Technische Universiteit
Eindhoven
University of Technology

Part 2: Lattice algorithms for solving the shortest (non-zero) vector problem

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Lecture for Cryptography I
(January 8, 2015)

Lattices

What is a lattice?

Lattices

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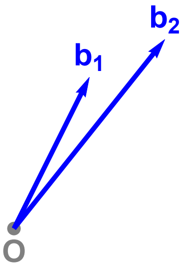

Lattices

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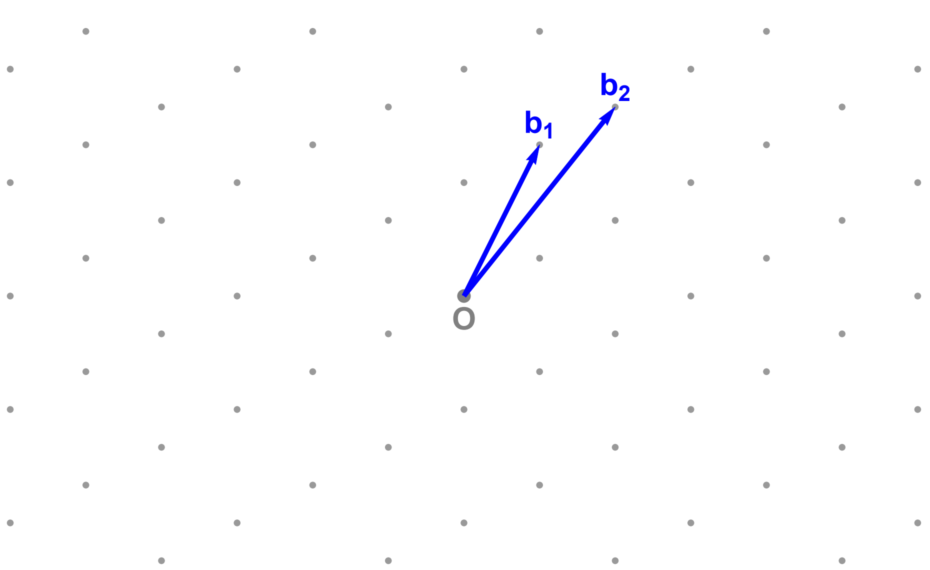
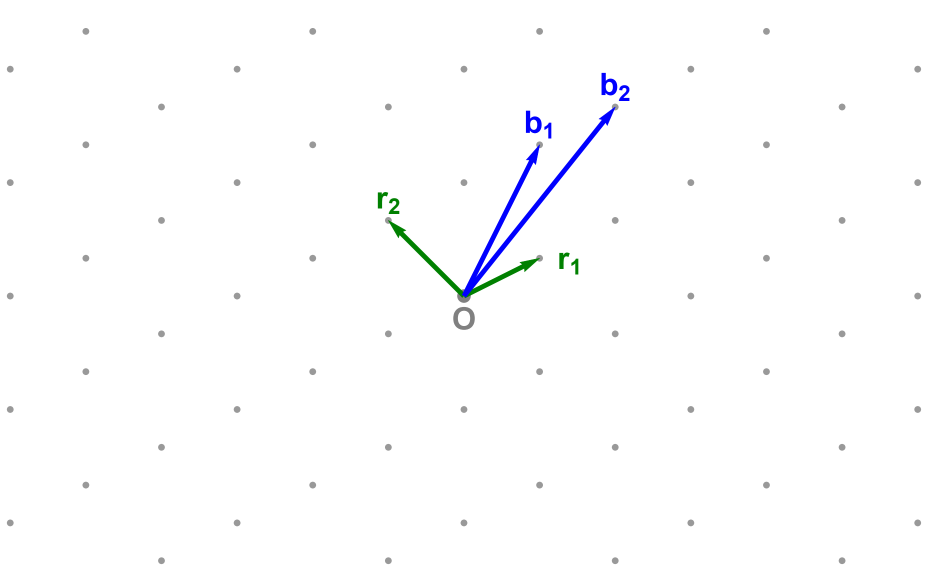


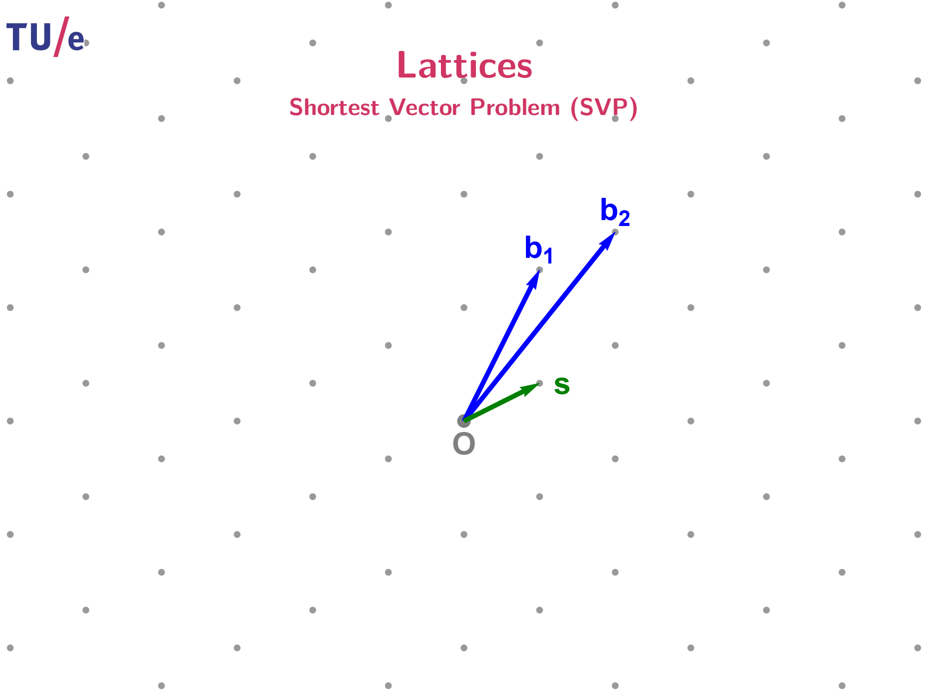
Lattices

Lattice basis reduction



Lattices

Shortest Vector Problem (SVP)



Outline

Enumeration algorithms

- Fincke-Pohst enumeration
- Kannan enumeration
- Pruning the enumeration tree

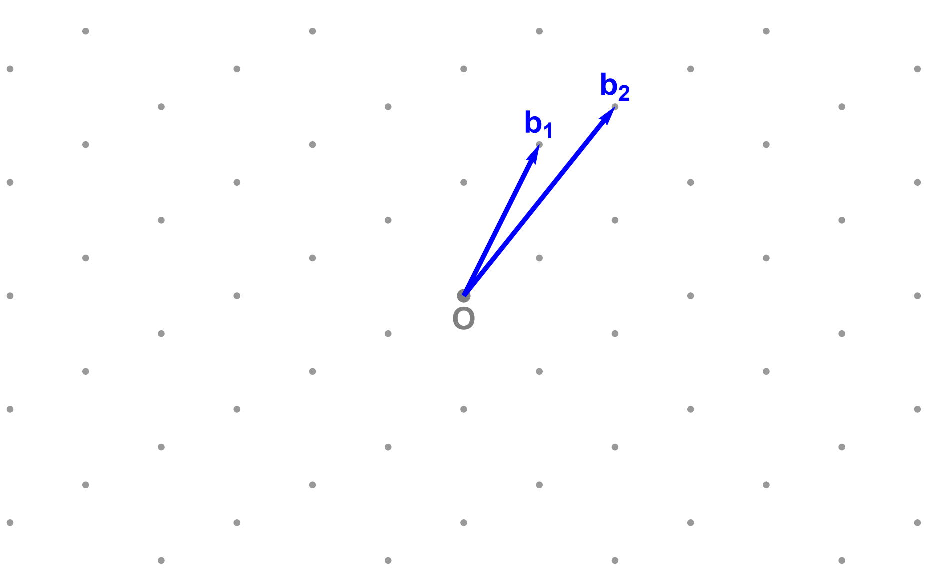
The Voronoi cell algorithm

Sieving algorithms

- Nguyen-Vidick sieve
- Multiple levels
- GaussSieve
- Locality-sensitive hashing

Fincke-Pohst enumeration

1. Determine possible coefficients of b_2



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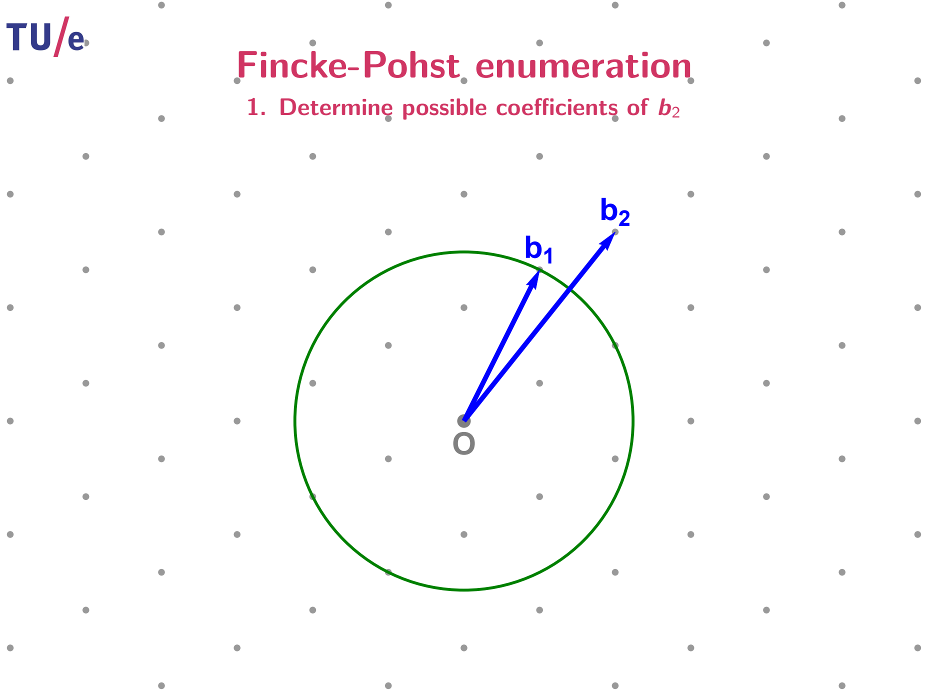

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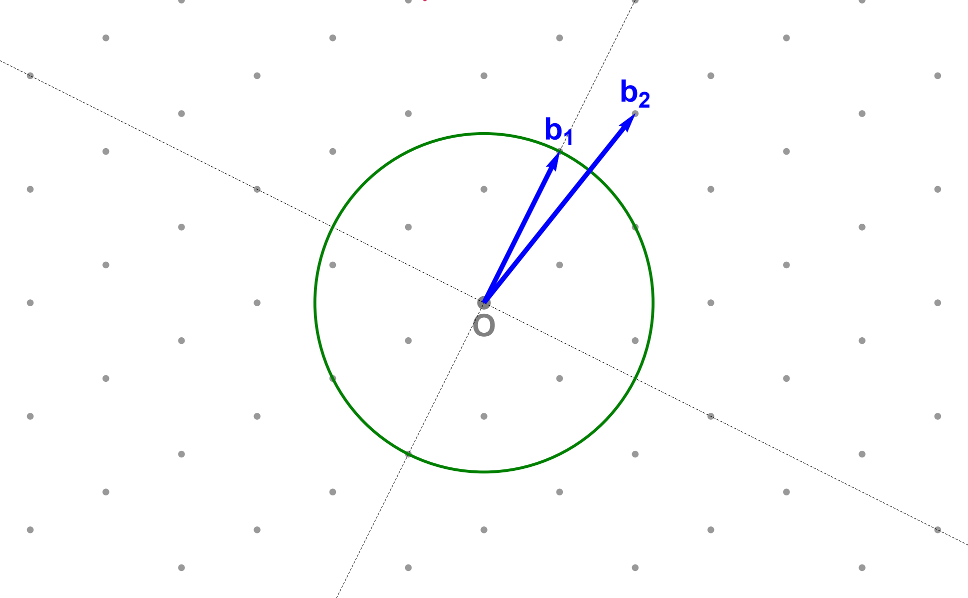

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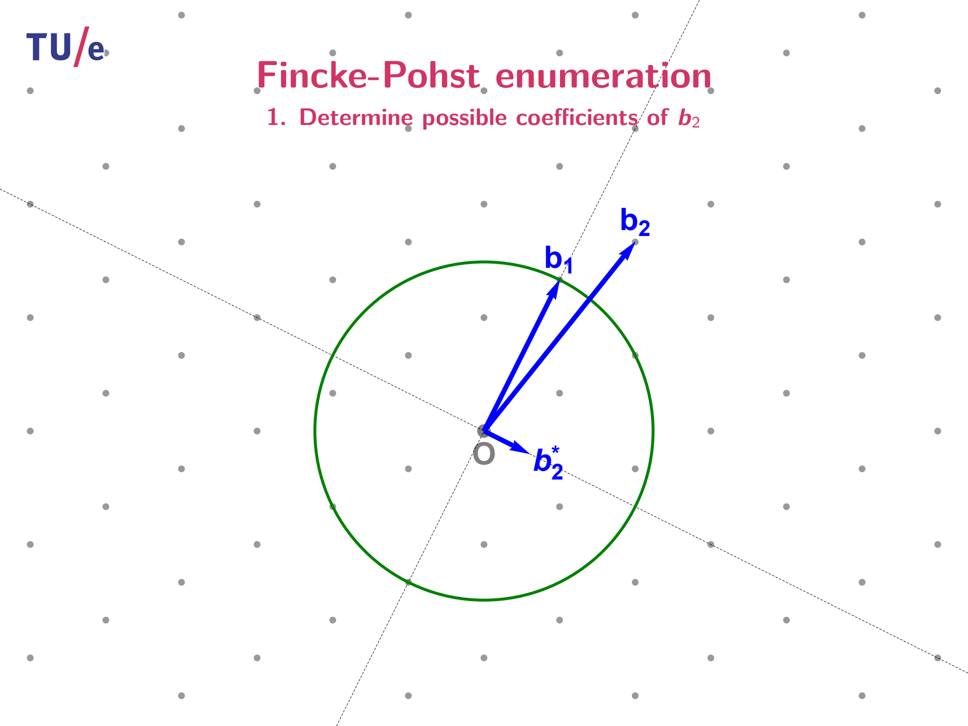

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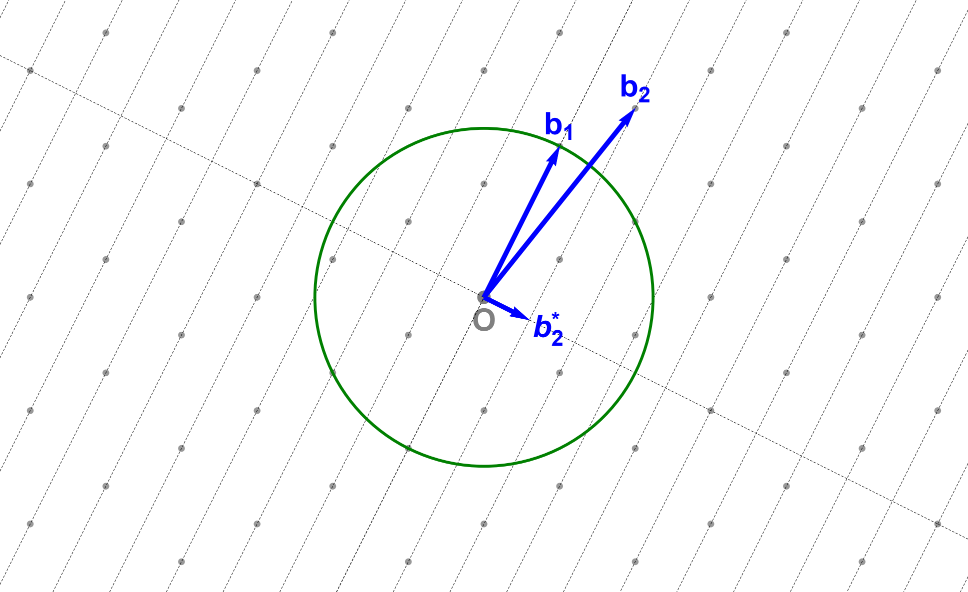
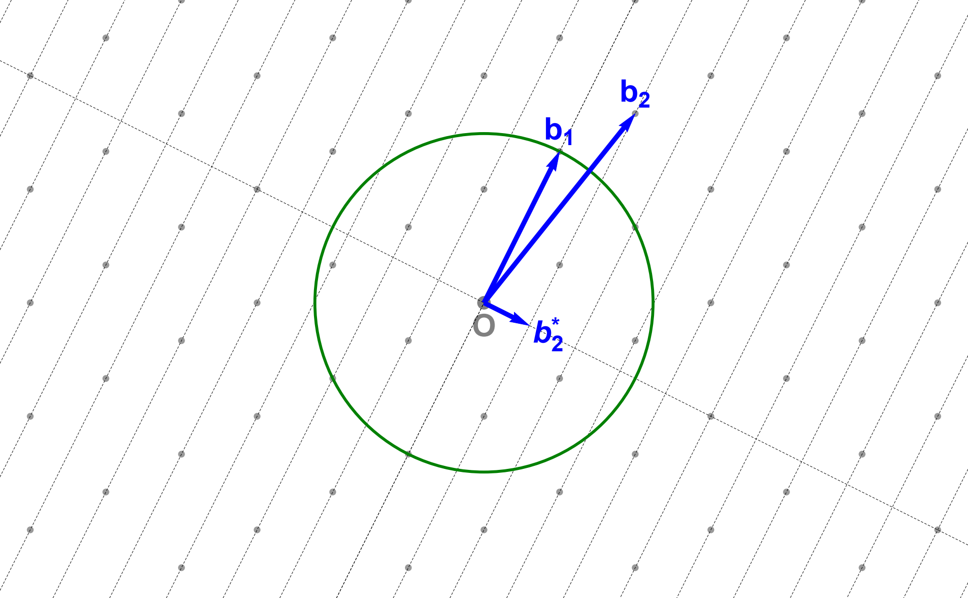


Fincke-Pohst enumeration

2. Find short vectors for each coefficient of b_2



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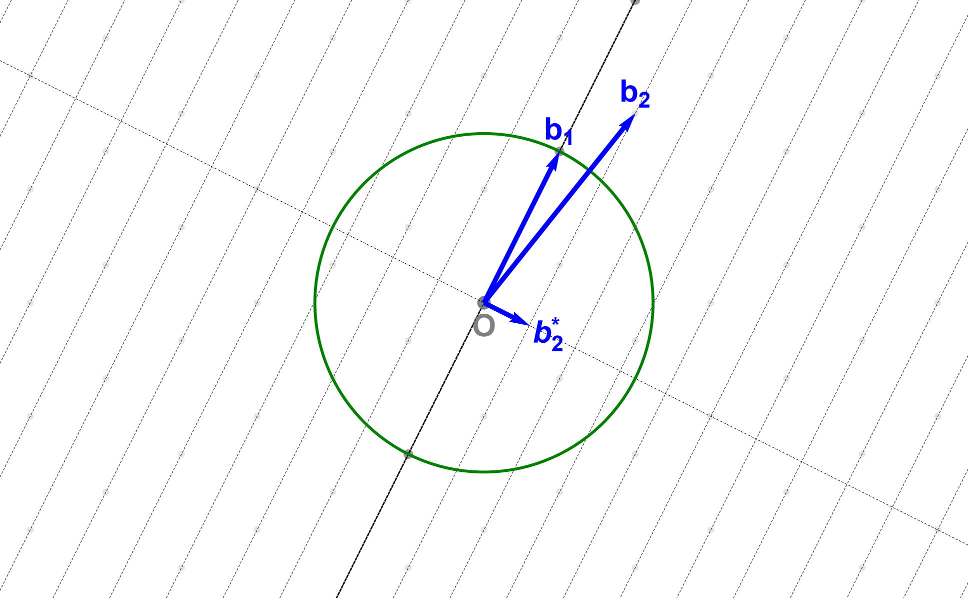

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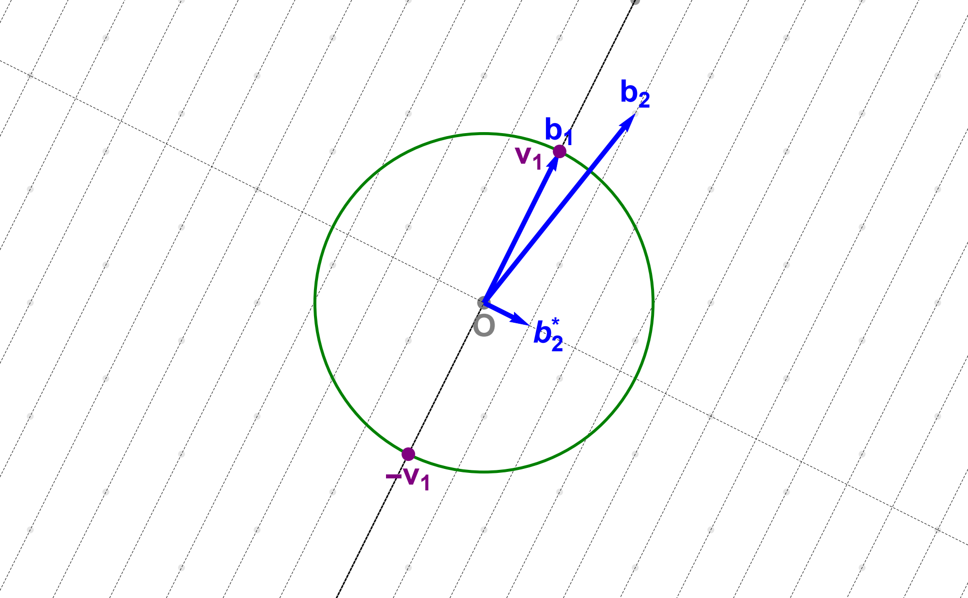

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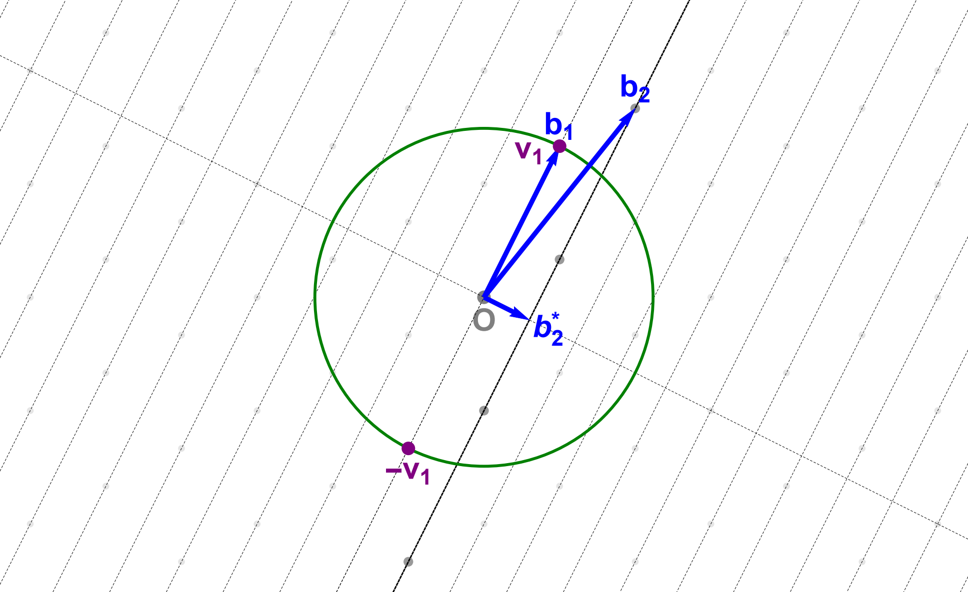

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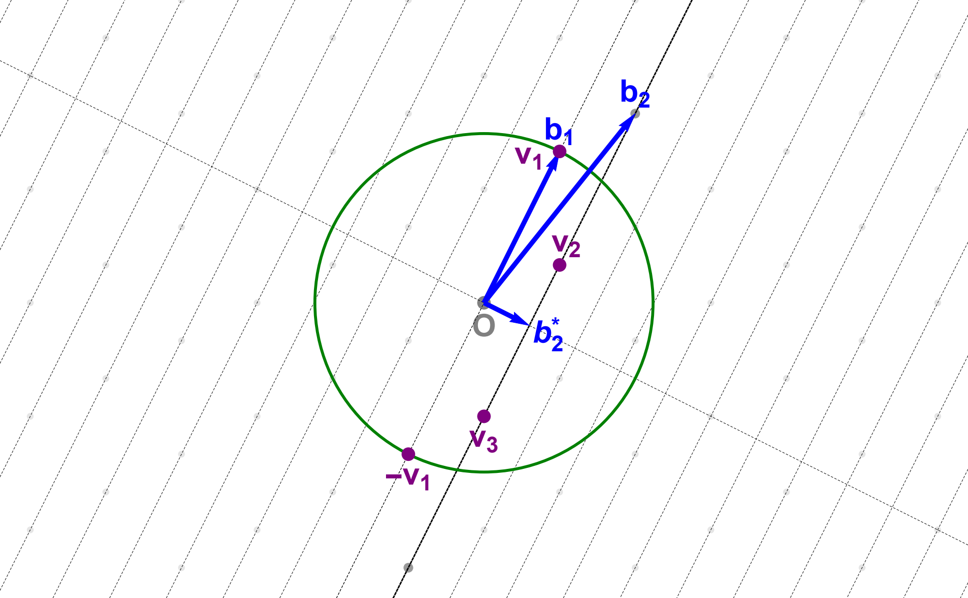

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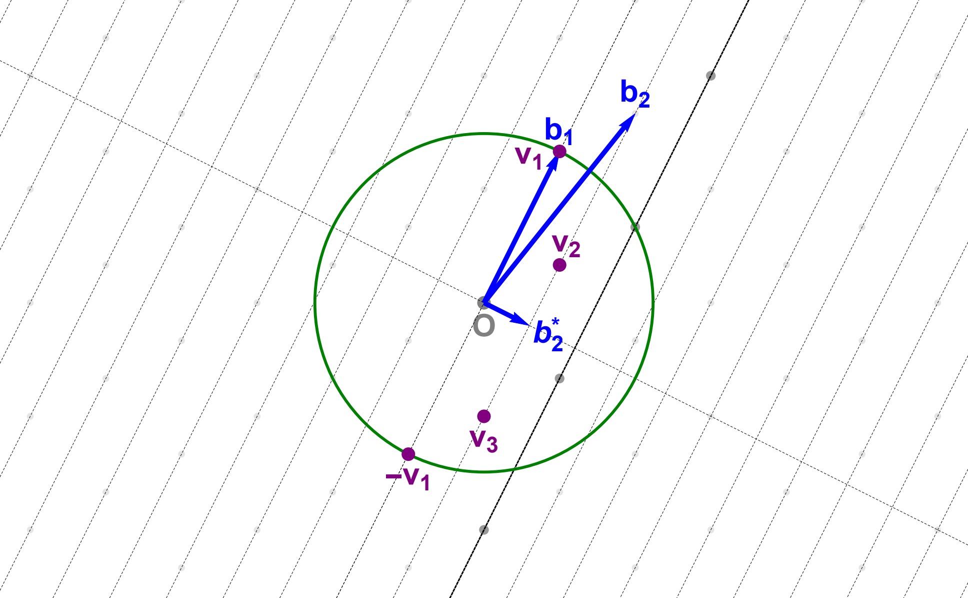

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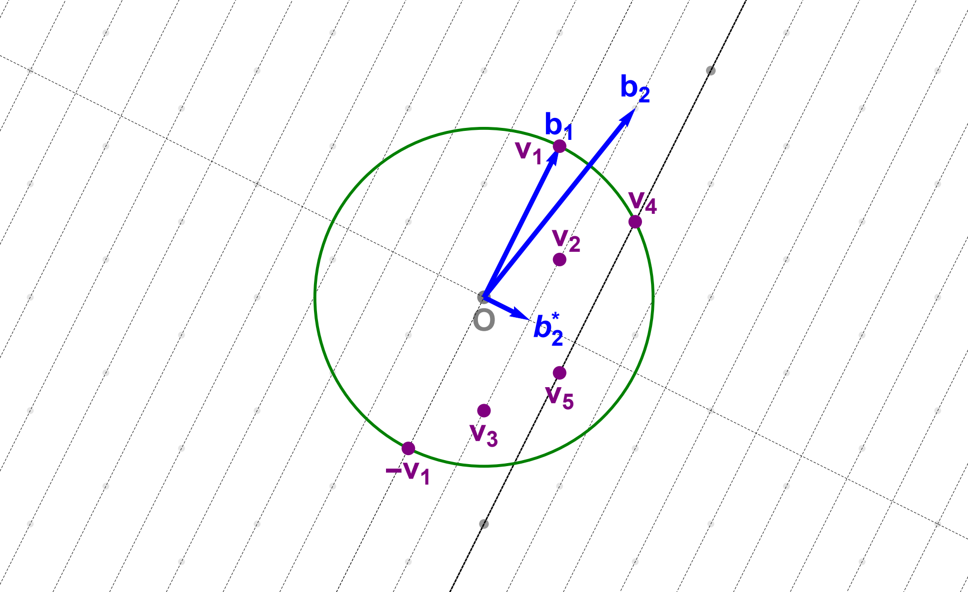

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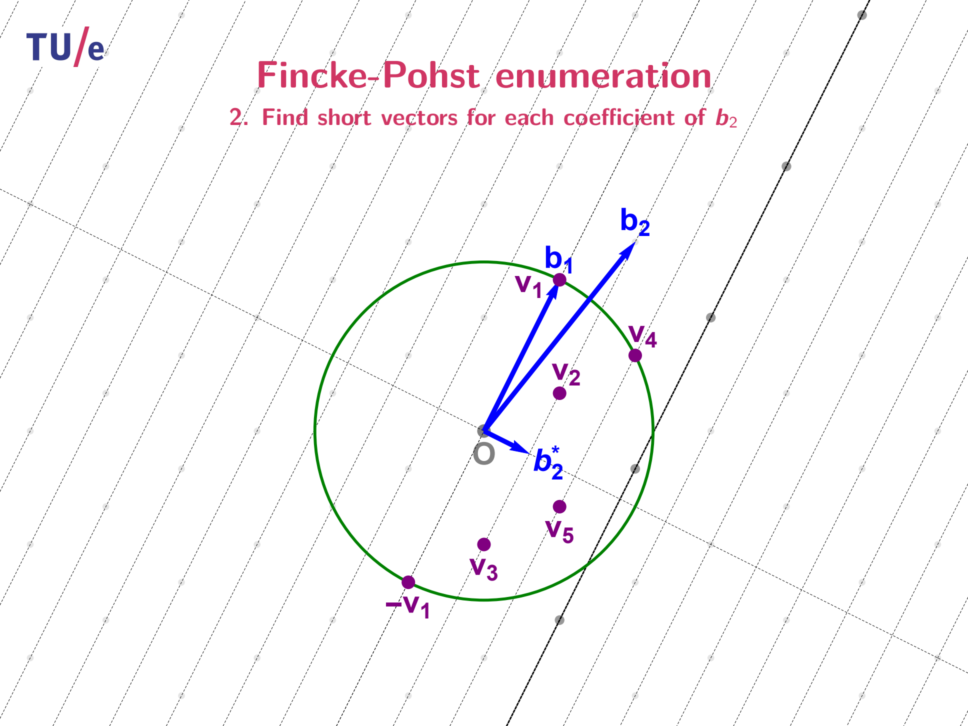

