Technische Universiteit
Eindhoven
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Approximate Voronoi cells for lattices, revisited

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(September 6, 2019)

Lattices

Basics

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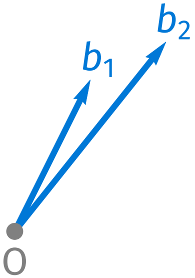

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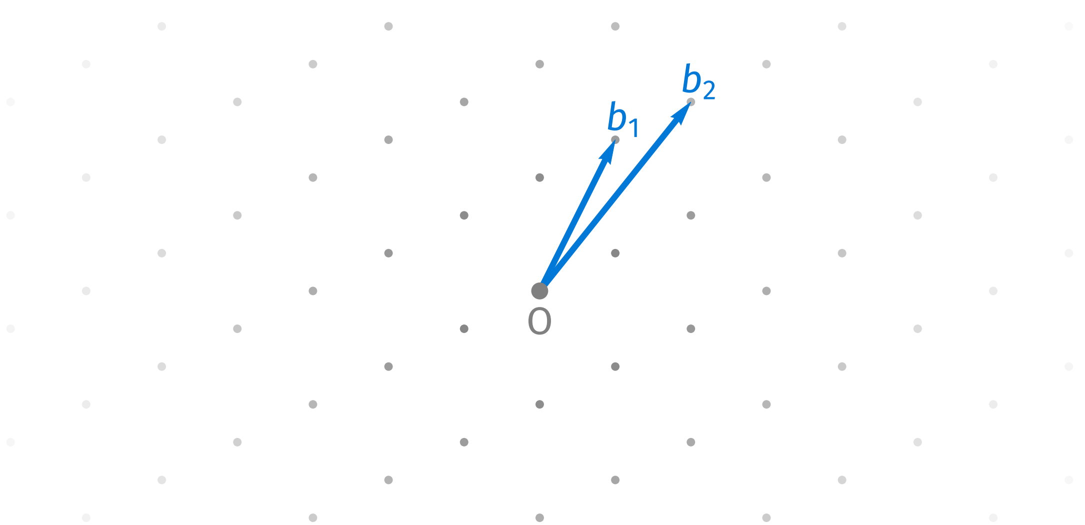
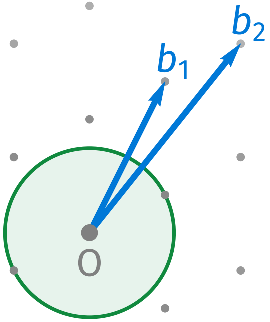


Lattice problems

Shortest Vector Problem (SVP)



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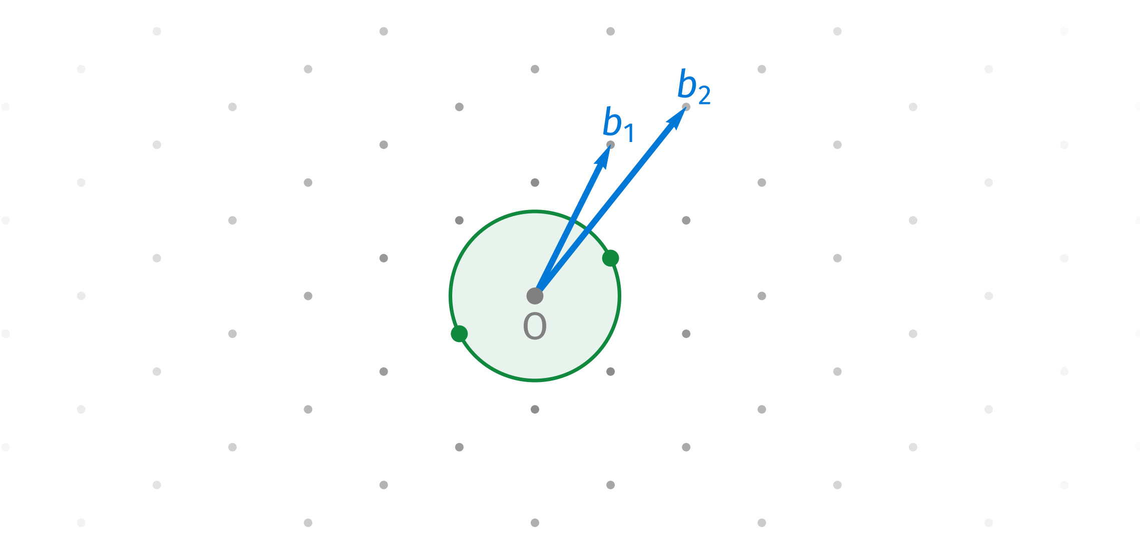

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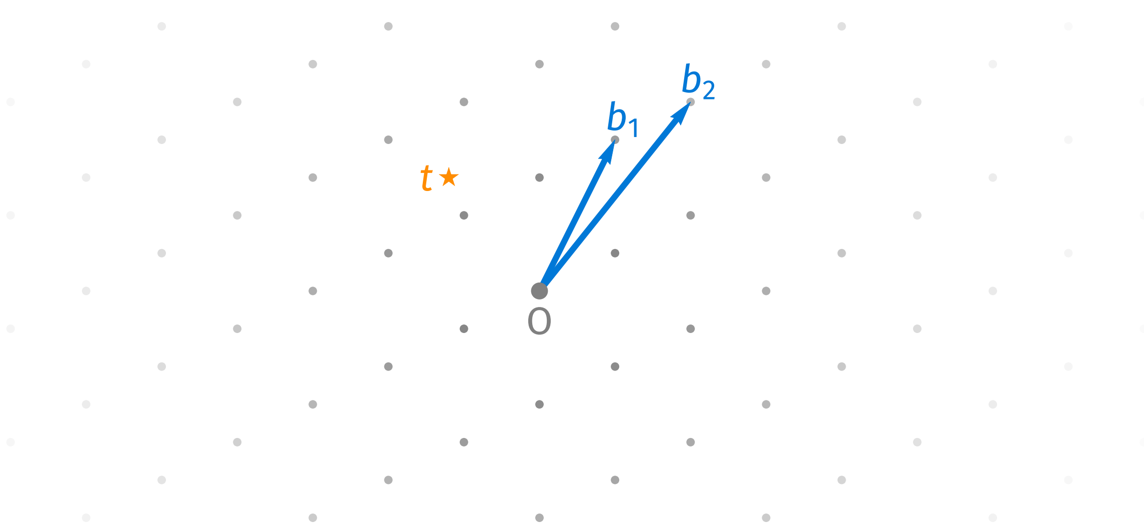

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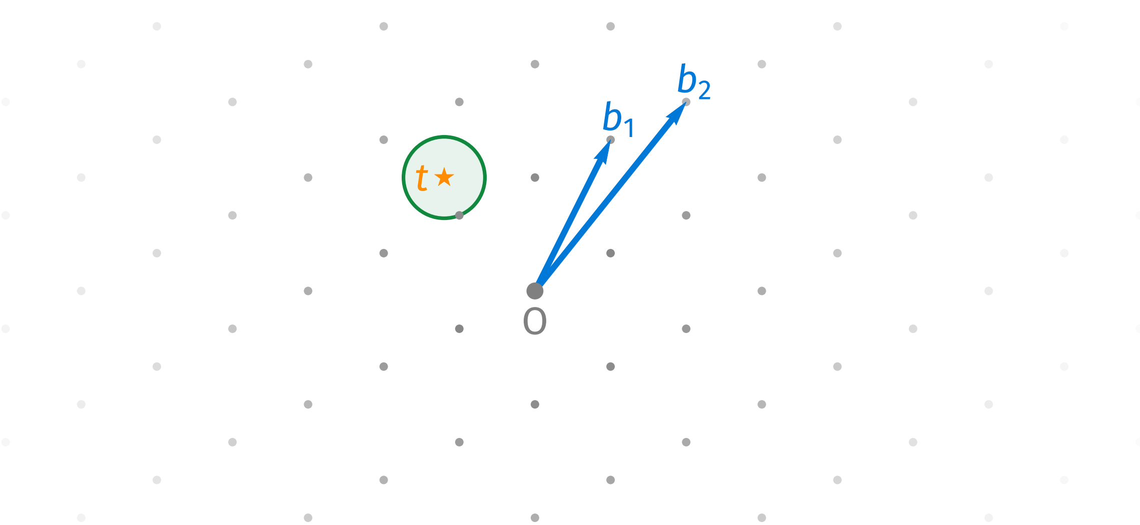

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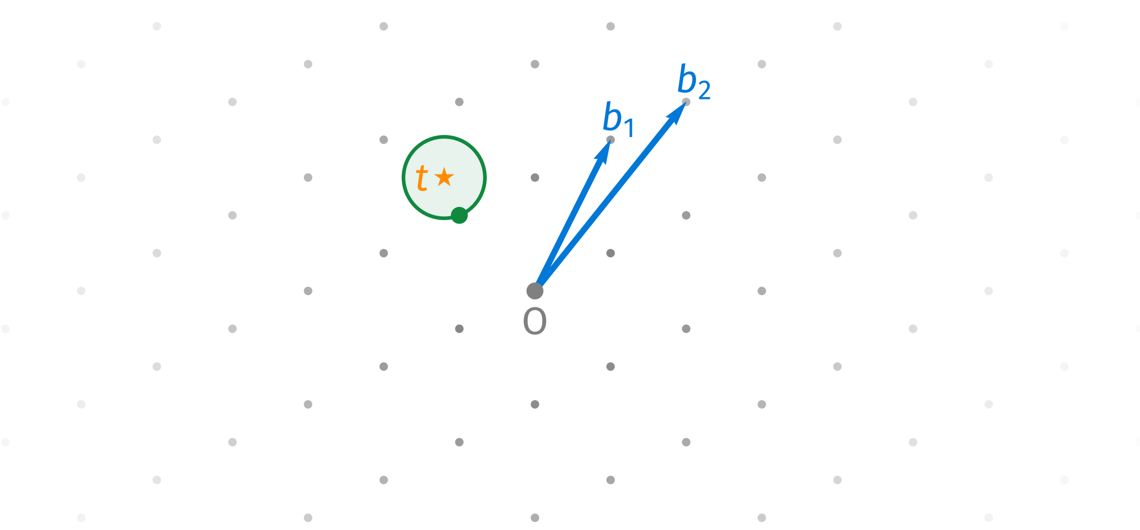


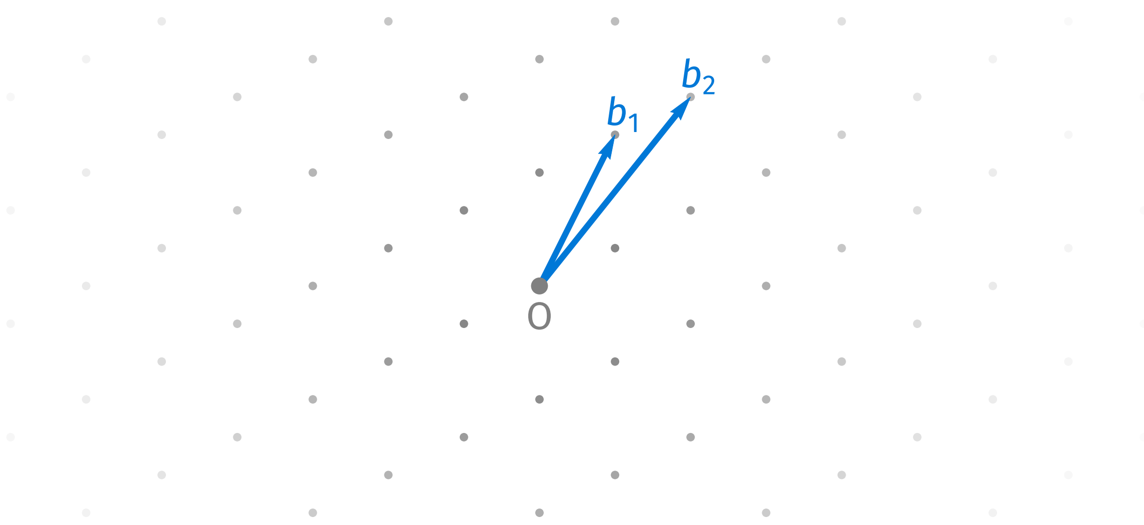
Lattice problems

Asymptotics for SVP and CVP

	Algorithm	$\log_2$(Time)	$\log_2$(Space)	Experiments
Worst-case SVP	Enumeration [Poh81, Kan83, FP85, . . .]	$O(n \log n)$	$O(\log n)$	–
	AKS-sieve [AKS01, NV08, MV10, HPS11]	$3.398n$	$1.985n$	–
	Birthday sieves [PS09, HPS11]	$2.465n$	$1.233n$	–
	Enumeration/DGS hybrid [CCL17]	$2.048n$	$0.500n$	–
	Voronoi cell algorithm [AEVZ02, MV10b, BD15]	$2.000n$	$1.000n$	40
	Quantum sieve [LMP13, LMP15]	$1.799n$	$1.286n$	–
	Quantum enumeration/DGS hybrid [CCL17]	$1.256n$	**0.500n**	–
	Discrete Gaussian sampling [ADRS15, ADS15, AS18]	**1.000n**	$1.000n$	–
Average-case SVP	Enumeration [GNR10, MW15, AN17]	$O(n \log n)$	$O(\log n)$	**152**
	The Nguyen–Vidick sieve [NV08]	$0.415n$	$0.208n$	50
	GaussSieve [MV10, . . . , IKMT14, BNvdP16, YKYC17]	$0.415n$	$0.208n$	130*
	Triple sieve [BLS16, HK17]	$0.396n$	$0.189n$	80
	Kleinjung sieve [Kle14]	$0.379n$	$0.189n$	116
	Leveled sieving [WLTB11, ZPH13]	$0.378n$	$0.283n$	–
	Overlattice sieve [BGJ14]	$0.377n$	$0.293n$	90
	Triple sieve with NNS [HK17, HKL18]	$0.359n$	**0.189n**	76
	Single filters [DL17, ADH+19]	$0.349n$	$0.246n$	155
	Hyperplane LSH [Cha02, FBB+14, Laa15, . . . , LM18]	$0.337n$	$0.337n$	107
	May–Ozerov NNS [MO15, BGJ15]	$0.311n$	$0.311n$	–
	Quantum sieve [LMP13]	$0.311n$	$0.208n$	–
	Spherical LSH [AINR14, LdW15]	$0.297n$	$0.297n$	–
	Cross-polytope LSH [TT07, AILRS15, BL16, KW17]	$0.297n$	$0.297n$	80
	Spherical LSF [BDGL16, MLB17, ALRW17, DSvW19]	**0.292n**	$0.292n$	**157**
	Quantum NNS sieve [LMP15, Laa16]	**0.265n**	$0.265n$	–

Lattice problems

Closest Vector Problem with Preprocessing (CVPP)



