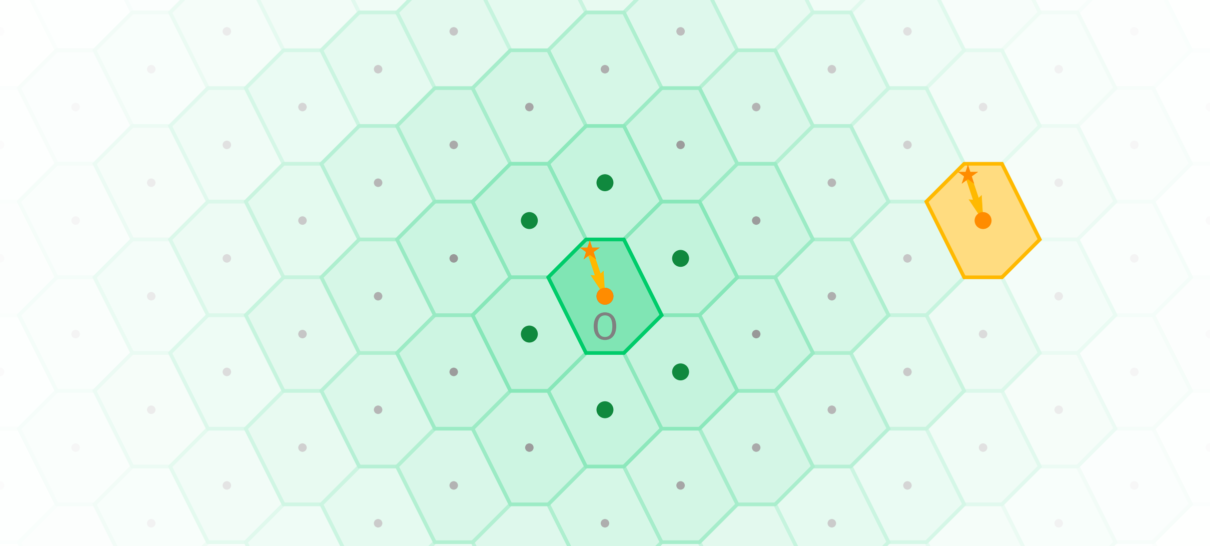

Voronoi cells

Overview

- *Preprocessing*: find the relevant vectors ($2^{2n+o(n)}$ time, $2^{n+o(n)}$ space [MV10])

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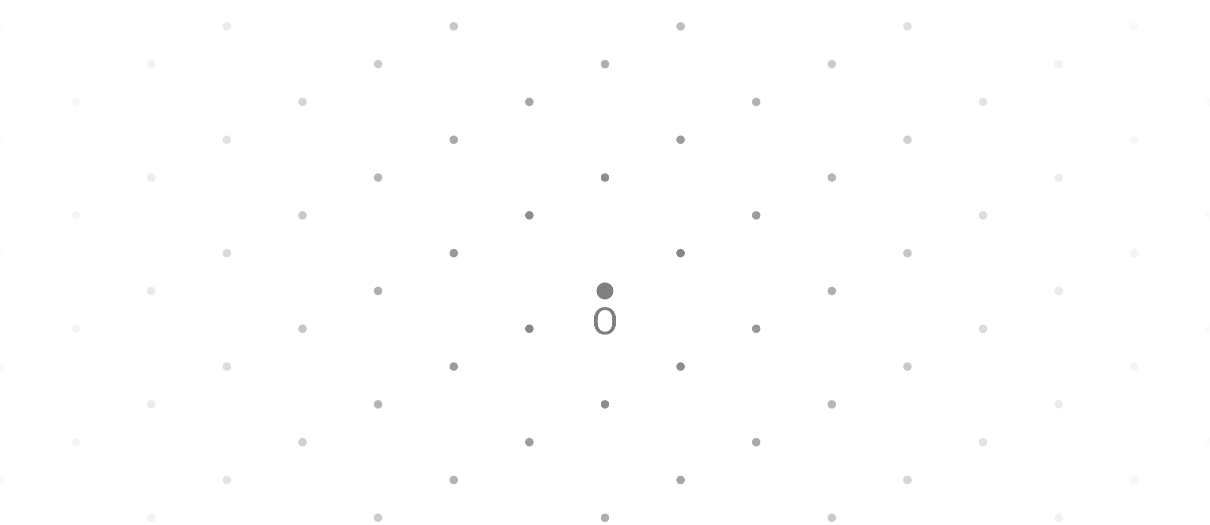
- *Query*: reduce with the relevant vectors ($2^{n+o(n)}$ time [BD15])
- *Strengths*: provably solves CVPP for arbitrary targets and lattices
- *Limitations*: large time and memory requirements

Approximate Voronoi cells

Decrease list size



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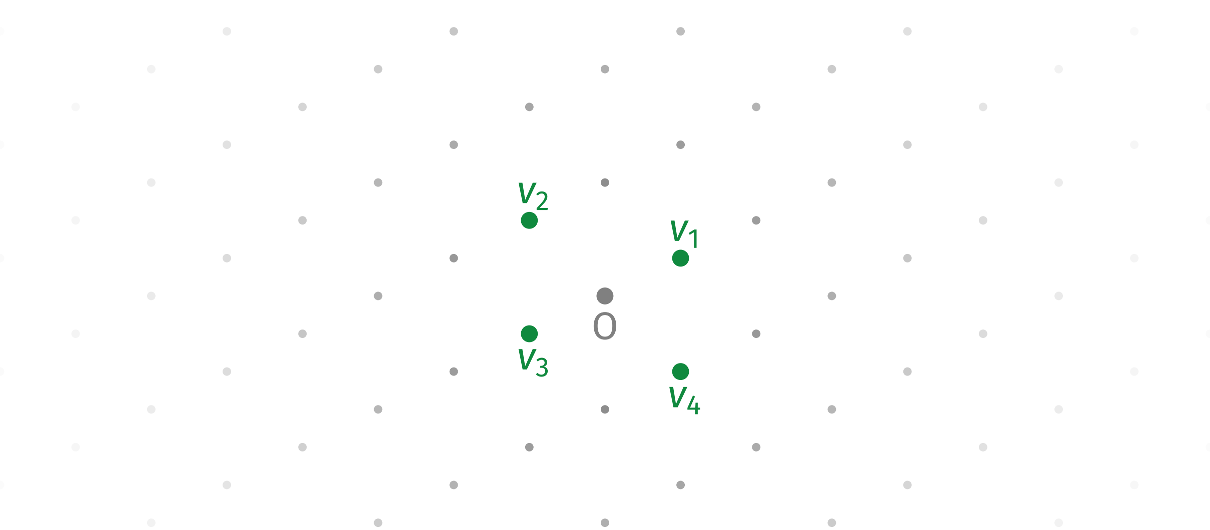

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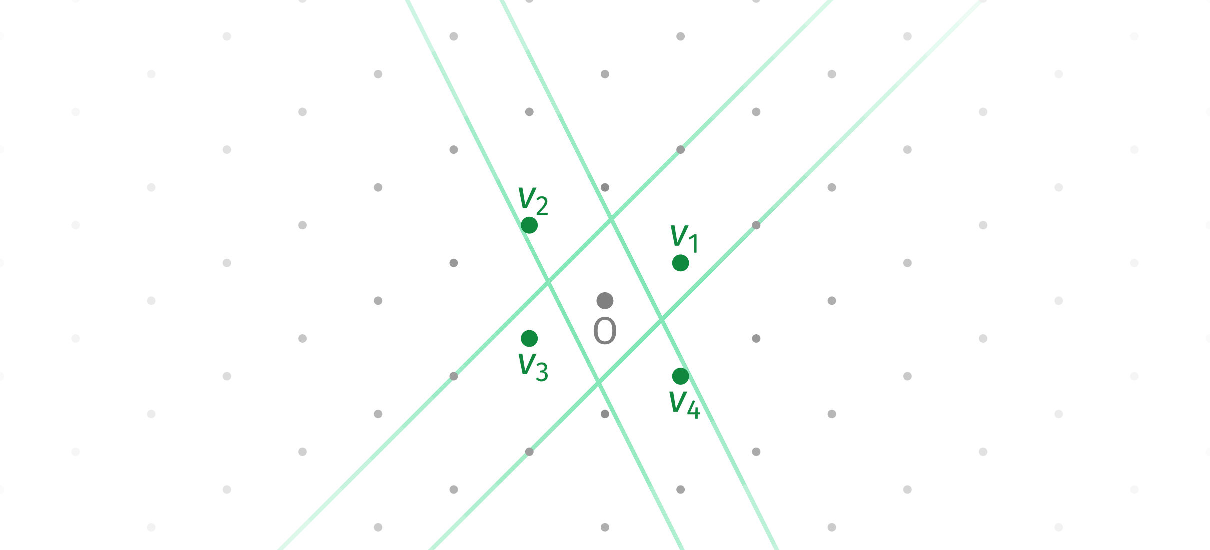