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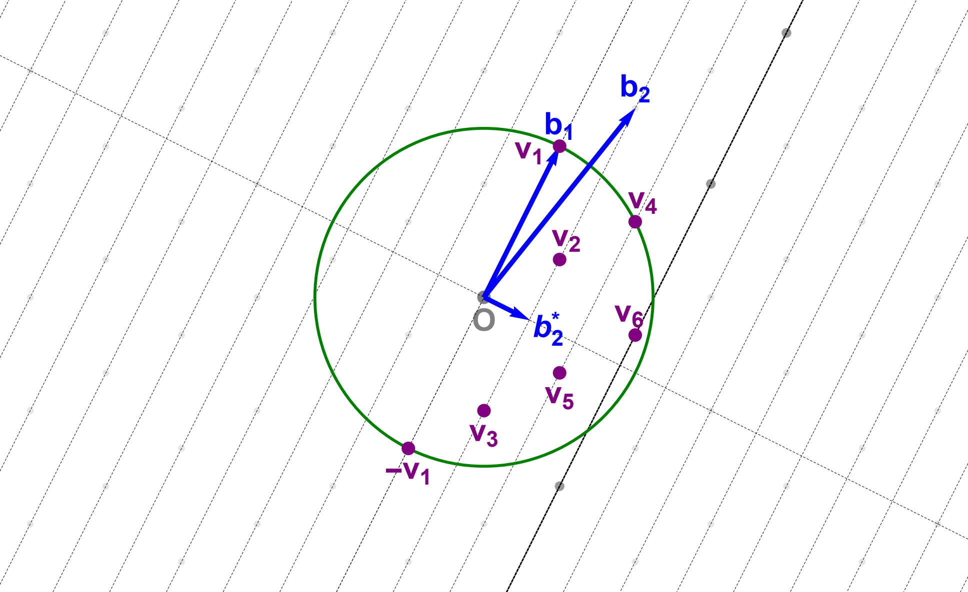

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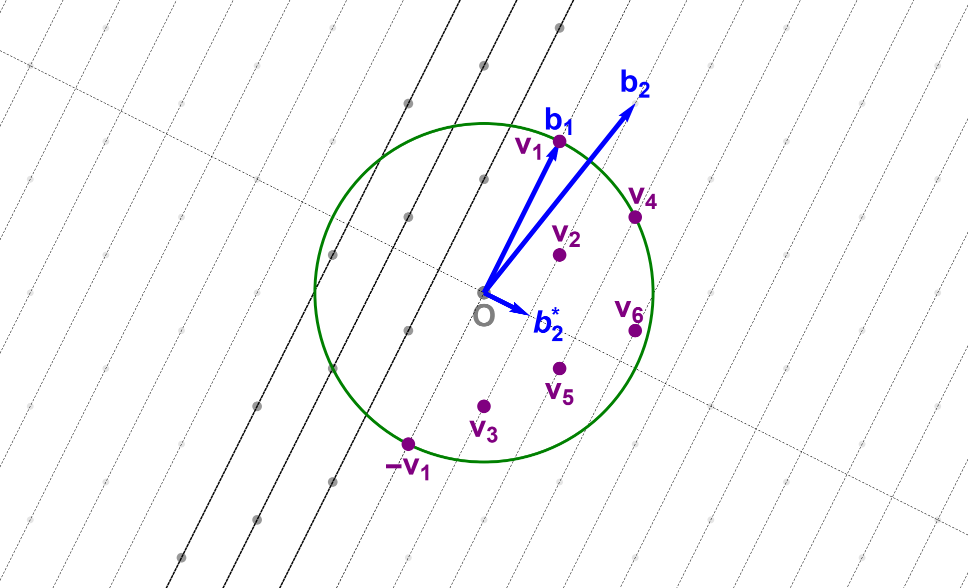

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Fincke-Pohst enumeration

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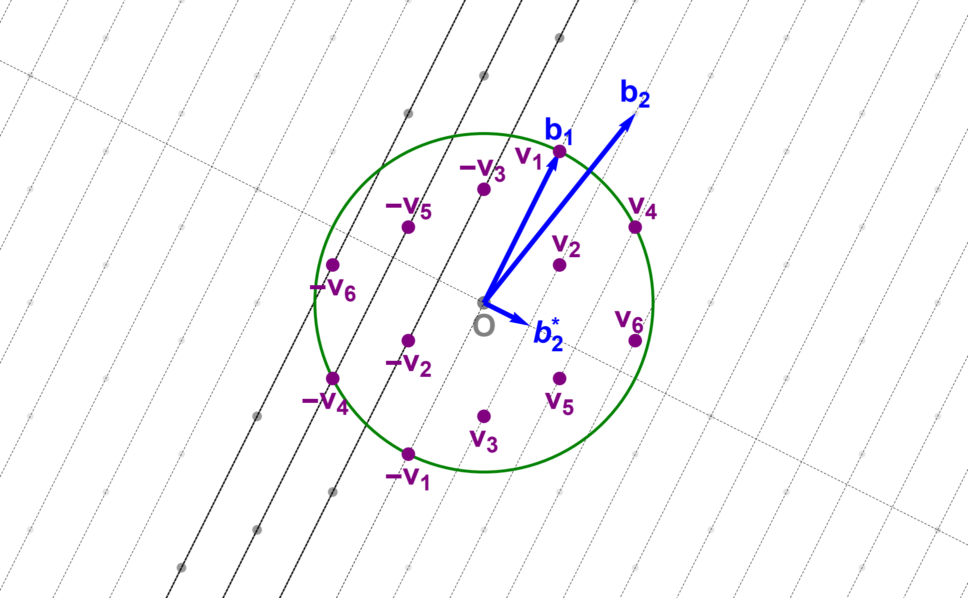

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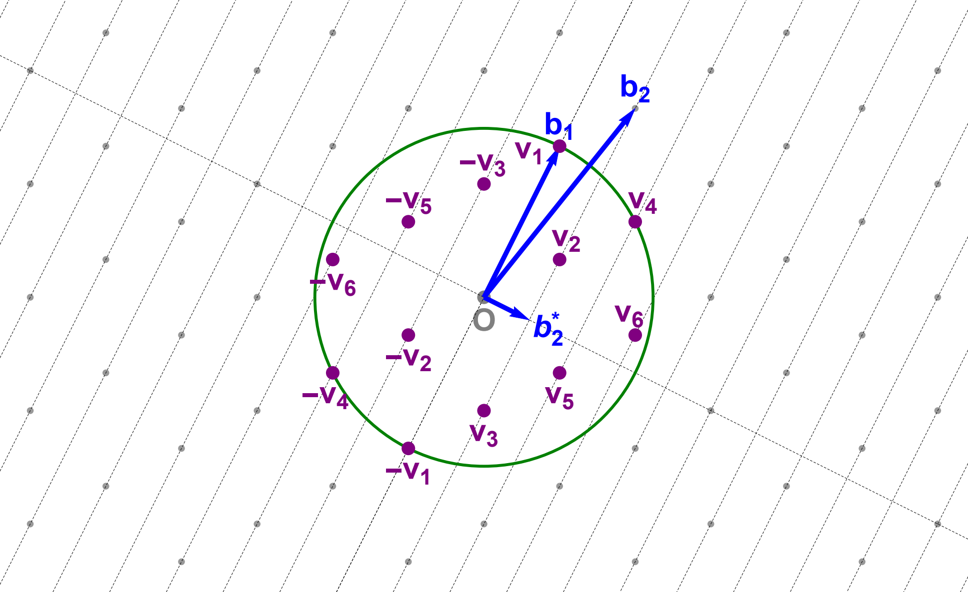
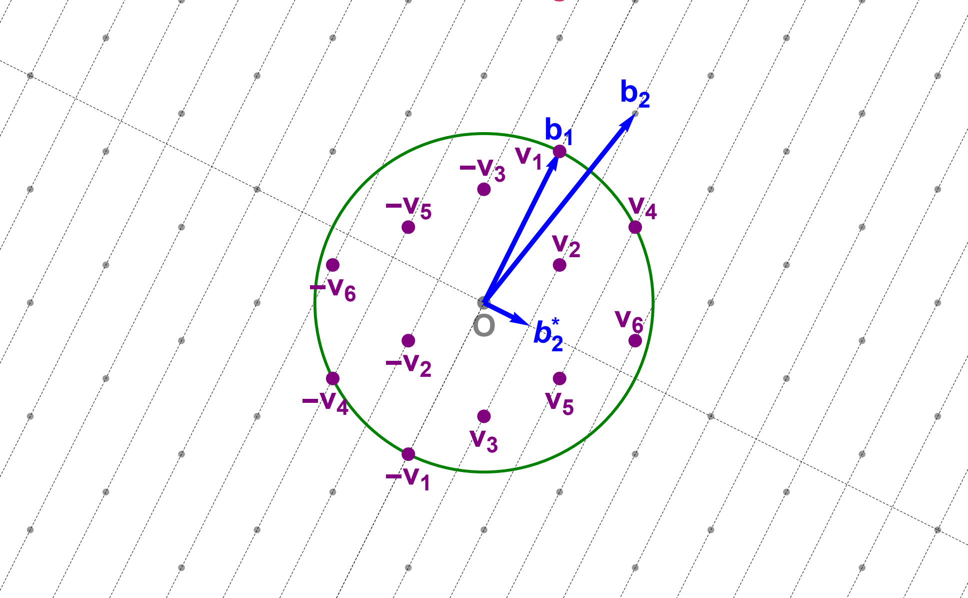


Fincke-Pohst enumeration

3. Find a shortest vector among all found vectors



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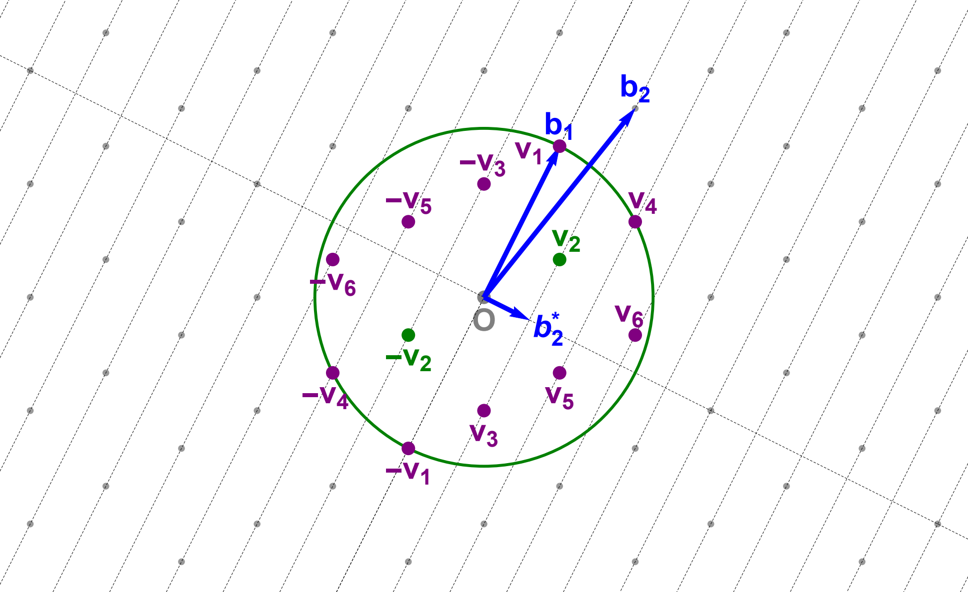

Fincke-Pohst enumeration

Overview

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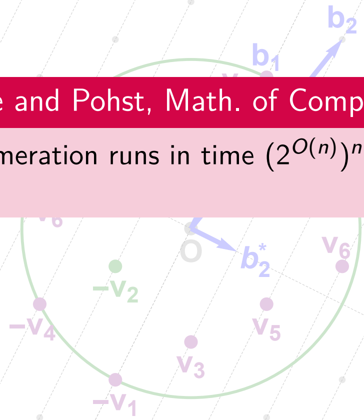
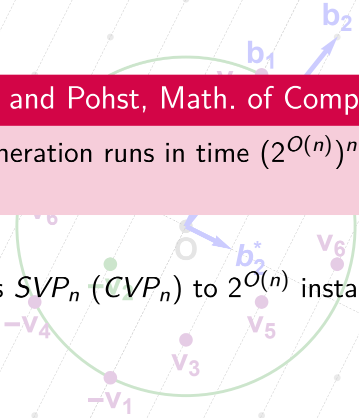


Theorem (Fincke and Pohst, Math. of Comp. '85)

Fincke-Pohst enumeration runs in time $(2^{O(n)})^n = 2^{O(n^2)}$ and space $\text{poly}(n)$.

Fincke-Pohst enumeration

Overview



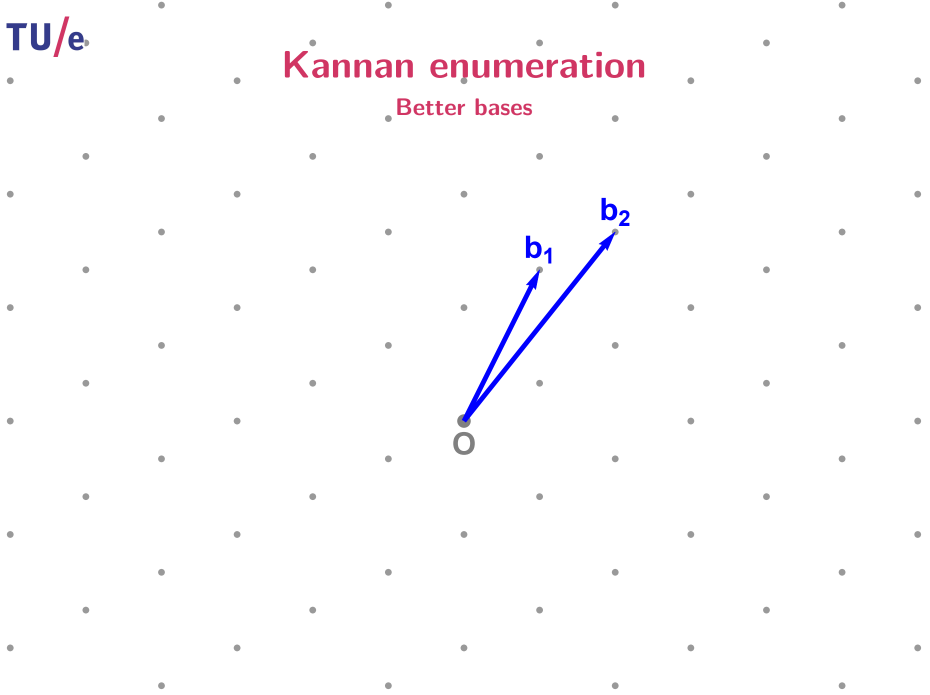
Theorem (Fincke and Pohst, Math. of Comp. '85)

Fincke-Pohst enumeration runs in time $(2^{O(n)})^n = 2^{O(n^2)}$ and space poly(n).

Essentially reduces SVP_n (CVP_n) to $2^{O(n)}$ instances of CVP_{n-1}

Kannan enumeration

Better bases



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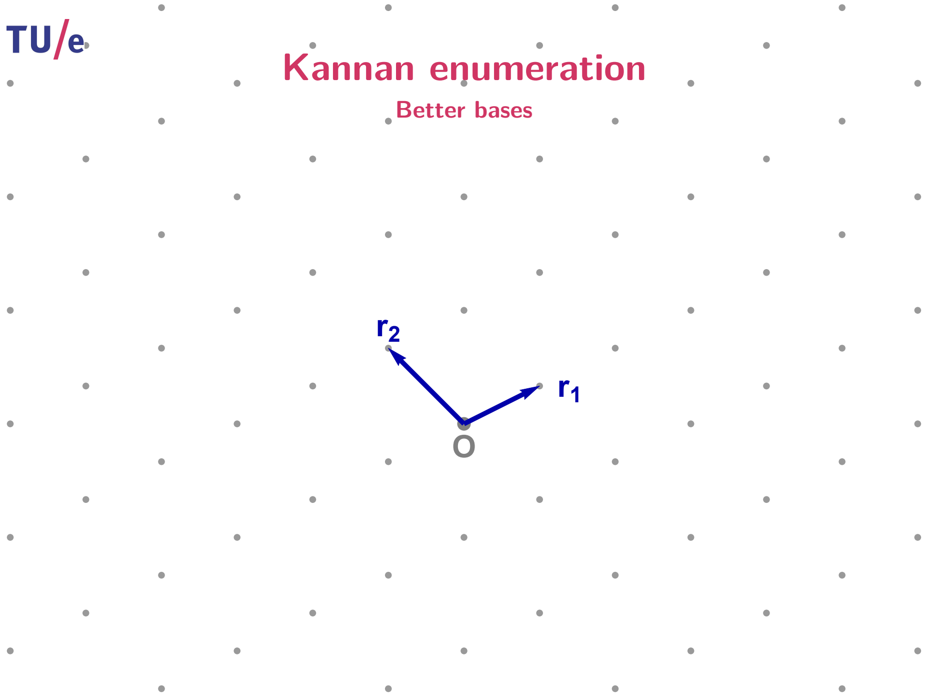

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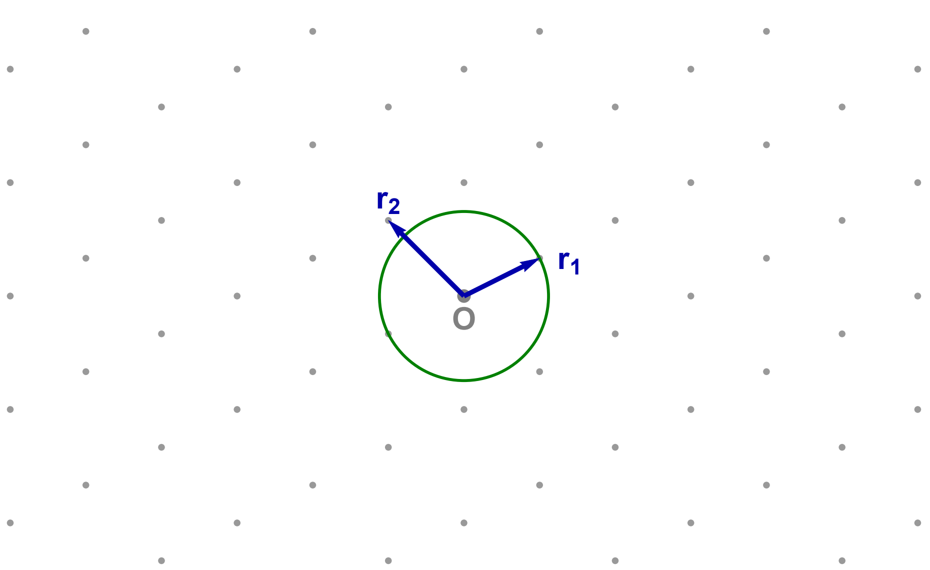

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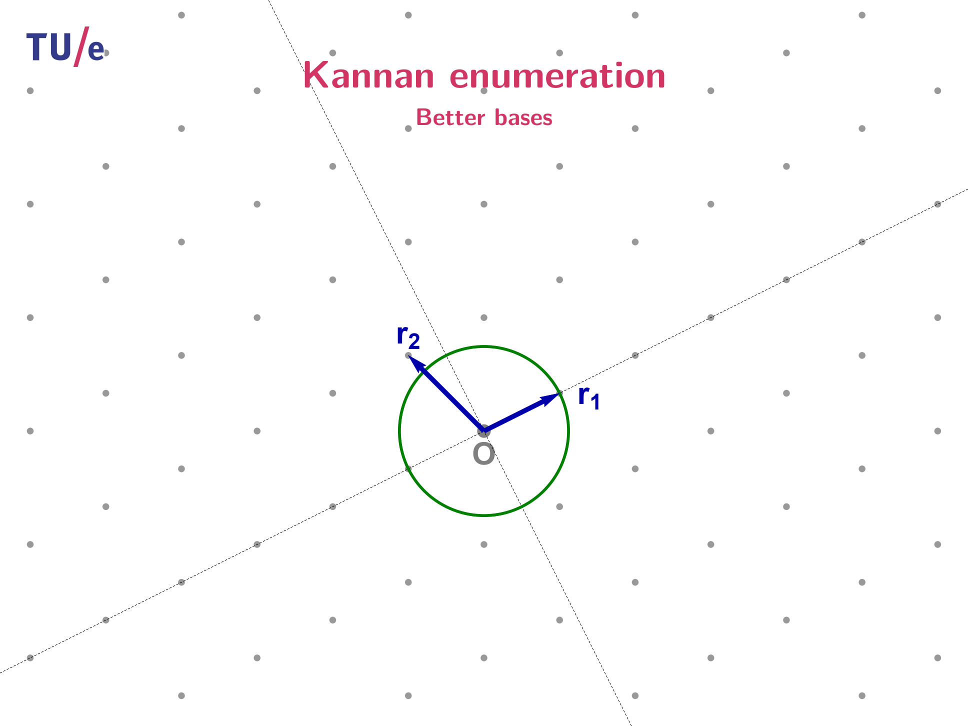

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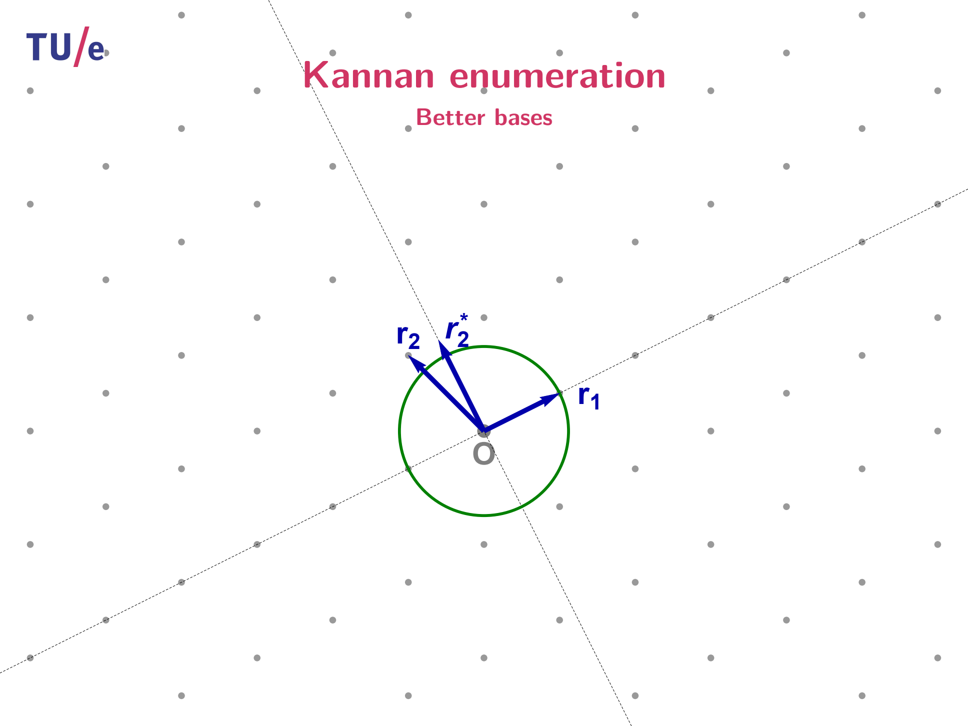

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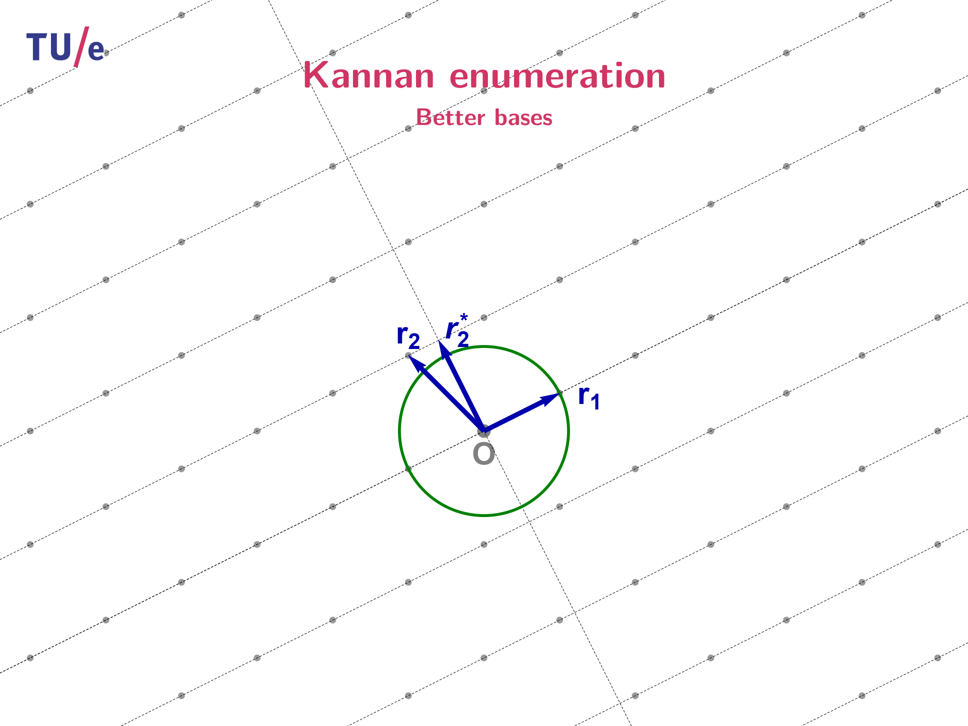

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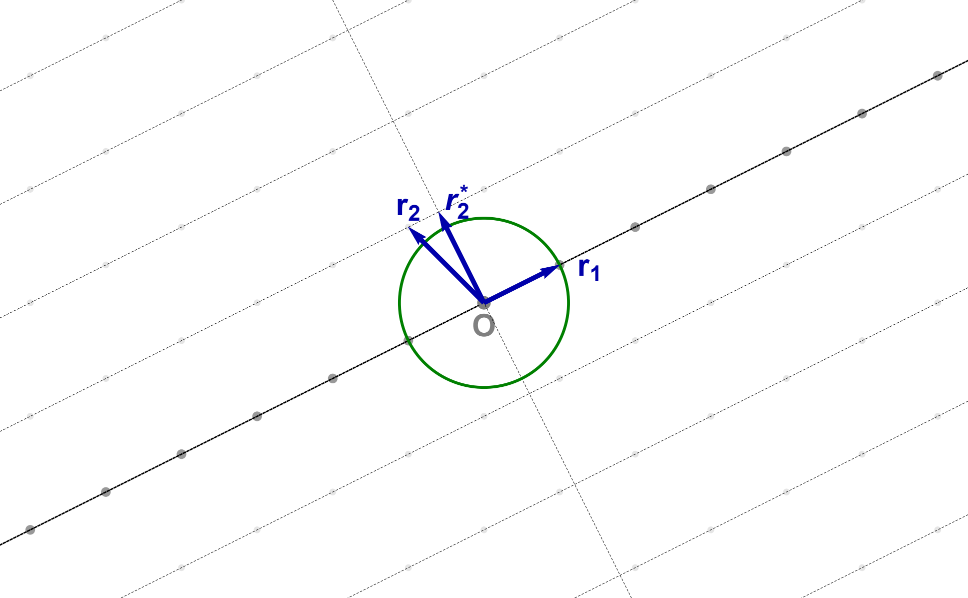

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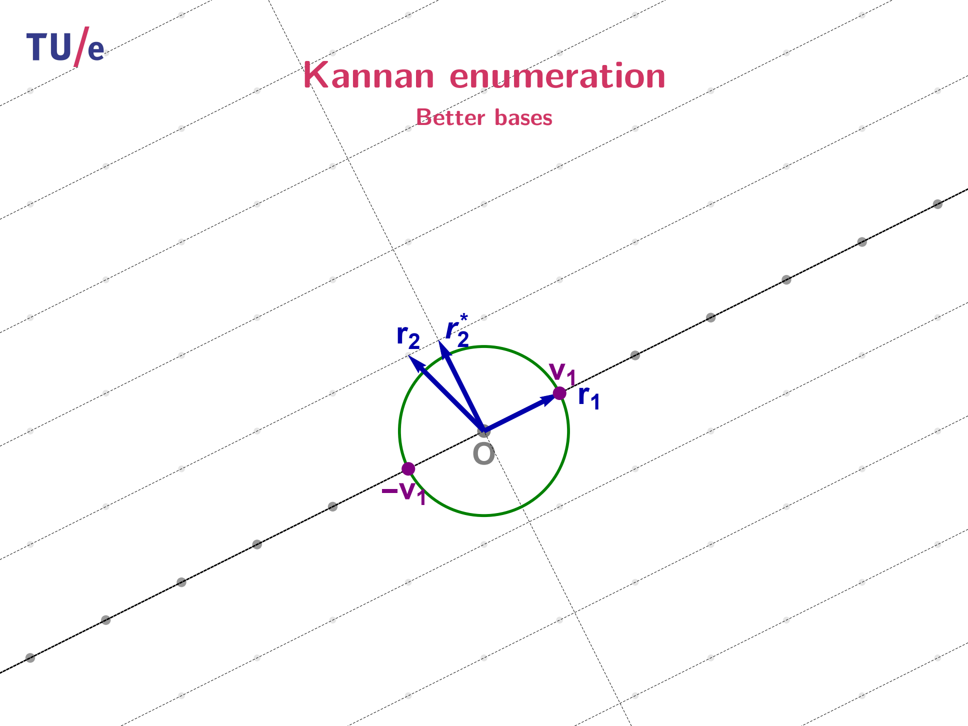

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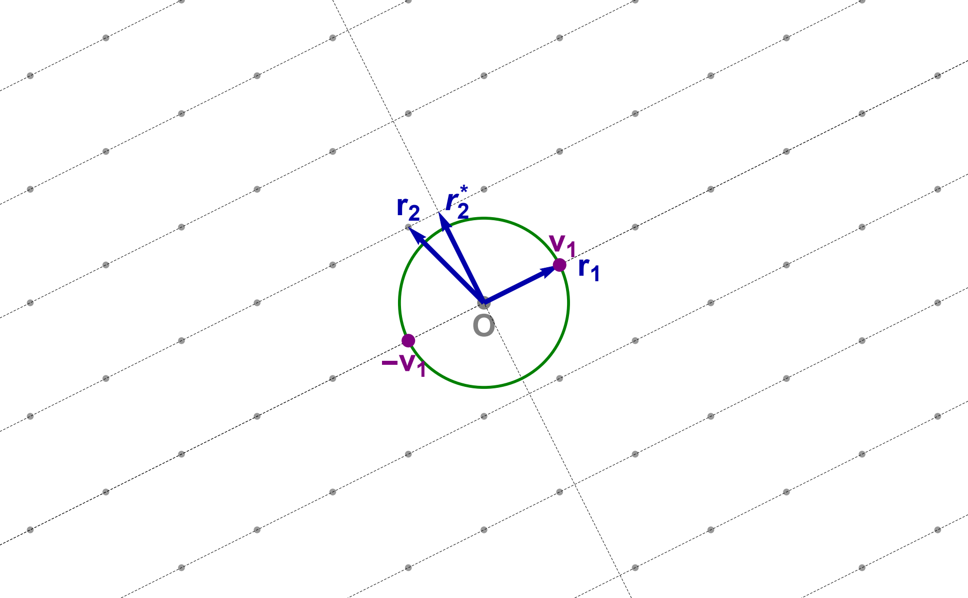

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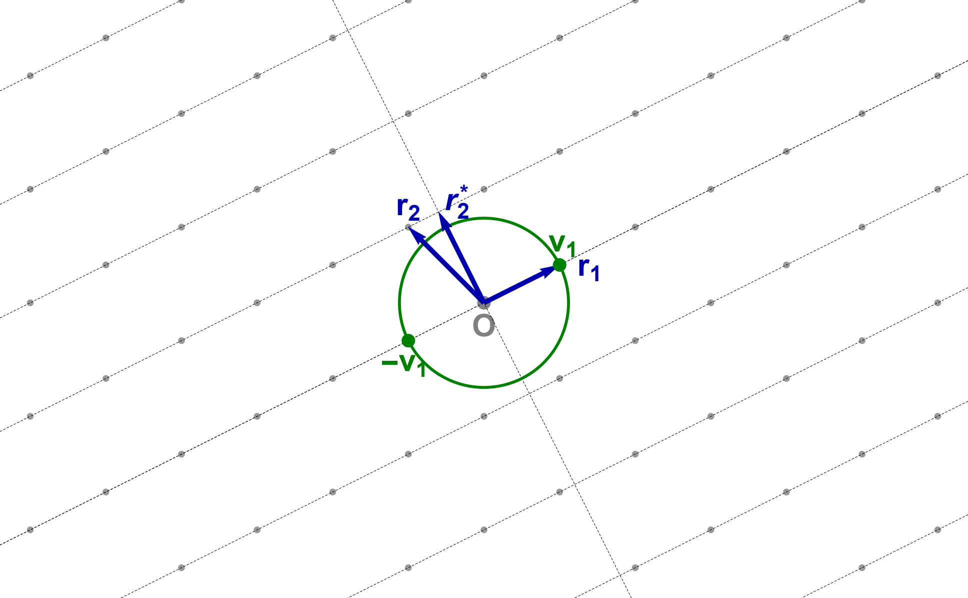

Kannan enumeration

Overview

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Theorem (Kannan, STOC'83)

Kannan enumeration runs in time $2^{O(n \log n)}$ and space $\mathrm{poly}(n)$.