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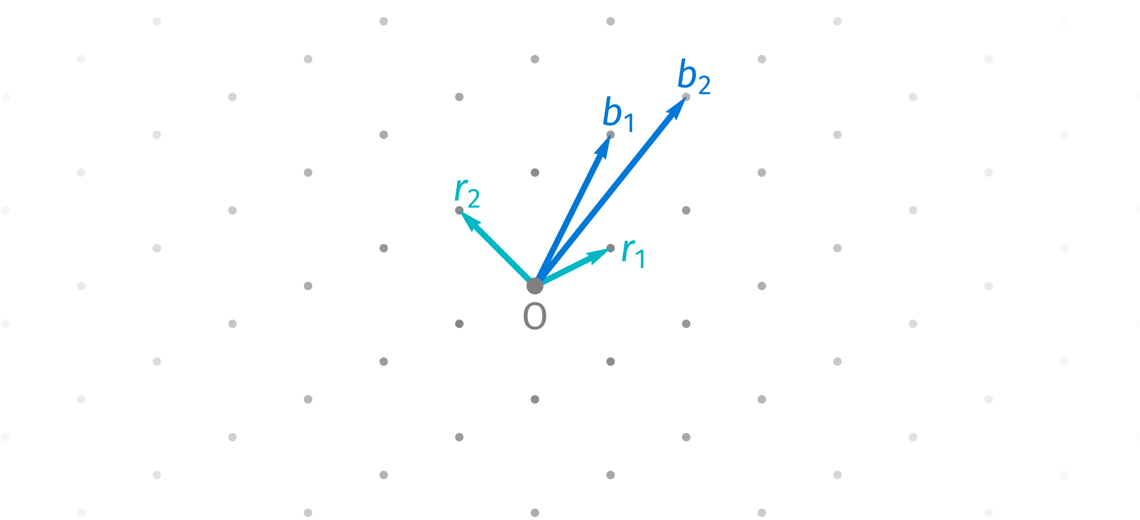

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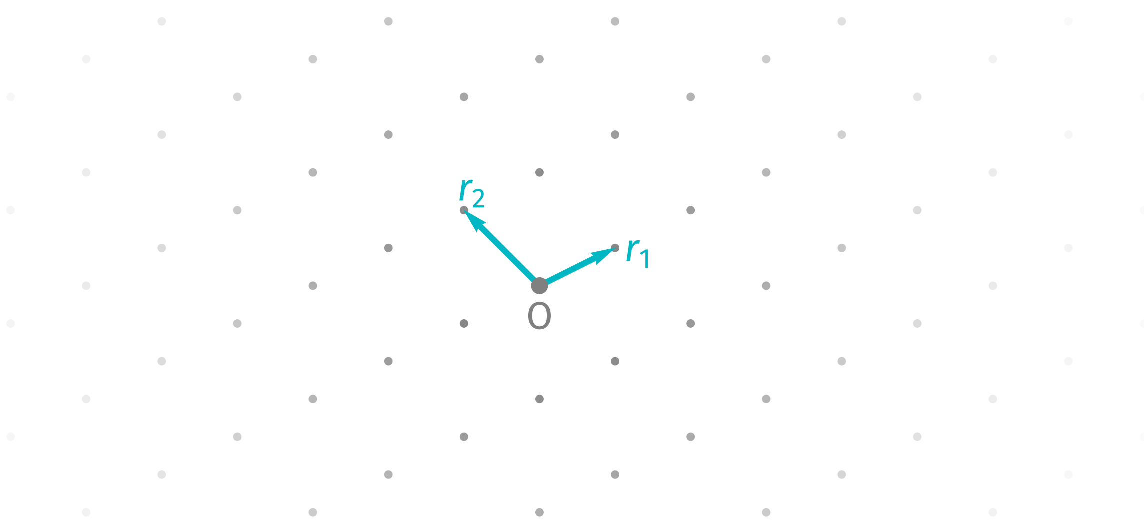

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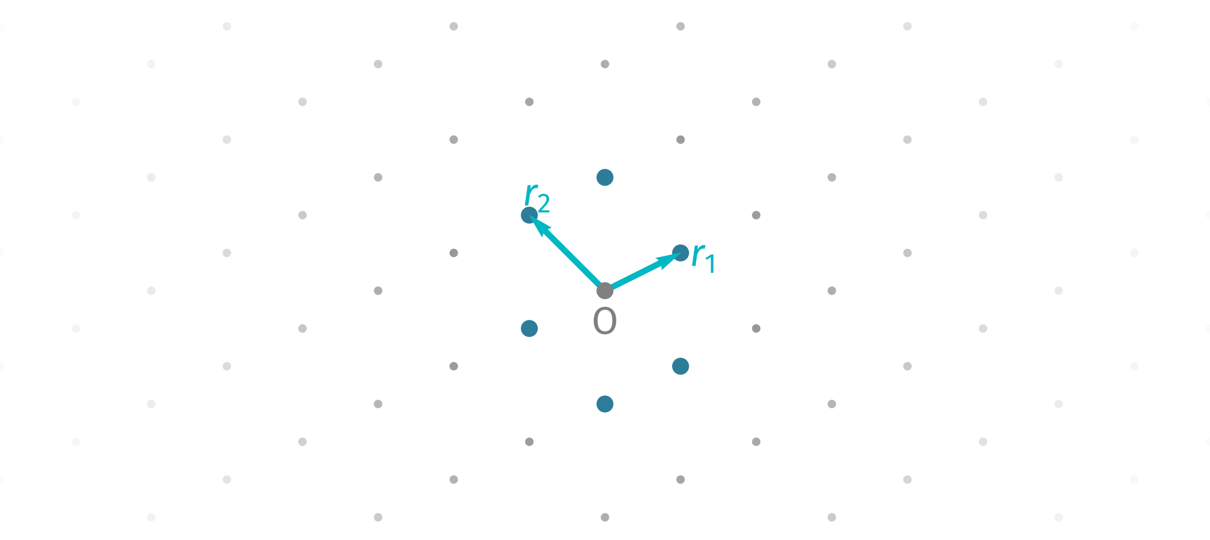

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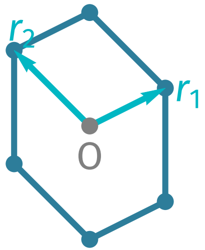

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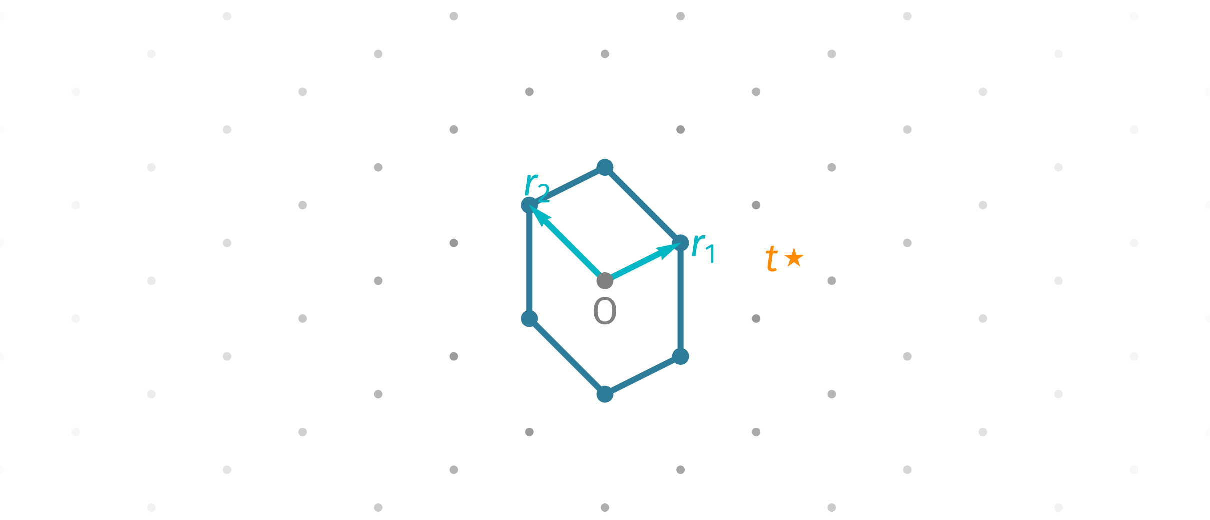

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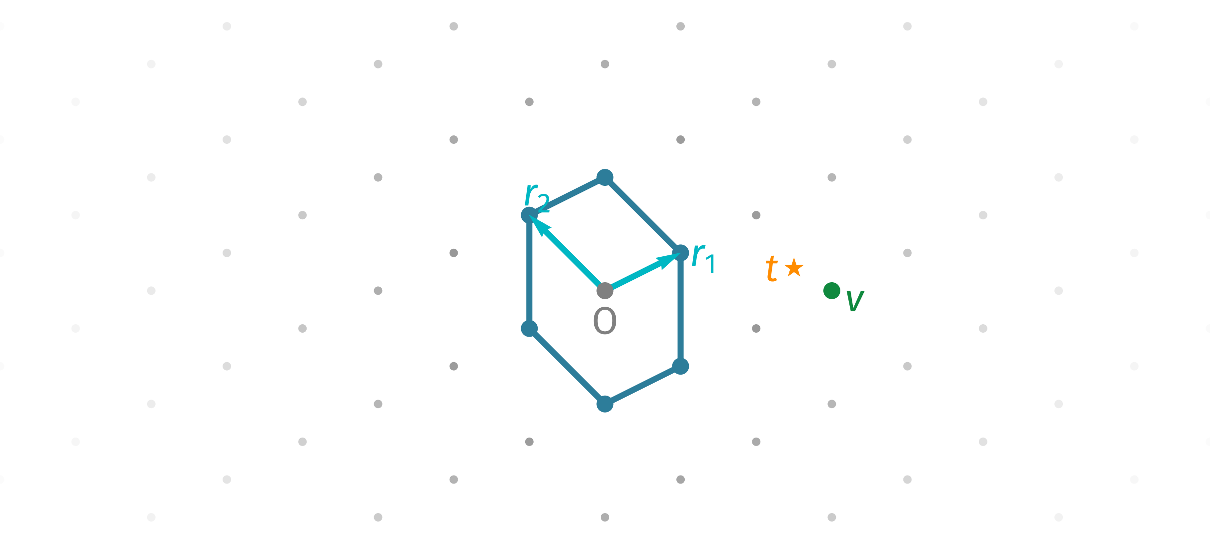

Lattice problems

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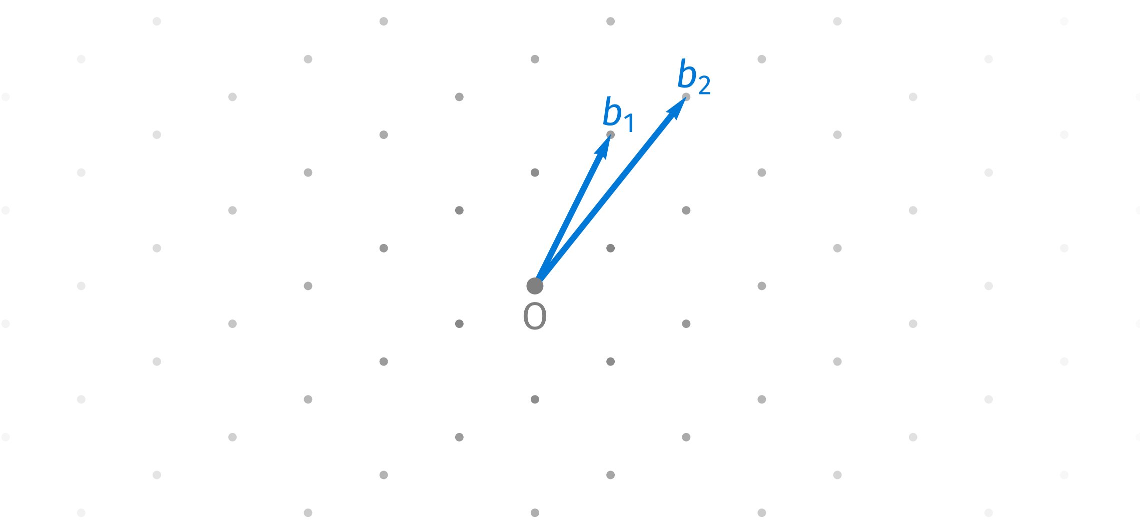

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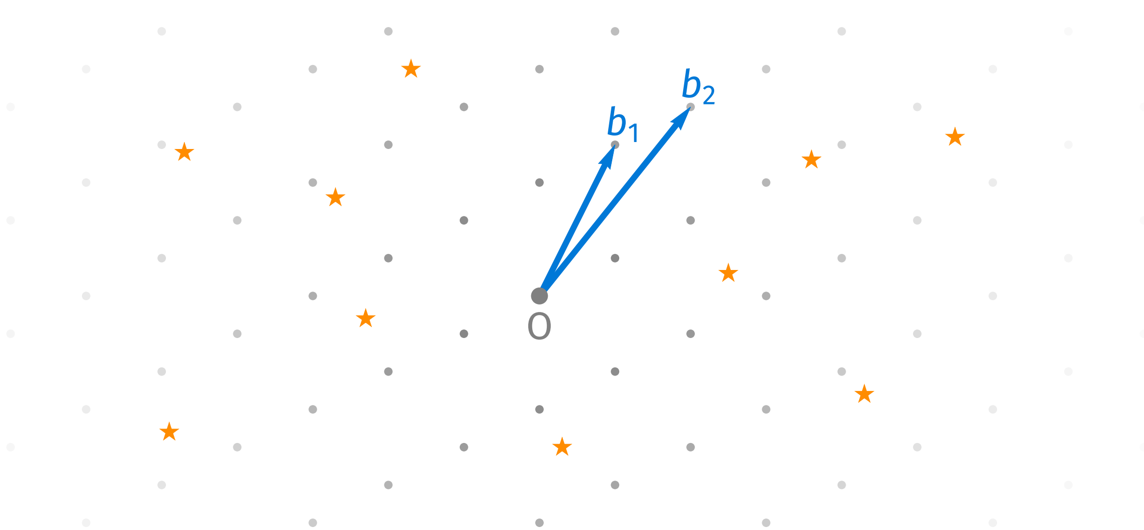

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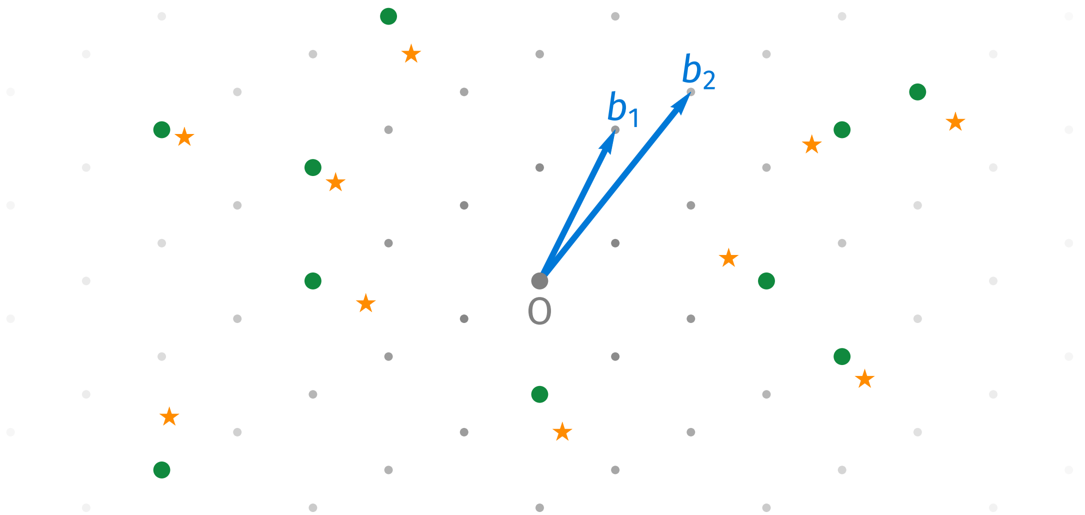


Lattice problems

Why study CVPP?