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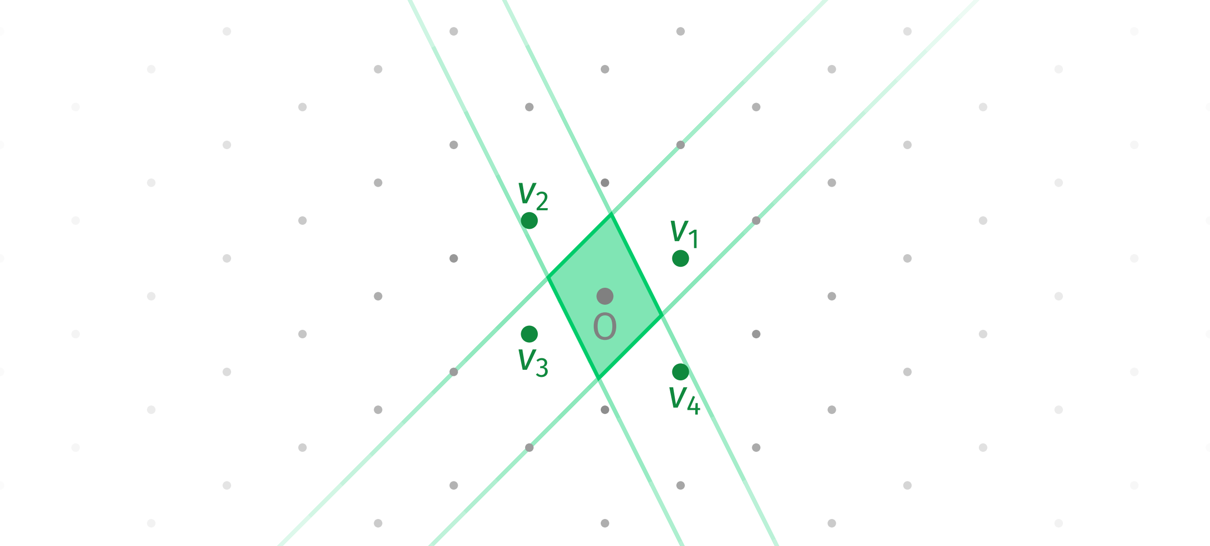

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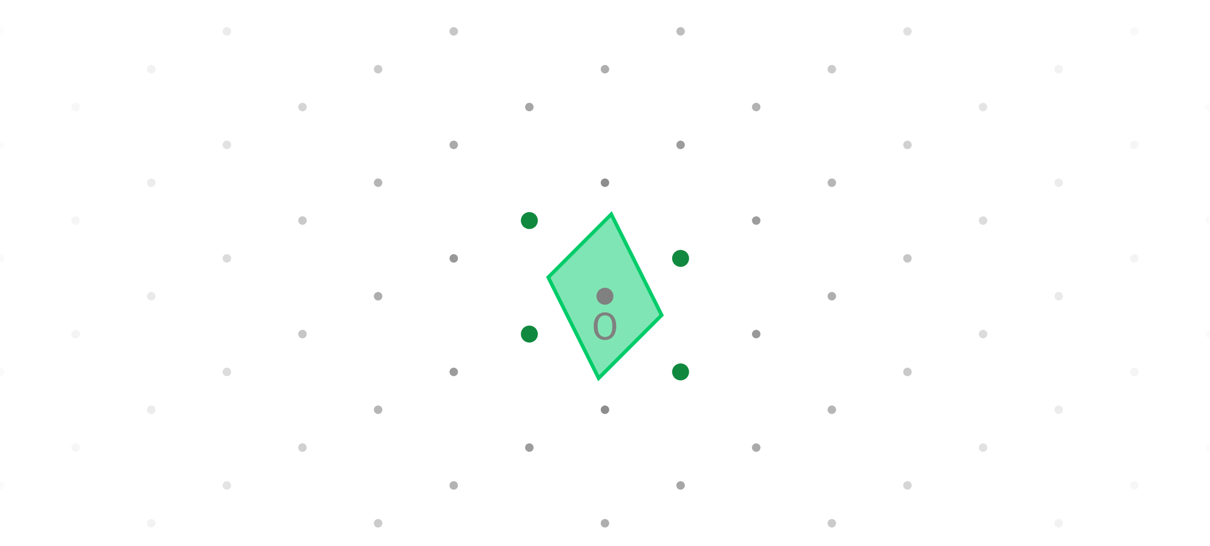
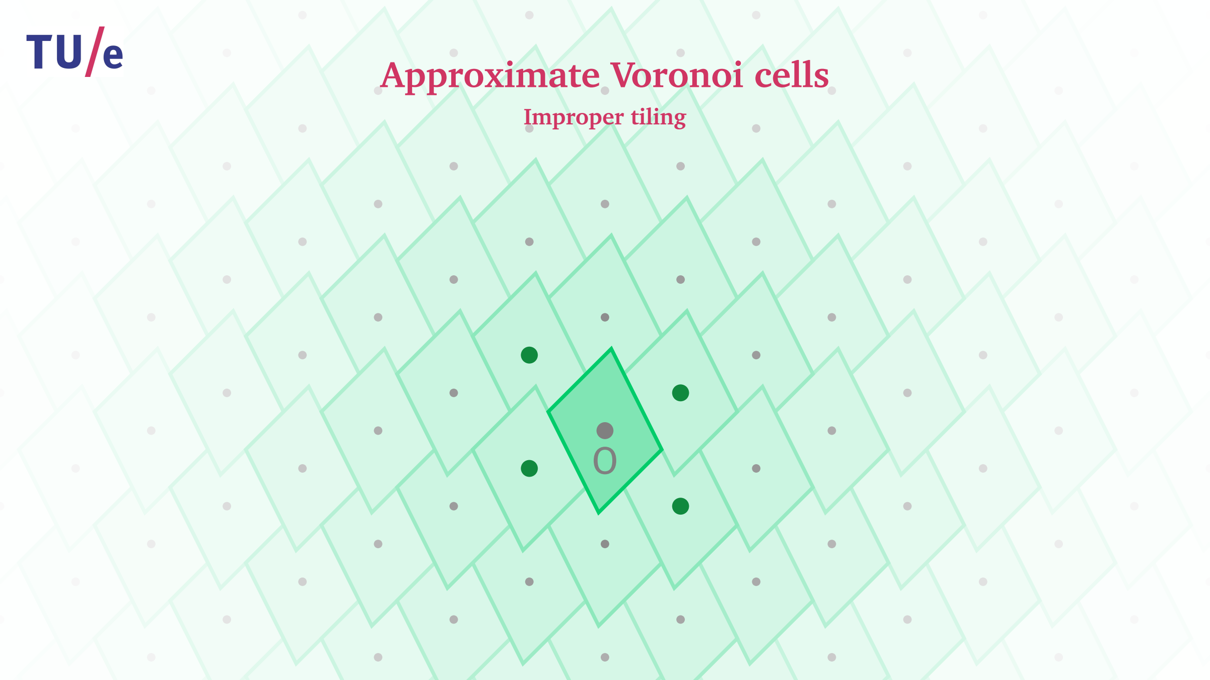


Approximate Voronoi cells

Improper tiling



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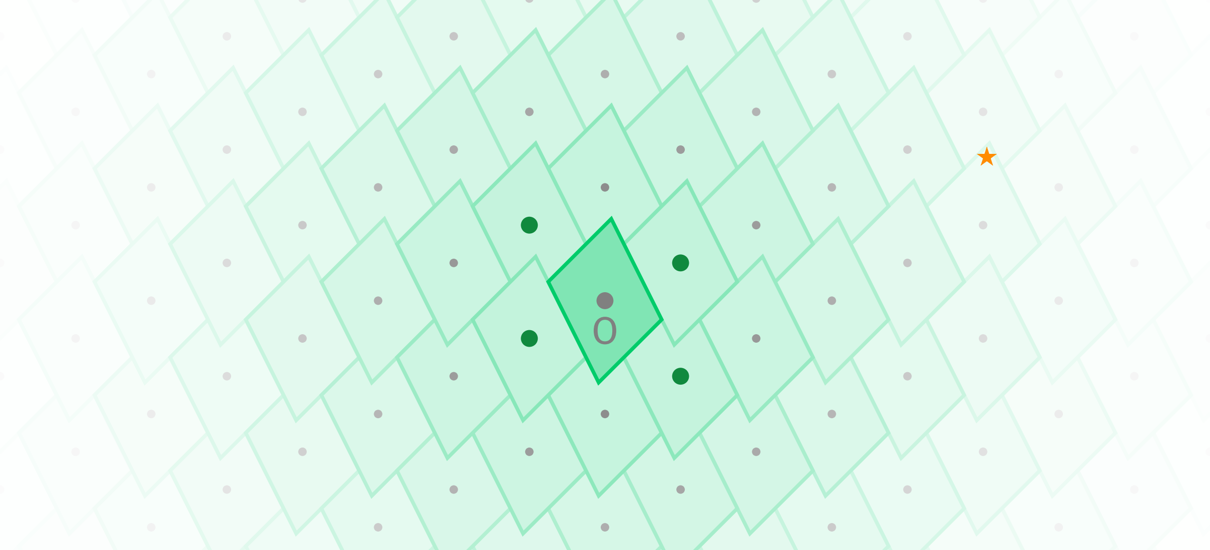