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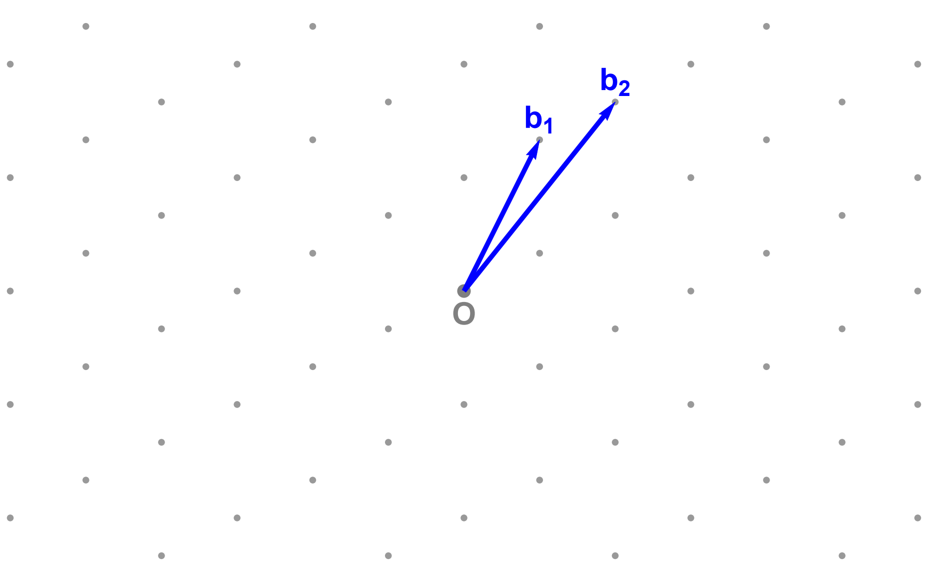
Theorem (Kannan, STOC'83)

Kannan enumeration runs in time $2^{O(n \log n)}$ and space poly(n).

> *"Our algorithm reduces an n-dimensional problem to polynomially many (instead of $2^{O(n)}$) $(n-1)$-dimensional problems. [...] The algorithm we propose, first finds a more orthogonal basis for a lattice in time $2^{O(n \log n)}$."*
>
> – Kannan, *STOC'83*

Pruned enumeration

Reducing the search space



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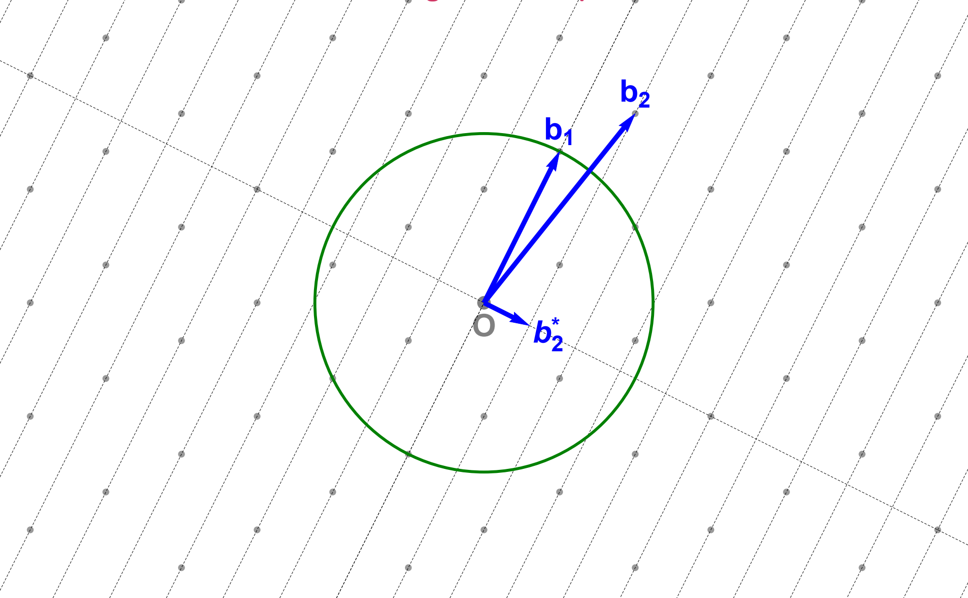

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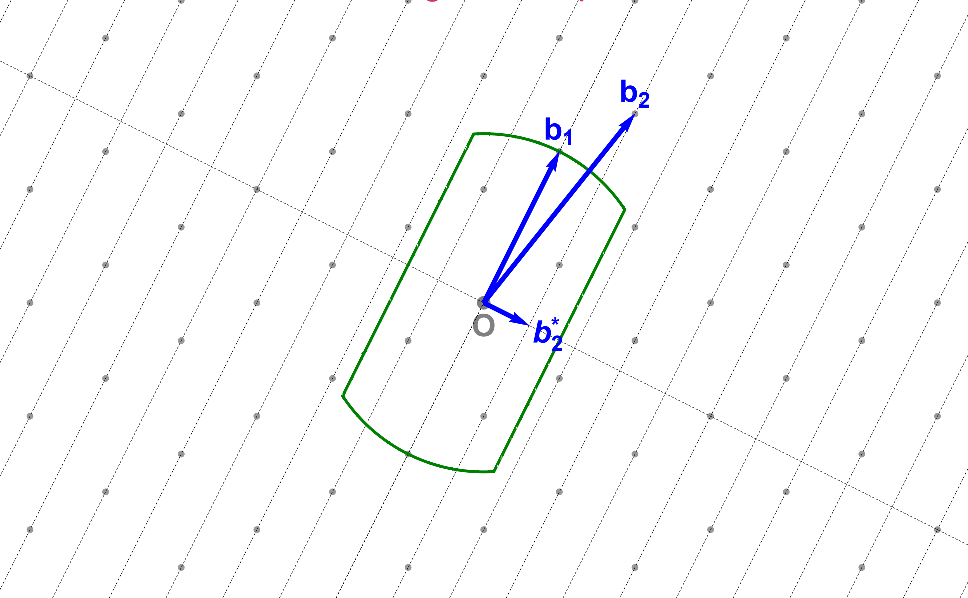

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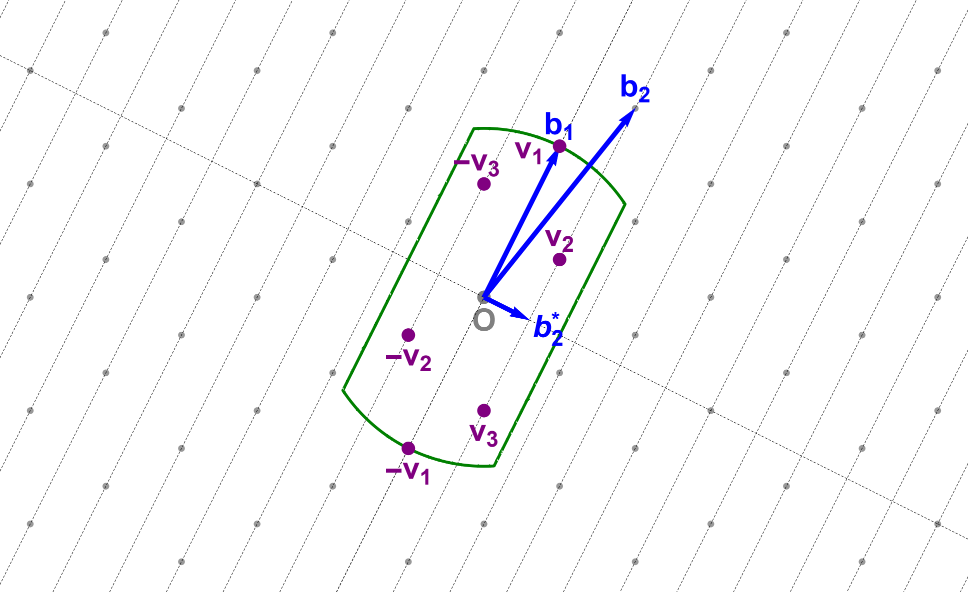

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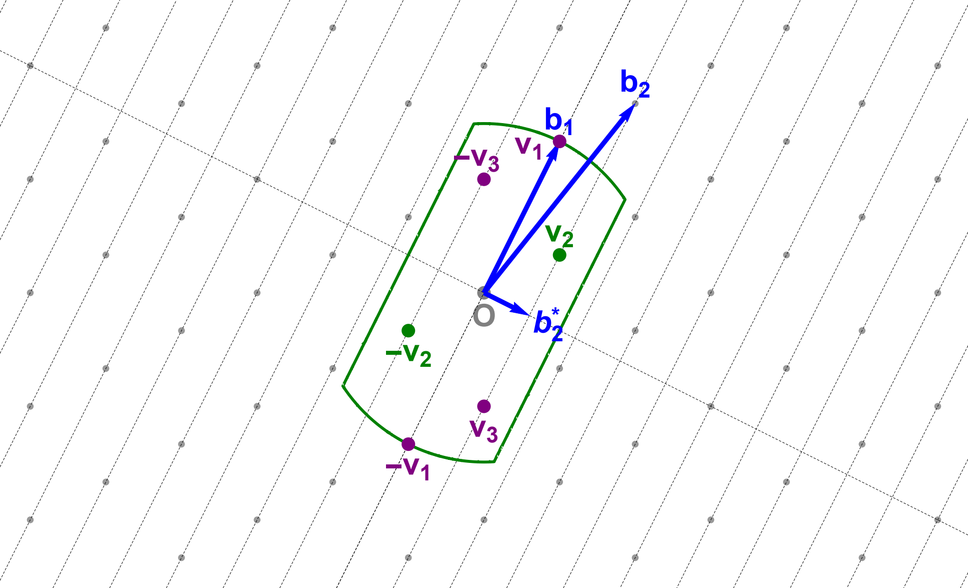

Pruned enumeration

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“Well-chosen bounding functions lead asymptotically to an exponential speedup of about $2^{n/4}$ over basic enumeration, maintaining a success probability $\geq 95\%$.”

– Gama et al., *EUROCRYPT’10*

Pruned enumeration

Overview

"Well-chosen bounding functions lead asymptotically to an exponential speedup of about $2^{n/4}$ *over basic enumeration, maintaining a success probability* $\geq 95\%$*."*

– Gama et al., *EUROCRYPT'10*

"With extreme pruning, the probability of finding the desired vector is actually rather low (say, 0.1%*), but surprisingly, the running time of the enumeration is reduced by a much more significant factor (say, much more than* 1000*)."*

– Gama et al., *EUROCRYPT'10*

Outline

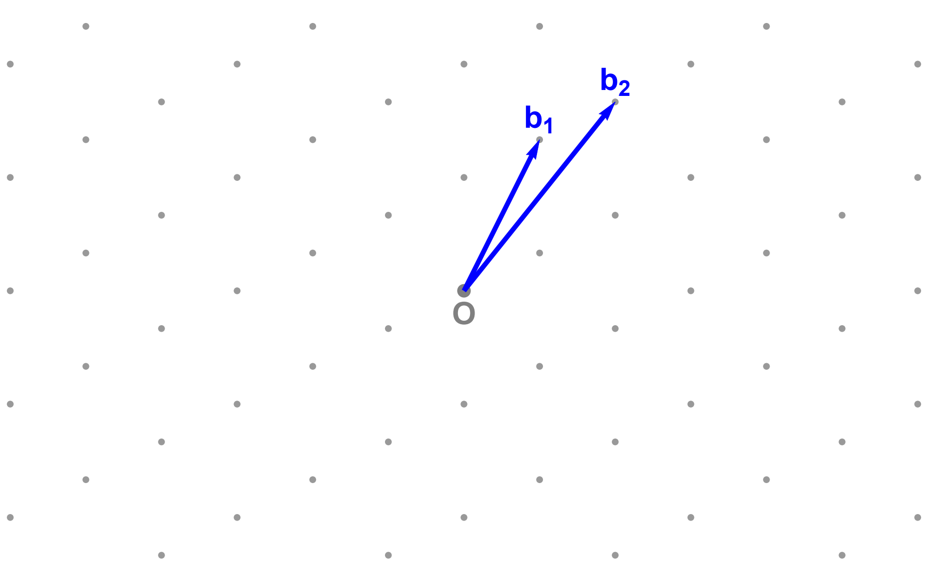
Enumeration algorithms
- Fincke-Pohst enumeration
- Kannan enumeration
- Pruning the enumeration tree

The Voronoi cell algorithm

Sieving algorithms
- Nguyen-Vidick sieve
- Multiple levels
- GaussSieve
- Locality-sensitive hashing

The Voronoi cell algorithm

1. Construct the Voronoi cell of $\mathcal{L}$



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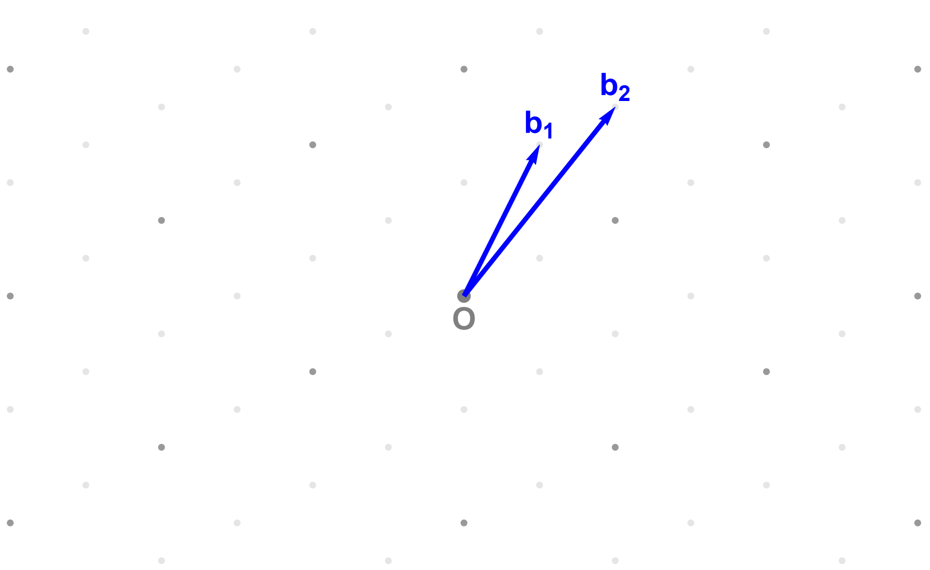

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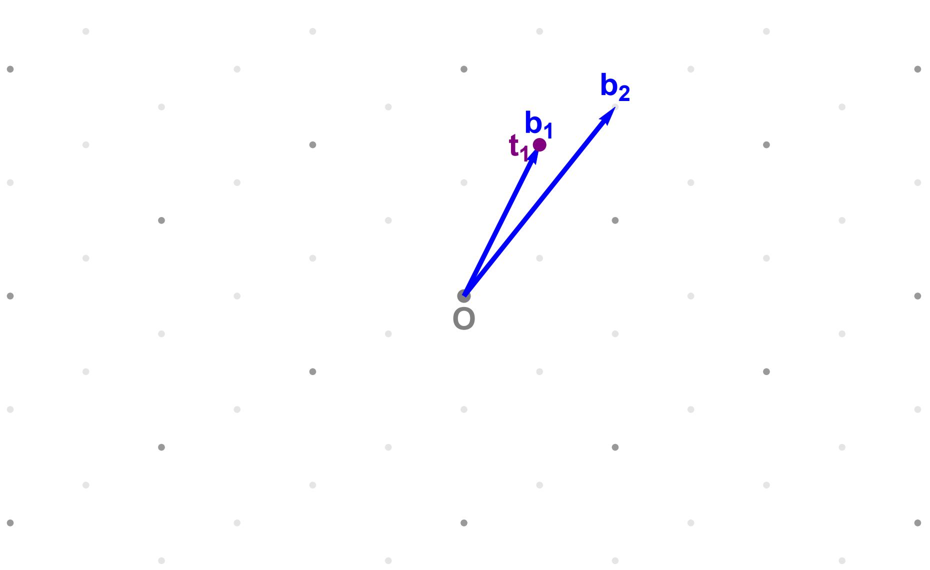

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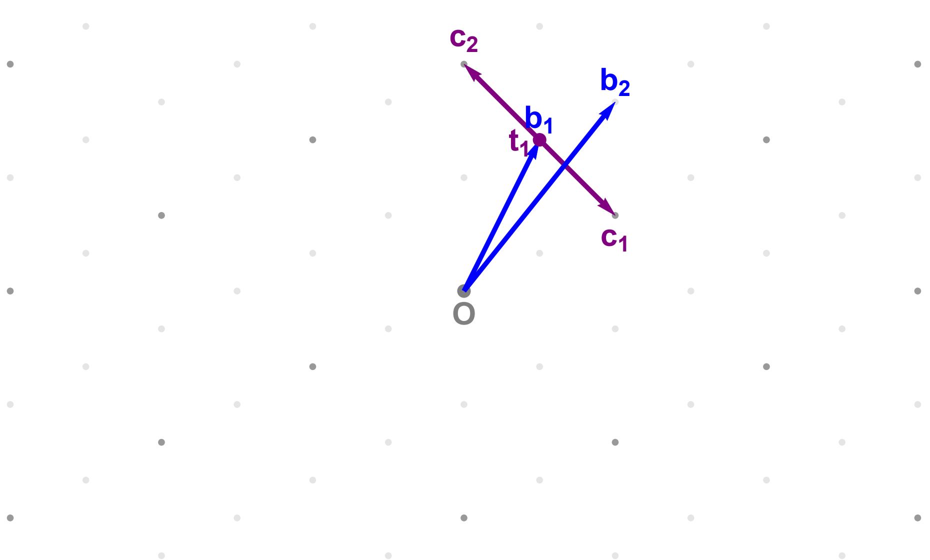

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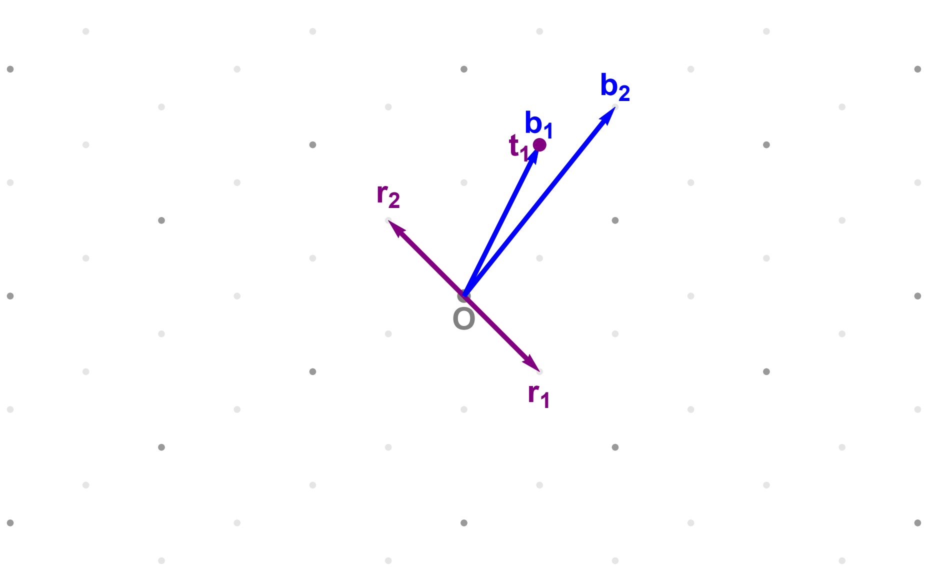

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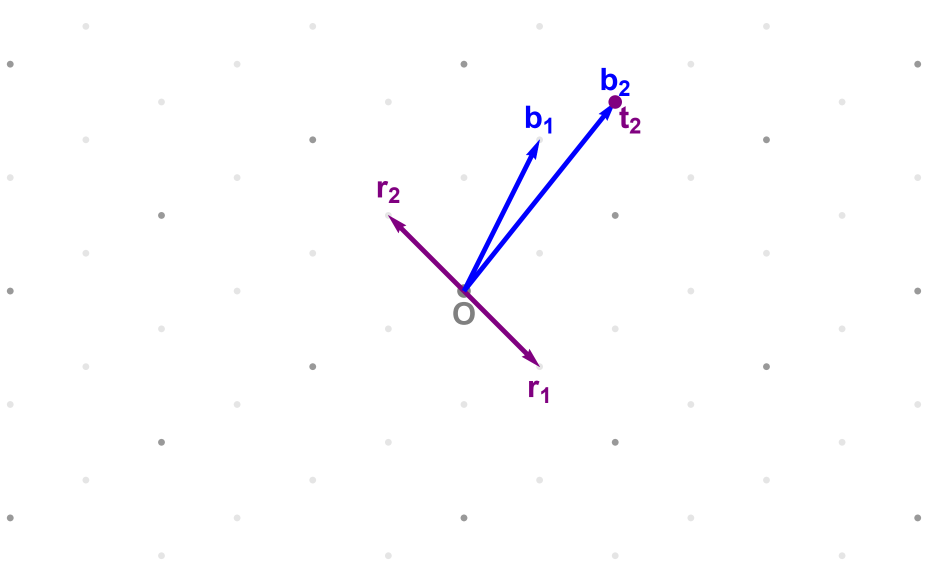

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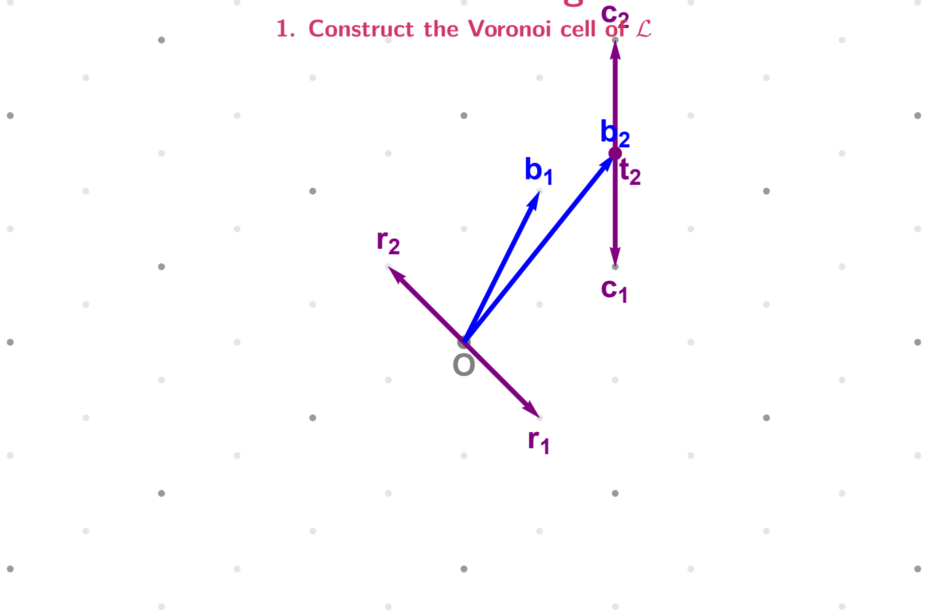

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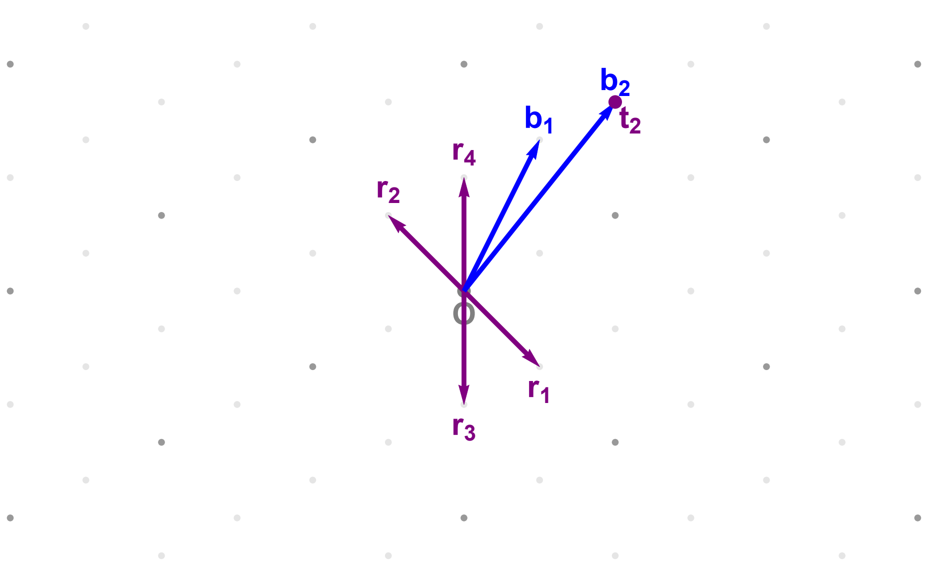

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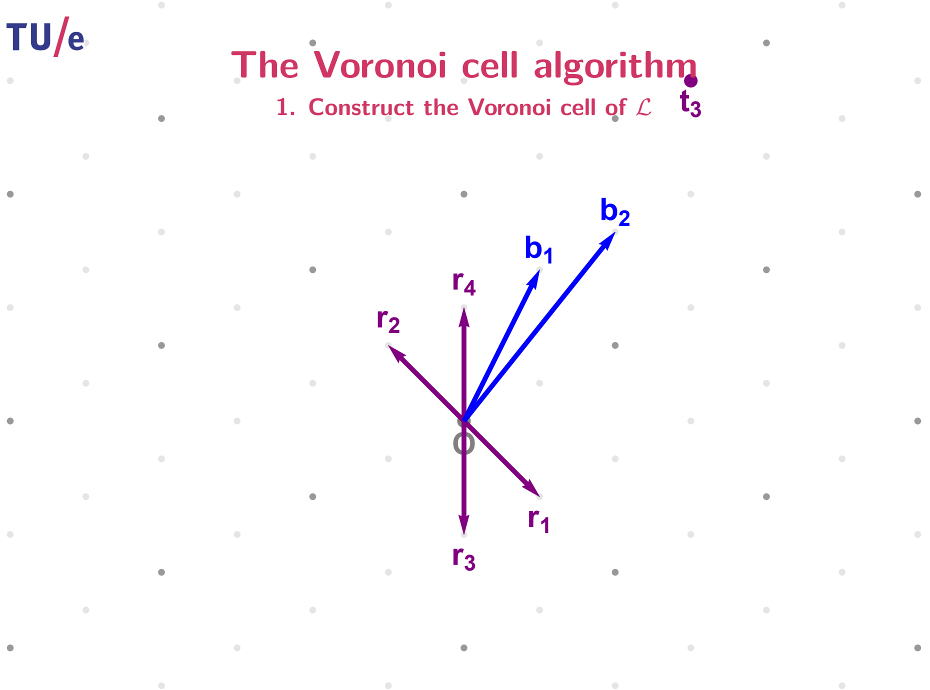

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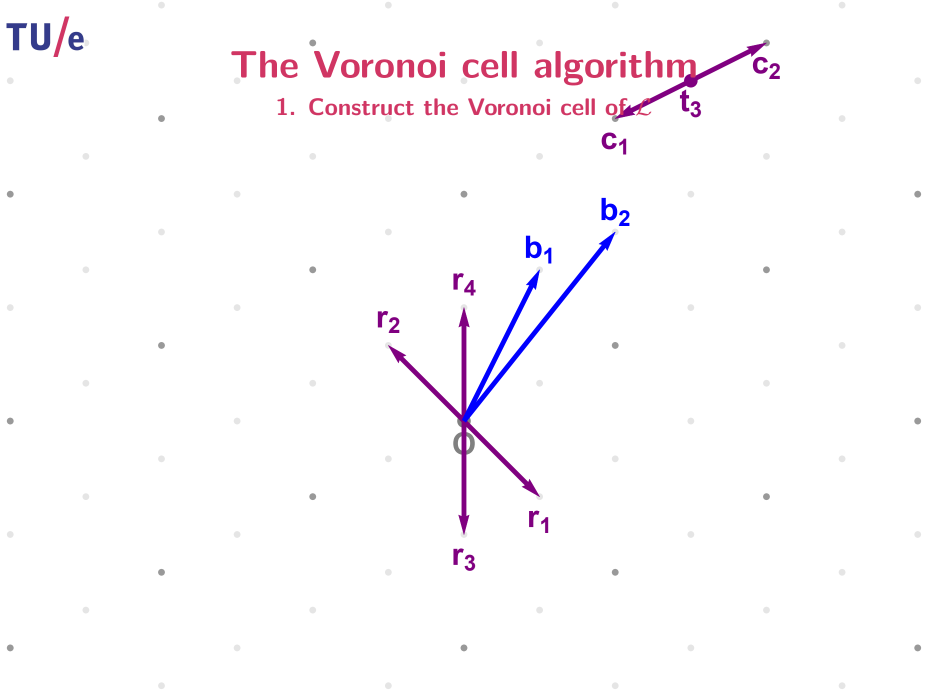

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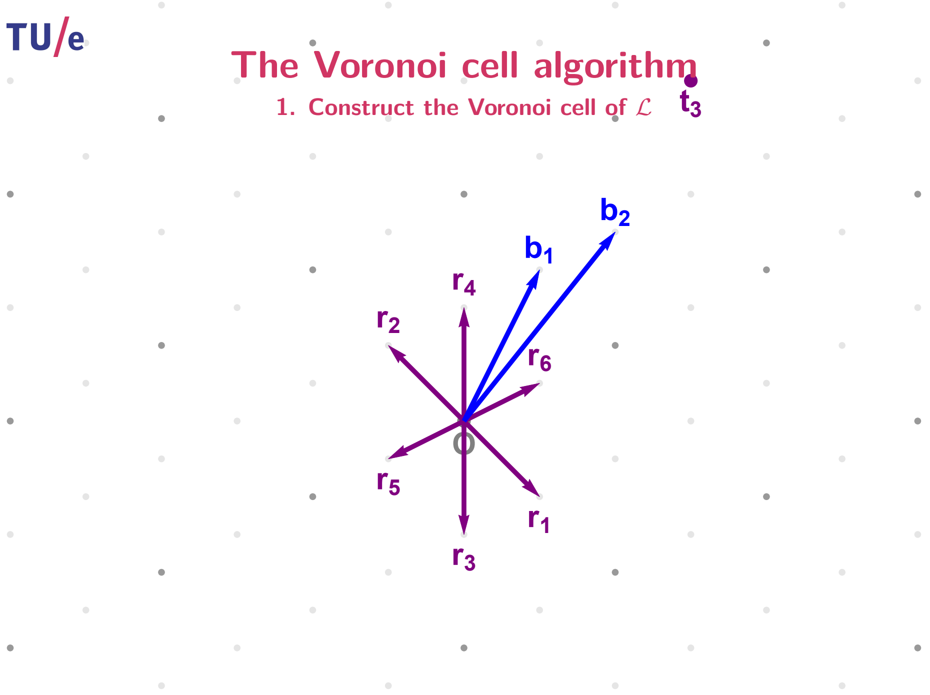