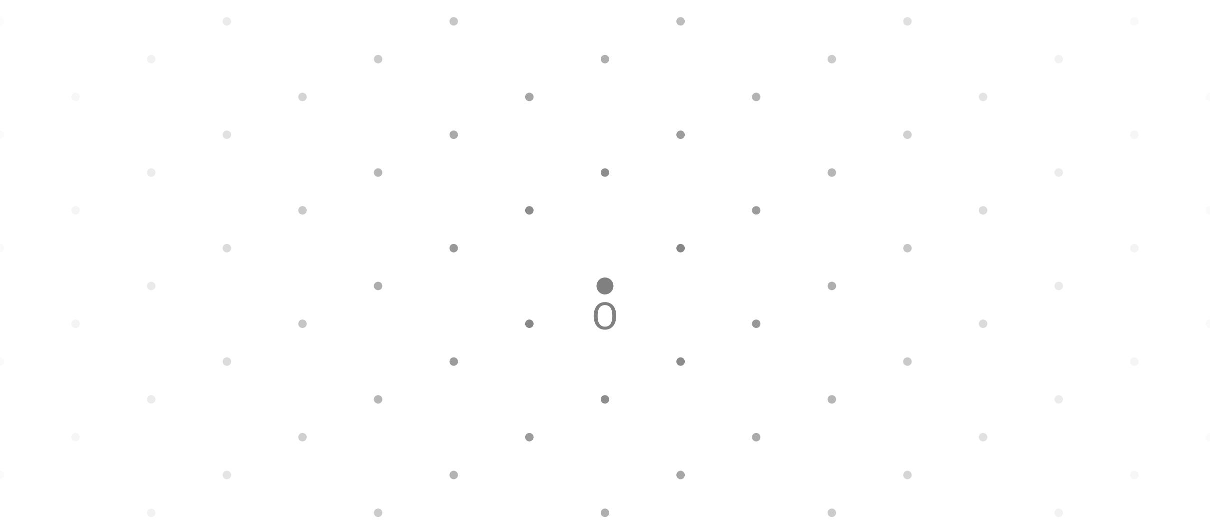
Concrete applications

- Speeding up lattice enumeration for SVP or CVP [GNR10]
- Solving approximate SVP on ideal lattices [PHS19]
- Computing class group actions in a relation lattice [BKV19]

Commonly a lattice basis (public key) is known long before the target vectors (encryptions, signatures)

Voronoi cells

Voronoi tiling



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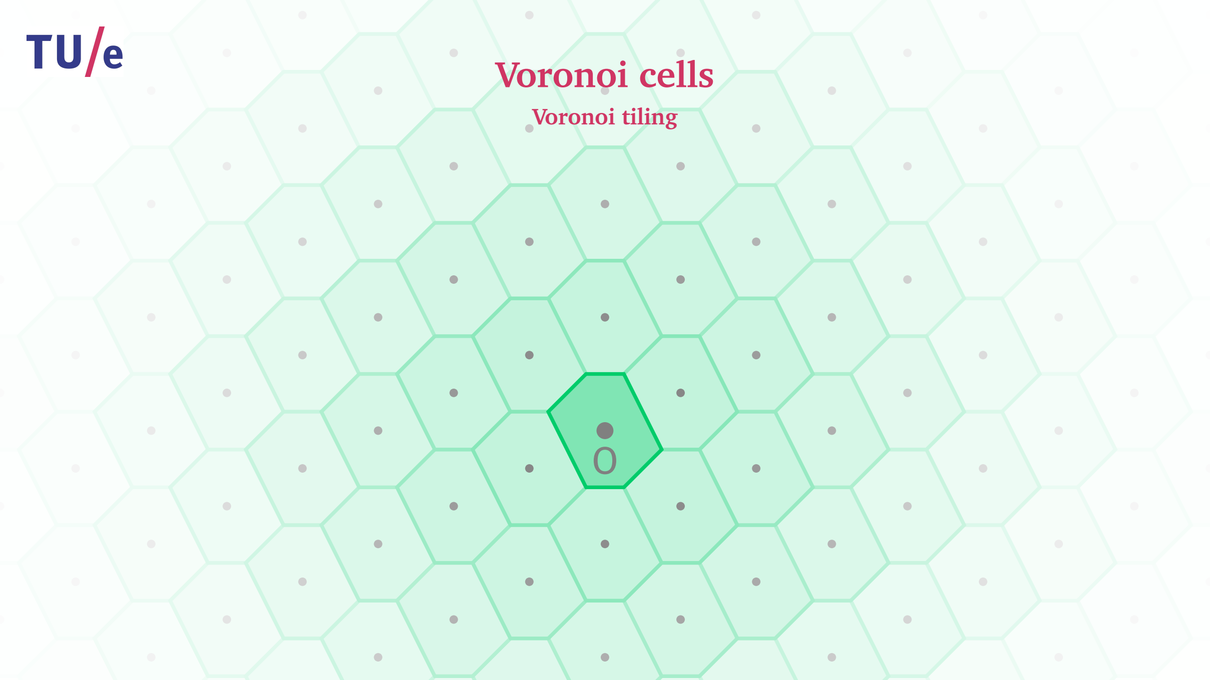
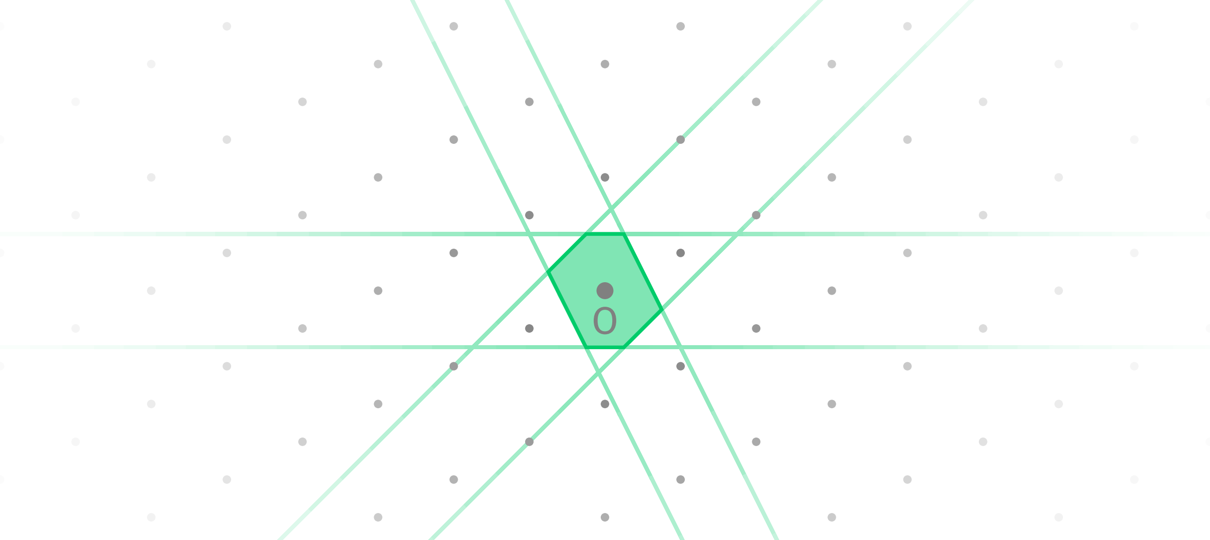


Voronoi cells

Relevant vectors



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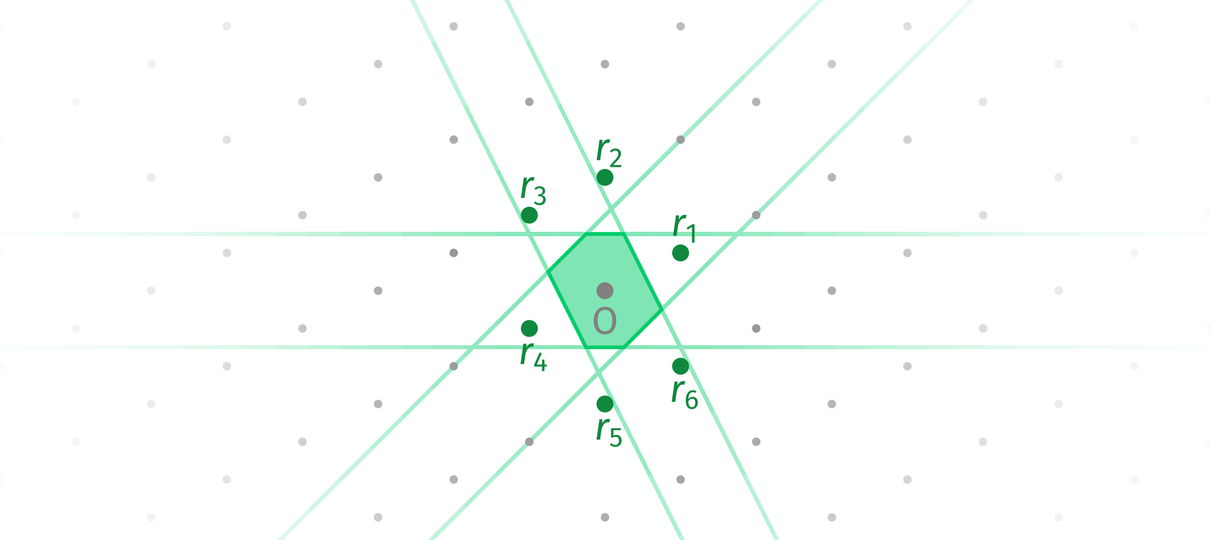

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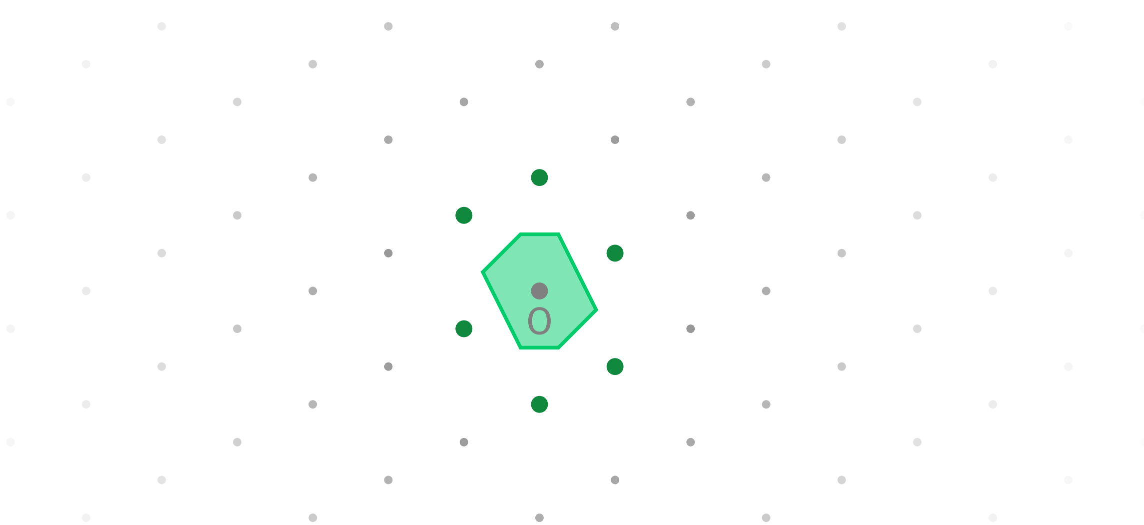

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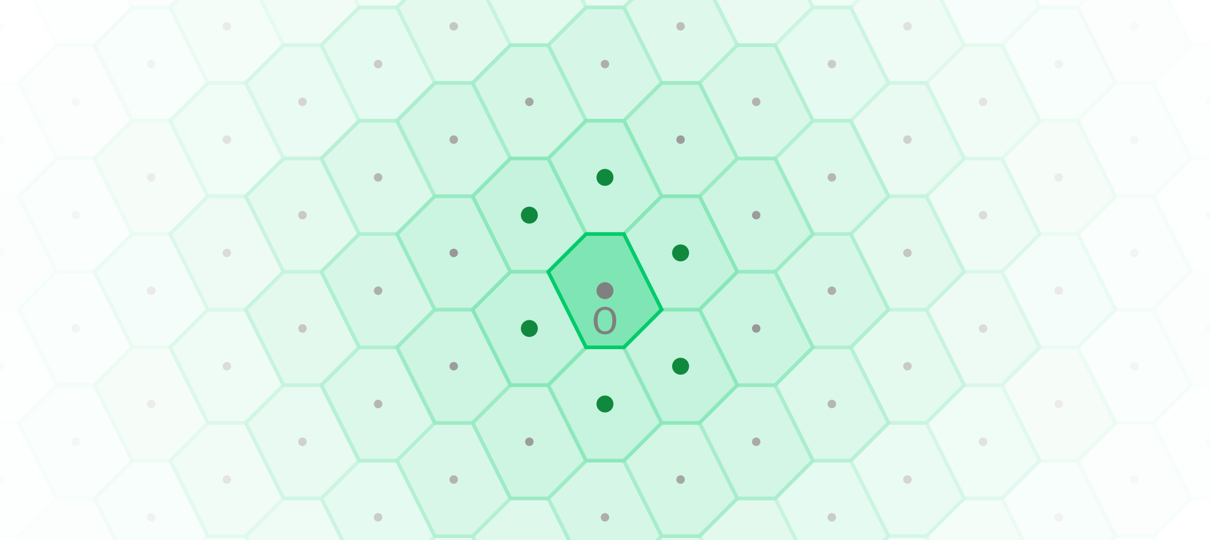
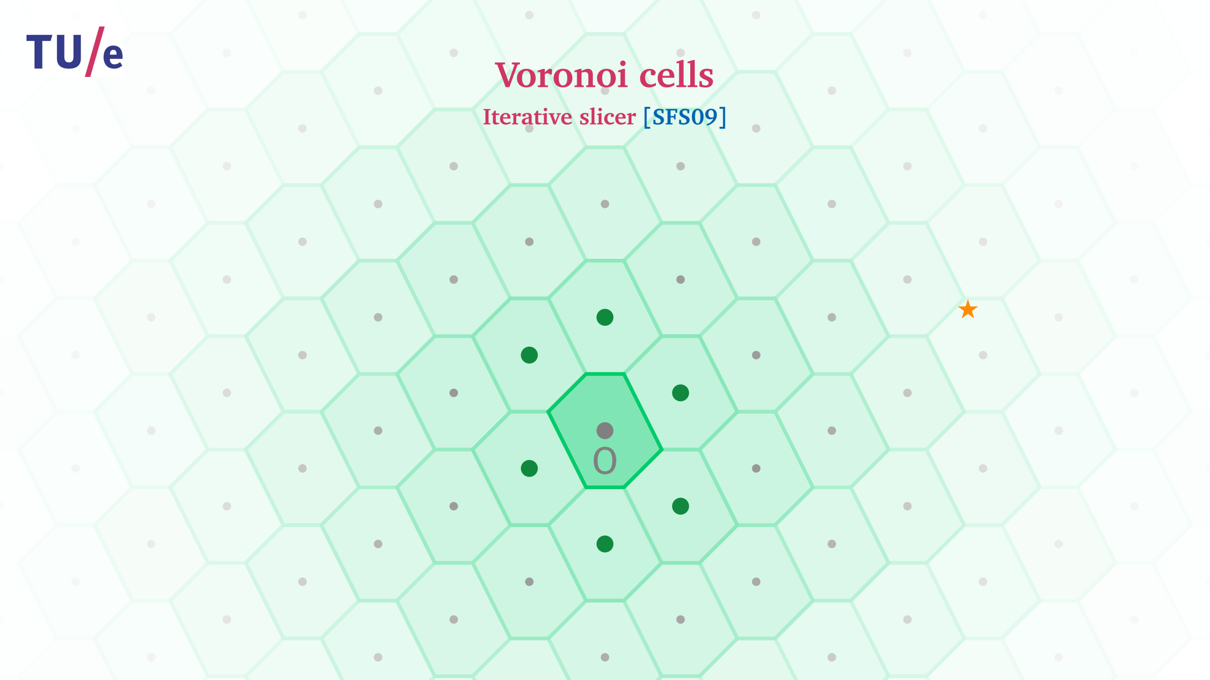


Voronoi cells

Iterative slicer [SFS09]