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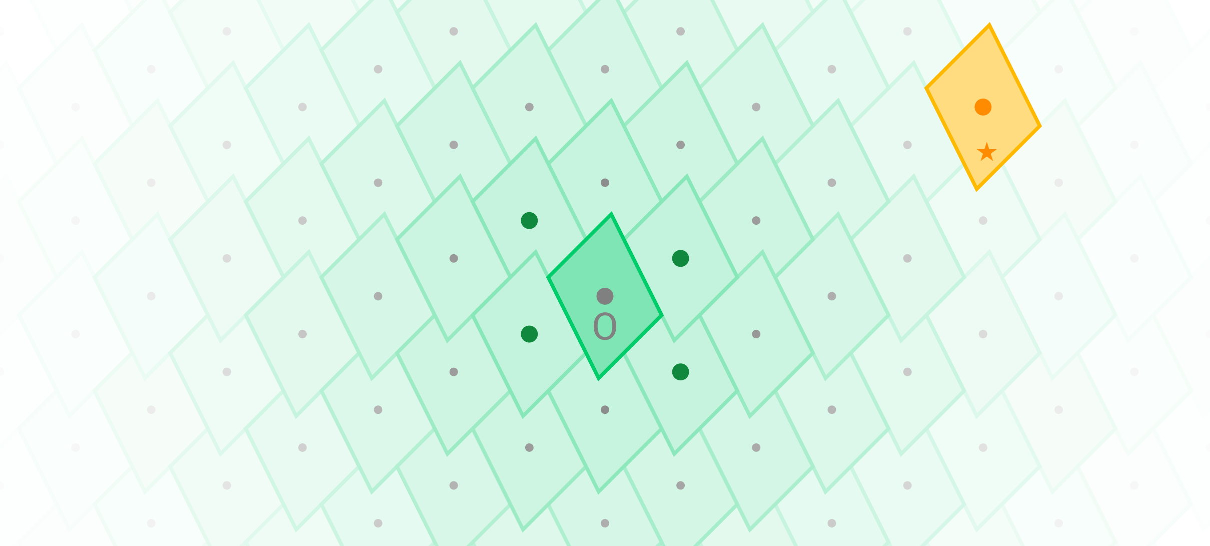

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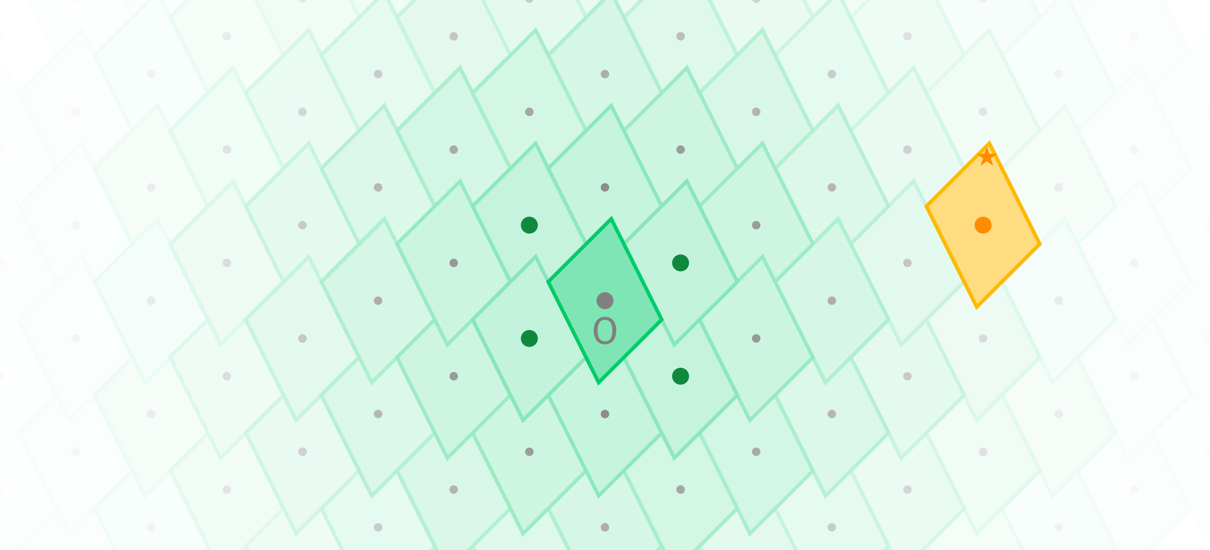

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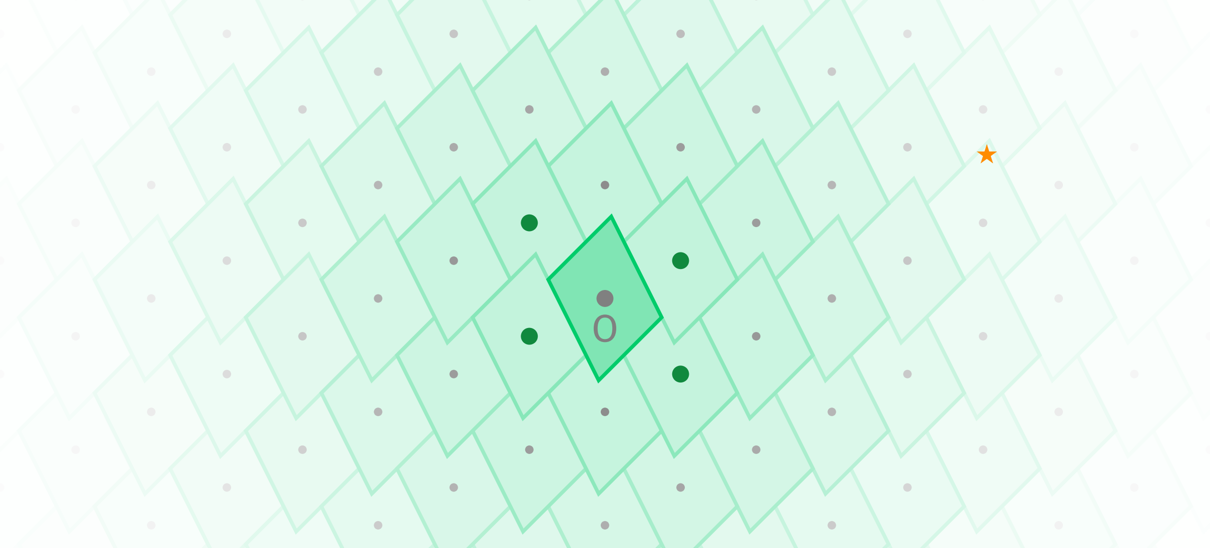
Iterative slicer [SFS09]