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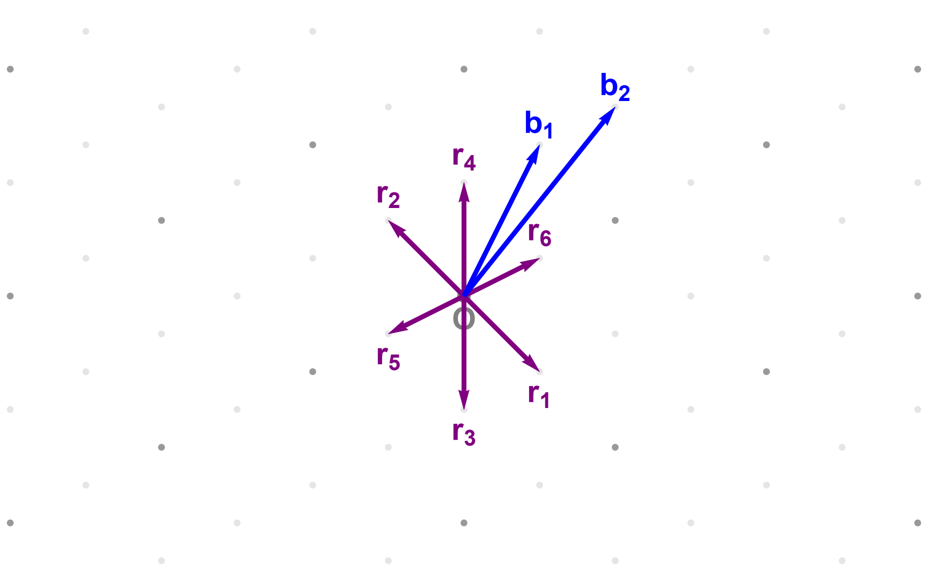

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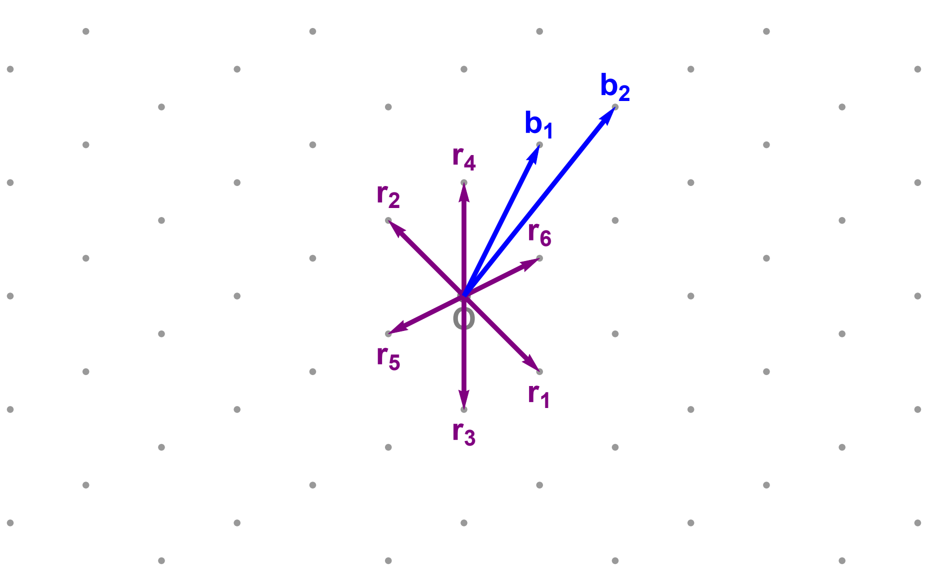

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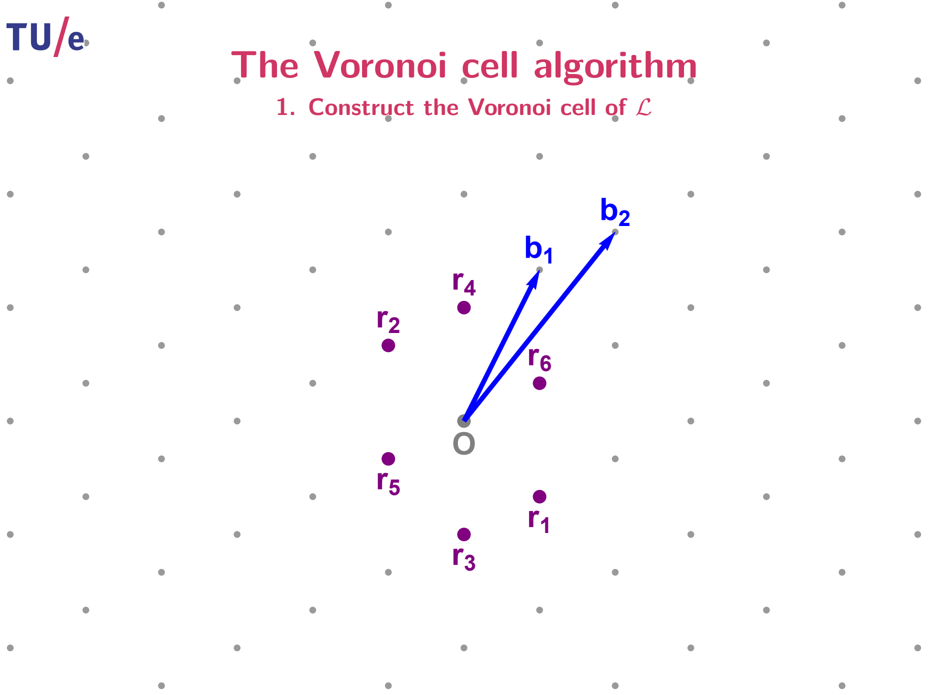

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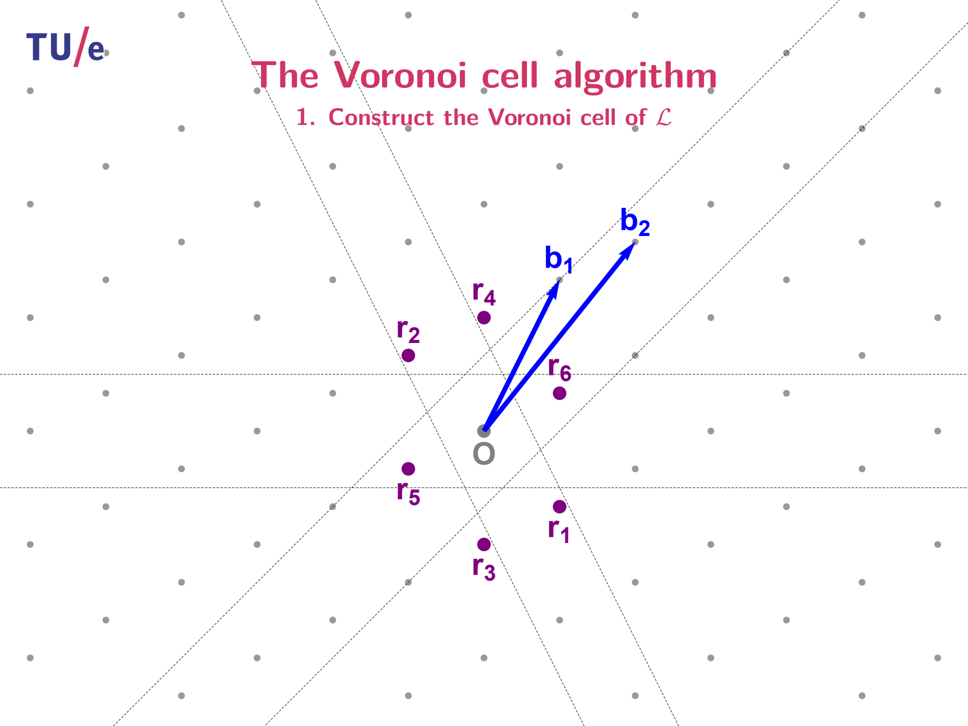

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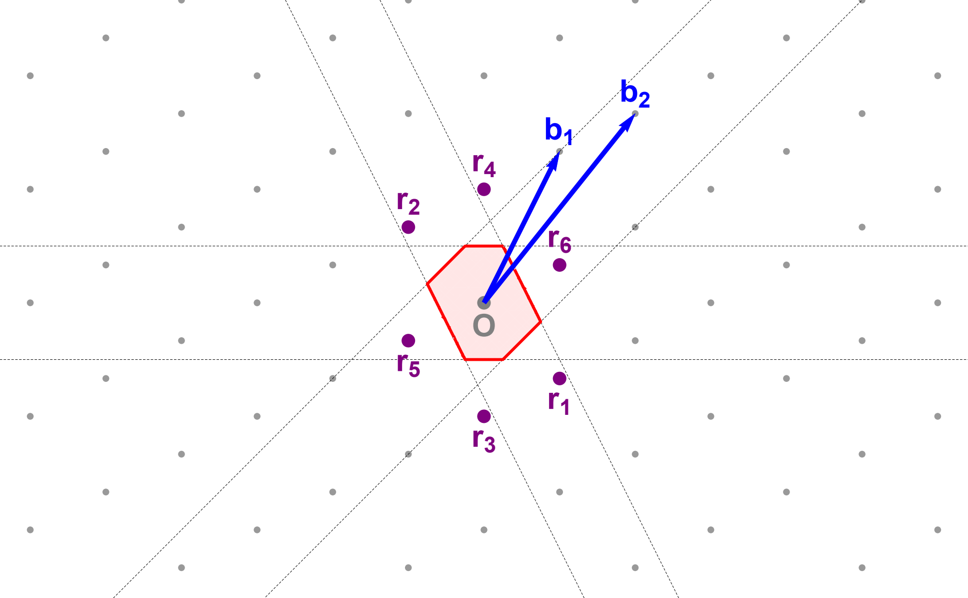
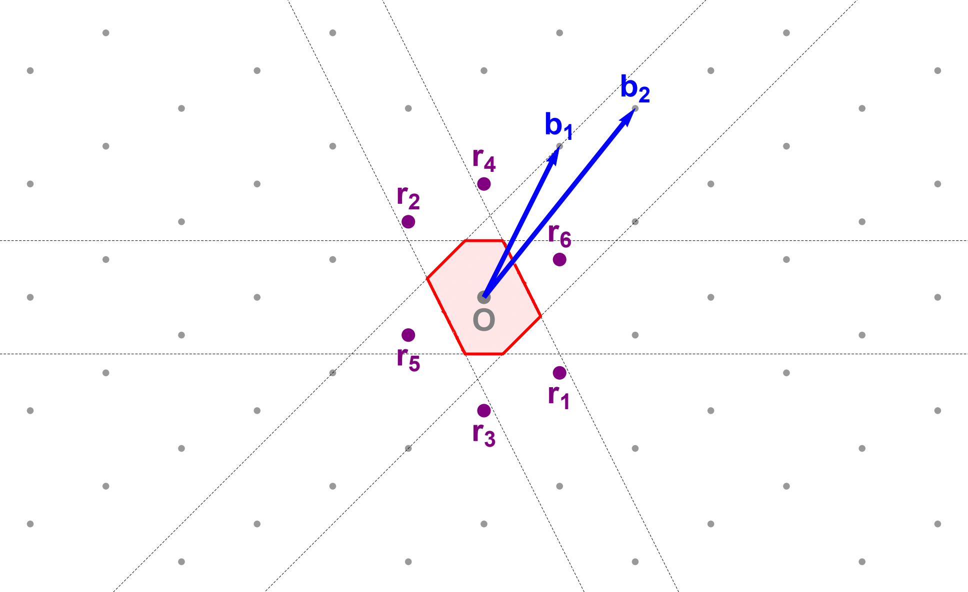


The Voronoi cell algorithm

2. Find a shortest vector among the relevant vectors



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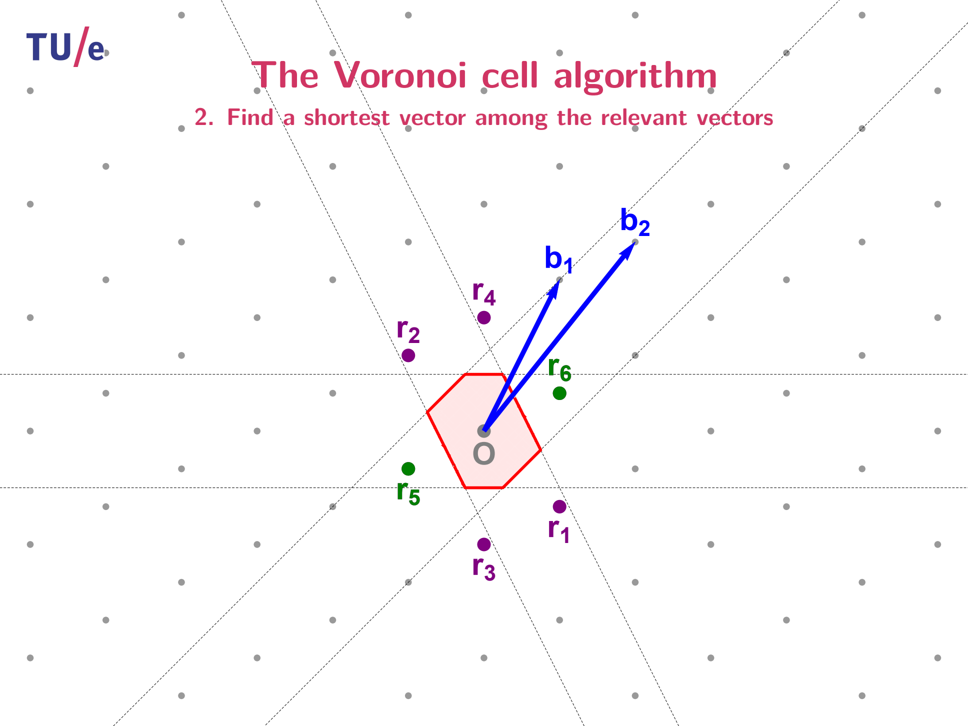


The Voronoi cell algorithm

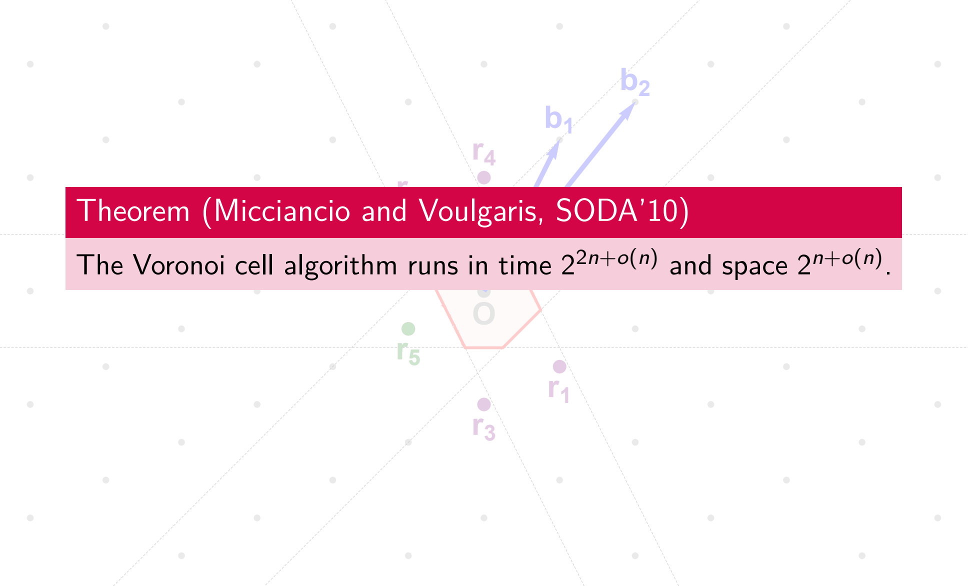
Overview

The Voronoi cell algorithm

Overview

Theorem (Micciancio and Voulgaris, SODA'10)

The Voronoi cell algorithm runs in time $2^{2n+o(n)}$ and space $2^{n+o(n)}$.



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- Locality-sensitive hashing

Nguyen-Vidick sieve

1. Sample a list L of random lattice vectors

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2. Split L into centers C and short vectors R

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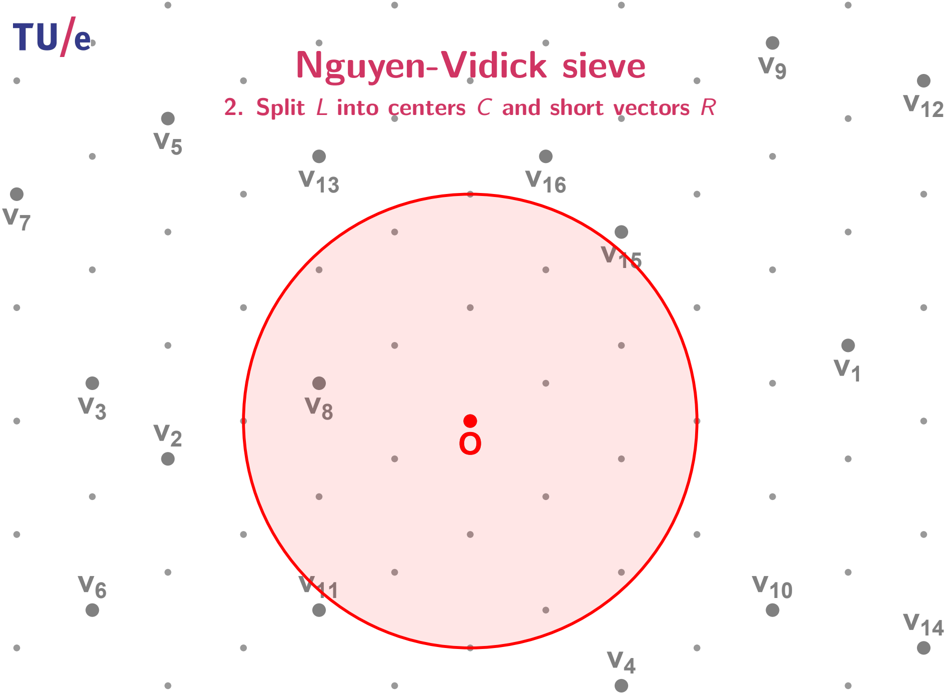

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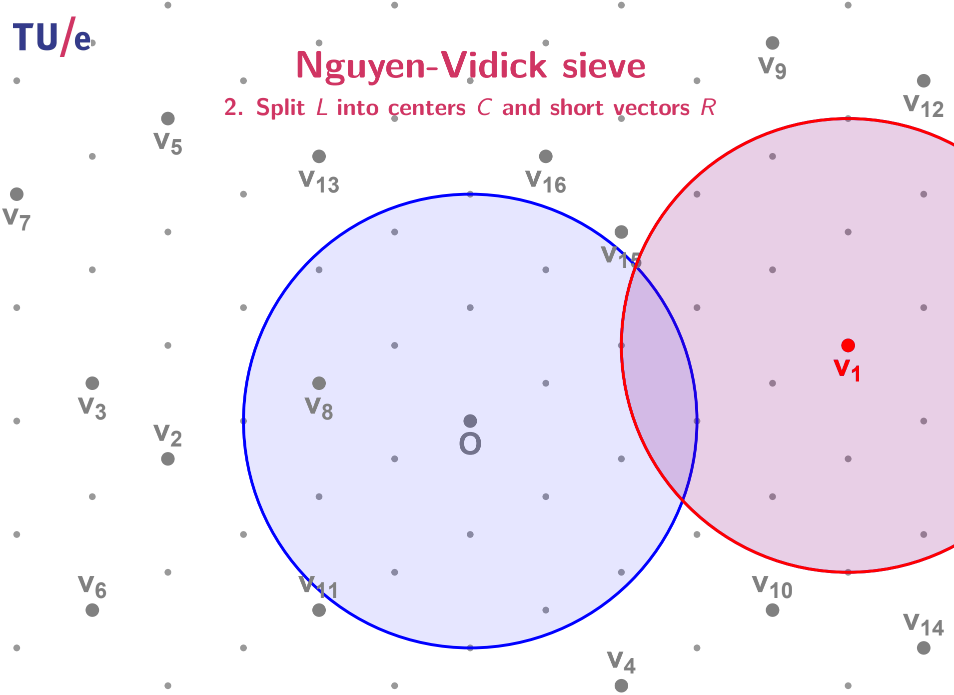

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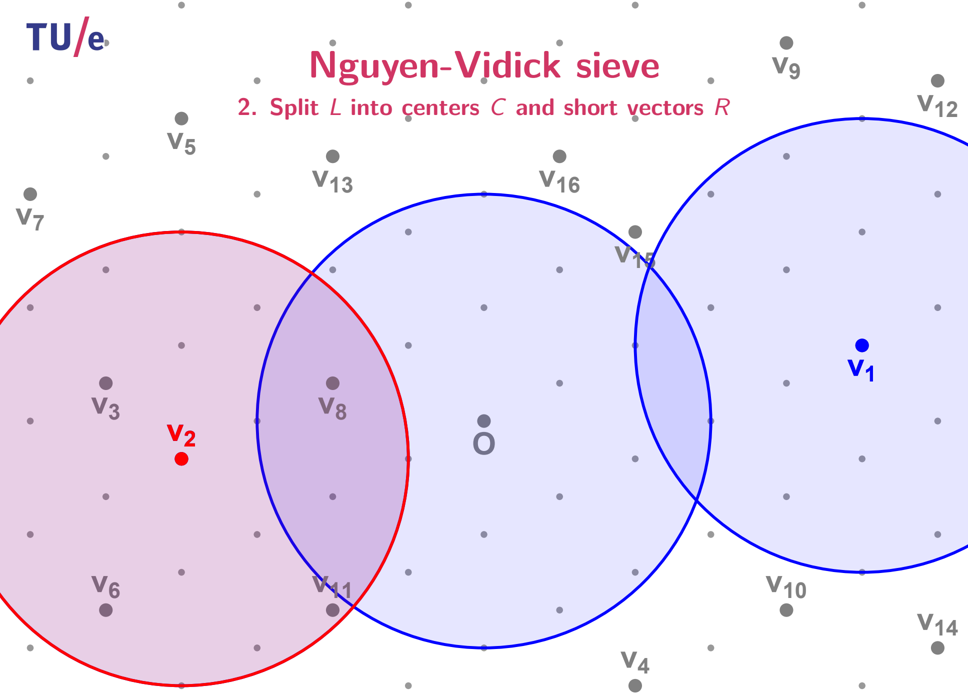

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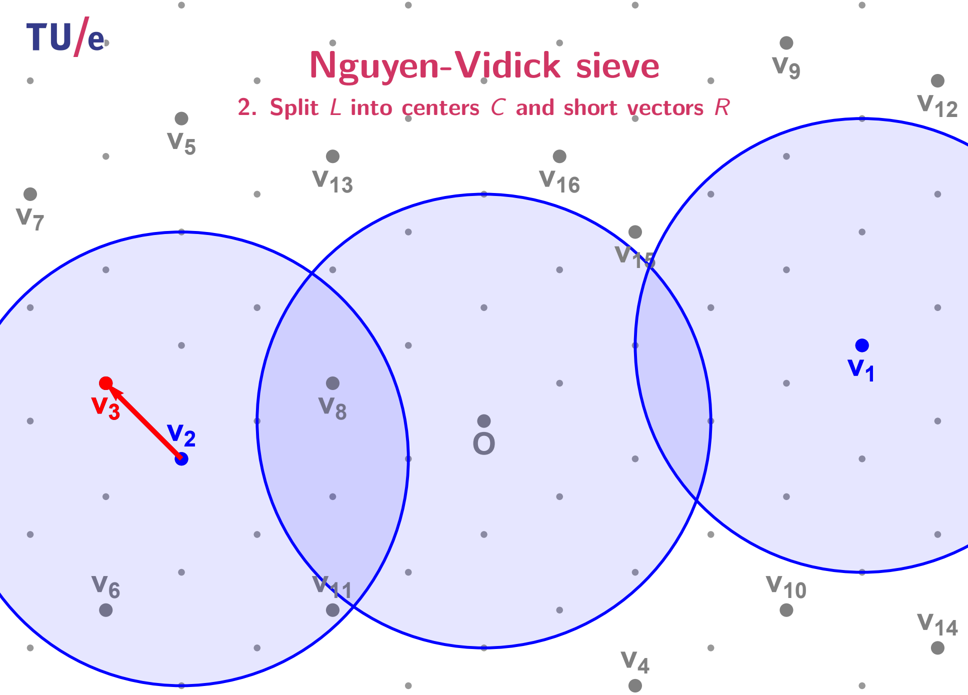

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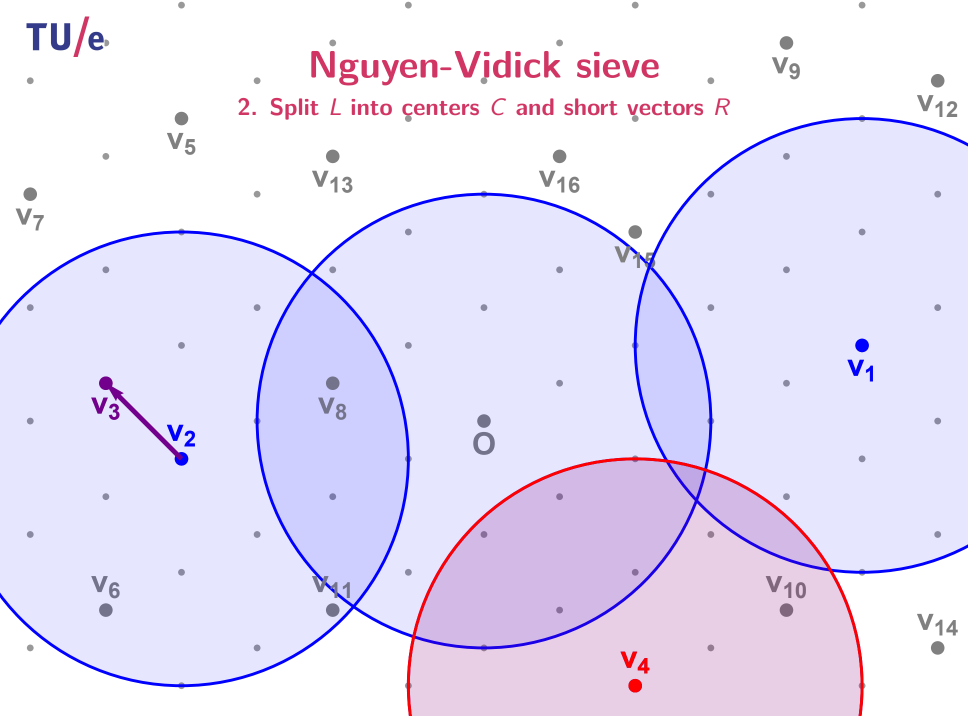

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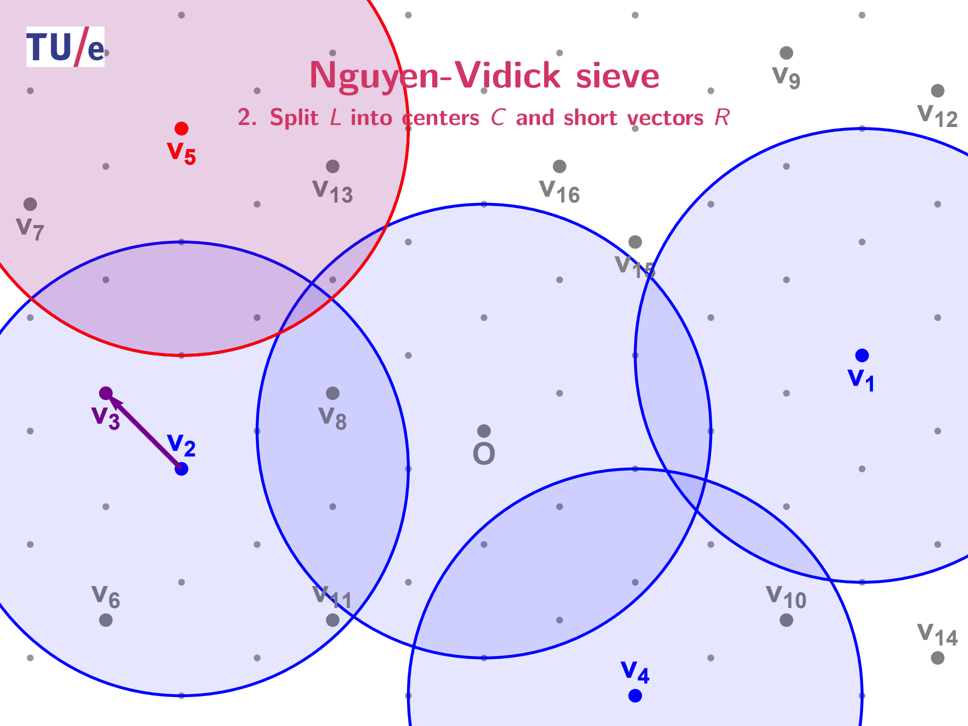

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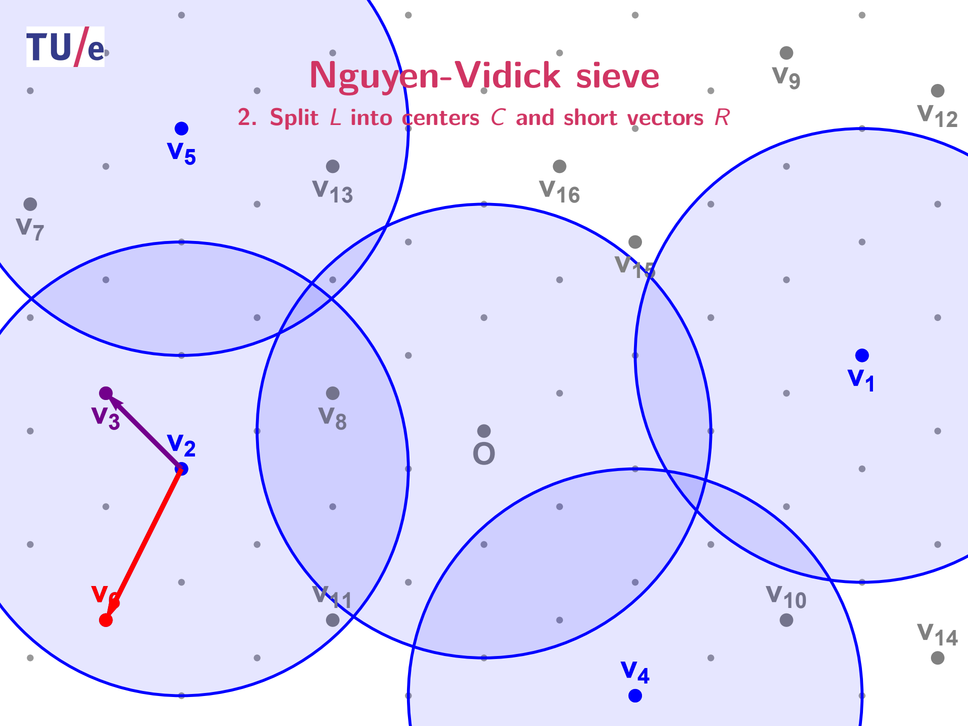