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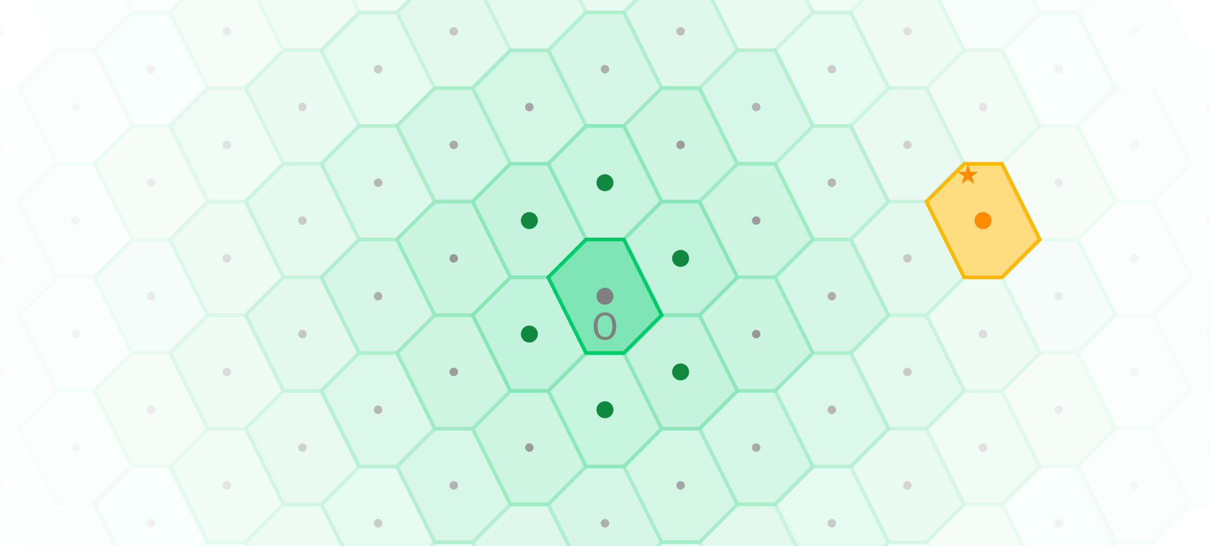

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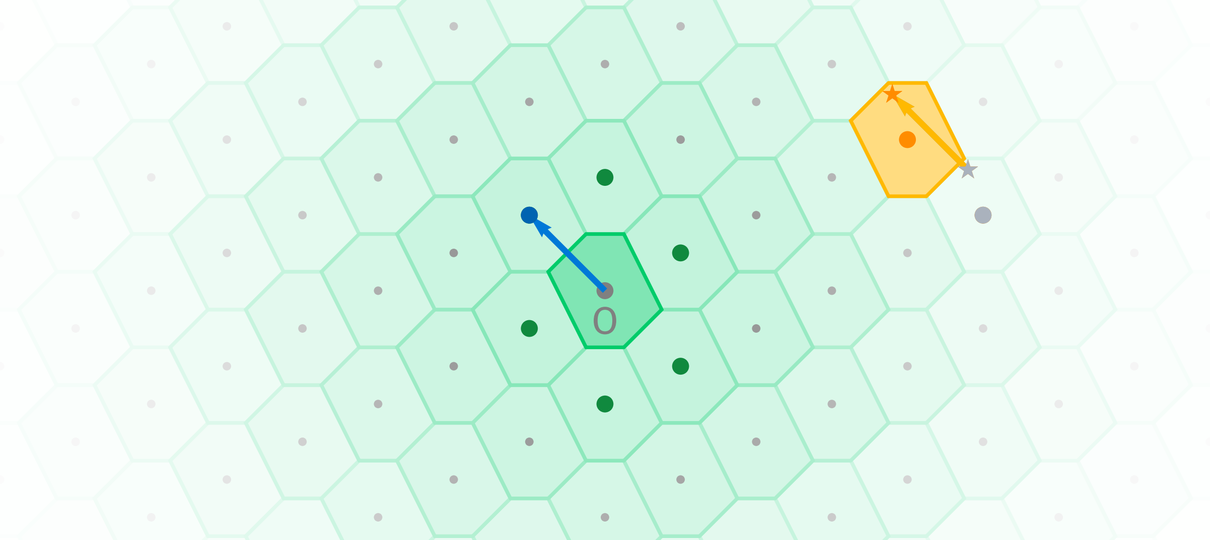

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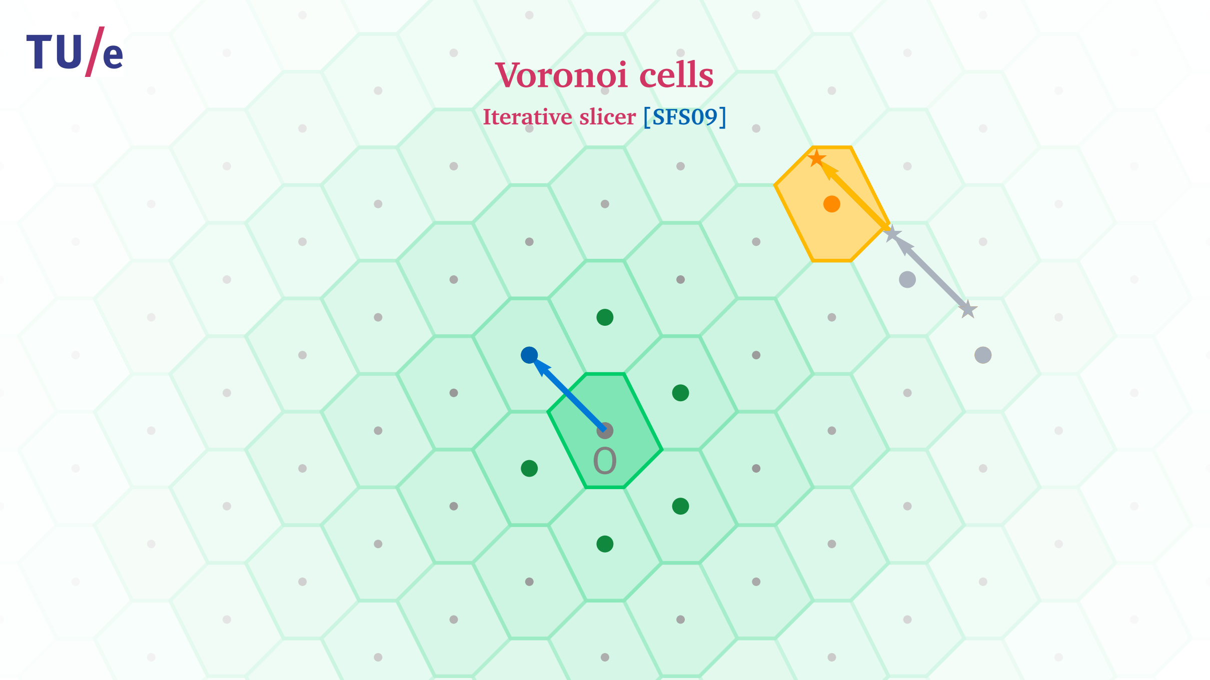

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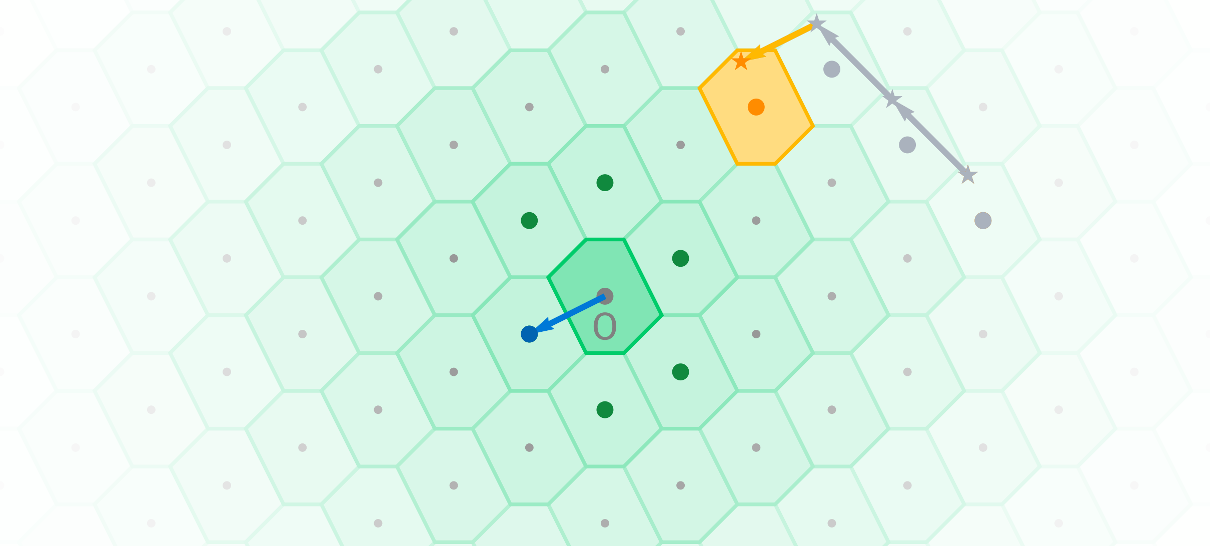

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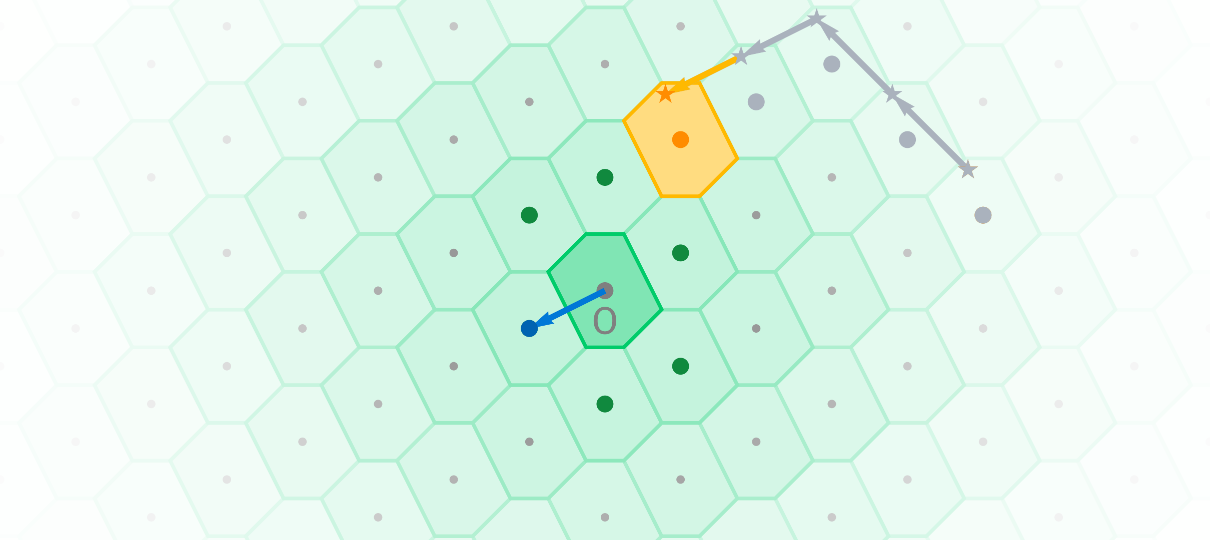

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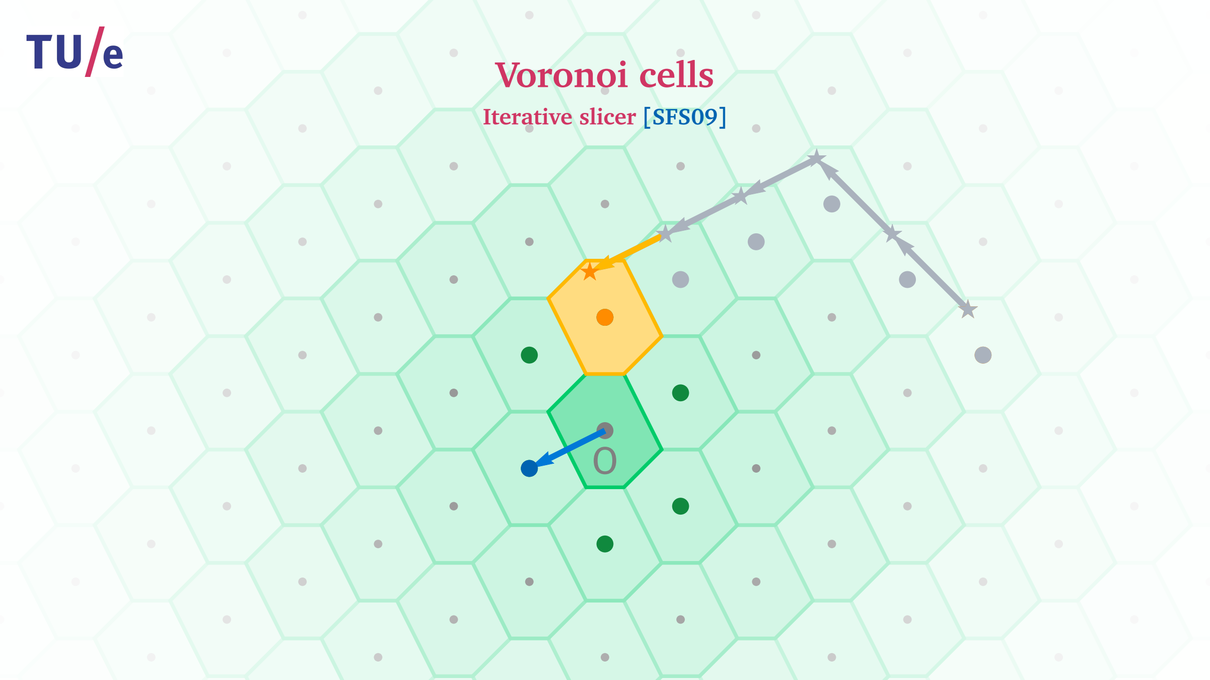

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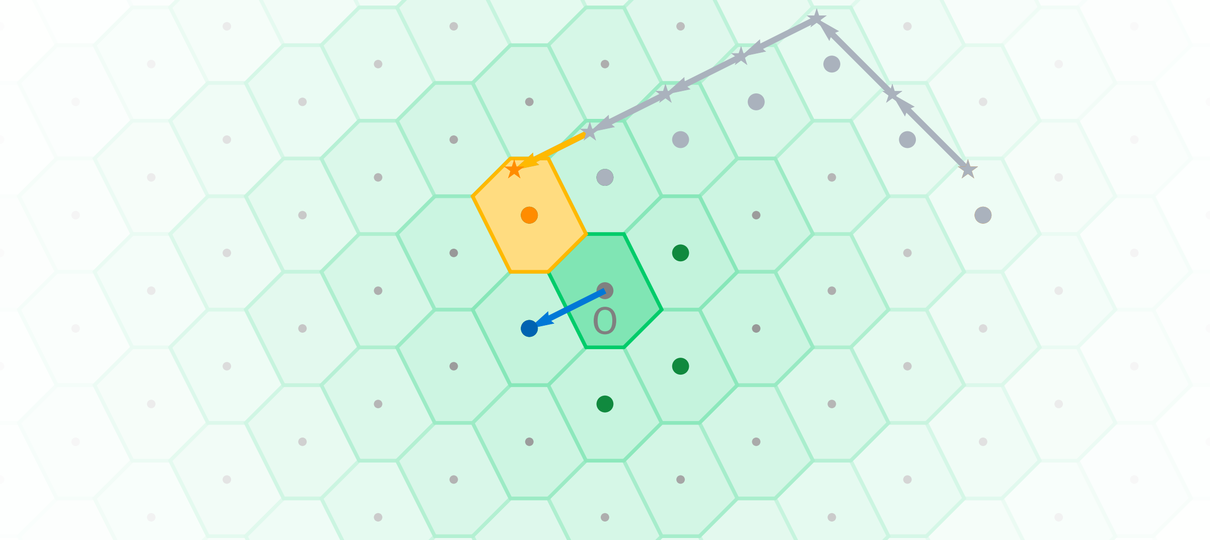