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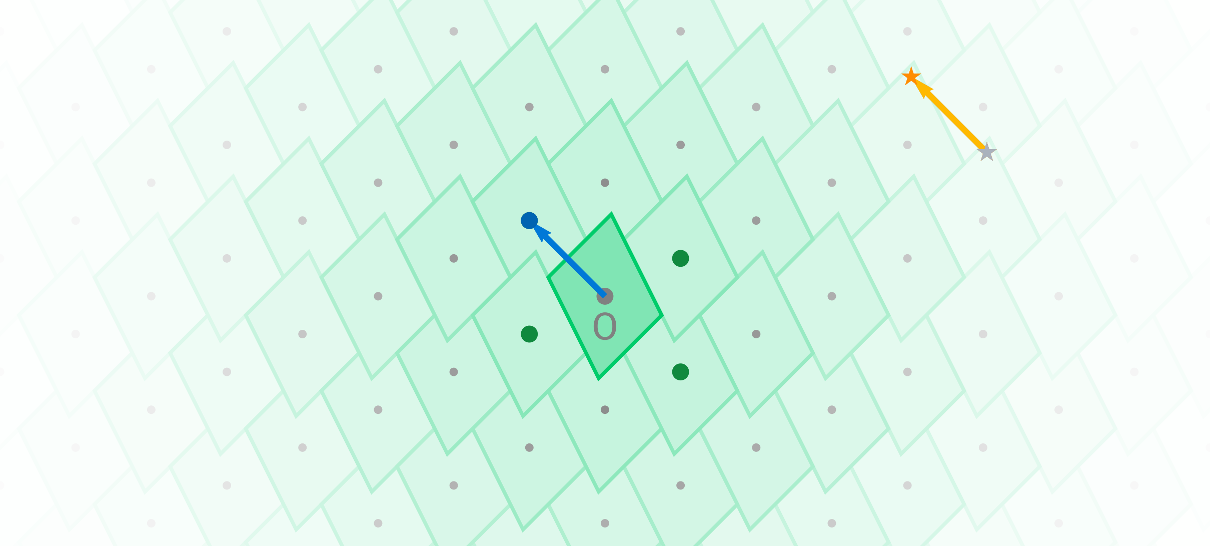

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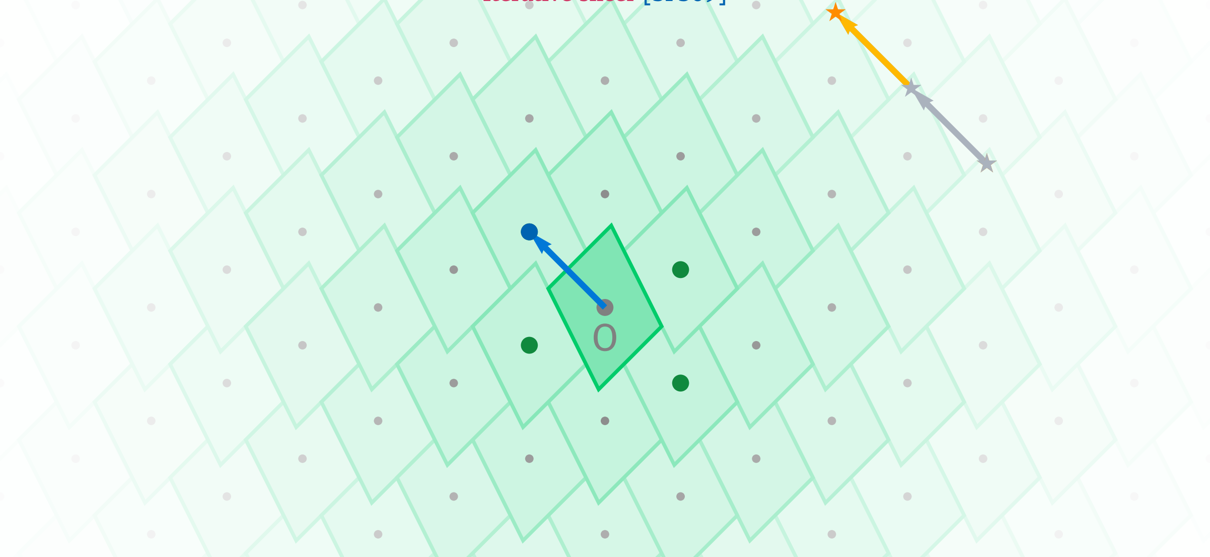

TU/e

Approximate Voronoi cells

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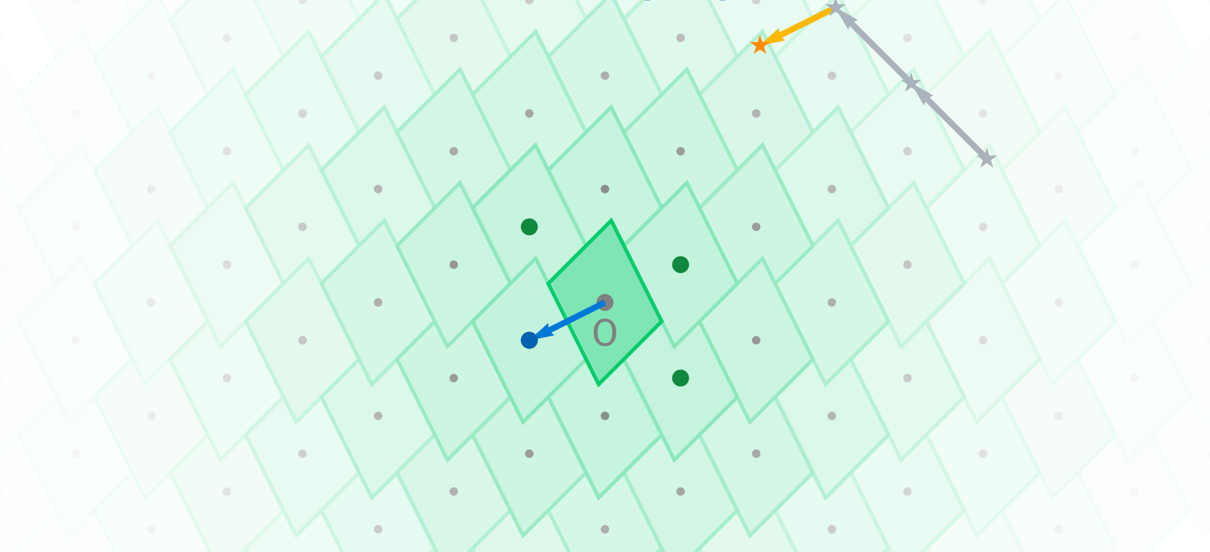

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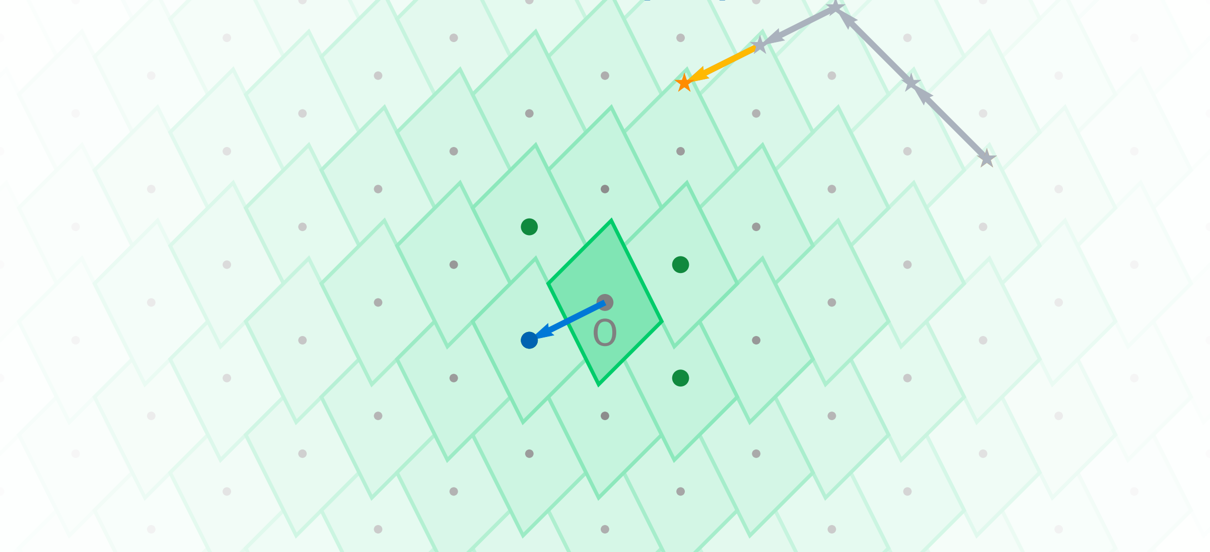