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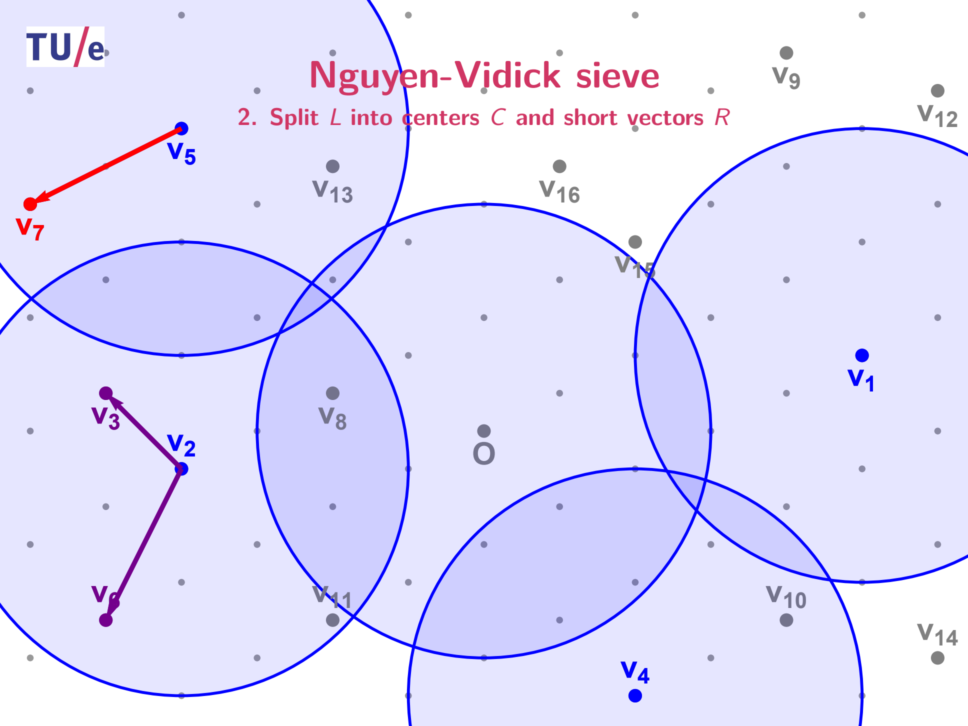

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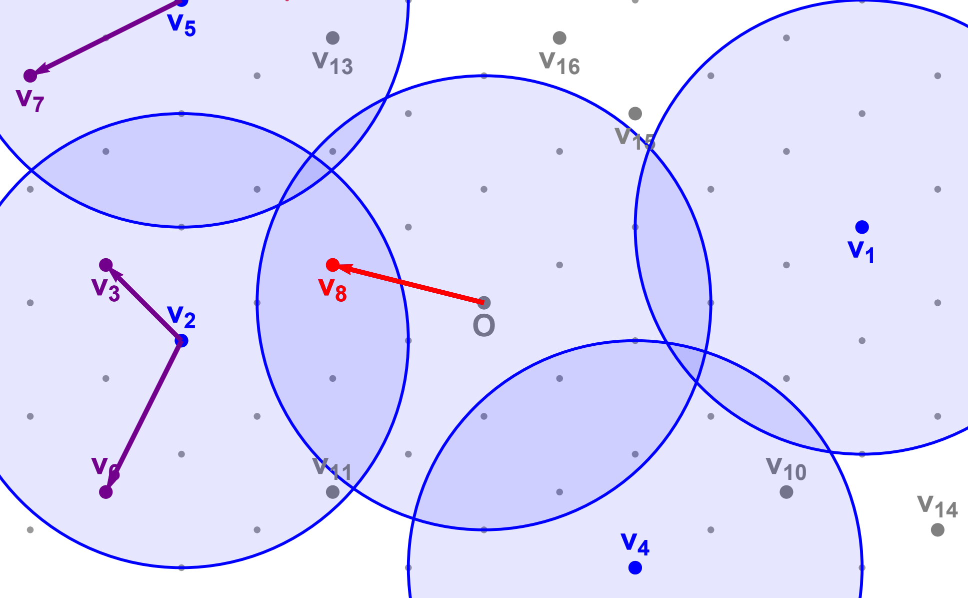

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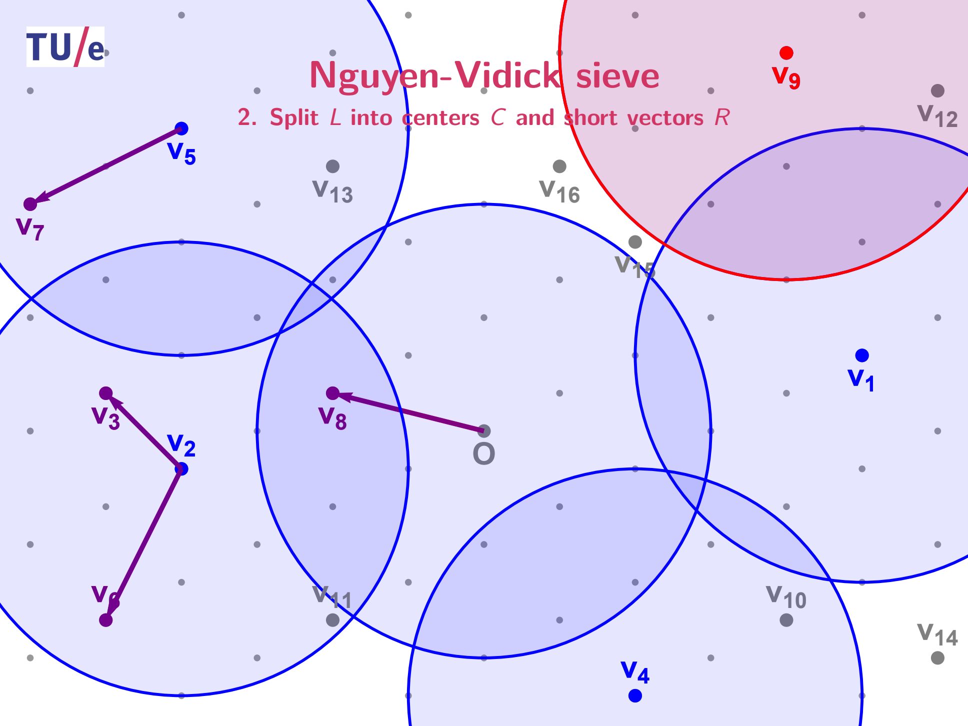

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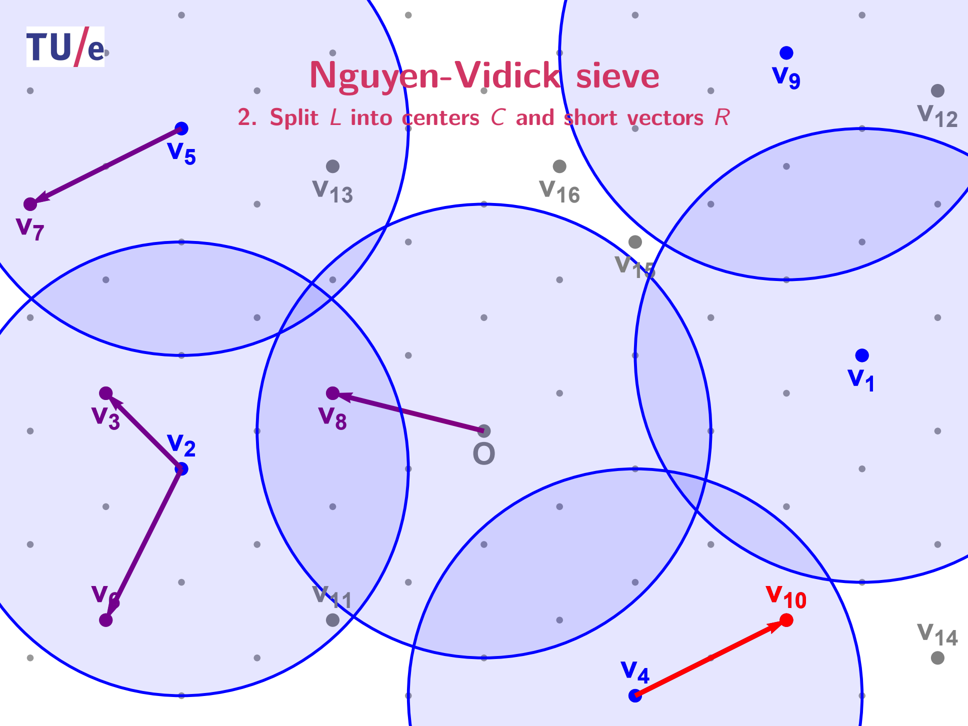

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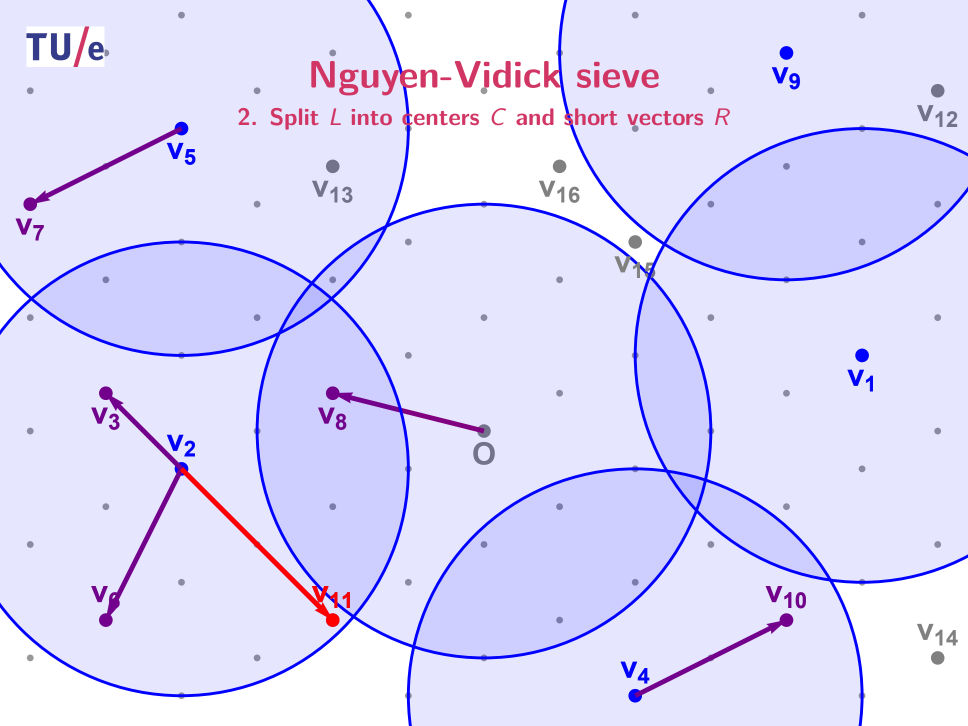

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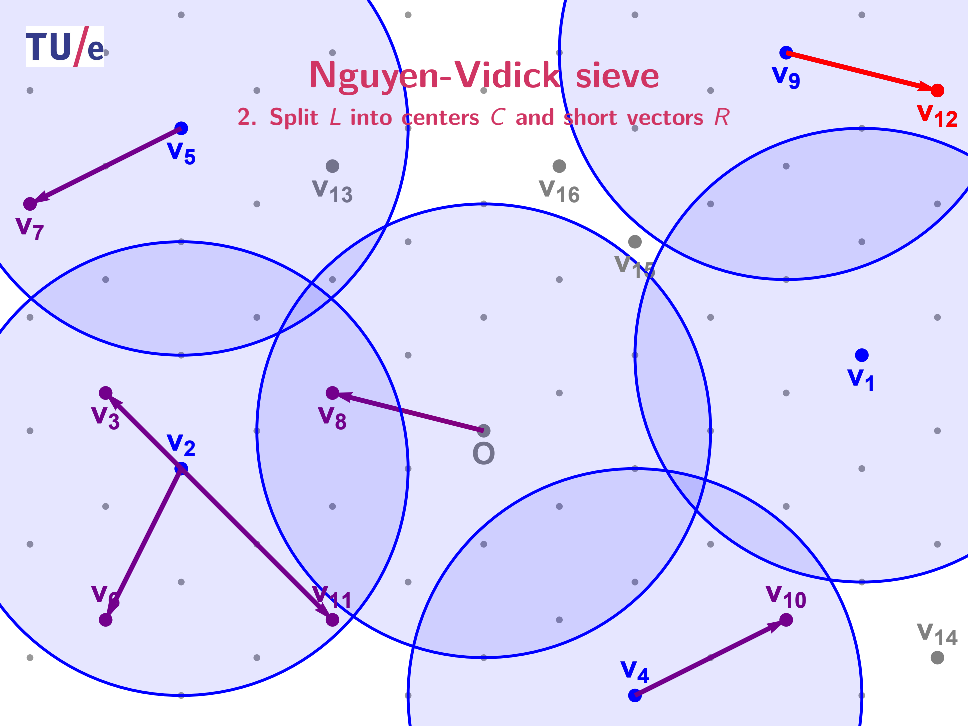

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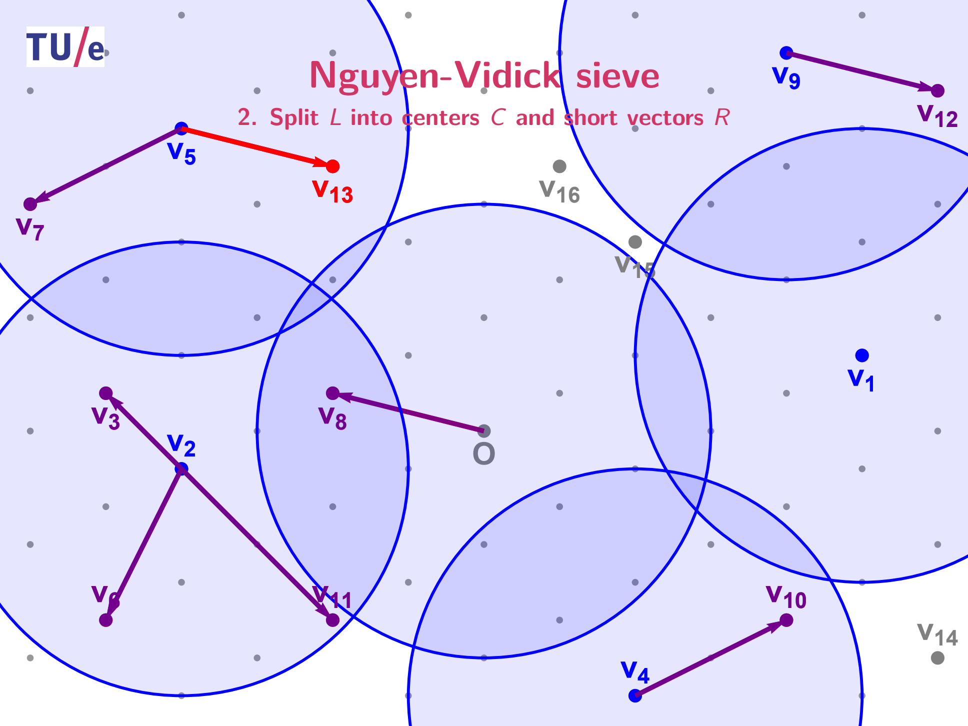

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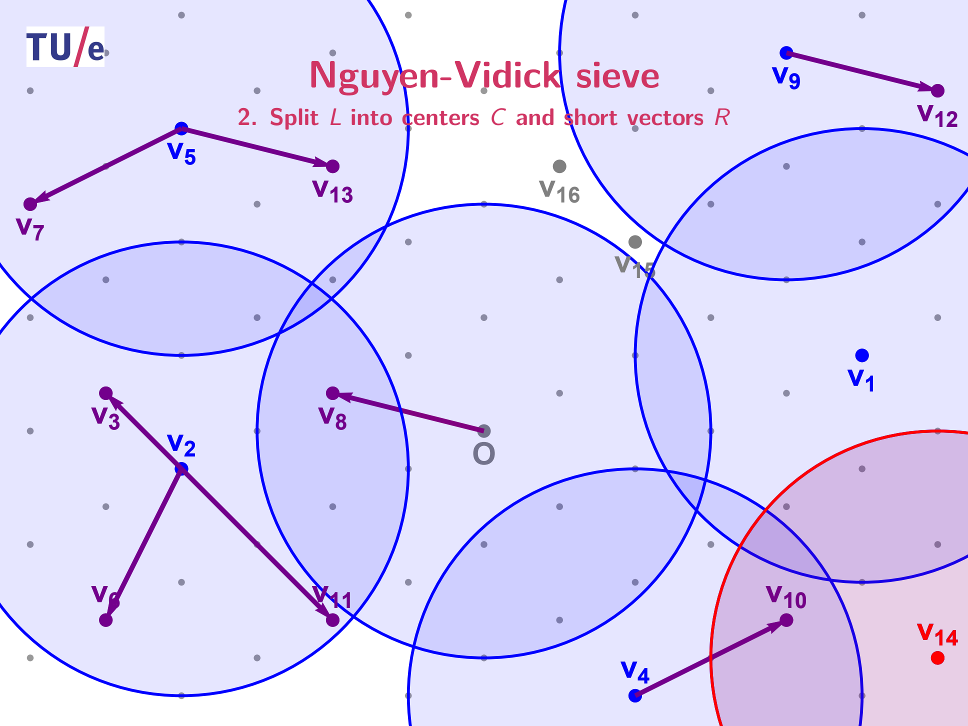

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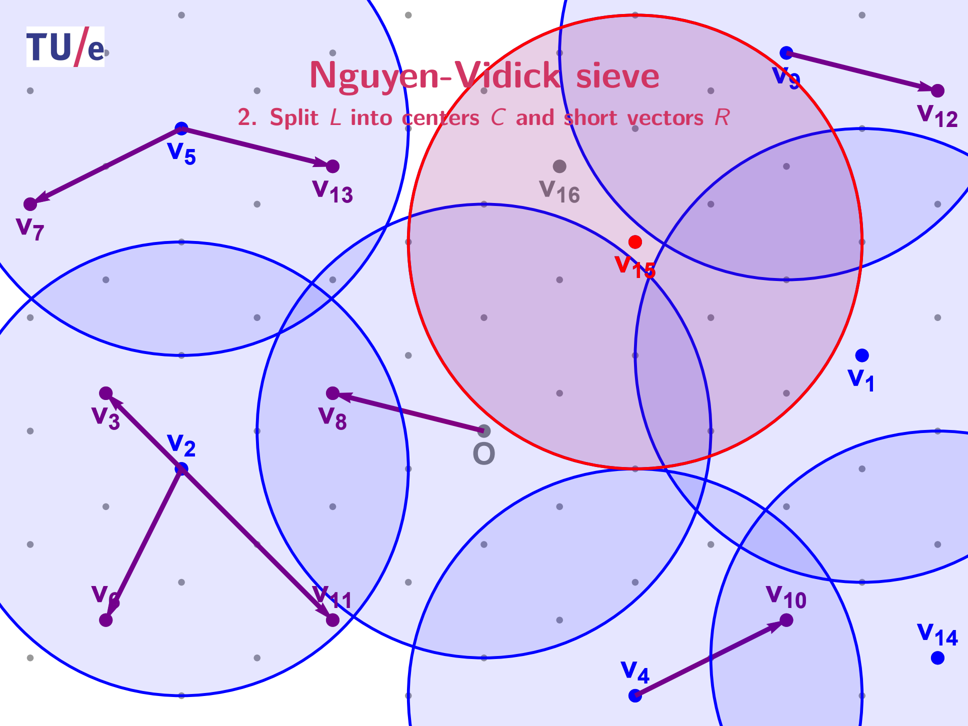

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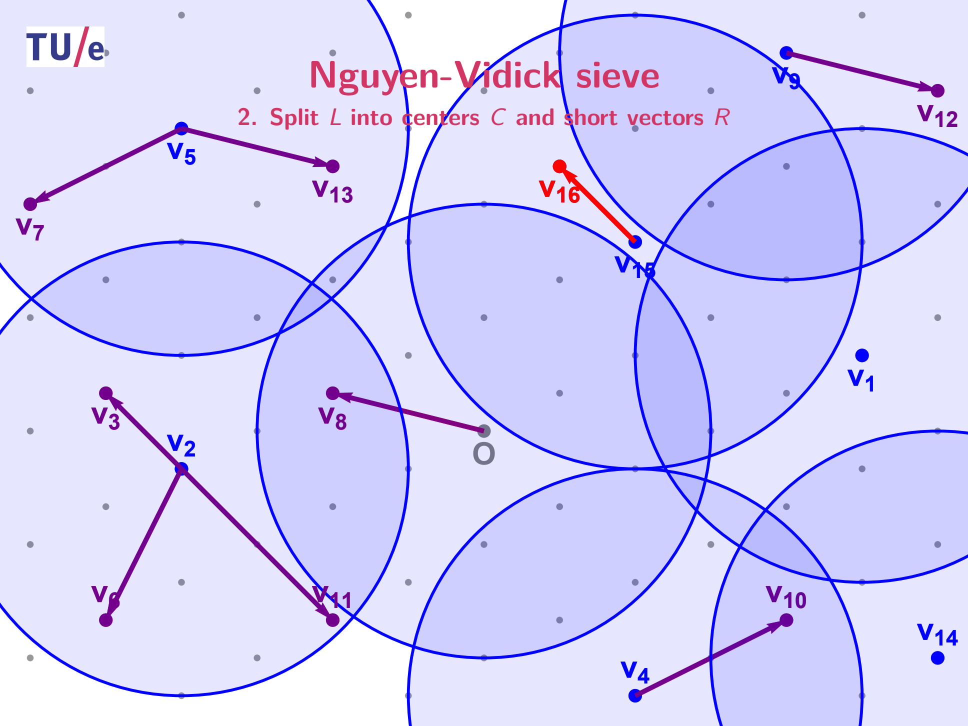

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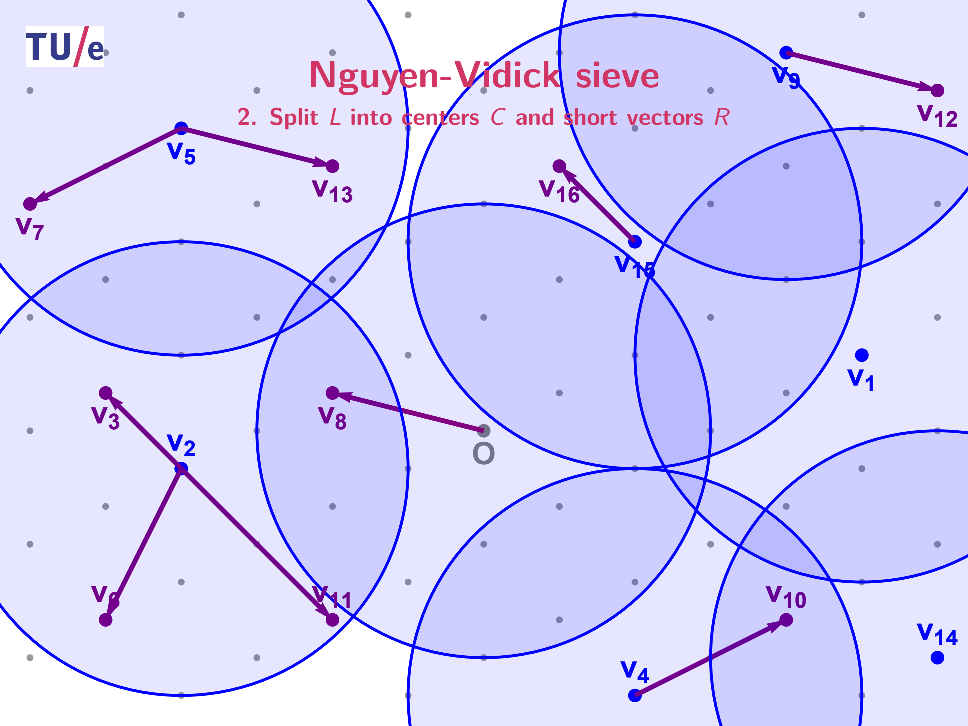
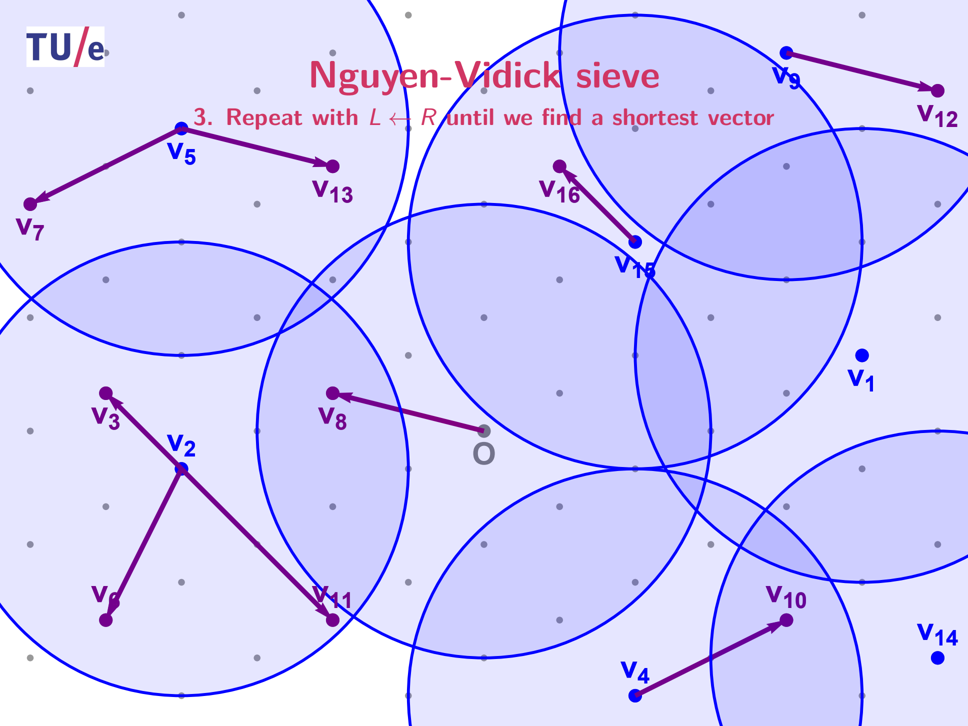


Nguyen-Vidick sieve

3. Repeat with $L \leftarrow R$ until we find a shortest vector



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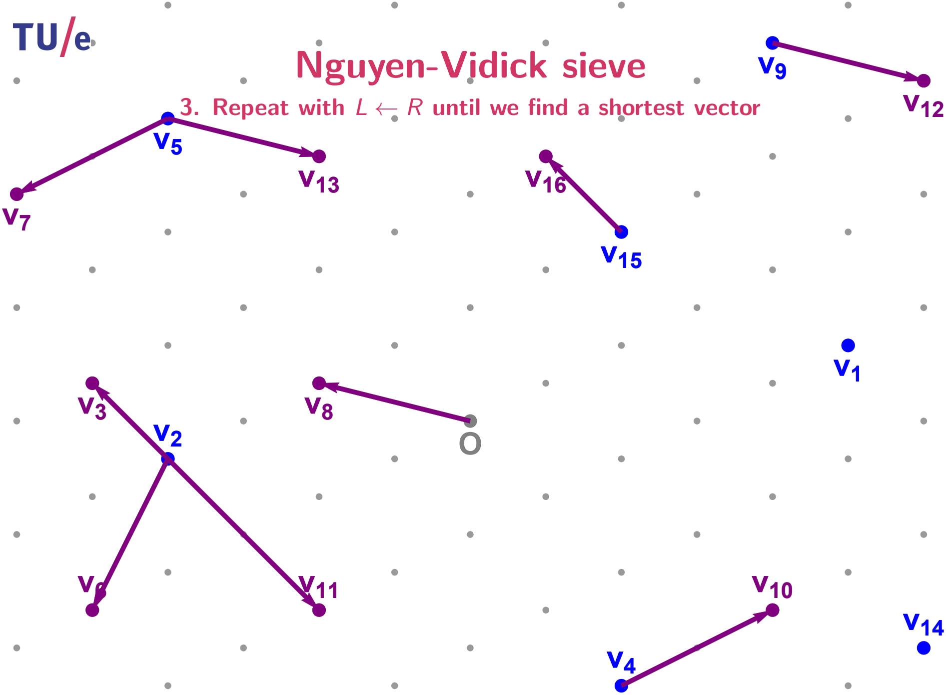

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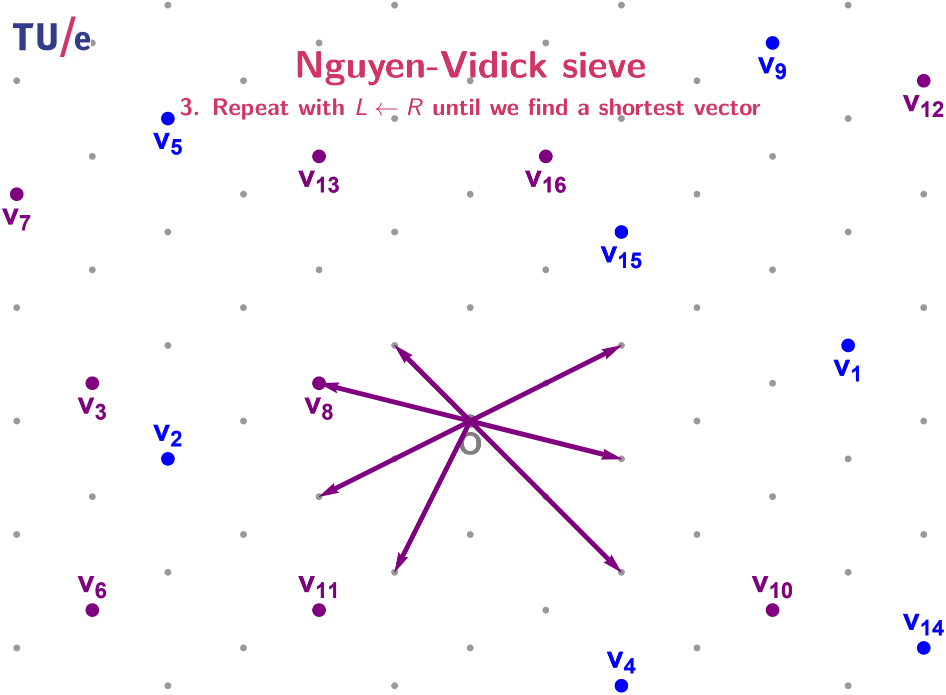

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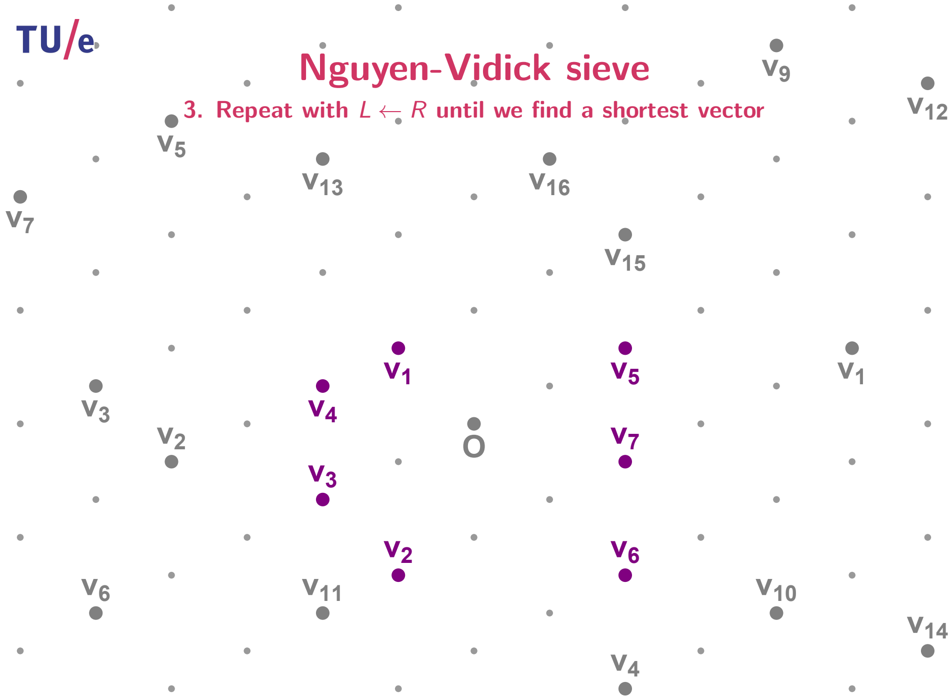

Nguyen-Vidick sieve

Overview

Nguyen-Vidick sieve

Overview

Heuristic (Nguyen and Vidick, J. Math. Crypt. '08)

The Nguyen-Vidick sieve runs in time $(4/3)^n$ and space $\sqrt{4/3}^n$.