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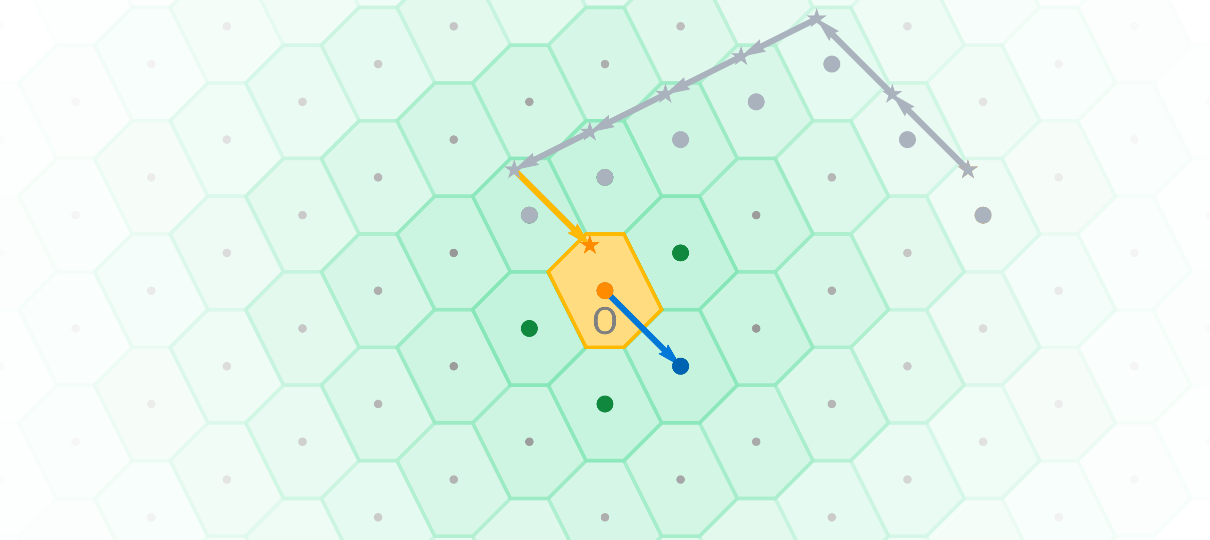

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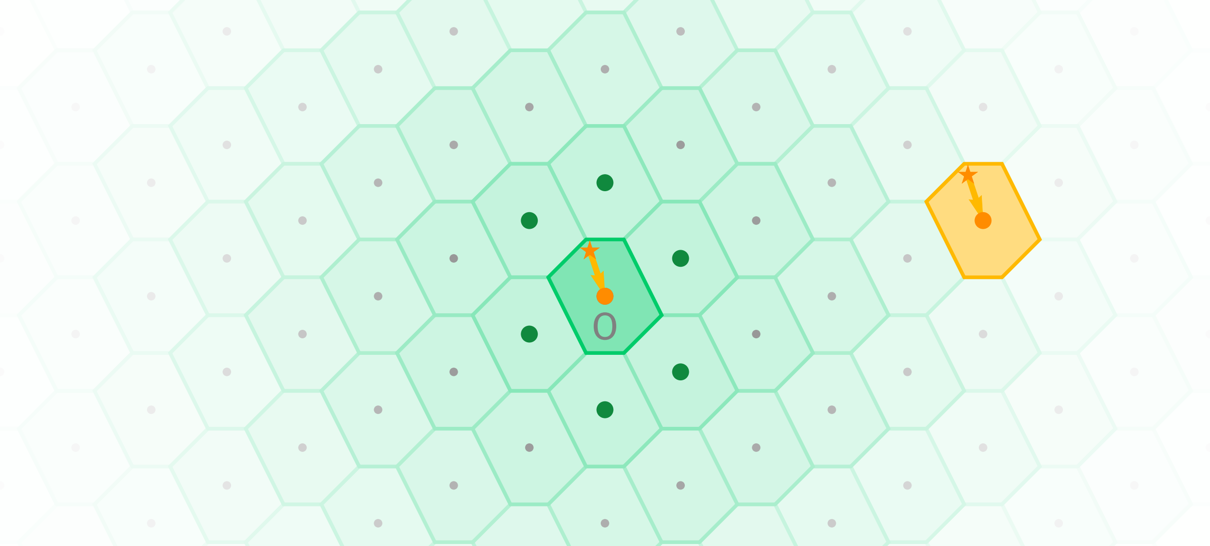
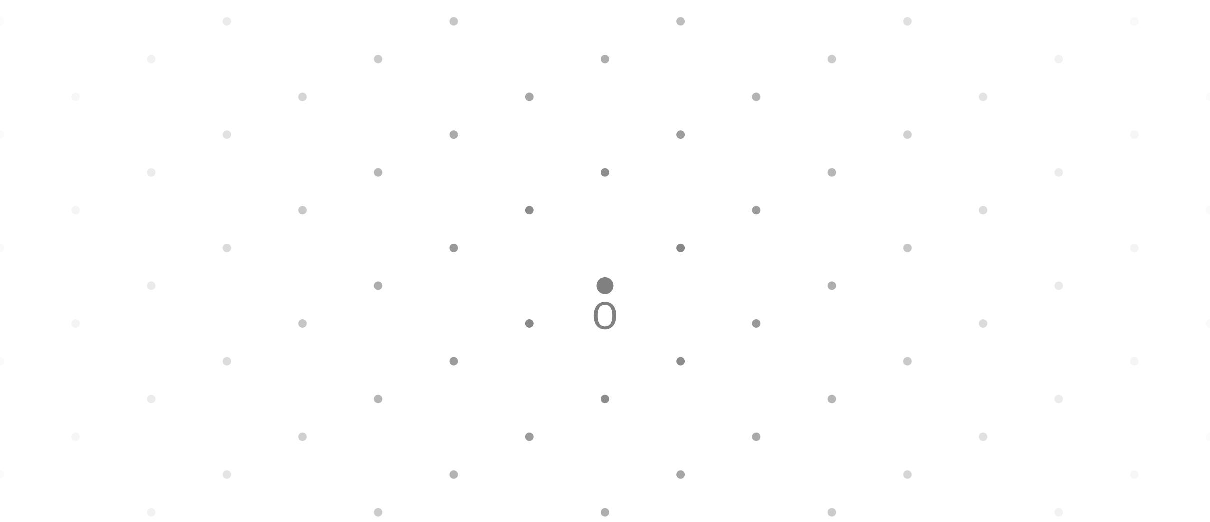


Approximate Voronoi cells

Decrease list size



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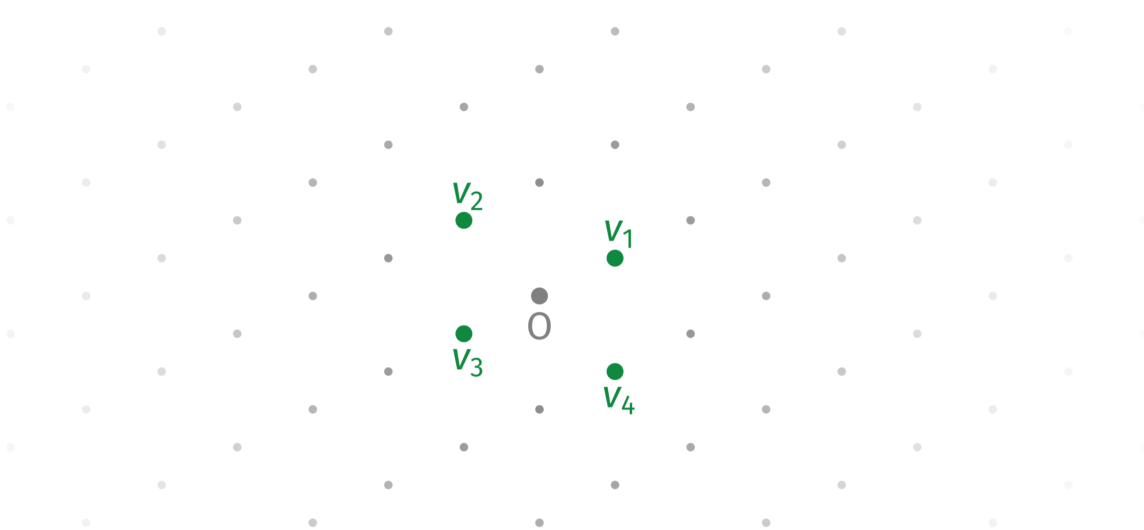

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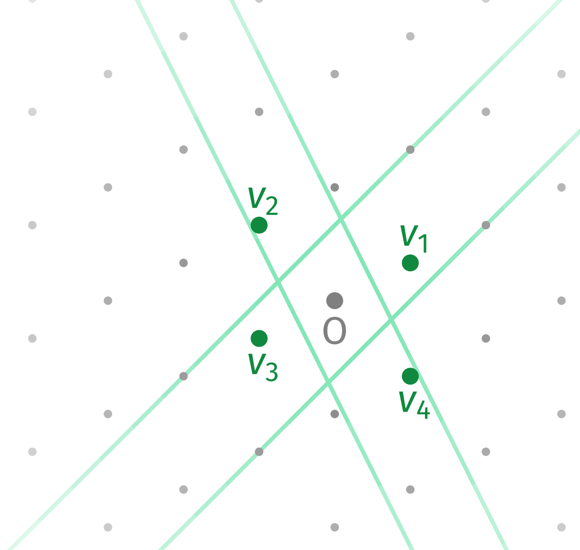

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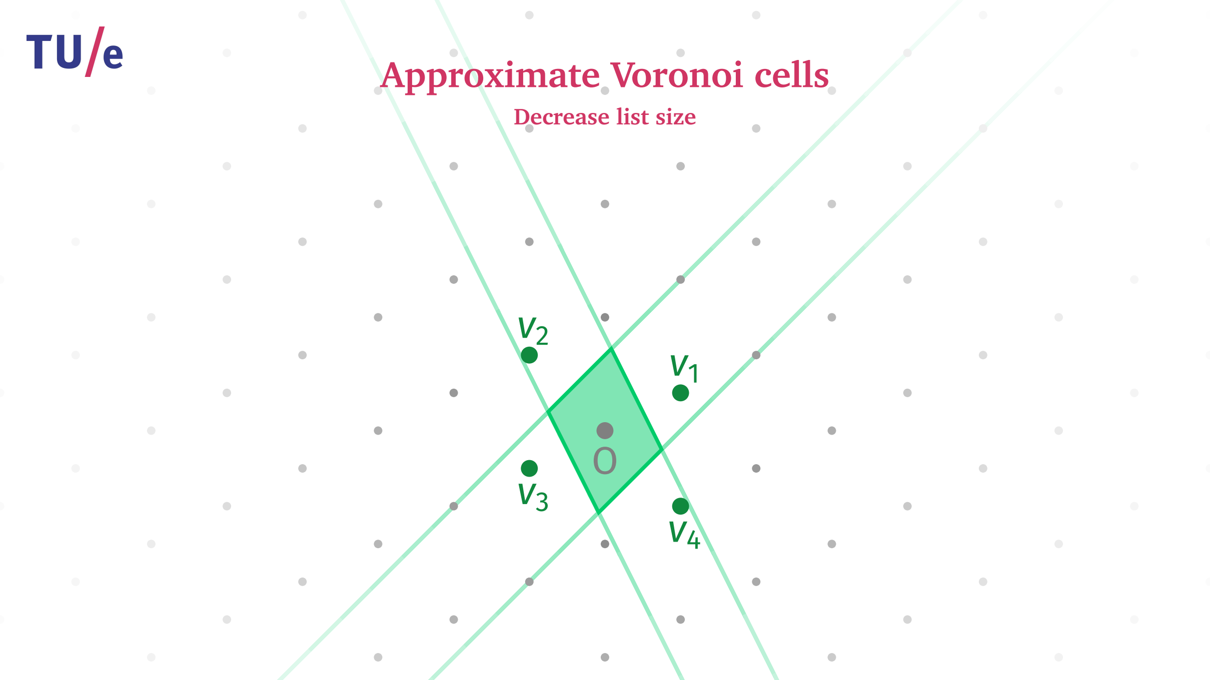

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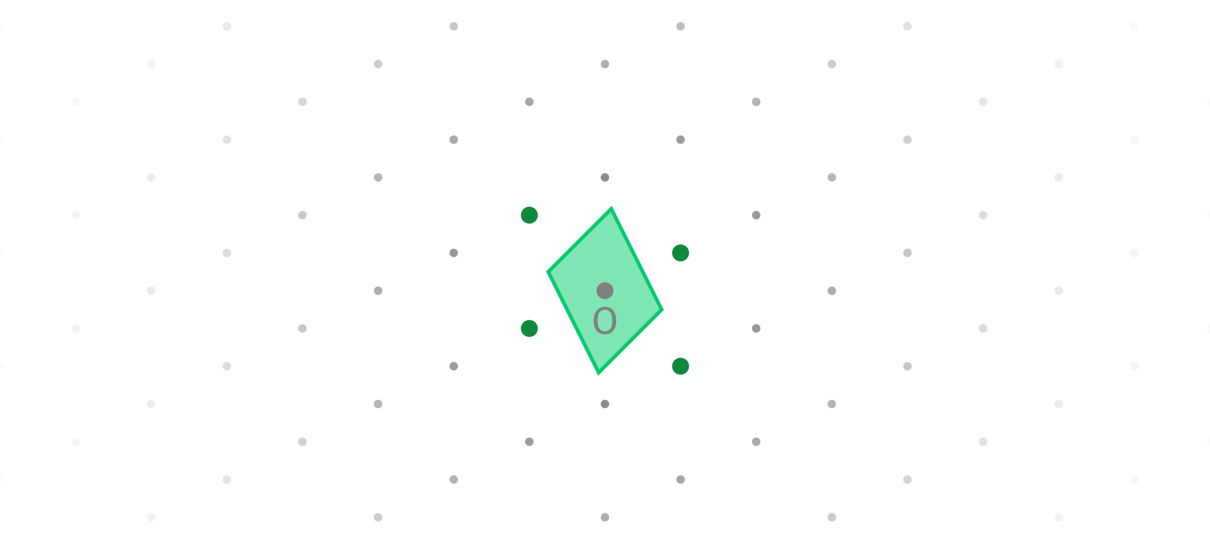