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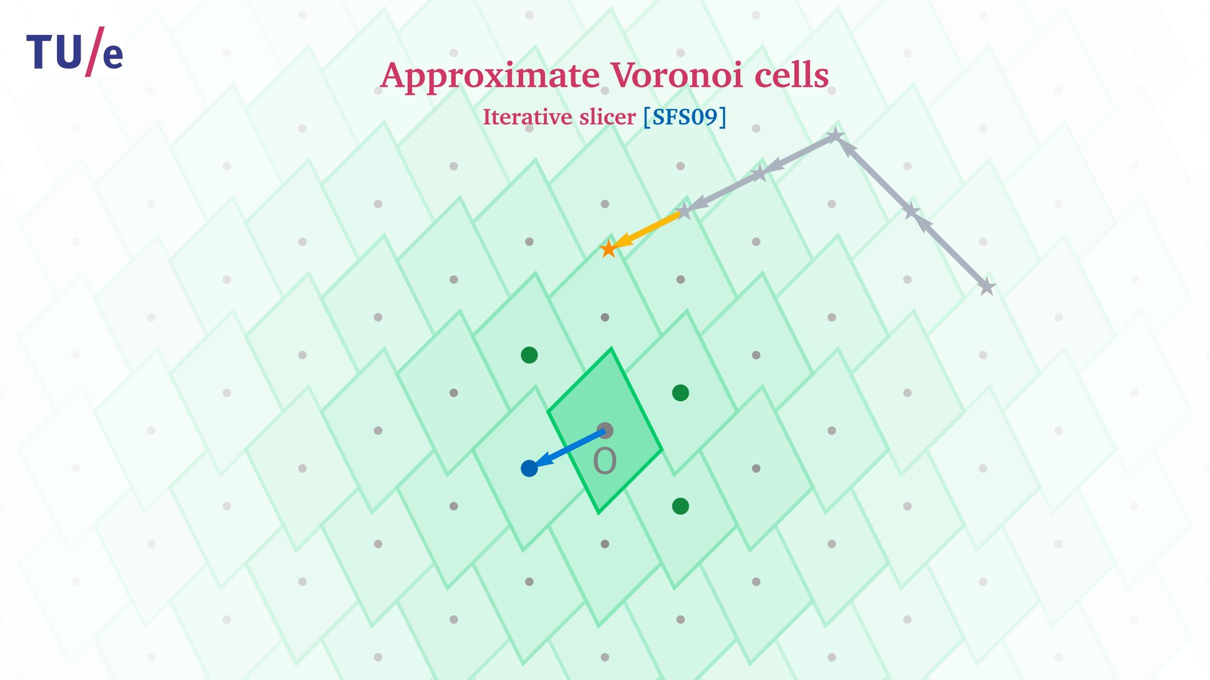

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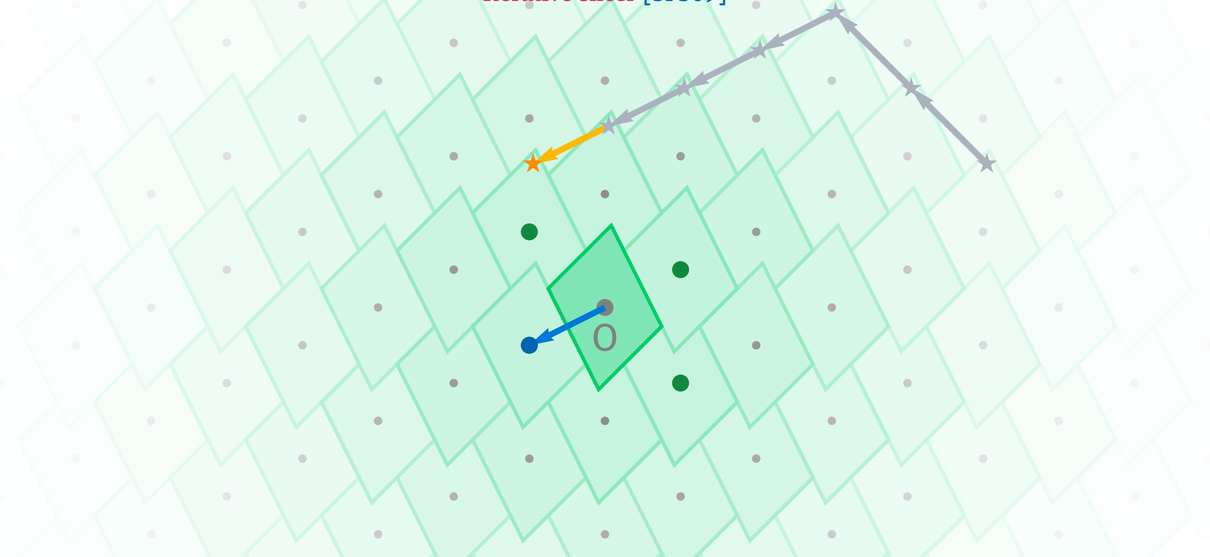

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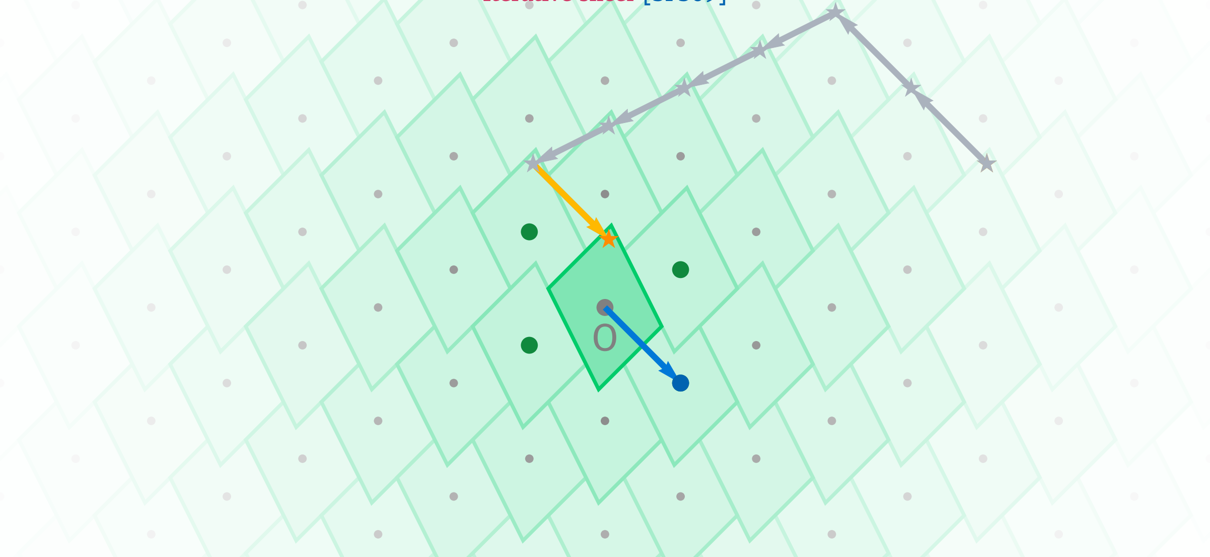

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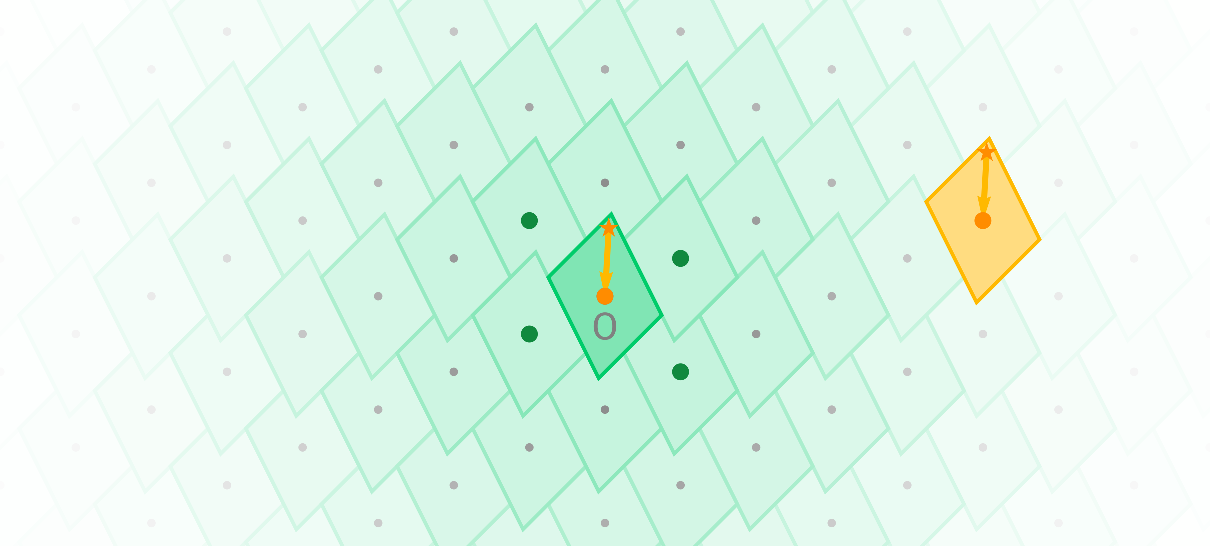
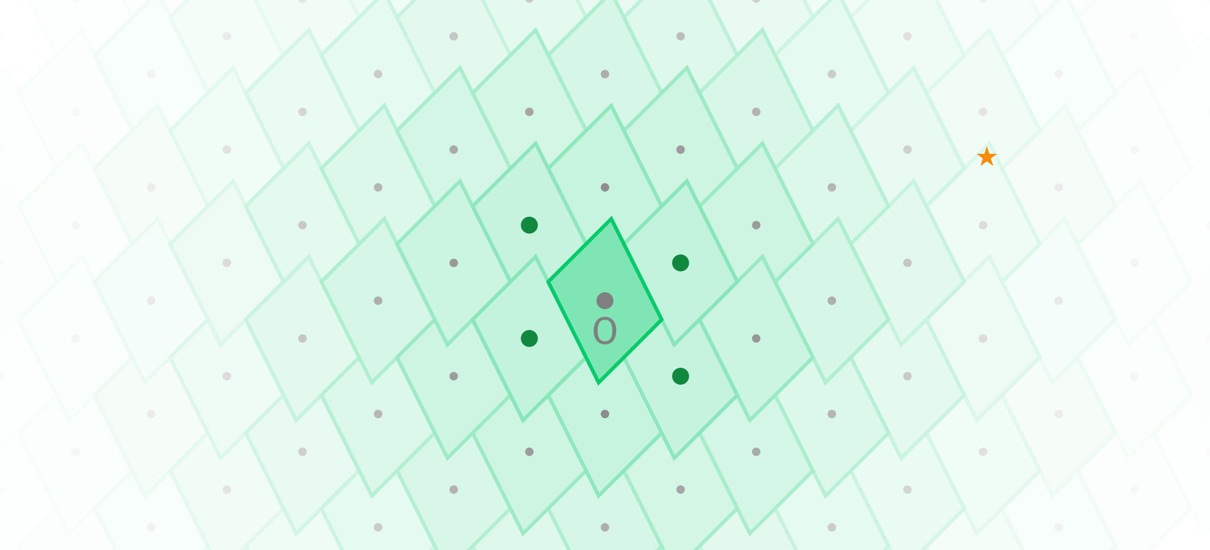