Two-level sieve

1. Sample a list L of random lattice vectors

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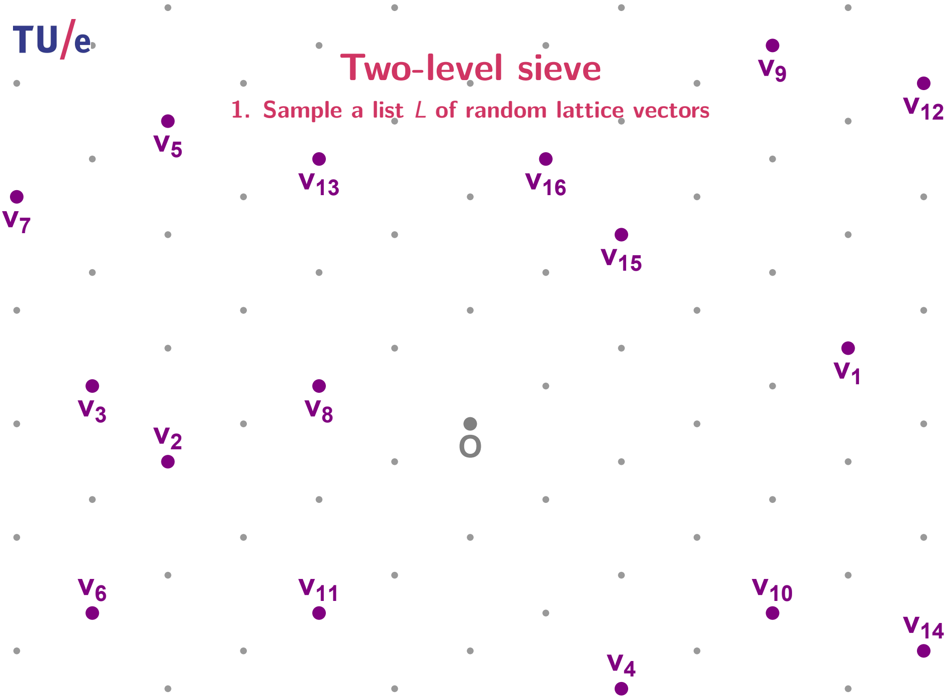
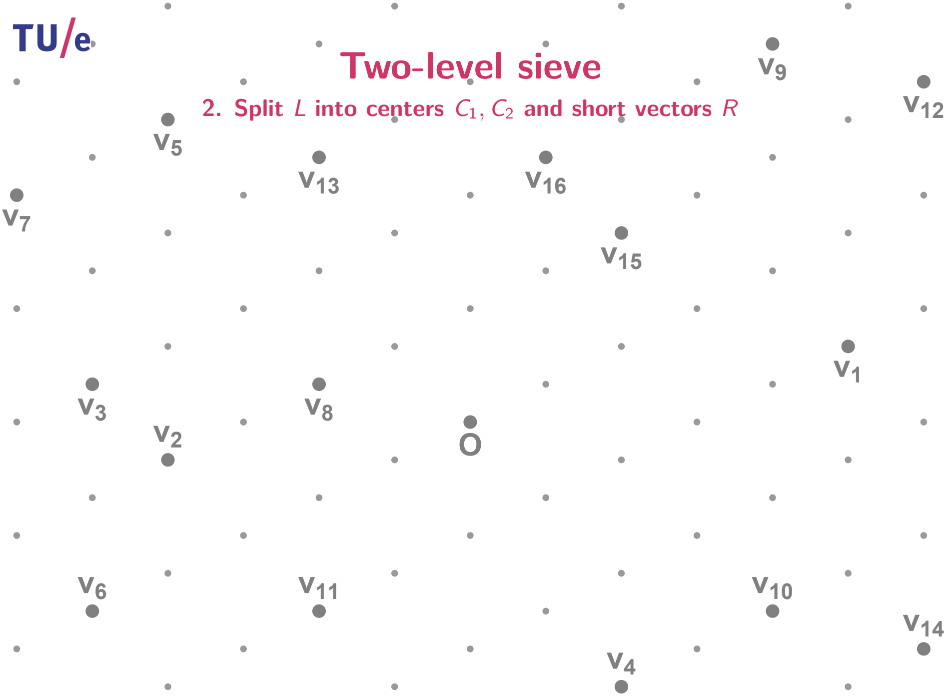


Two-level sieve

2. Split L into centers C_1, C_2 and short vectors R



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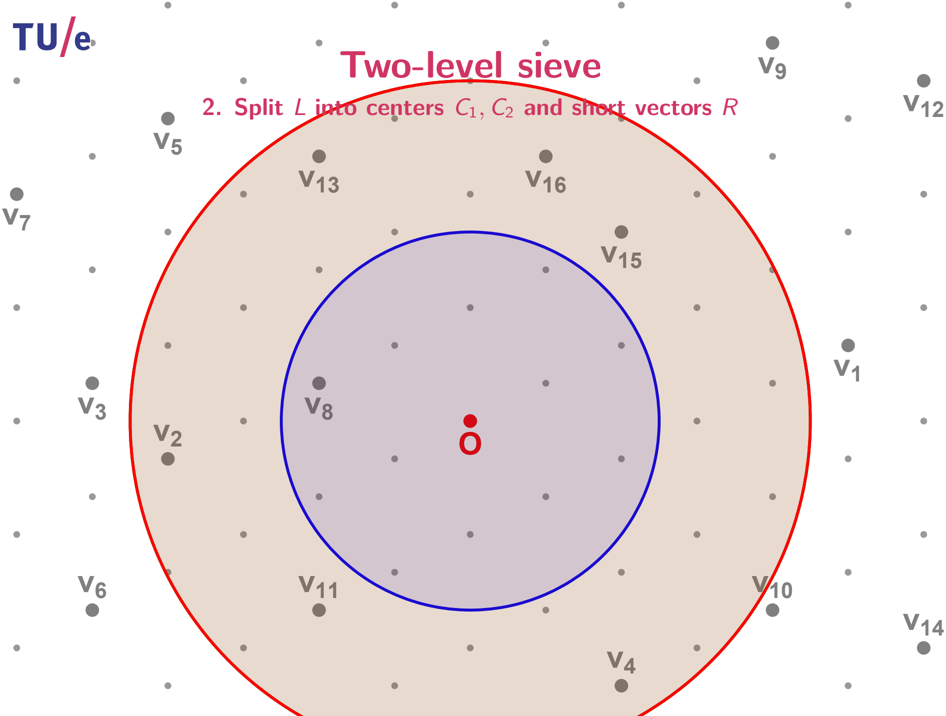

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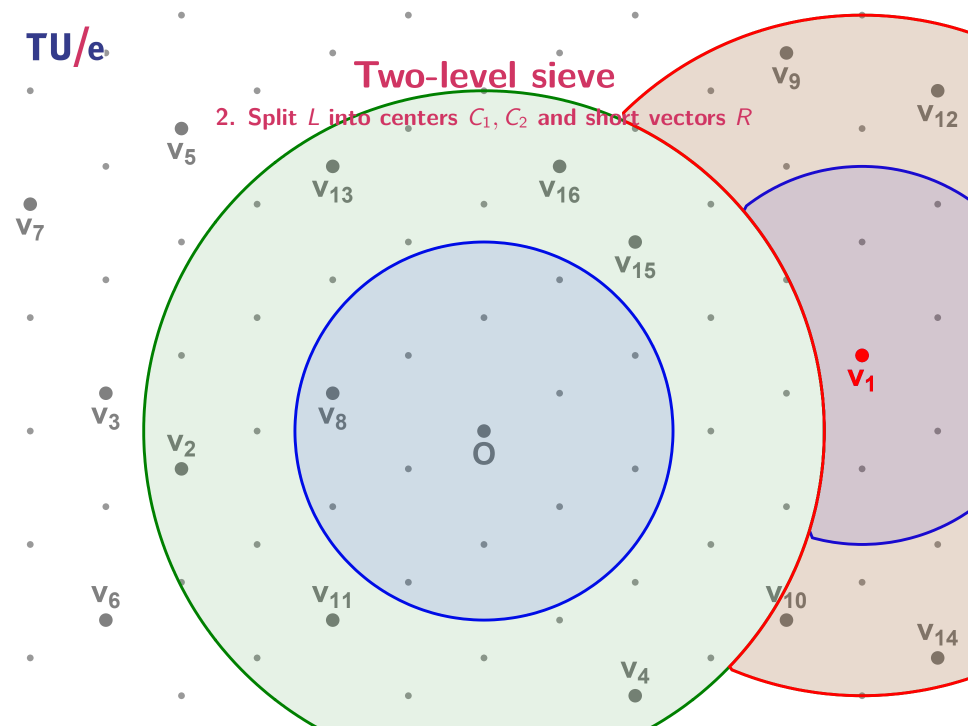

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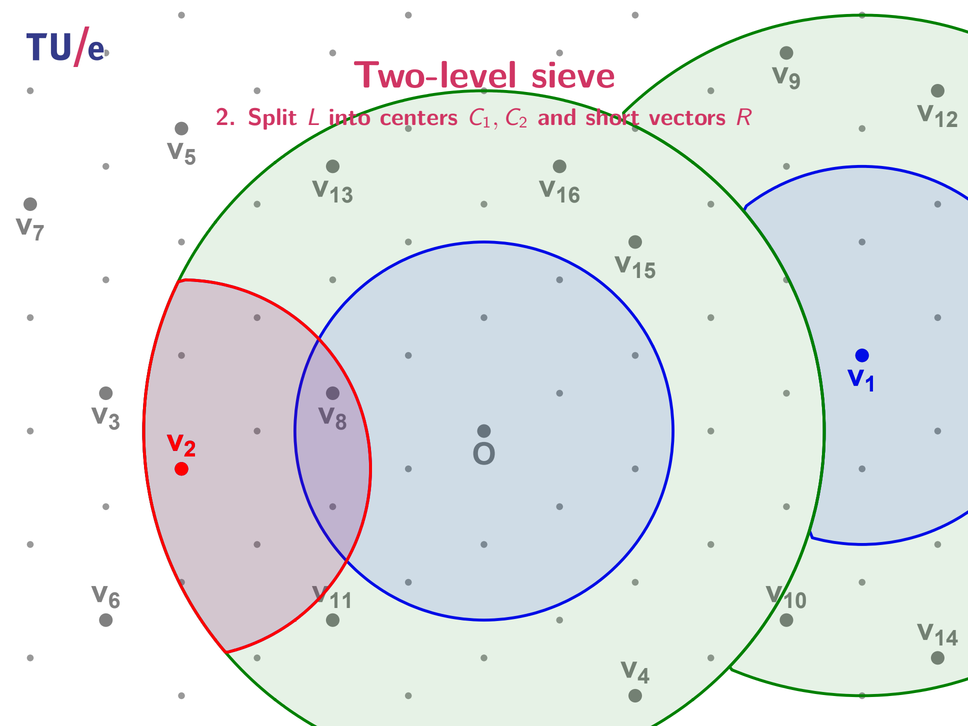

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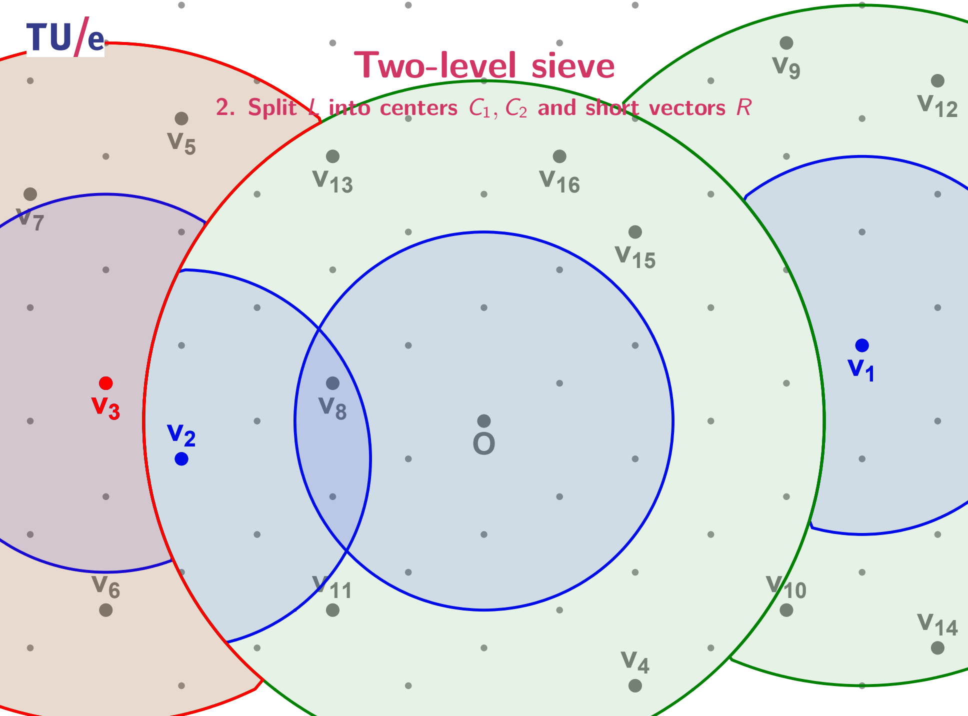

TU/e

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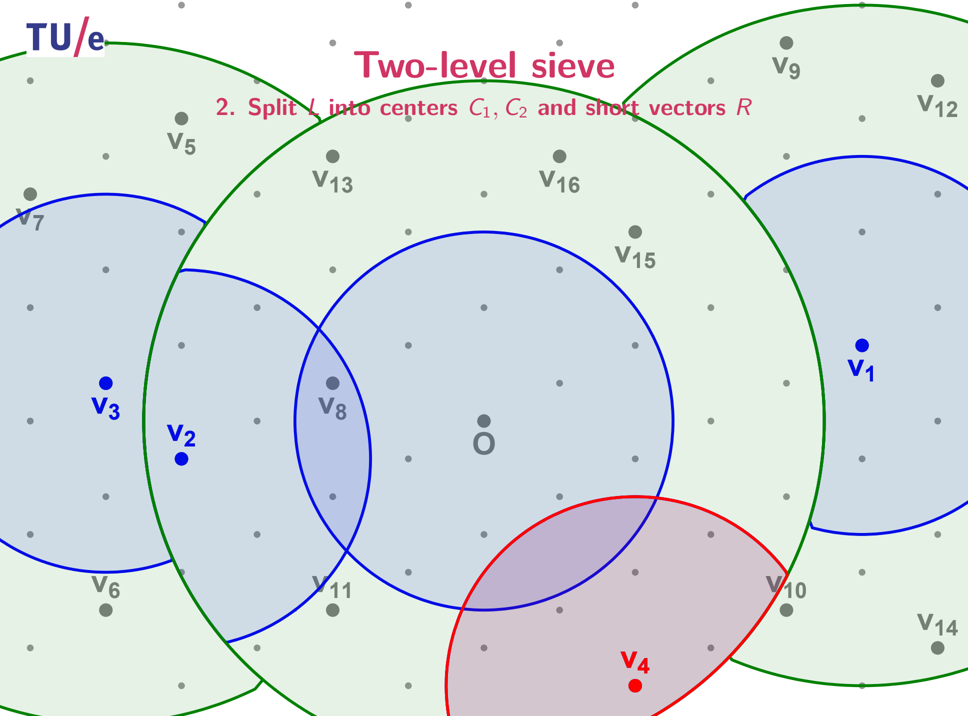

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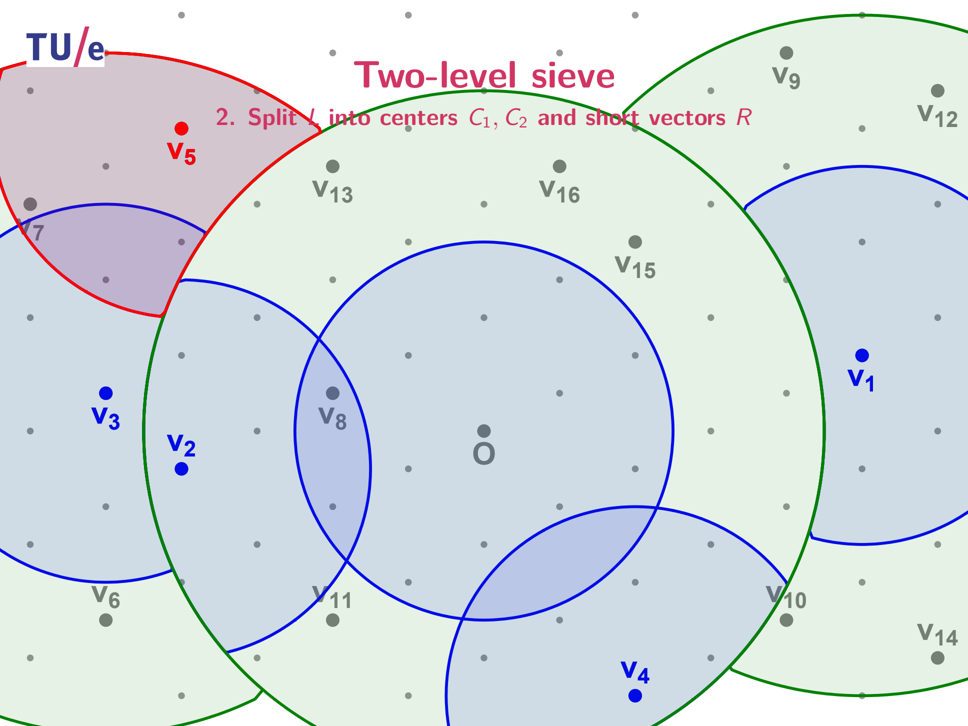

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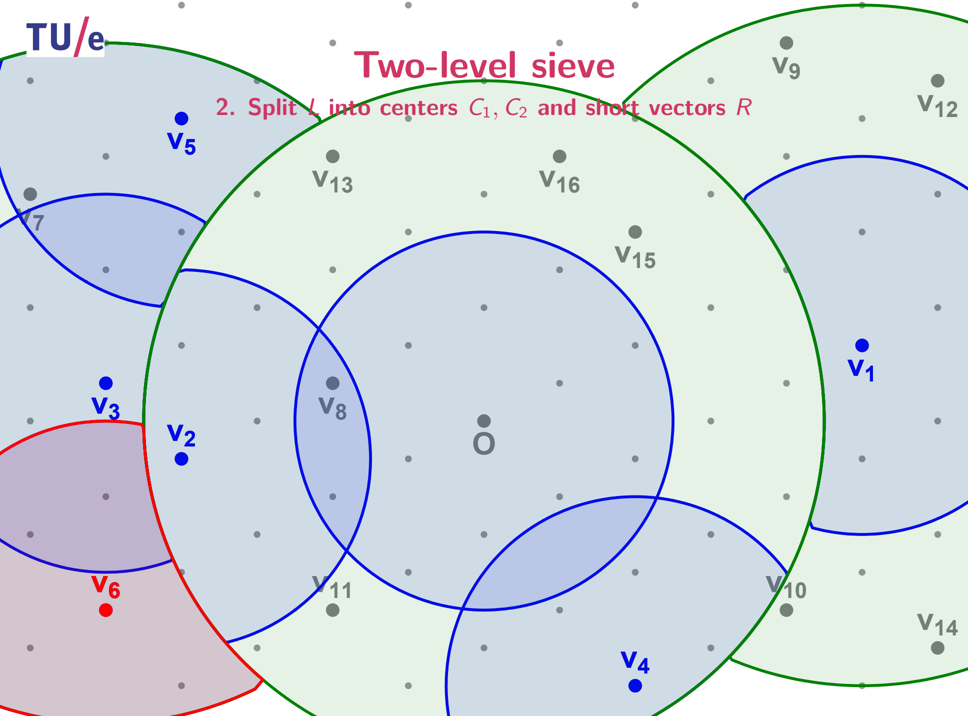

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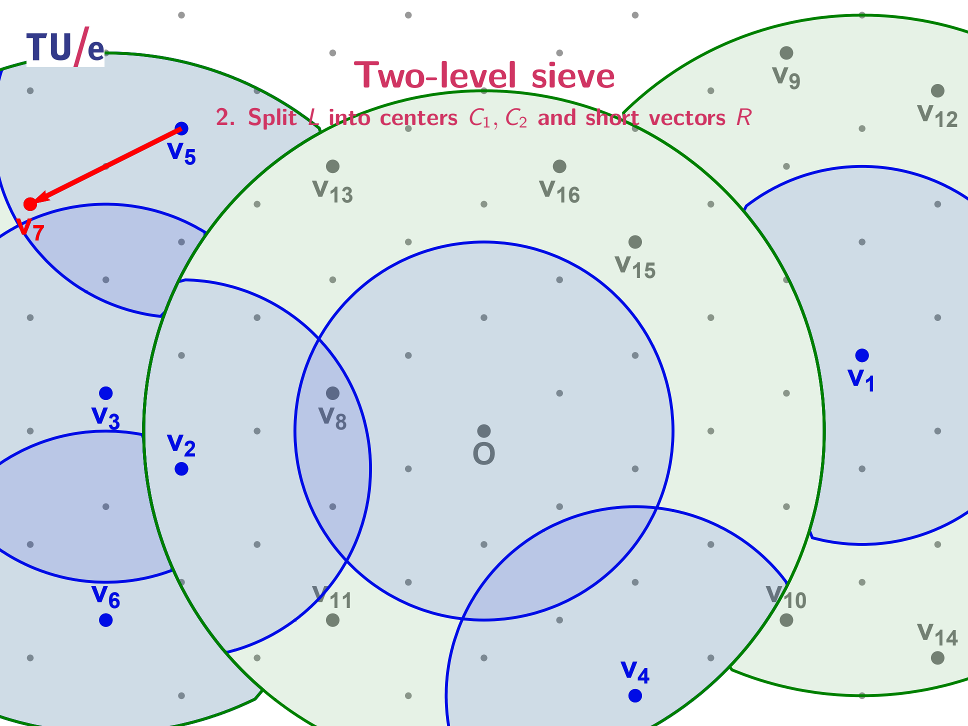

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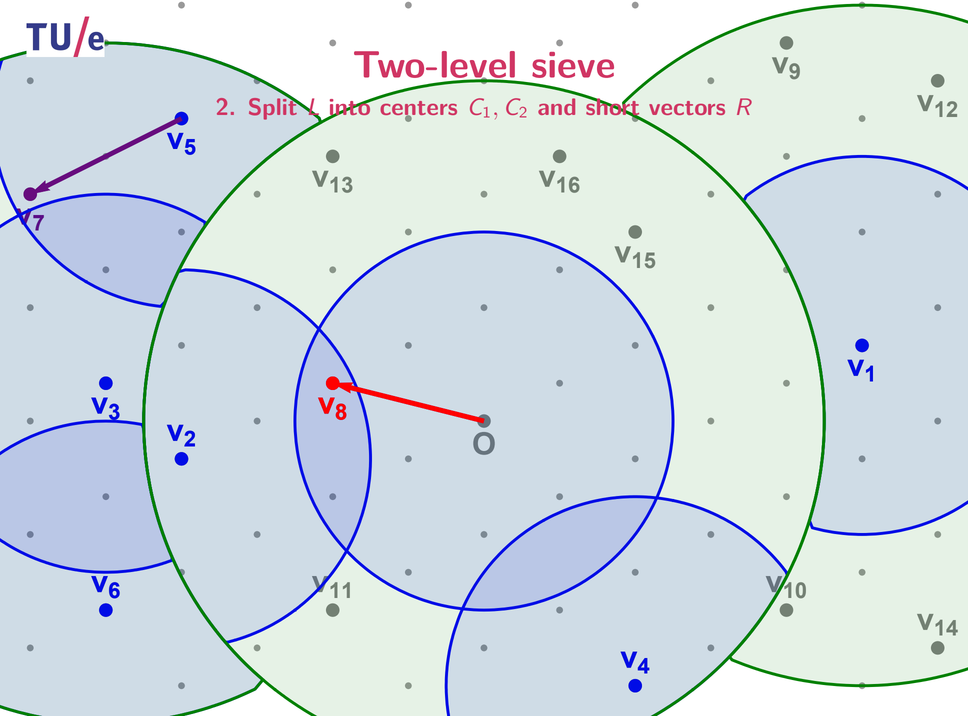

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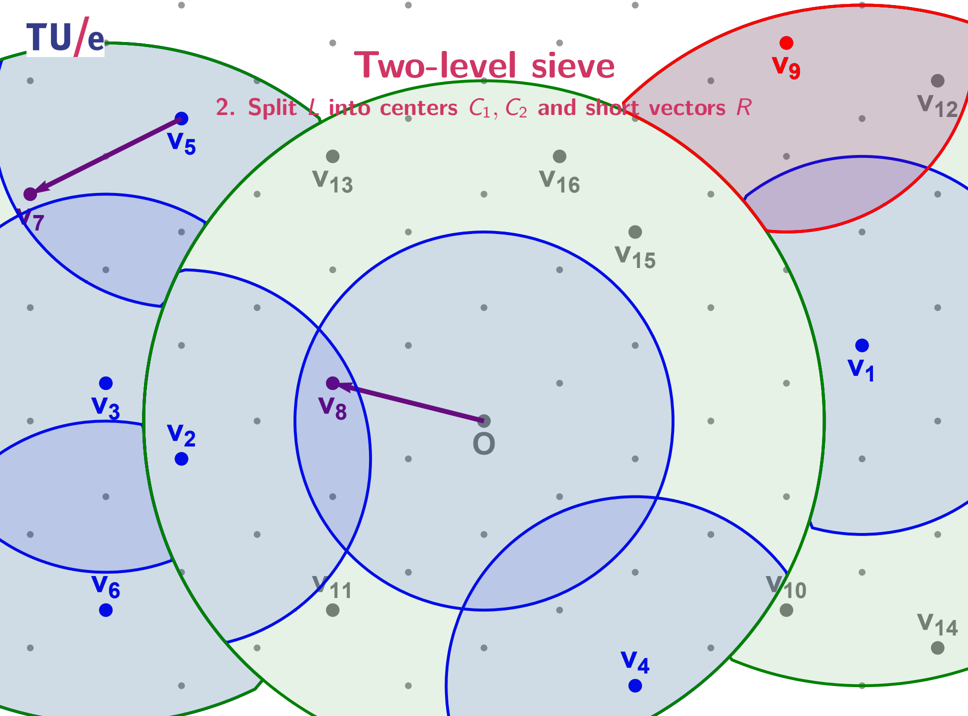

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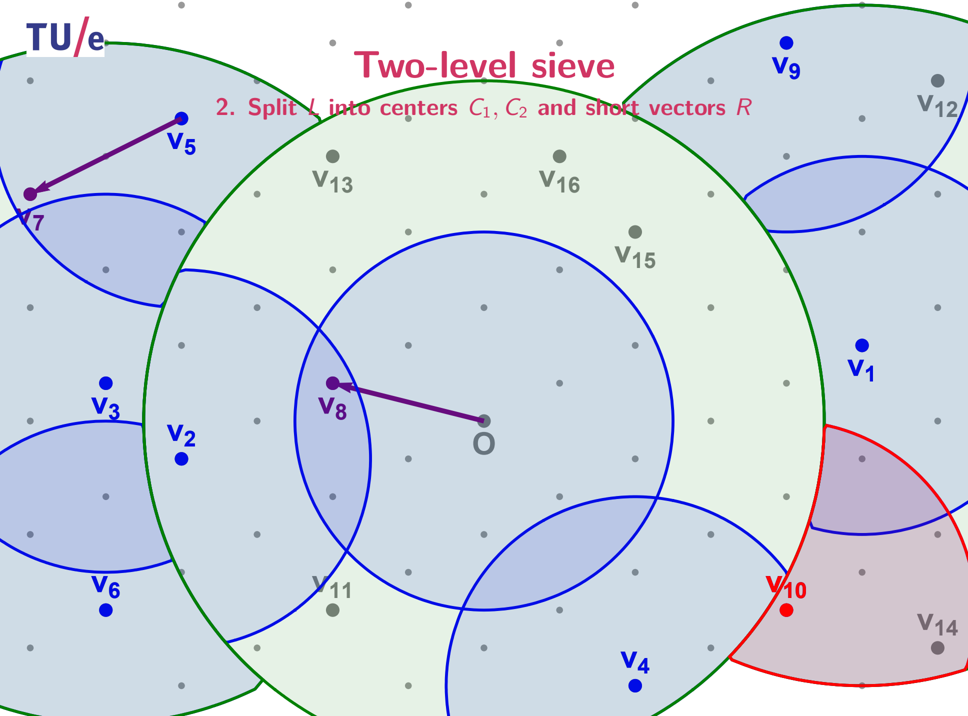

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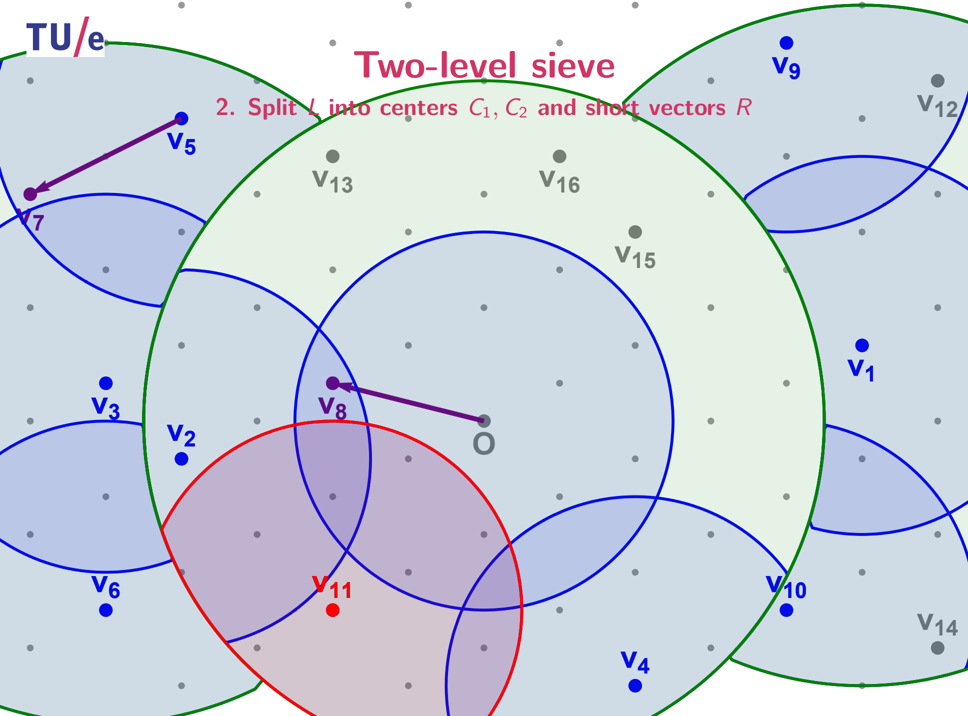