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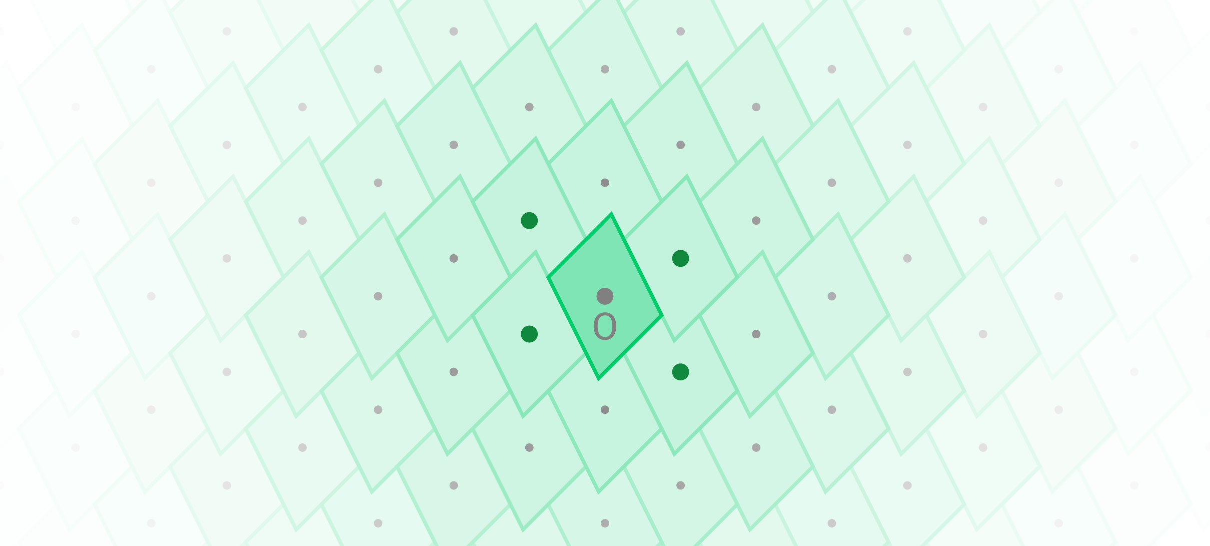
Improper tiling



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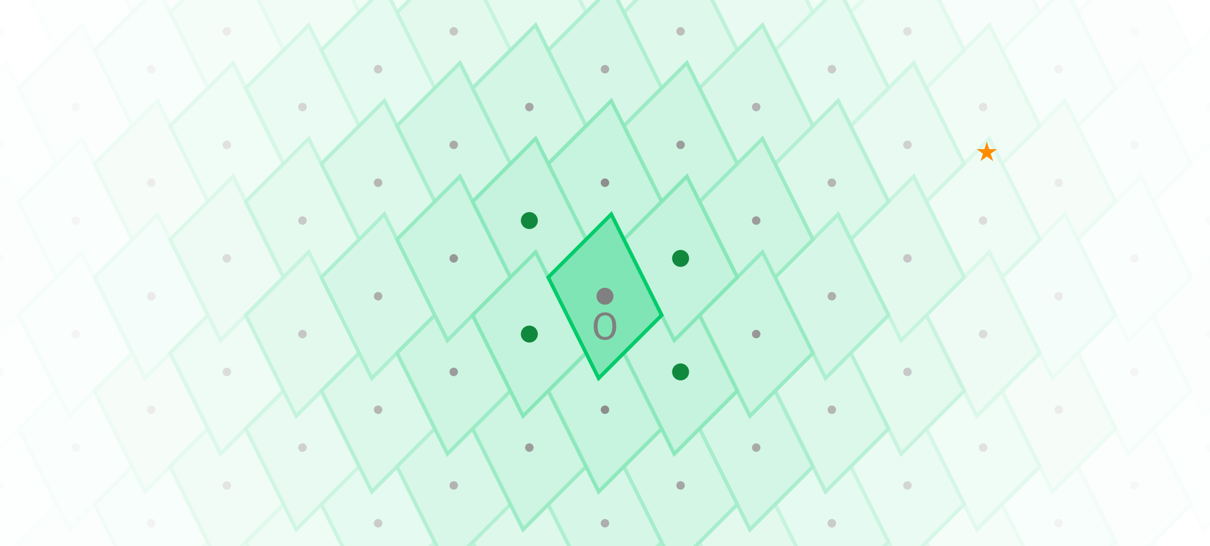

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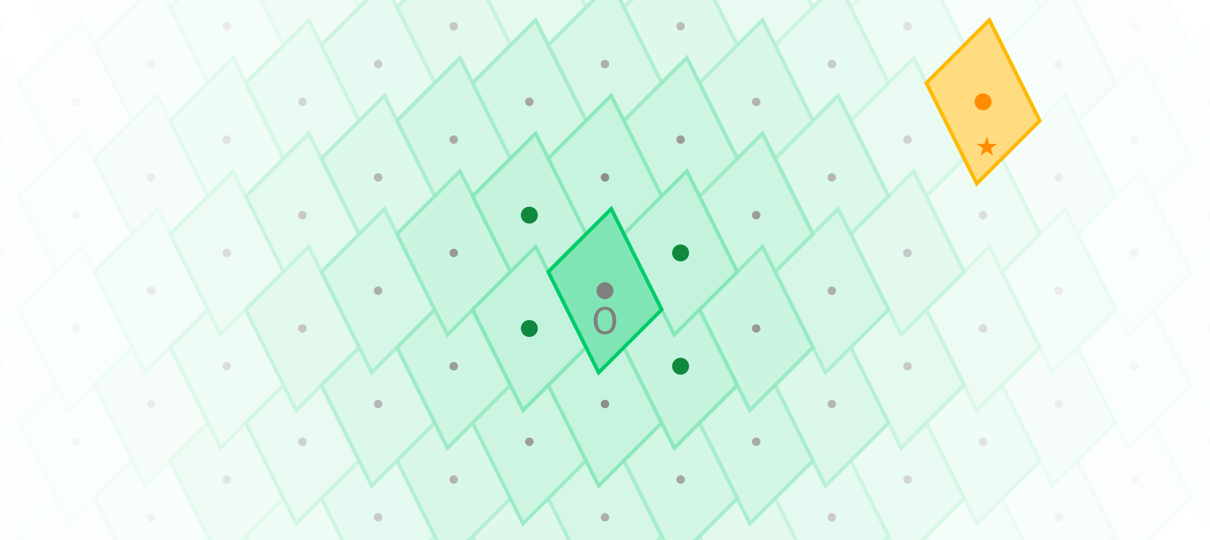

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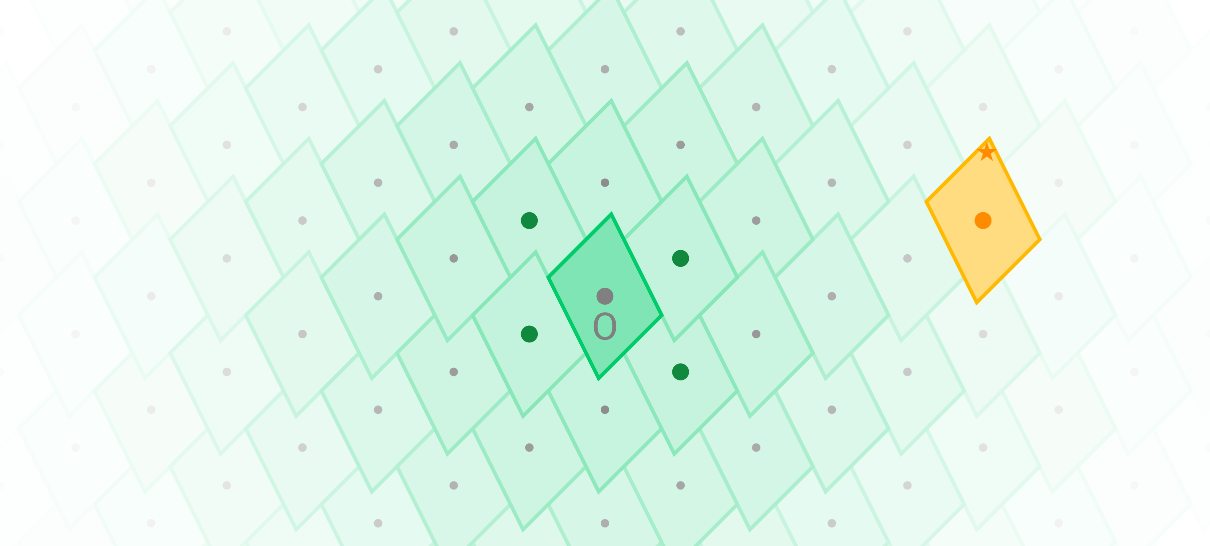

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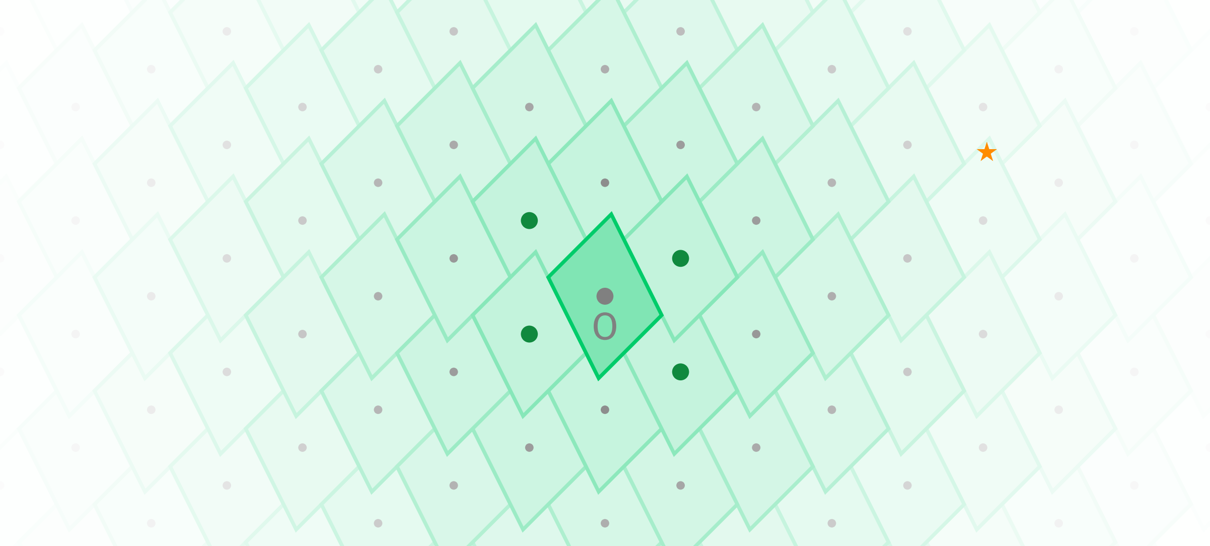

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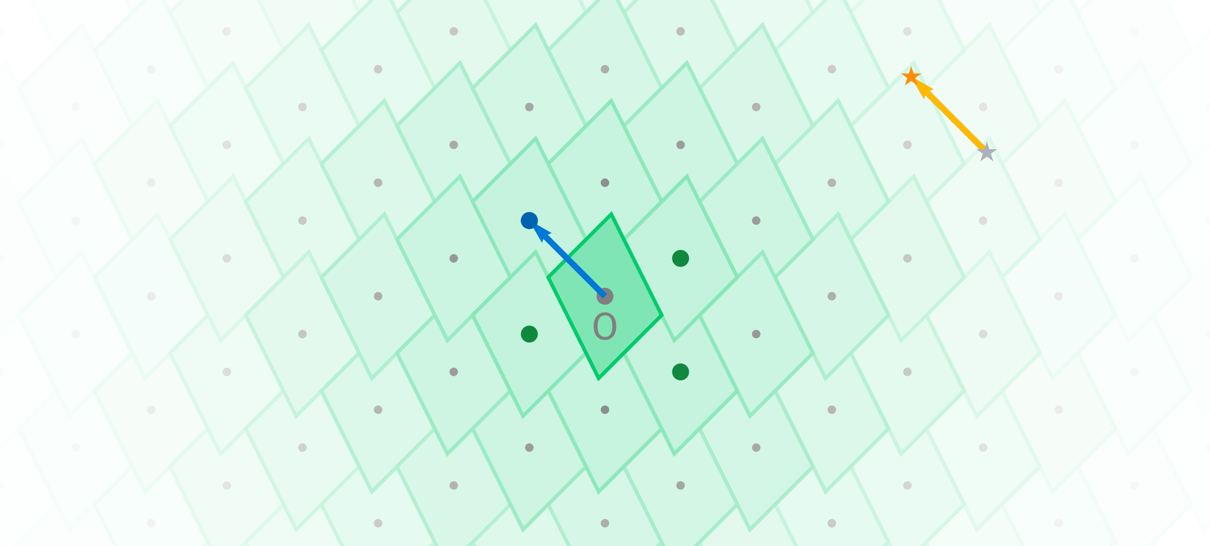

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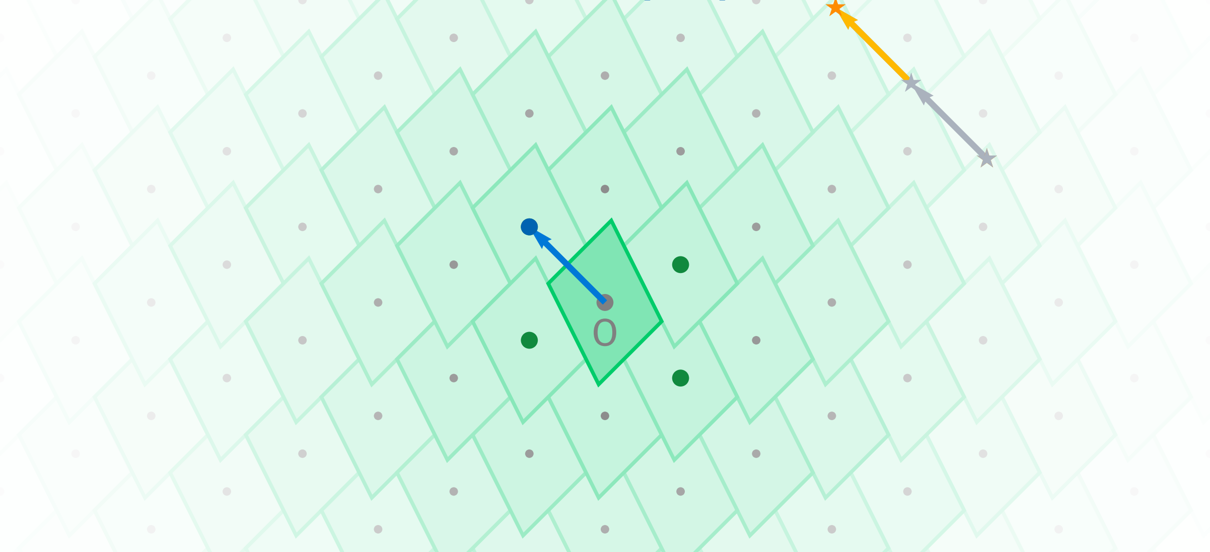

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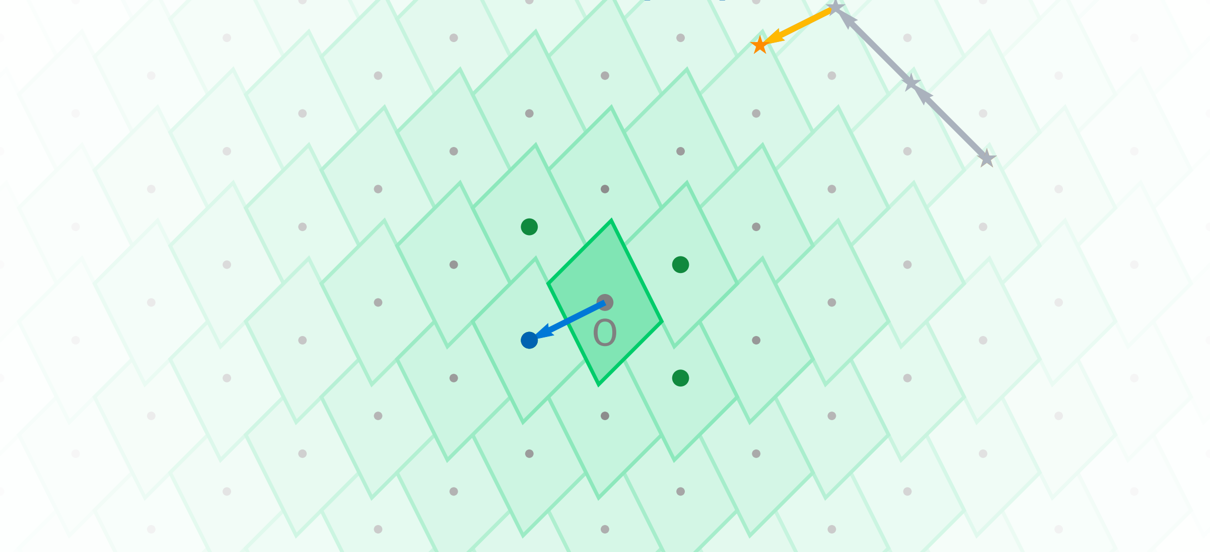