Approximate Voronoi cells

Randomized slicer



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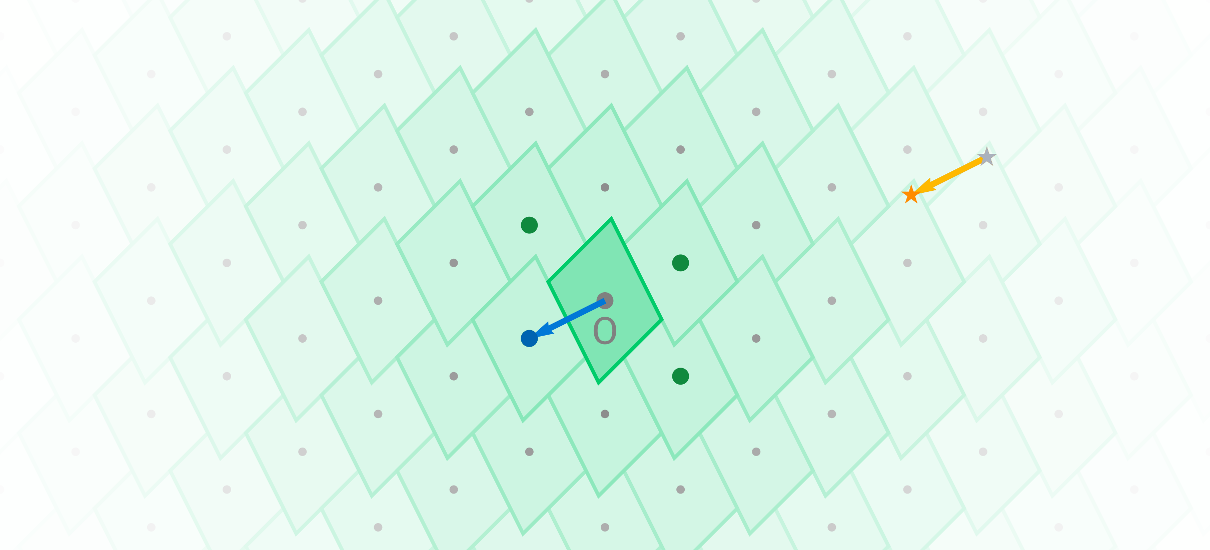

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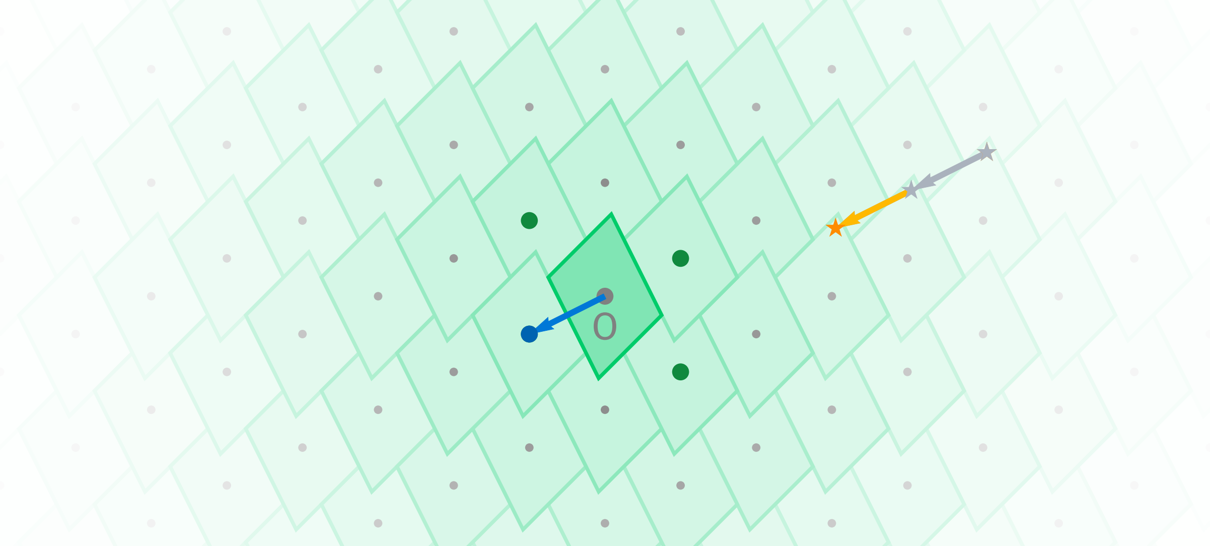

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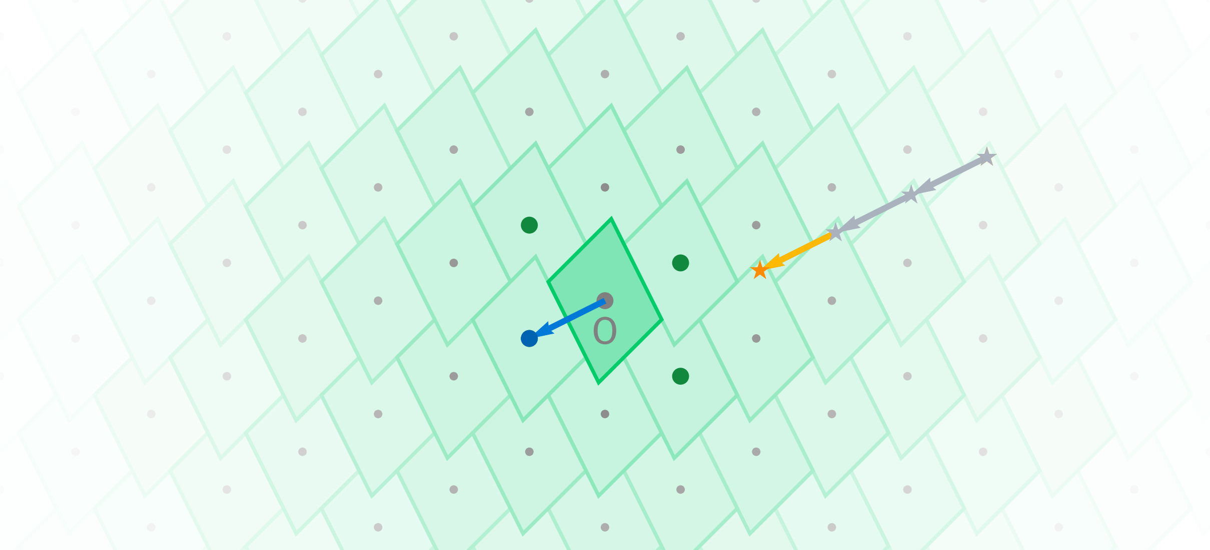