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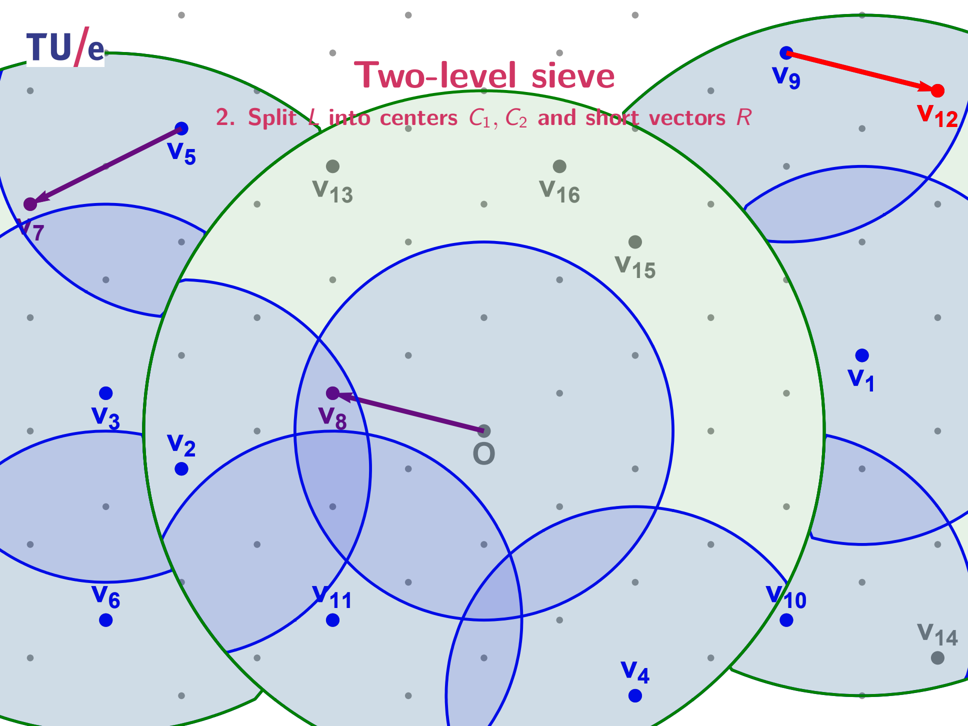

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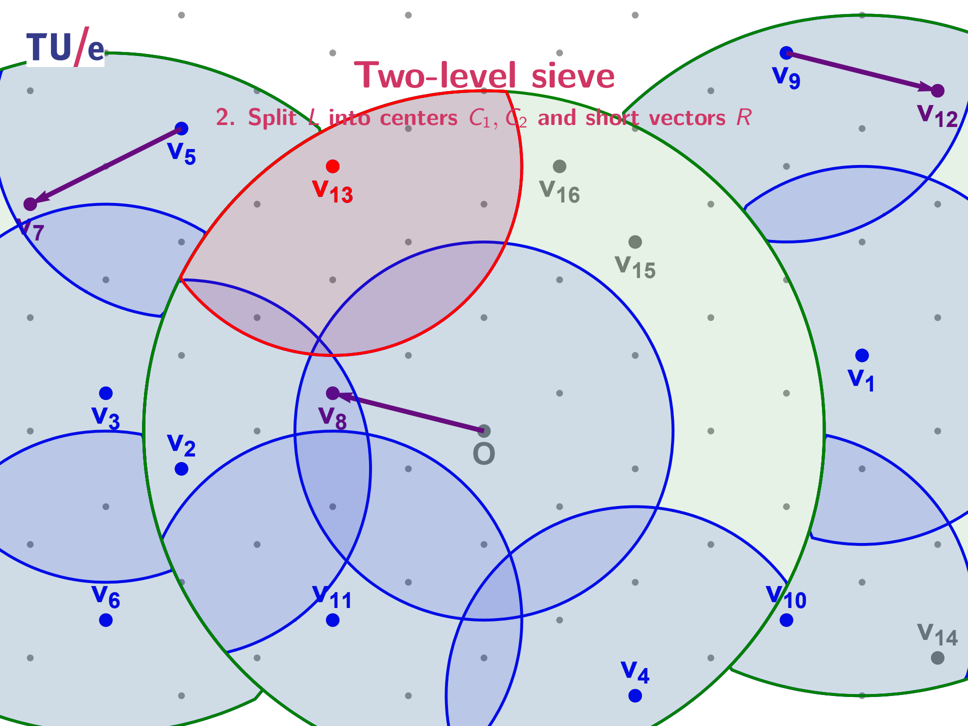

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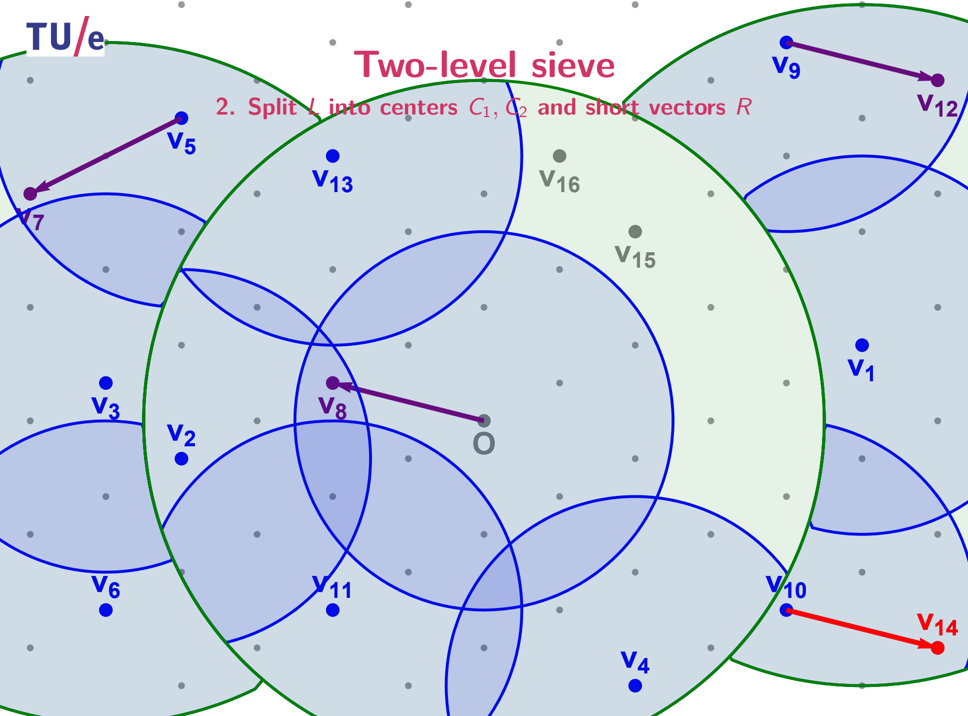

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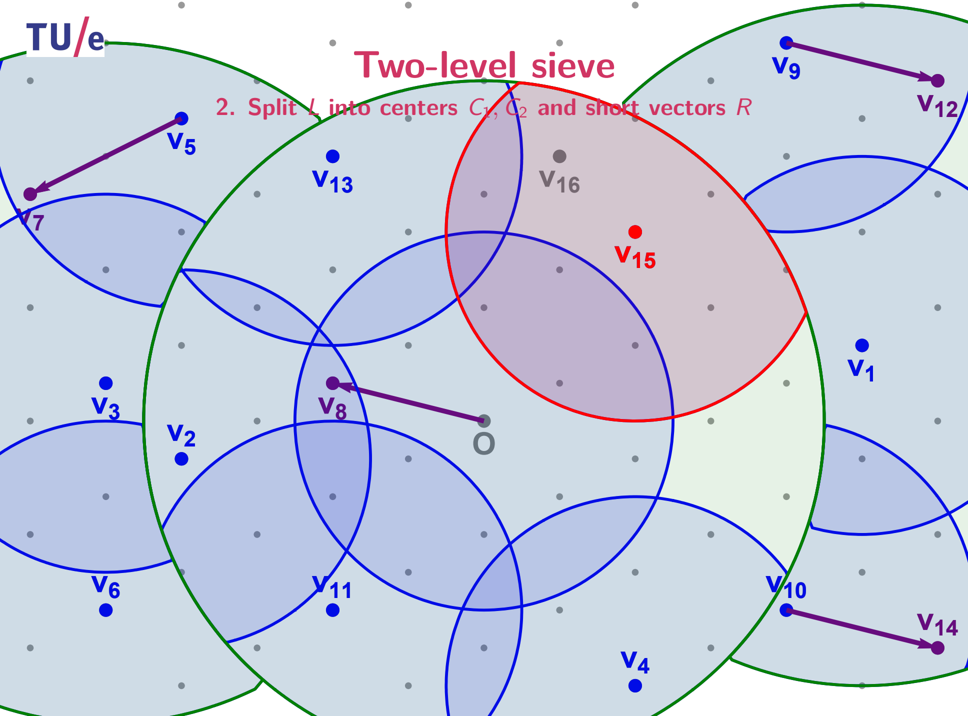

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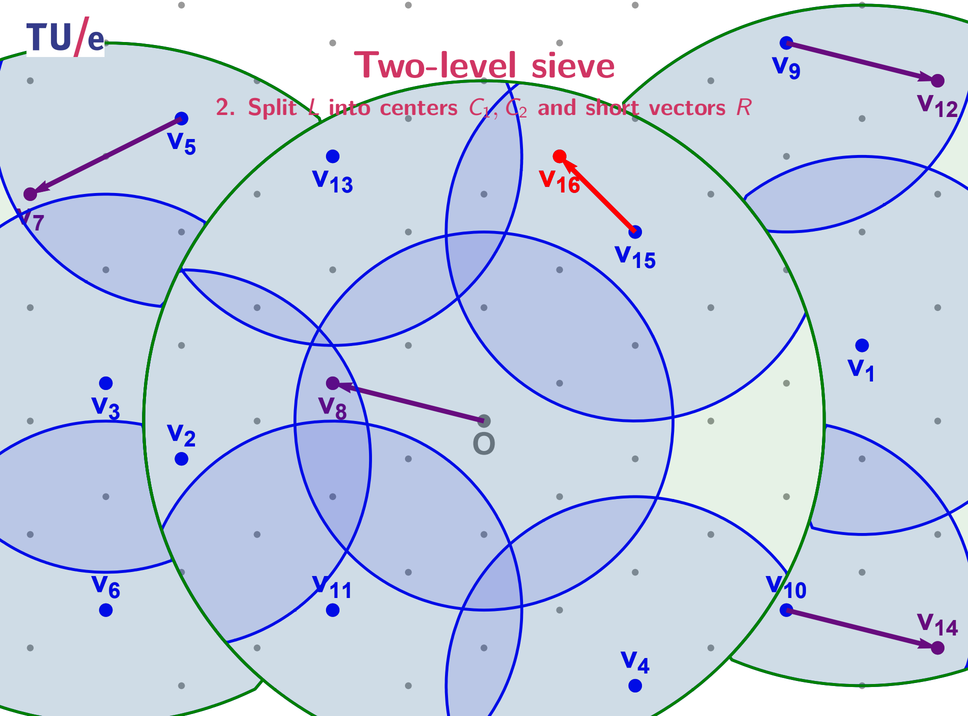

TU/e

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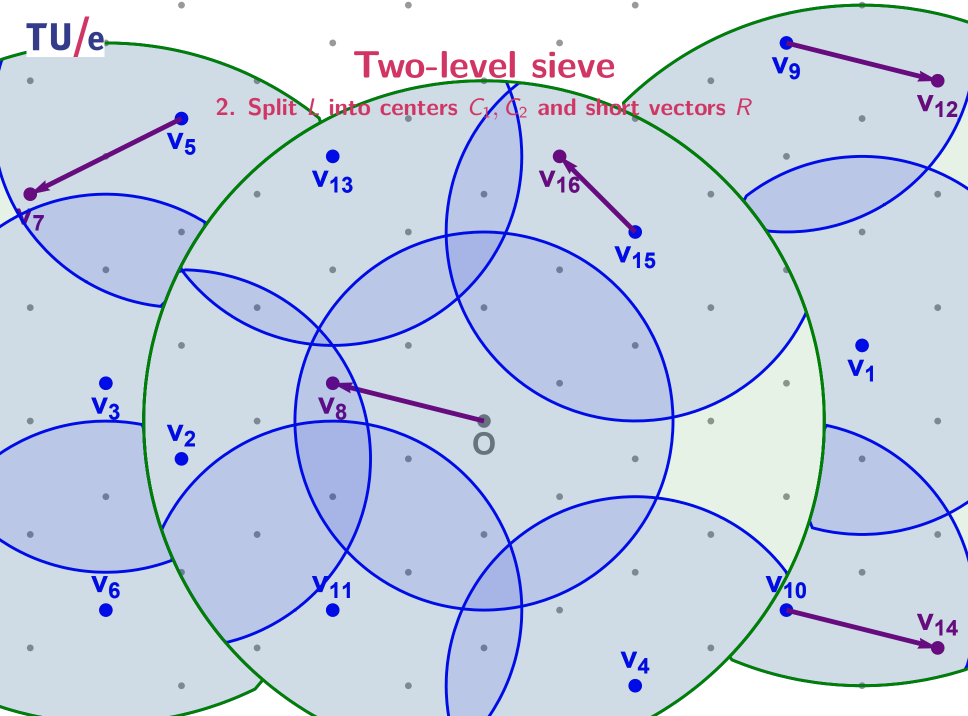
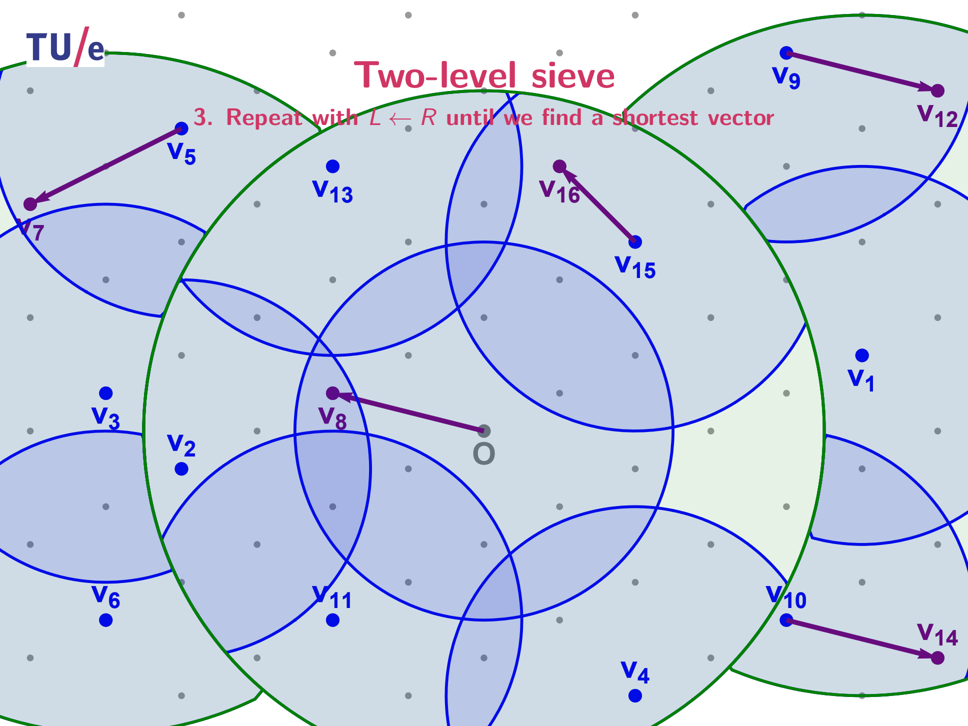


Two-level sieve

3. Repeat with $L \leftarrow R$ until we find a shortest vector



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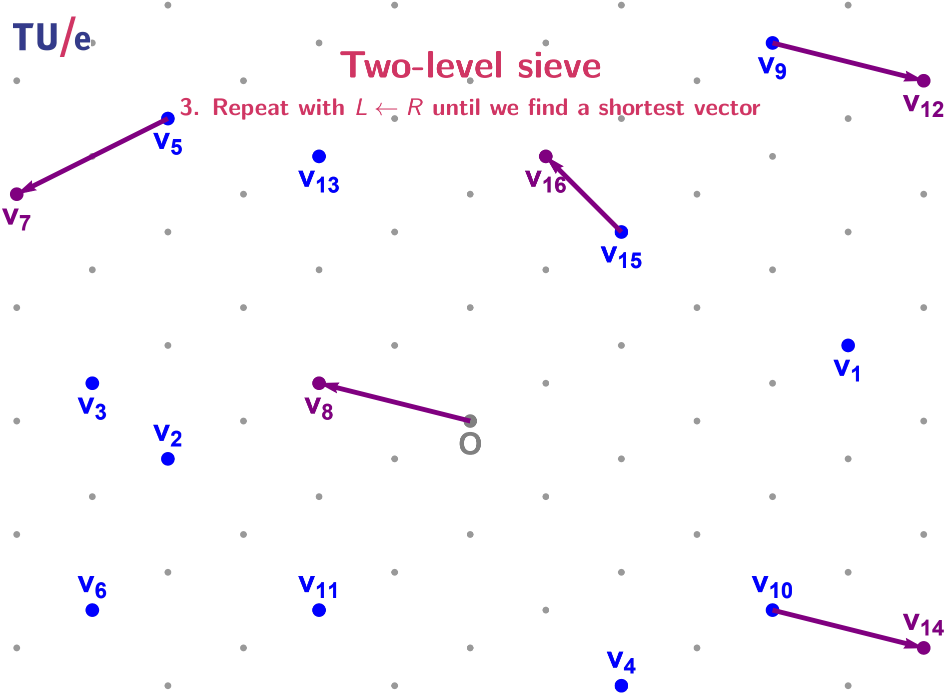

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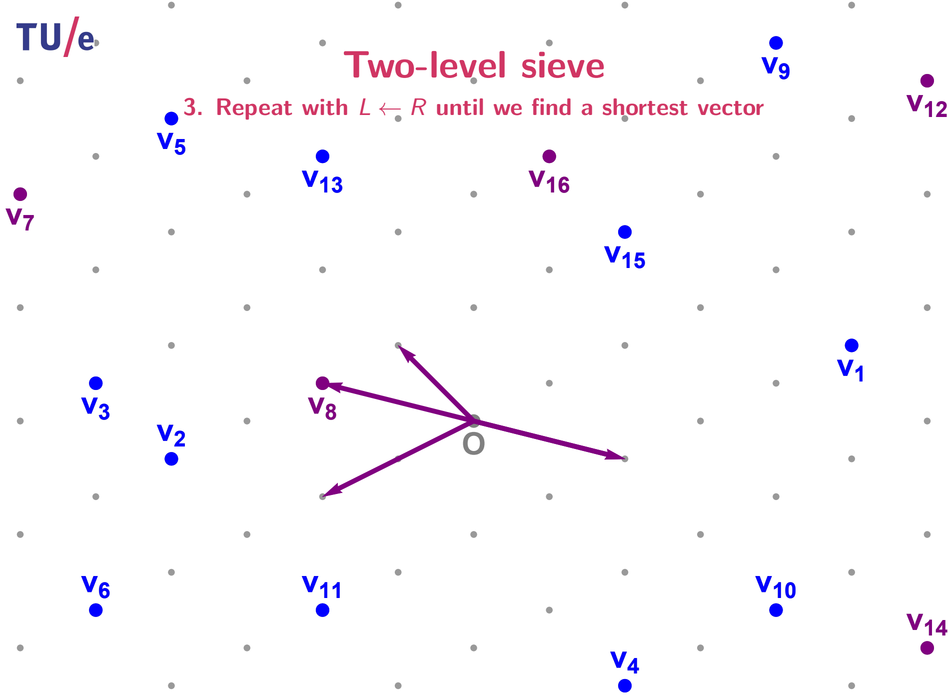

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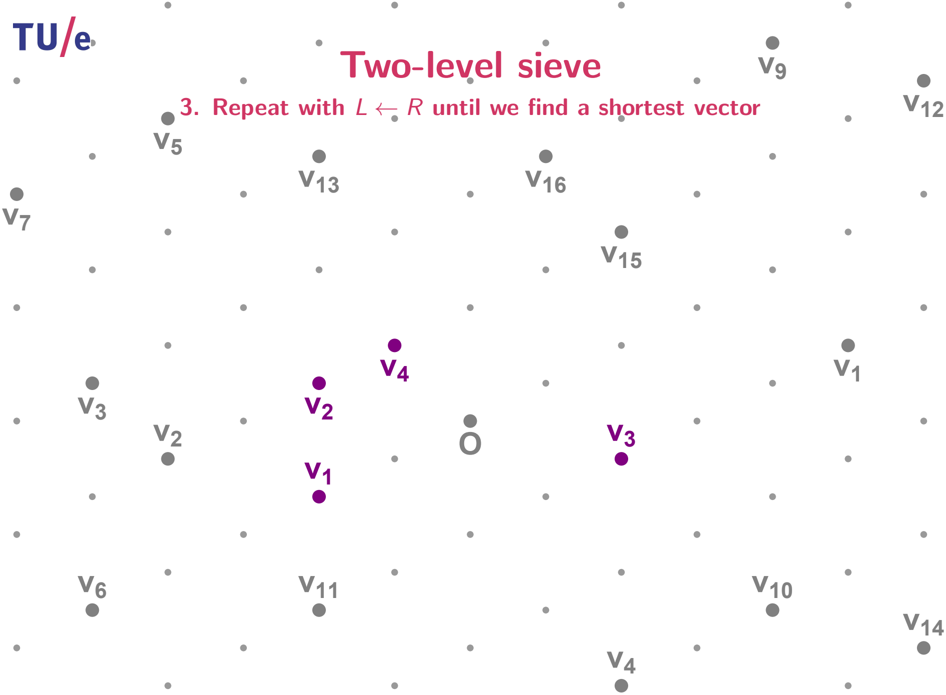
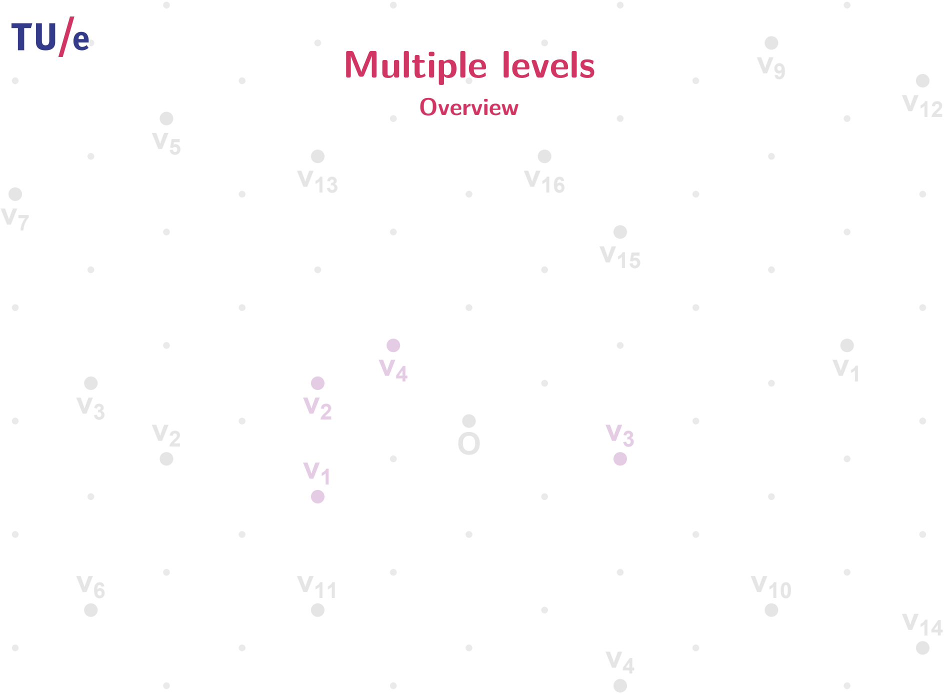


Multiple levels

Overview



Multiple levels

Overview

Heuristic (Nguyen and Vidick, J. Math. Crypt. '08)

The one-level sieve runs in time $2^{0.4150n}$ and space $2^{0.2075n}$.

Multiple levels

Overview

Heuristic (Nguyen and Vidick, J. Math. Crypt. '08)

The one-level sieve runs in time $2^{0.4150n}$ and space $2^{0.2075n}$.

Heuristic (Wang et al., ASIACCS'11)

The two-level sieve runs in time $2^{0.3836n}$ and space $2^{0.2557n}$.

Multiple levels

Overview

Heuristic (Nguyen and Vidick, J. Math. Crypt. '08)

The one-level sieve runs in time $2^{0.4150n}$ and space $2^{0.2075n}$.

Heuristic (Wang et al., ASIACCS'11)

The two-level sieve runs in time $2^{0.3836n}$ and space $2^{0.2557n}$.

Heuristic (Zhang et al., SAC'13)

The three-level sieve runs in time $2^{0.3778n}$ and space $2^{0.2833n}$.

GaussSieve

1. Generate random lattice vectors

GaussSieve

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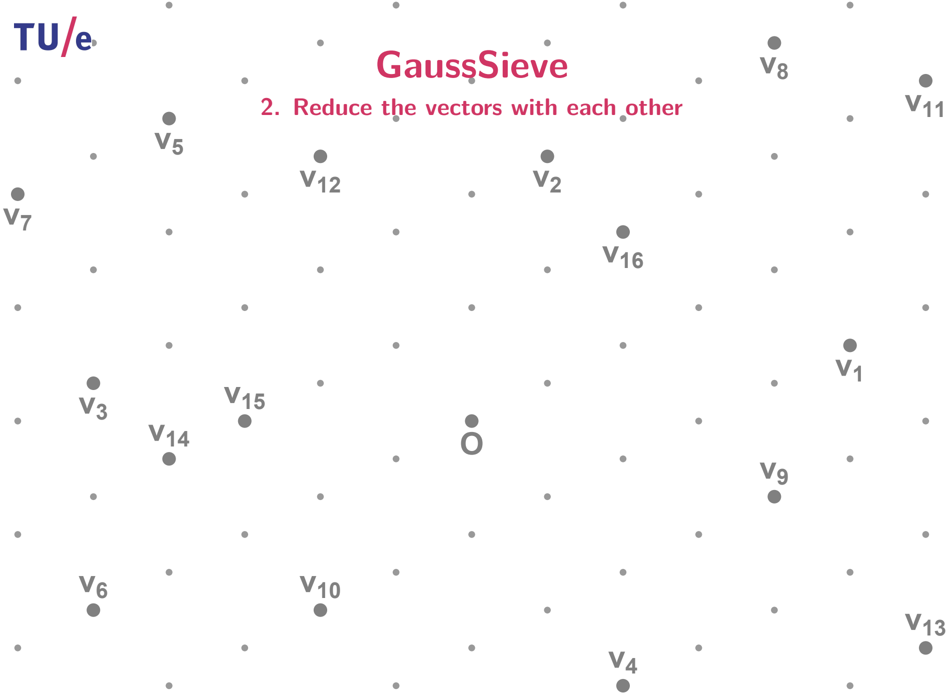
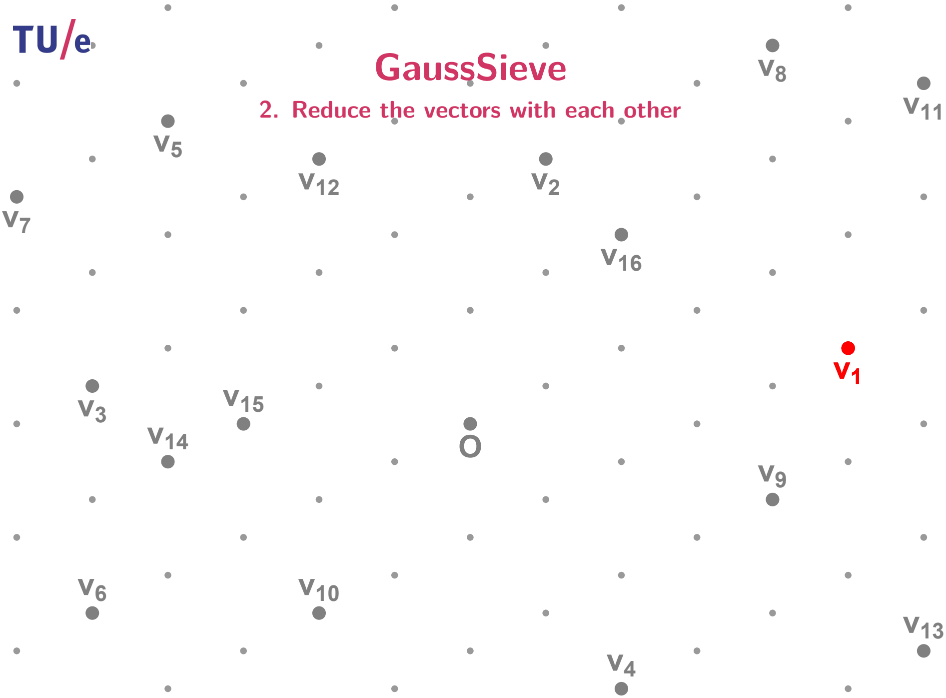
GaussSieve

2. Reduce the vectors with each other



GaussSieve

2. Reduce the vectors with each other



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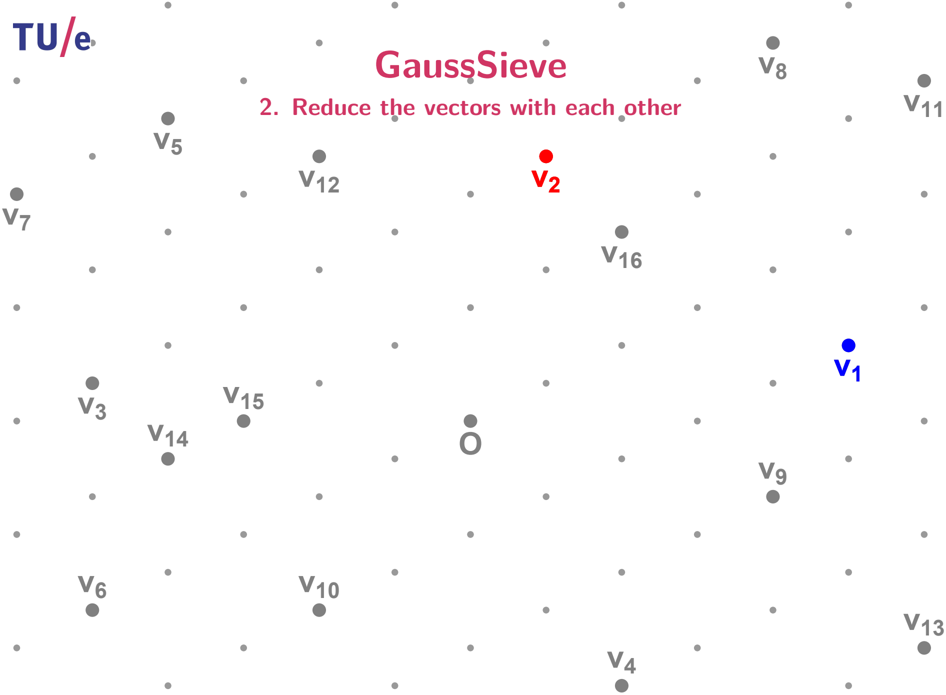