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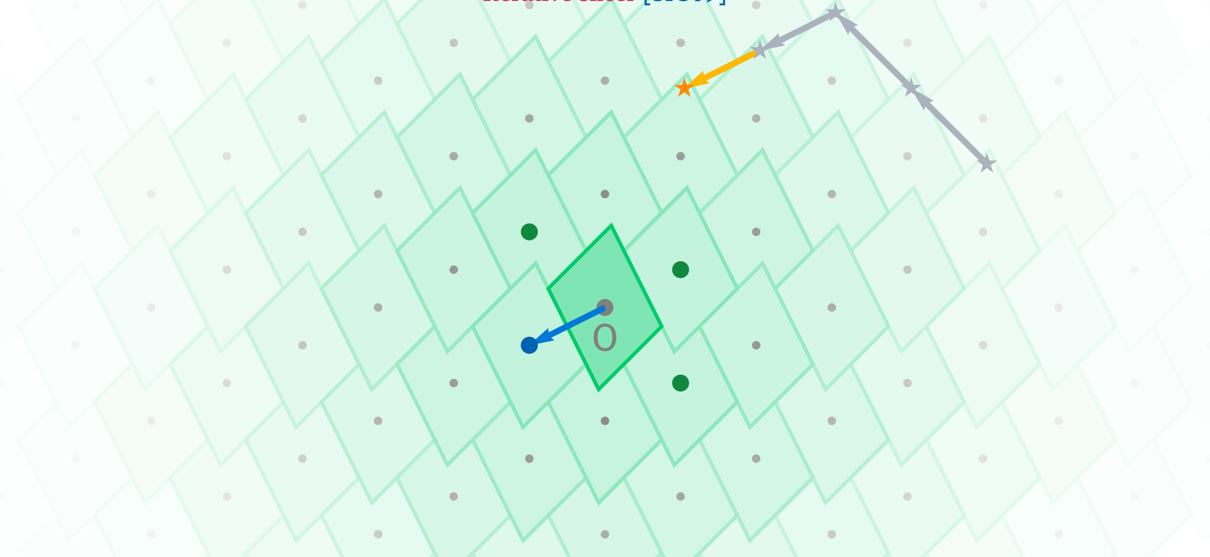

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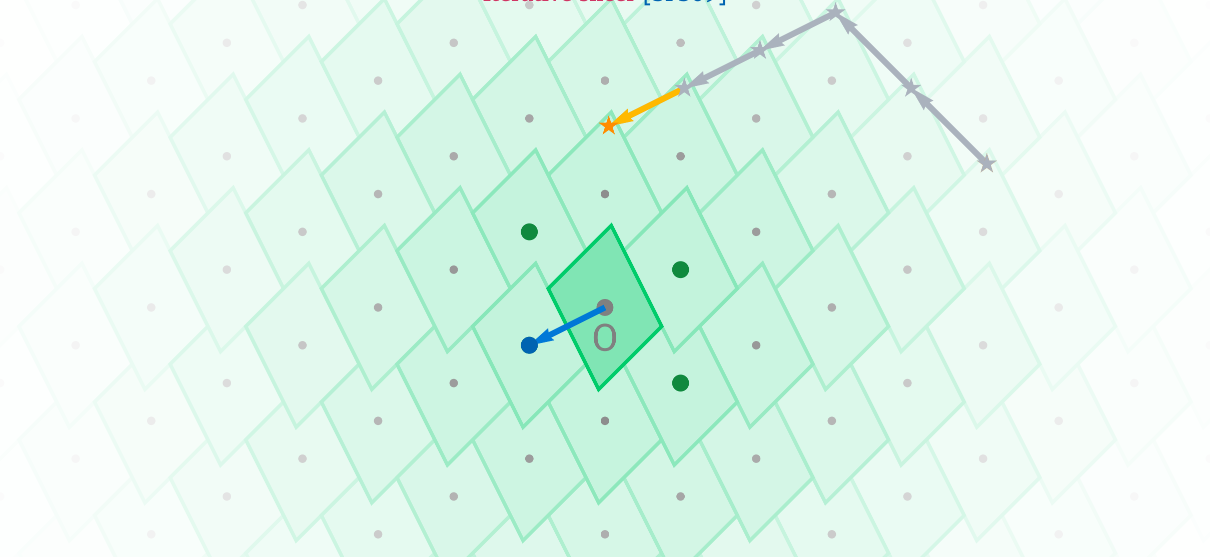

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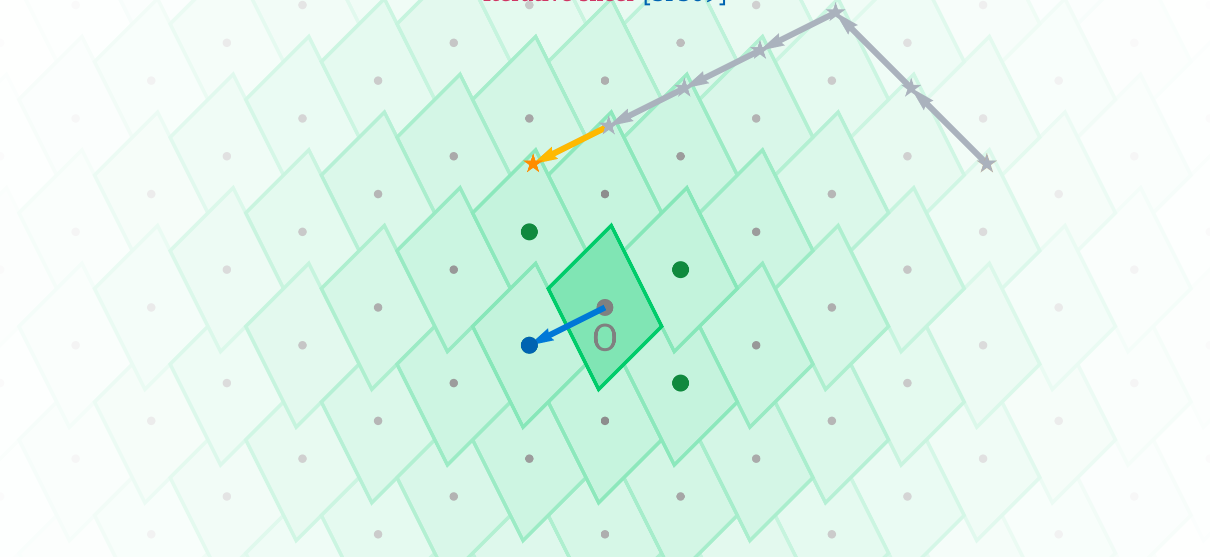

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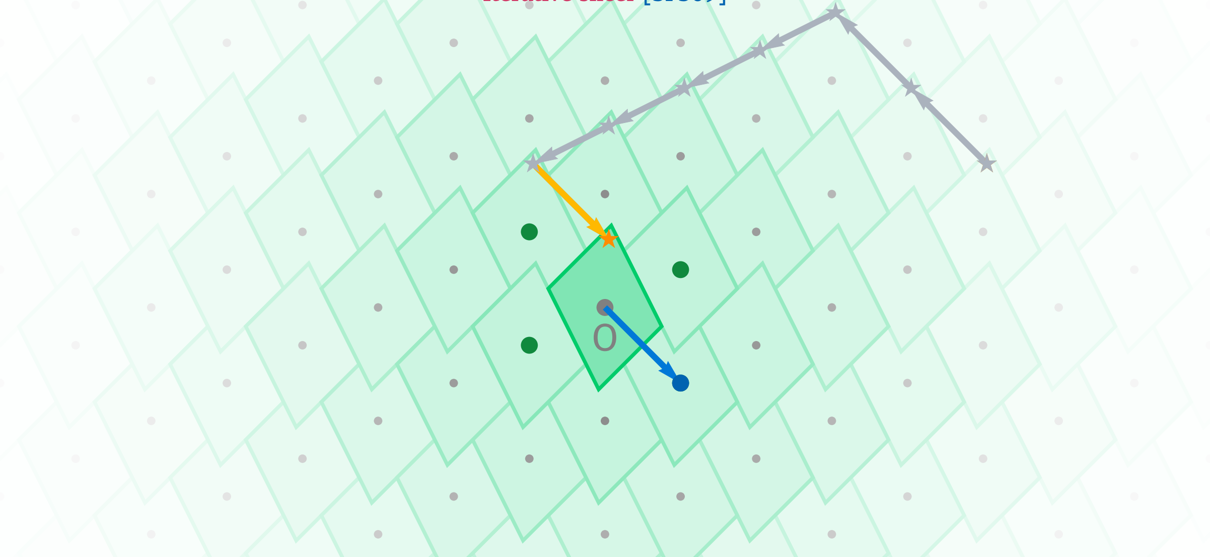

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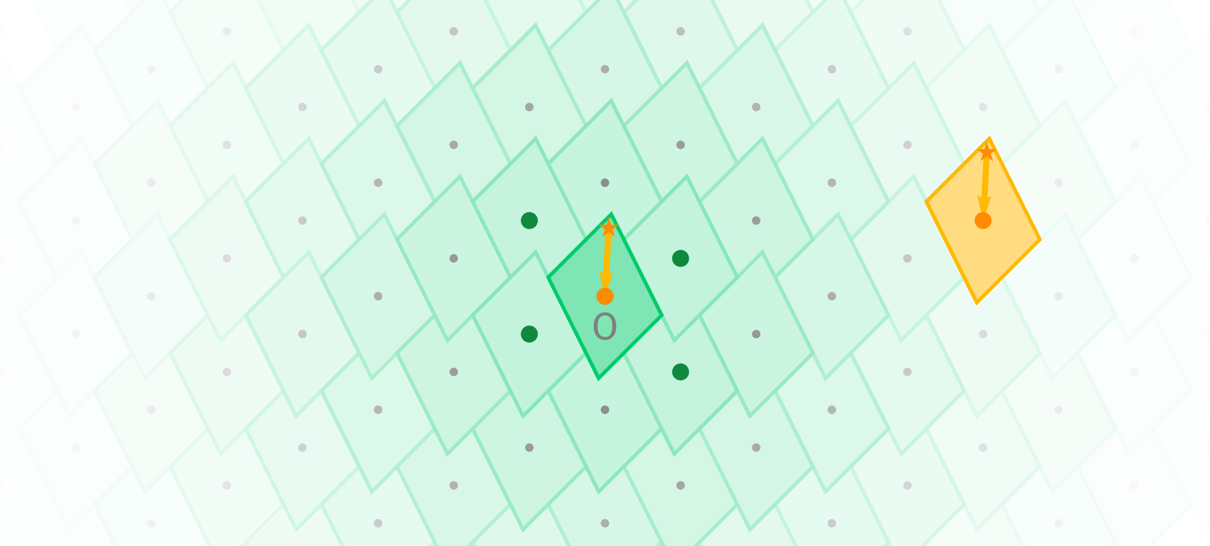

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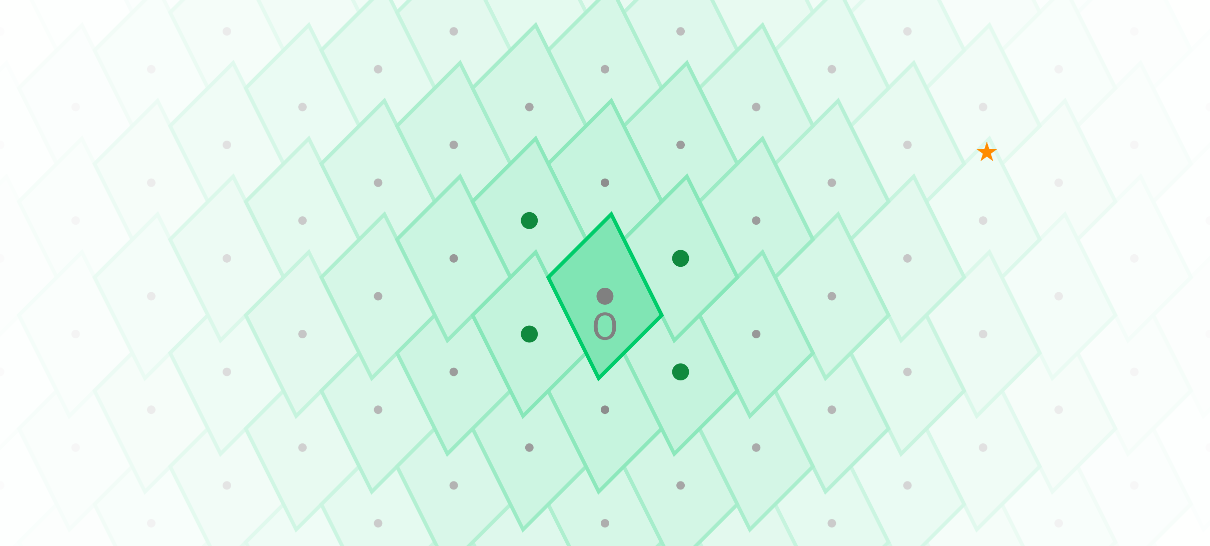
Randomized slicer [DLW19]



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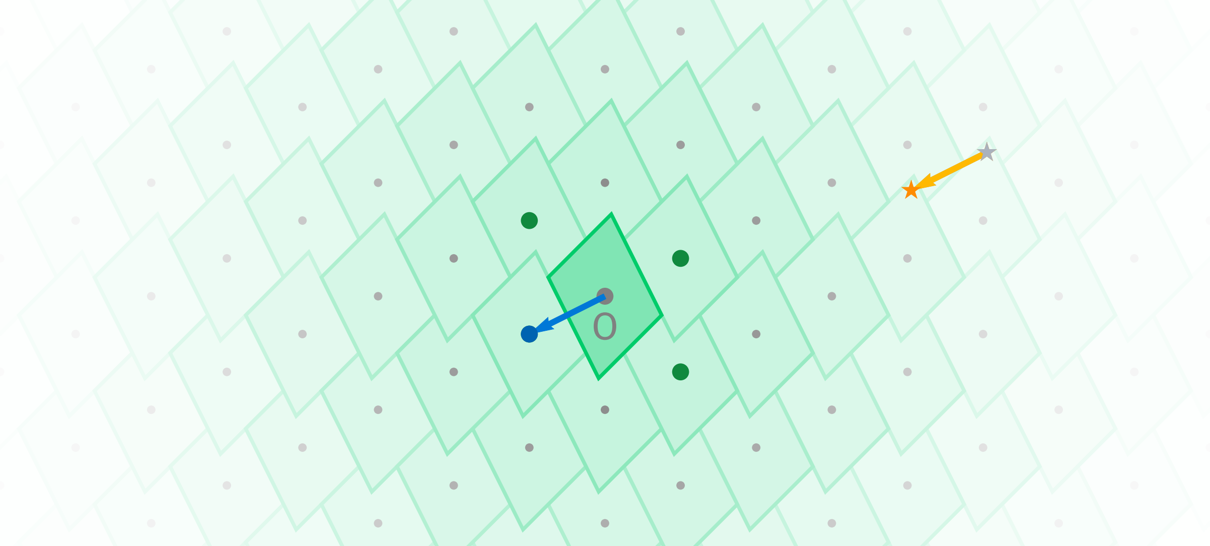