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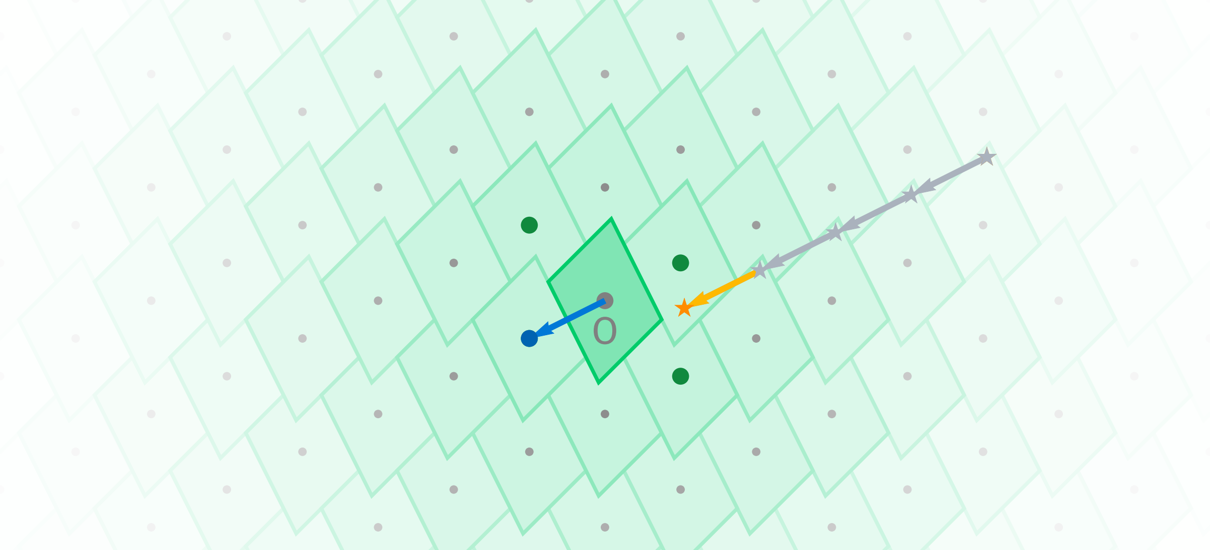

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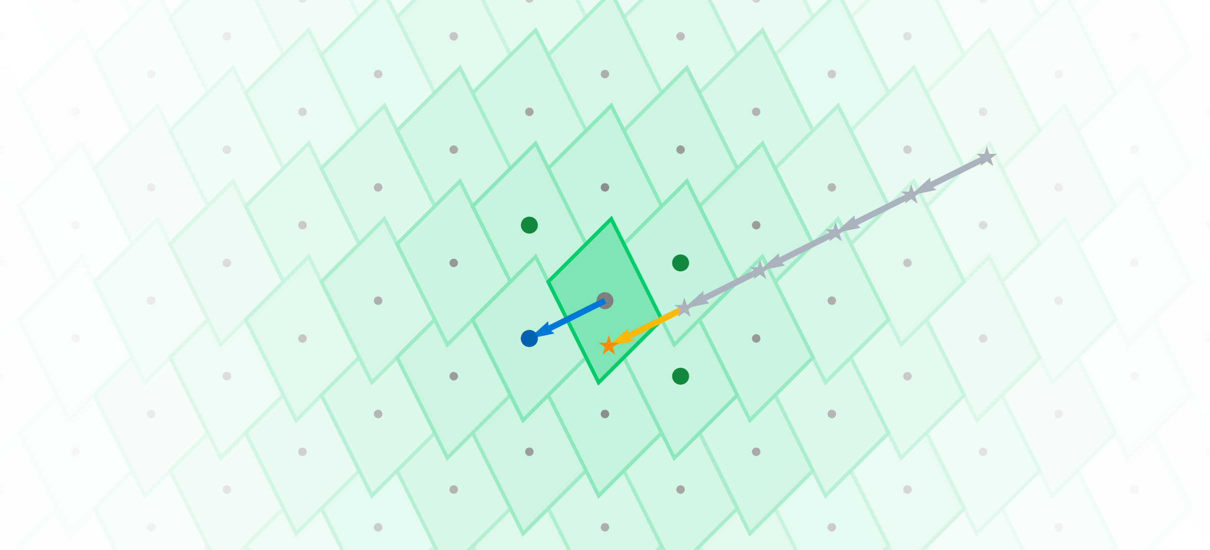

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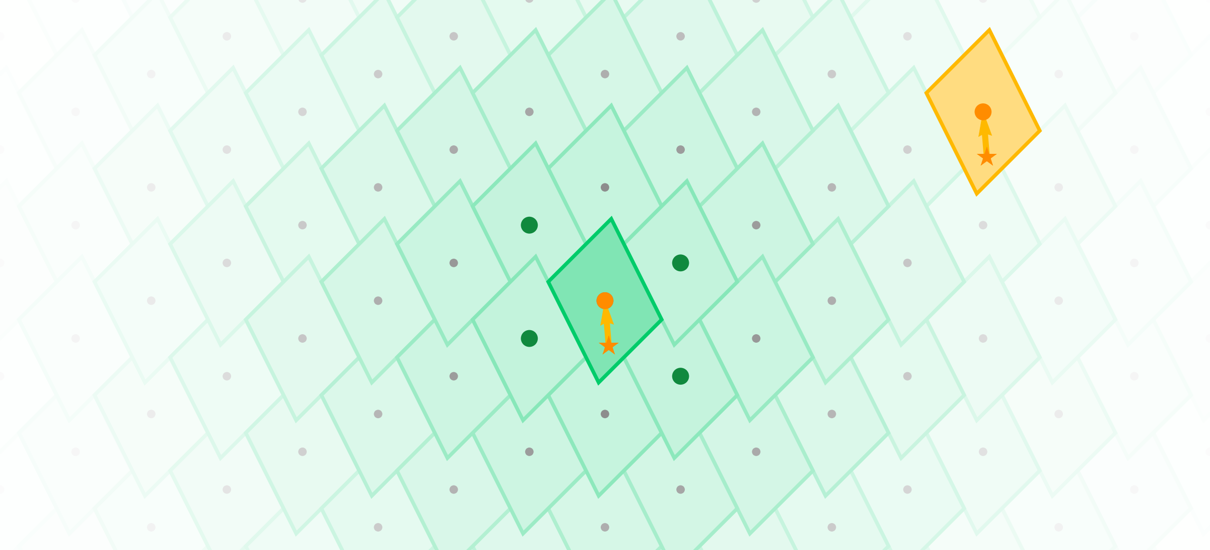


Approximate Voronoi cells

Estimating the volume [Laa16, DLW19]

Lemma (Good approximations, with heuristics)

Let L consist of the $\alpha^{n+o(n)}$ shortest vectors of a lattice $\mathcal{L}$, with $\alpha \geq \sqrt{2} + o(1)$. Then:

$$\frac{\operatorname{vol}(\mathcal{V}_L)}{\operatorname{vol}(\mathcal{V})} = 1 + o(1). \tag{1}$$

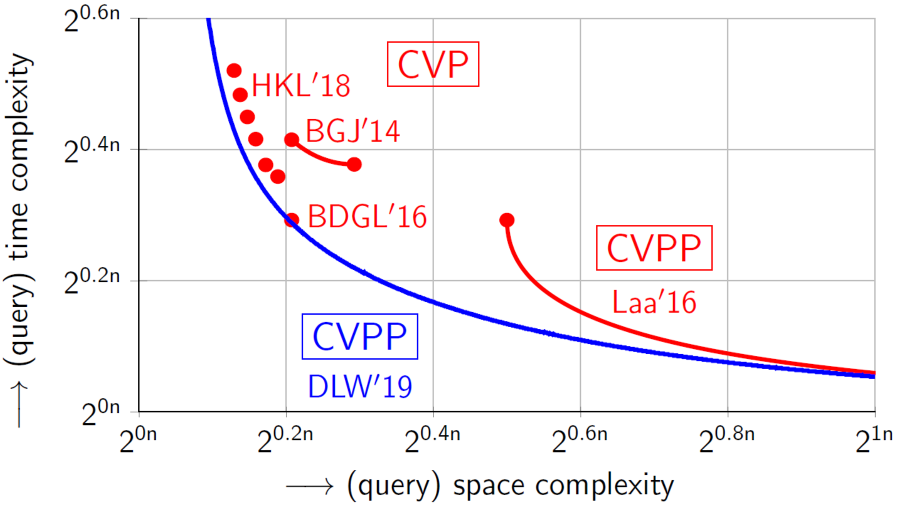
Lemma (Arbitrary approximations, with heuristics)