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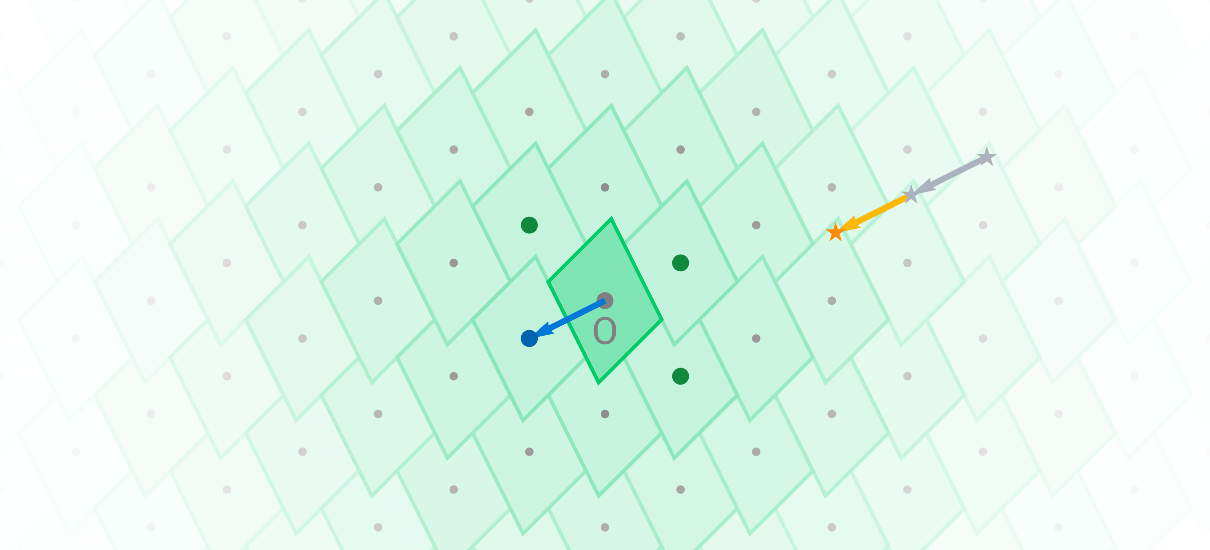

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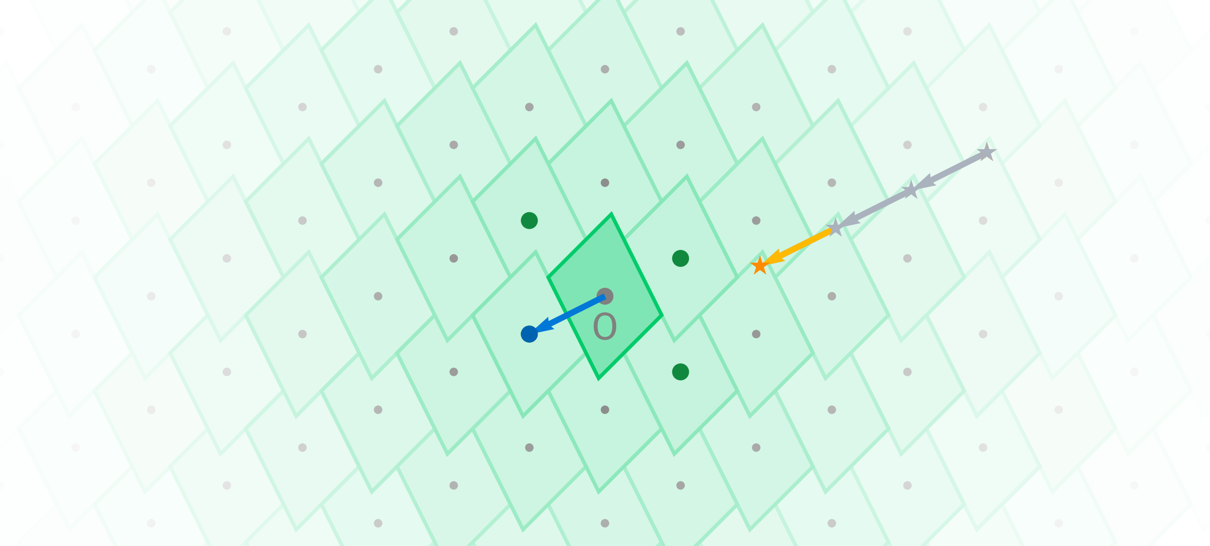

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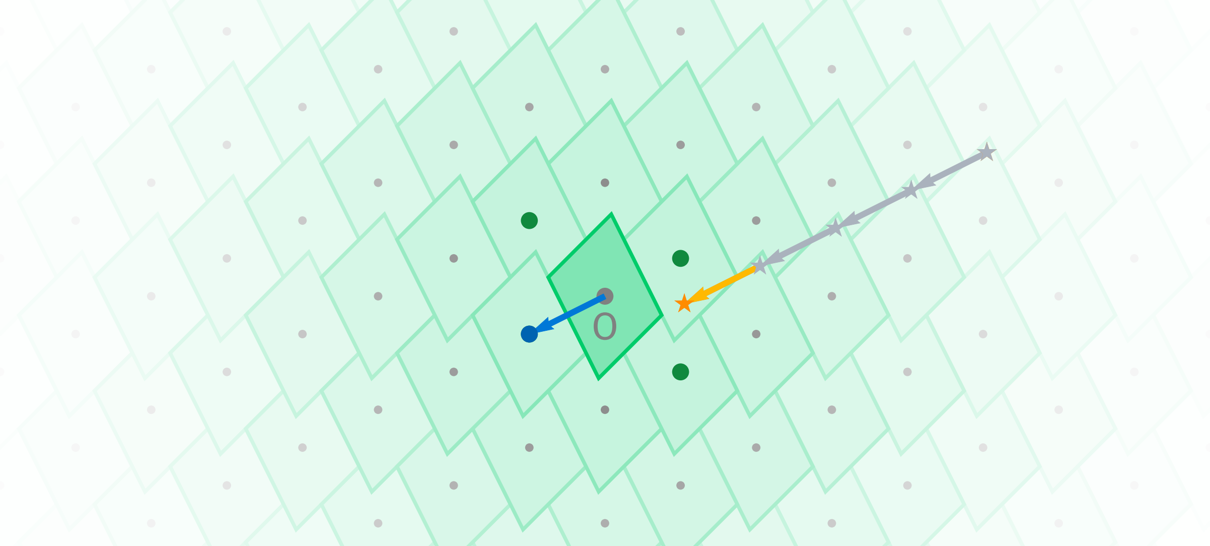

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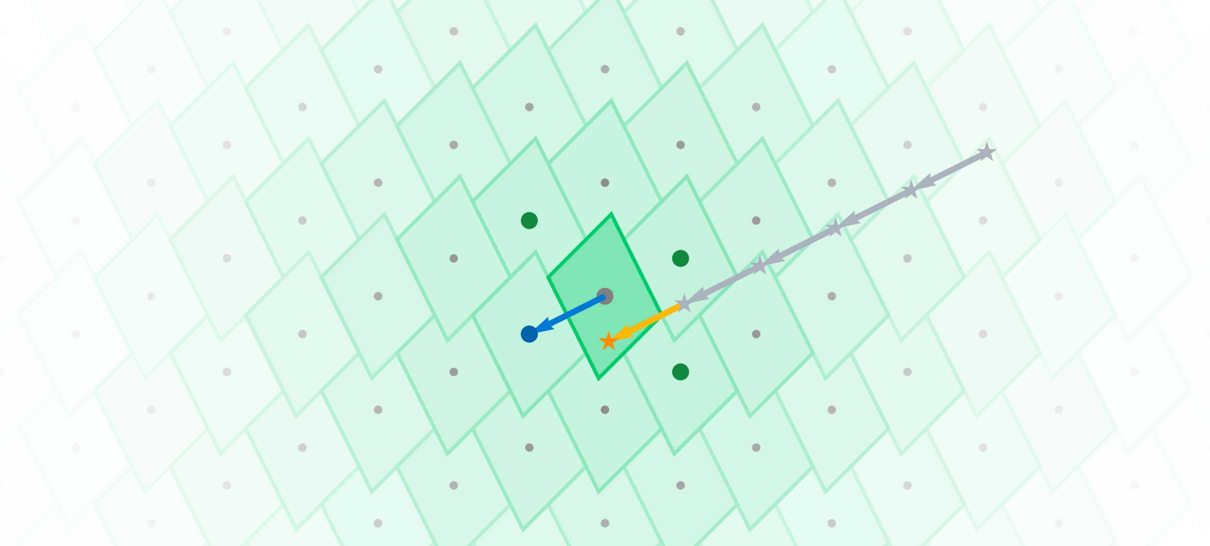

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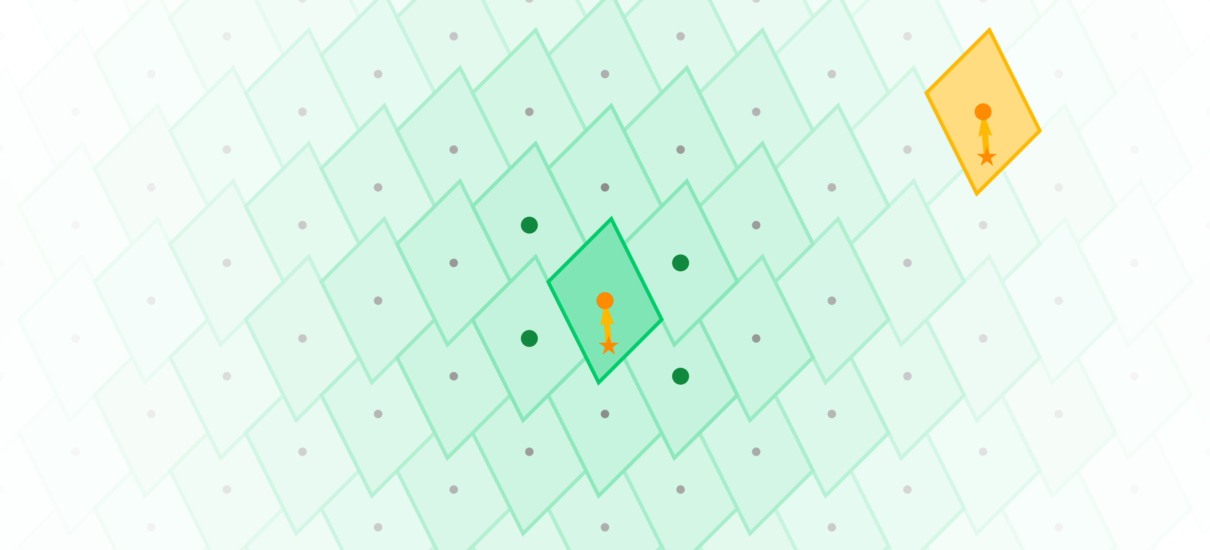

Approximate Voronoi cells

Success probability estimation

Main problem: Success probability p of the iterative slicer?

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Success probability estimation

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Attempt 1: [DLW19]

- Directly obtained a lower bound on p via the slicer
- Conjectured that p is exactly proportional to $\mathrm{vol}(\mathcal{V})/\mathrm{vol}(\mathcal{V}_L)$
- Open problem: obtain a tight analysis, perhaps via $\mathrm{vol}(\mathcal{V})/\mathrm{vol}(\mathcal{V}_L)$

Approximate Voronoi cells

Success probability estimation

Main problem: Success probability p of the iterative slicer?

Attempt 1: [DLW19]

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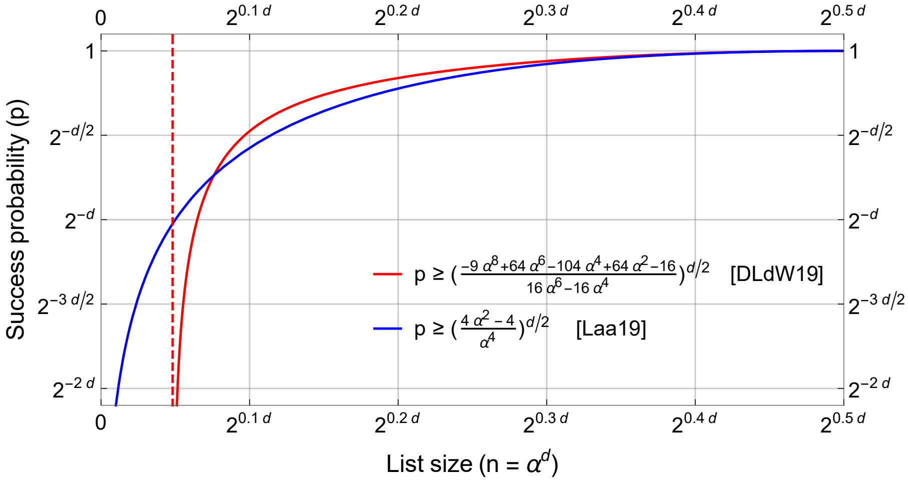
Attempt 2: [Laa19]

- Proved tight bounds on $\mathrm{vol}(\mathcal{V})/\mathrm{vol}(\mathcal{V}_L)$ under the Gaussian heuristic
- Results show that p **cannot** be (exactly) proportional to $\mathrm{vol}(\mathcal{V})/\mathrm{vol}(\mathcal{V}_L)$
- From $p \geq \mathrm{vol}(\mathcal{V})/\mathrm{vol}(\mathcal{V}_L)$ we obtain new lower bounds on p
- No nonsensical asymptote at $2^{0.05d}$ memory anymore

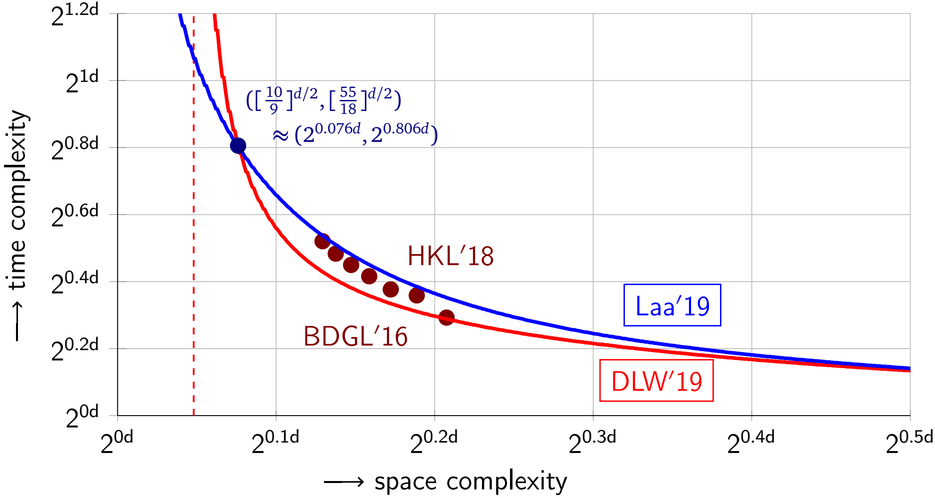
Approximate Voronoi cells

Lower bounds on success probability



Approximate Voronoi cells

Time–space trade-offs for CVPP



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- Offers heuristic alternative to exact Voronoi cells
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  - Original analysis did not appear to be tight
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  - New results obtained tight bounds on the ratio $\mathrm{vol}(\mathcal{V})/\mathrm{vol}(\mathcal{V}_L)$
  - Resulted in better CVPP complexities for low-memory regime
  - Unfortunately, approach via $\mathrm{vol}(\mathcal{V})/\mathrm{vol}(\mathcal{V}_L)$ is not tight either

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Open problems

- Obtain truly tight bounds (ongoing work with Leo Ducas, Wessel van Woerden)
- Find an efficient BDDP-version of this CVPP algorithm