Let L consist of the $\alpha^{n+o(n)}$ shortest vectors of a lattice $\mathcal{L}$, with $\alpha \in (1.03396, \sqrt{2})$. Then:

$$\frac{\operatorname{vol}(\mathcal{V}_L)}{\operatorname{vol}(\mathcal{V})} \leq \left(\frac{16\alpha^4 \left(\alpha^2 - 1\right)}{-9\alpha^8 + 64\alpha^6 - 104\alpha^4 + 64\alpha^2 - 16} \right)^{n/2+o(n)}. \tag{2}$$

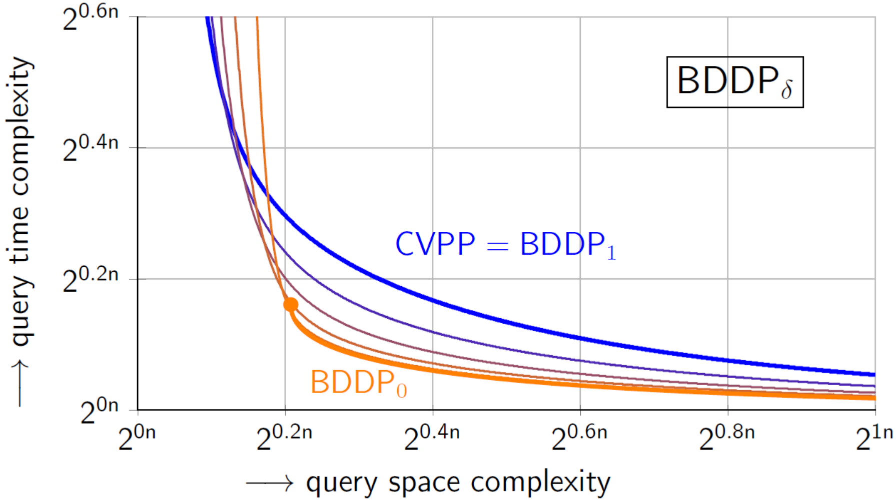
Approximate Voronoi cells

Results for CVPP



Approximate Voronoi cells

Results for BDDP



Approximate Voronoi cells

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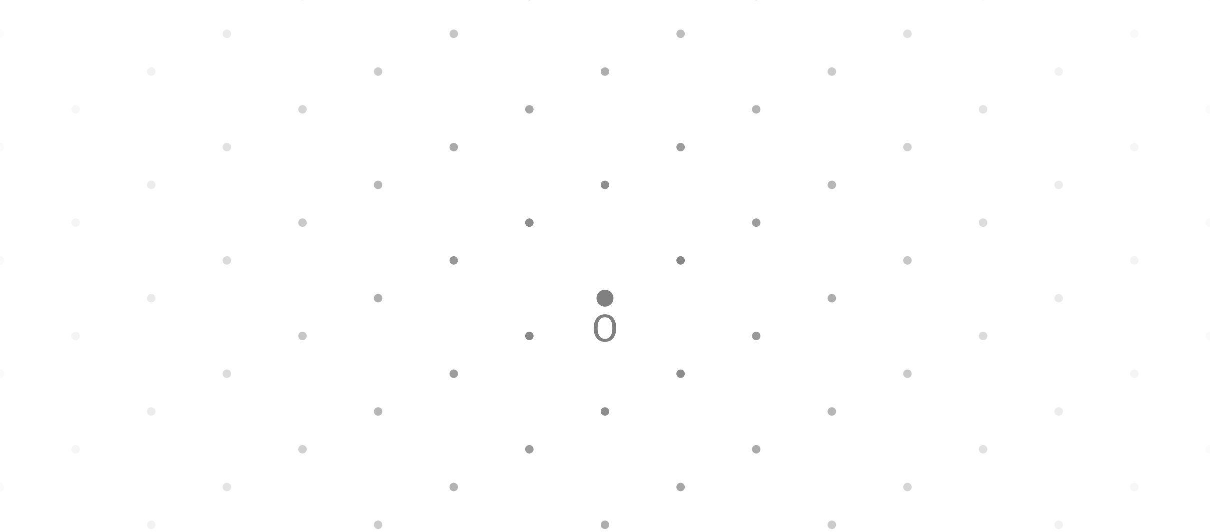
- *Limitations*: does not scale well for BDDP instances

Dual approach

Dual lattices



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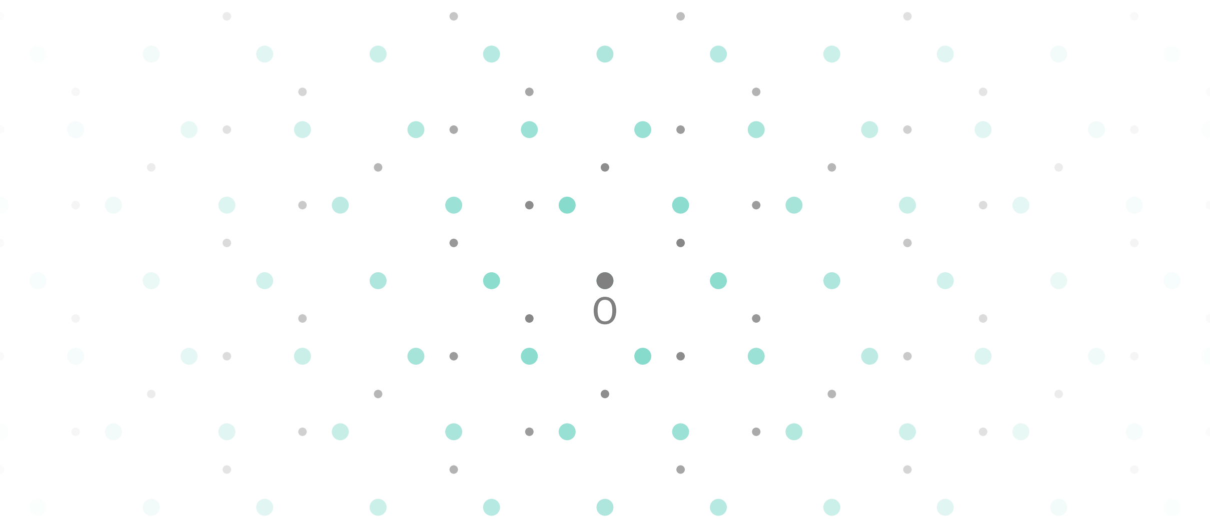

Dual approach

Distinguisher

$\mathcal{L}^* = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{x}, \boldsymbol{v} \rangle \in \mathbb{Z}, \forall\, \boldsymbol{v} \in \mathcal{L}\}$

- Primal target vector $\boldsymbol{t} = \boldsymbol{v} + \boldsymbol{e}$ with $\boldsymbol{v} \in \mathcal{L}$
- Short dual vector $\boldsymbol{v}^* \in \mathcal{L}^*$
- Distinguisher:

$$\begin{cases} \langle \boldsymbol{t}, \boldsymbol{v}^* \rangle \mod 1 = 0 & \text{if } \|\boldsymbol{e}\| = 0; \\ \langle \boldsymbol{t}, \boldsymbol{v}^* \rangle \mod 1 \approx 0 & \text{if } \|\boldsymbol{e}\| \approx 0 \text{ and } \|\boldsymbol{v}^*\| \text{ small}; \\ \langle \boldsymbol{t}, \boldsymbol{v}^* \rangle \mod 1 \sim U(-\frac{1}{2}, \frac{1}{2}) & \text{if } \|\boldsymbol{e}\| \gg 0. \end{cases}$$

Dual approach

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Dual approach

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- *Query*: distinguish based on dot products modulo 1
- *Strengths*: smooth trade-offs for BDDP
- *Limitations*: traditionally only solves decisional BDD(P)

Conclusion

Summary

Babai's algorithms

- Fast and simple algorithms
- Targets must lie close to the lattice

Voronoi cells

- Provable, deterministic algorithm
- Requires $2^{n+o(n)}$ time and space

Approximate Voronoi cells

- Heuristic alternative to exact Voronoi cells
- Nearest neighbor speed-ups
- Does not scale well for BDDP

Dual approach

- Distinguisher using short dual vectors
- Works better when target is somewhat close to lattice
- Traditionally only solves decisional problem

Conclusion

Open problems / Work in progress

Approximate Voronoi cells

- Eliminate lower bound on space complexity
- Improve upper bound on volume ratio
- Apply other nearest neighbor techniques

Dual approach

- Analyze method heuristically
- Efficient conversion to search-CVPP
- Find cross-over point with other methods