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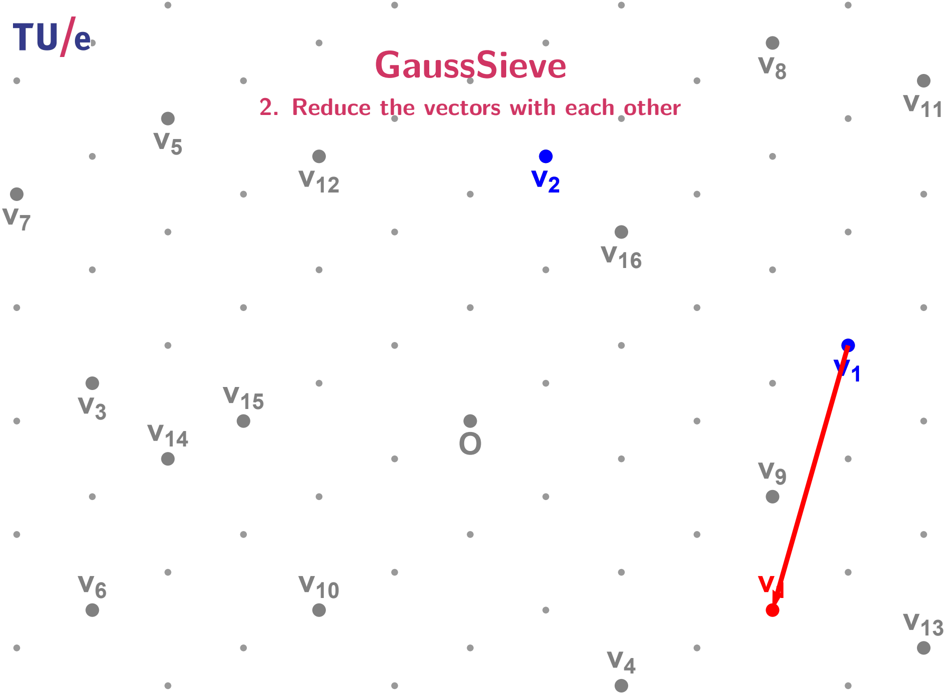

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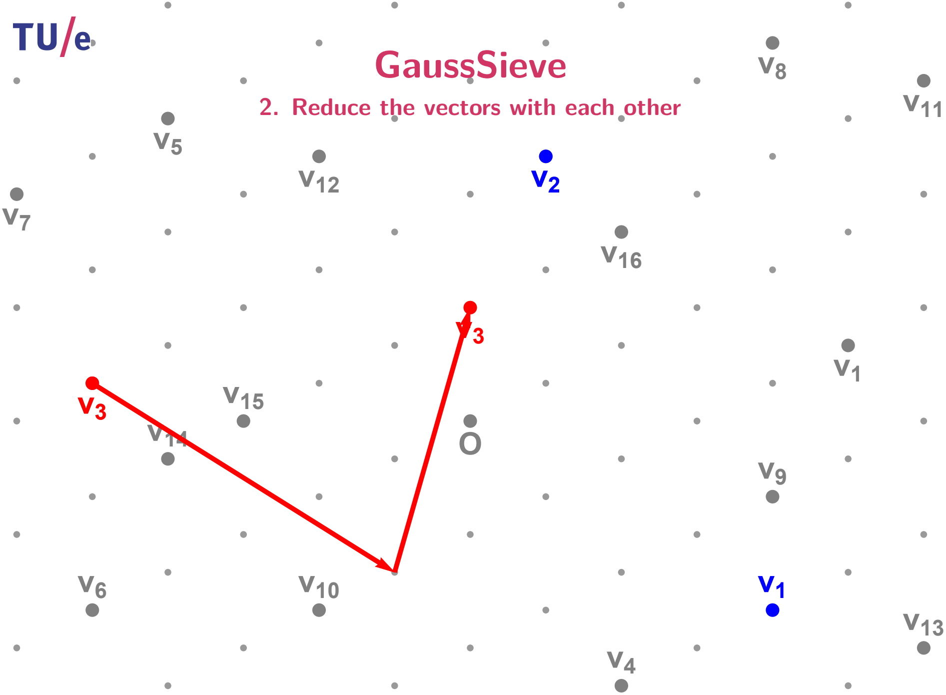

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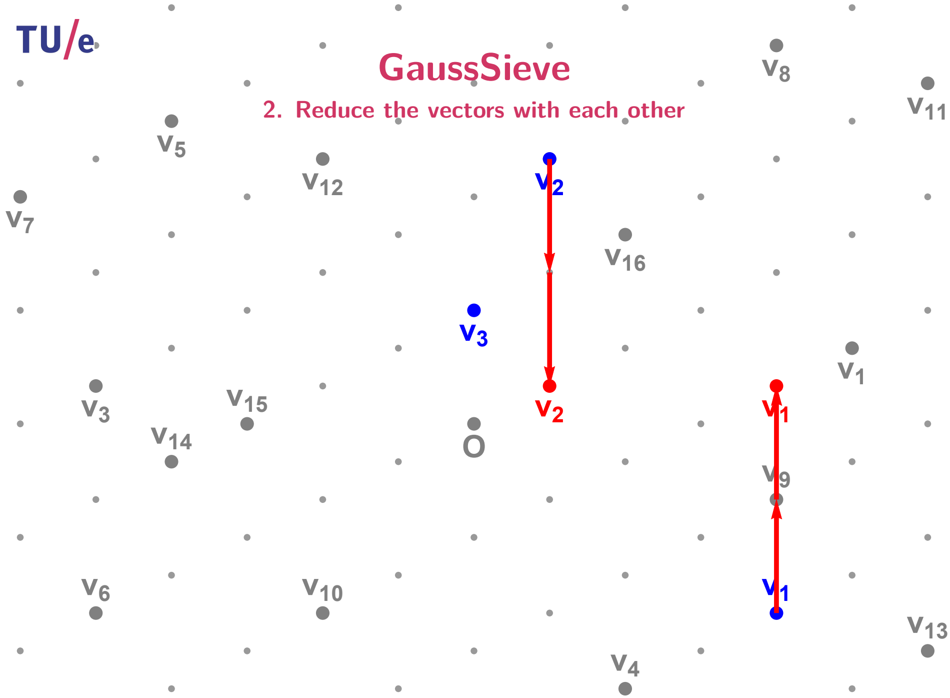

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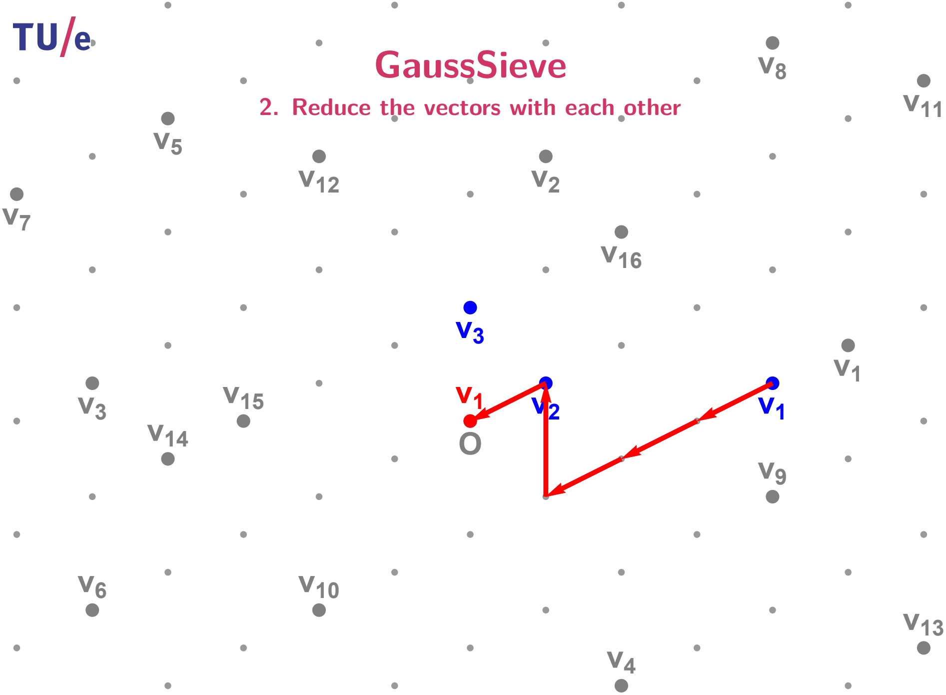

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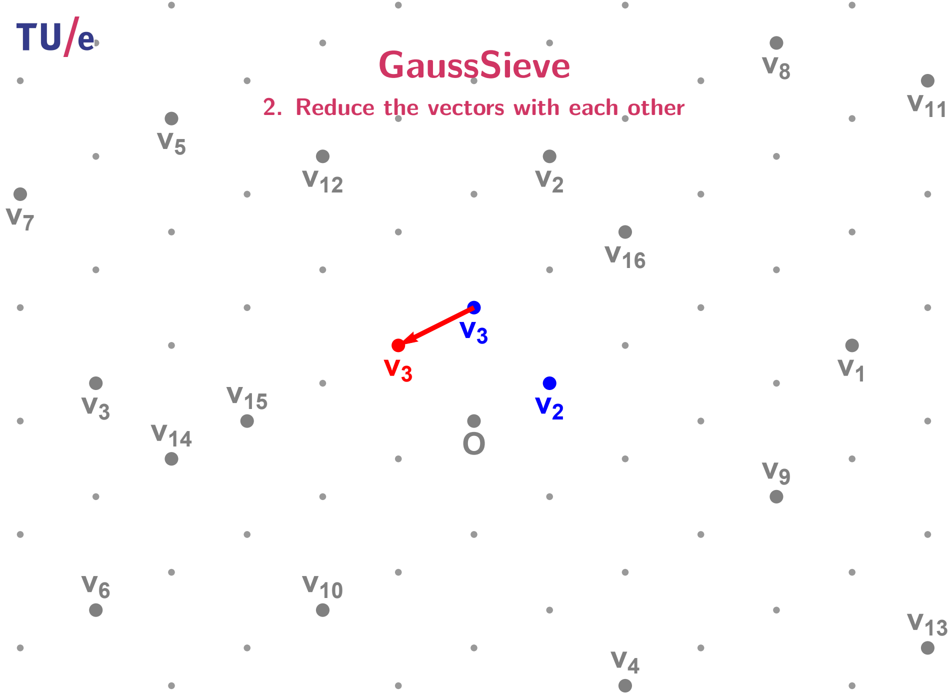

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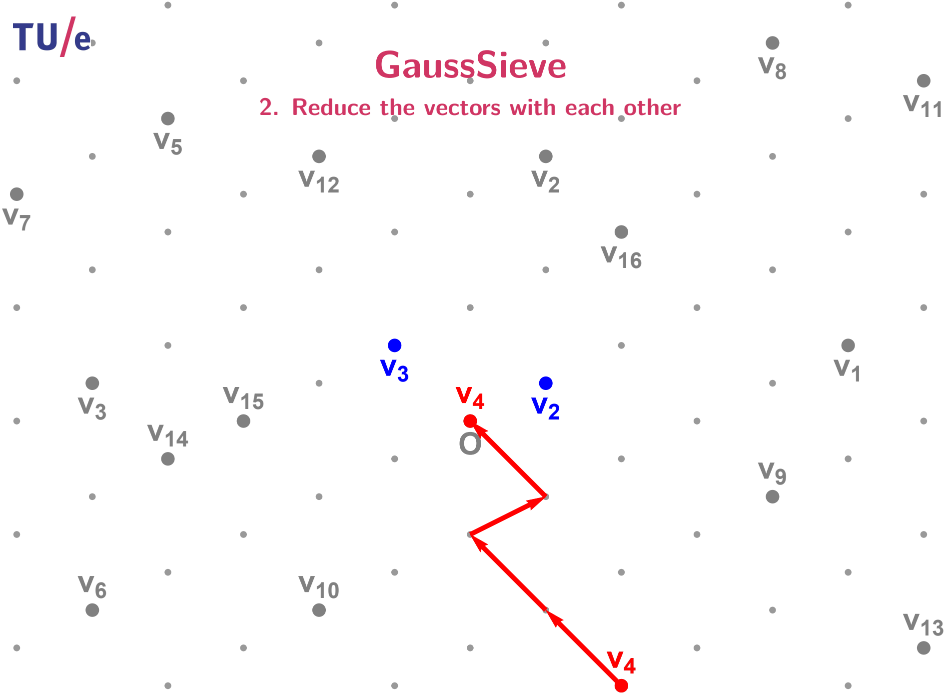

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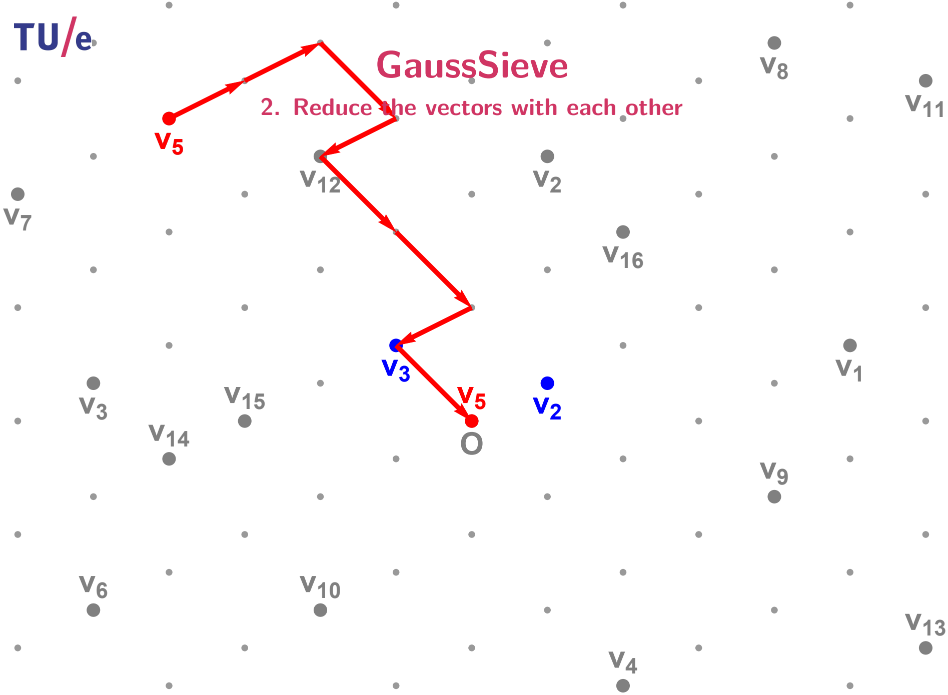

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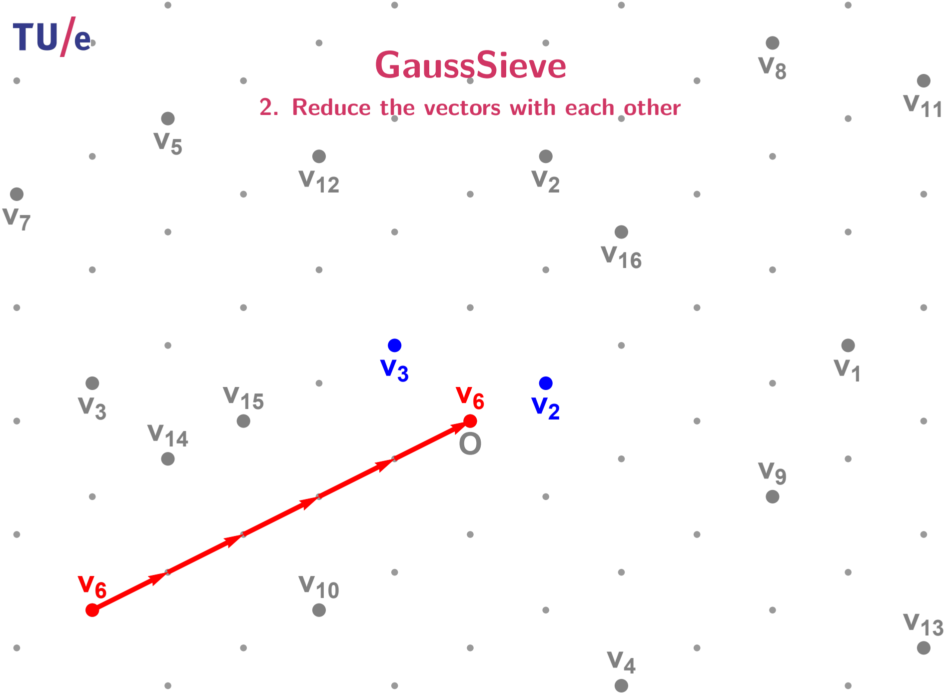

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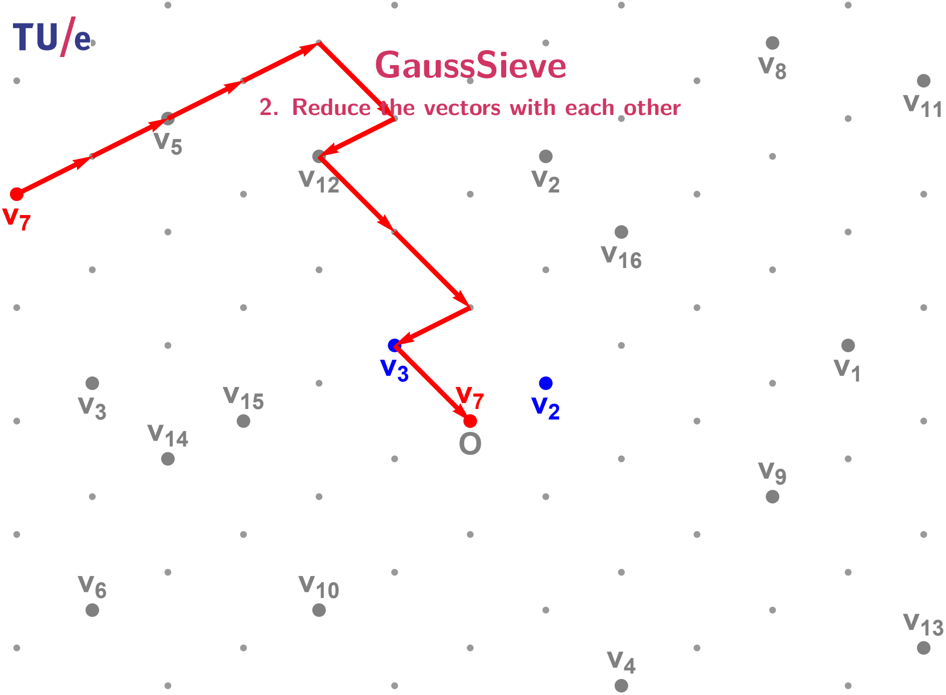

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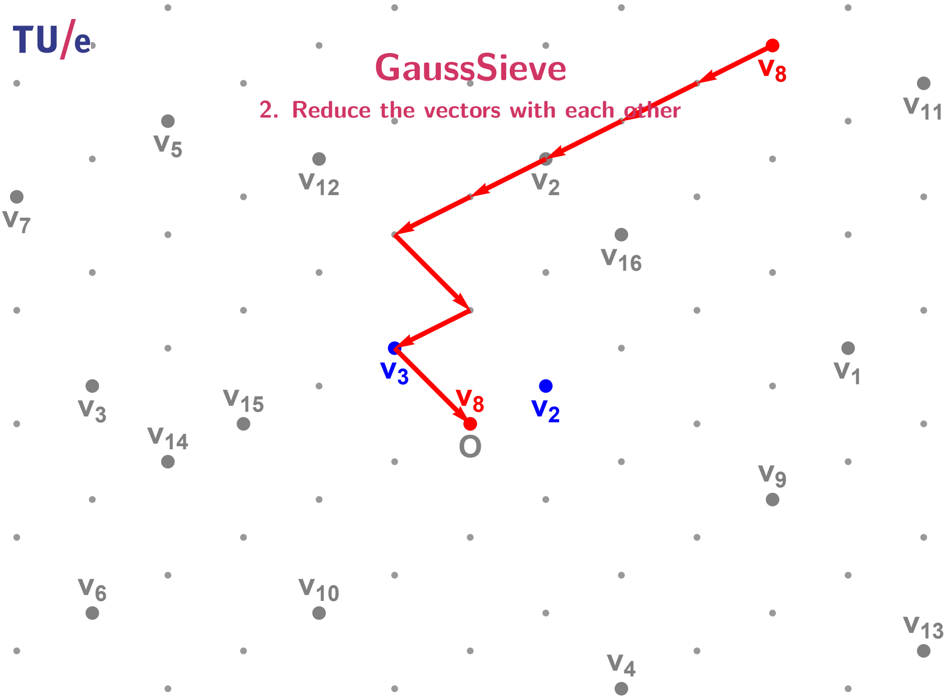

GaussSieve

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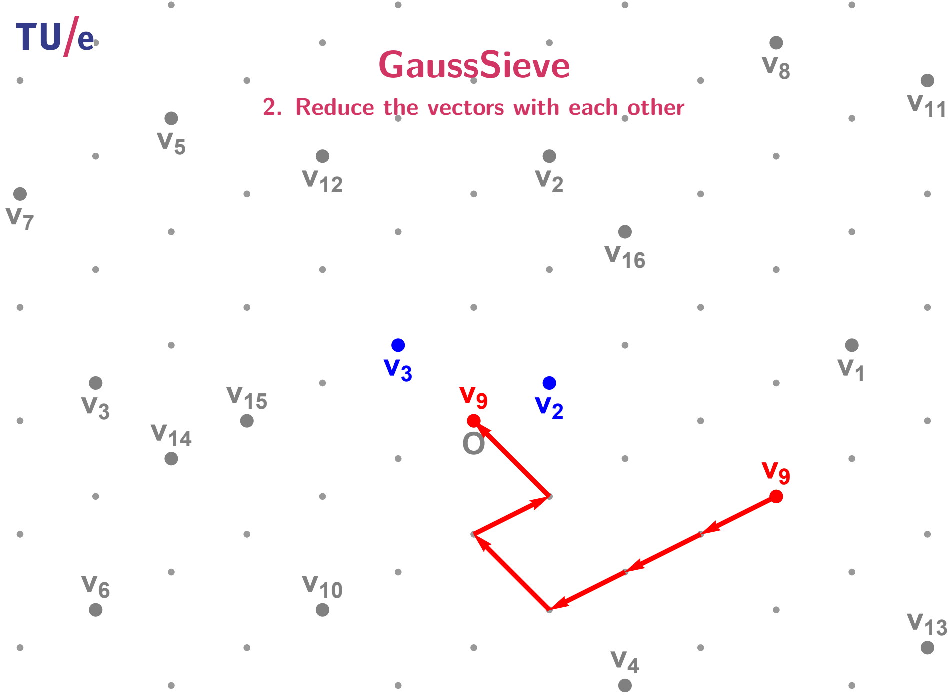

GaussSieve

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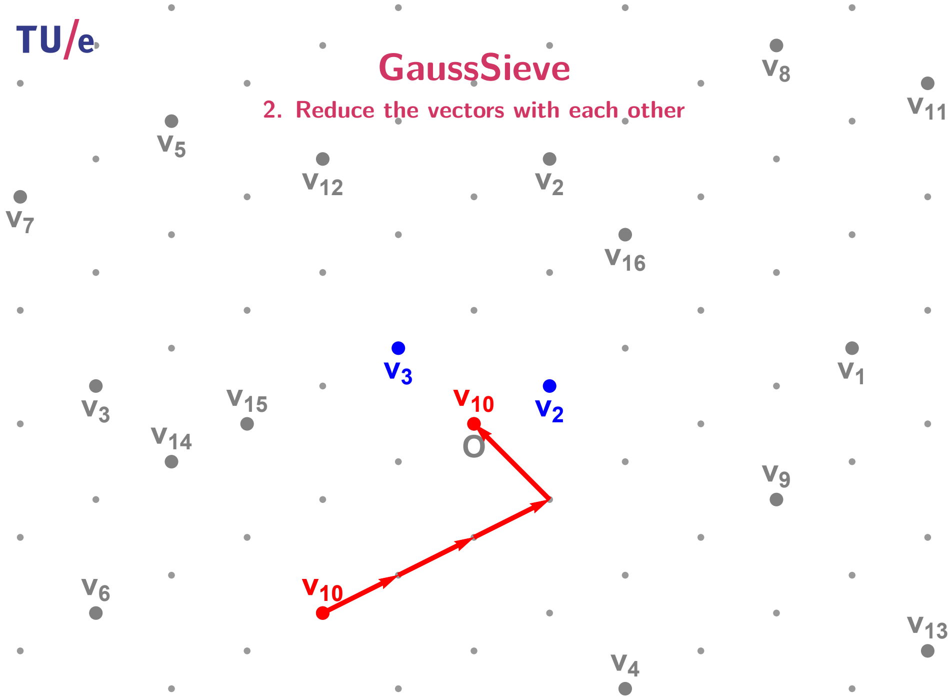

GaussSieve

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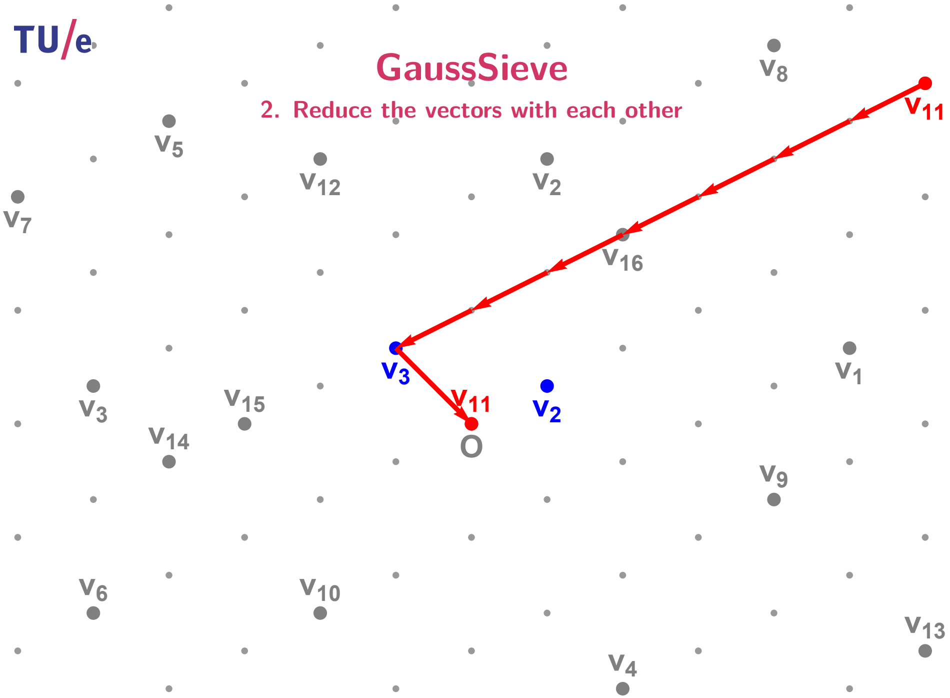

GaussSieve

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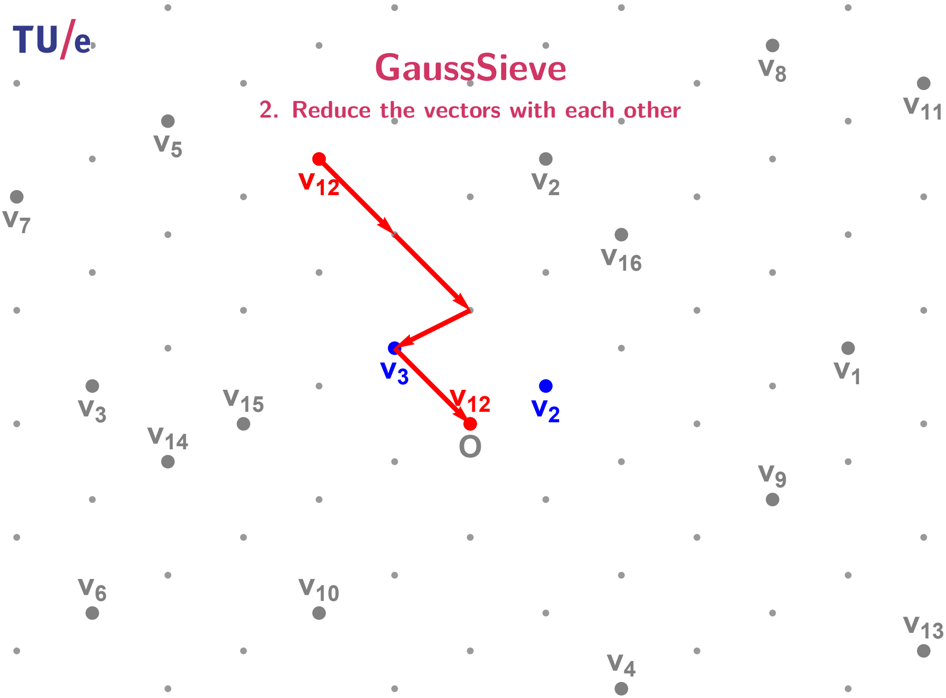

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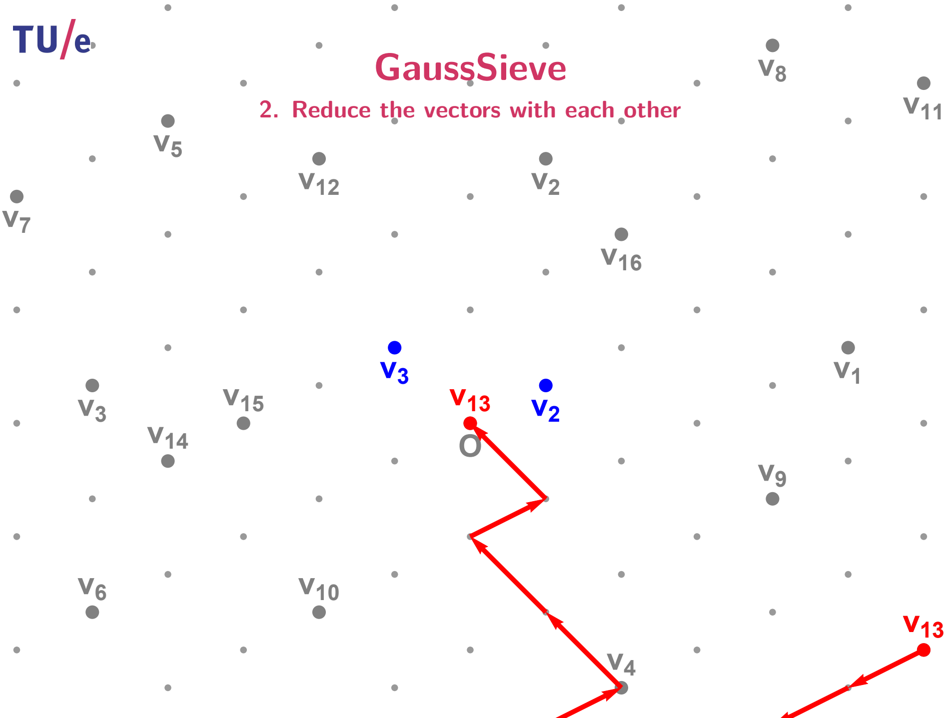

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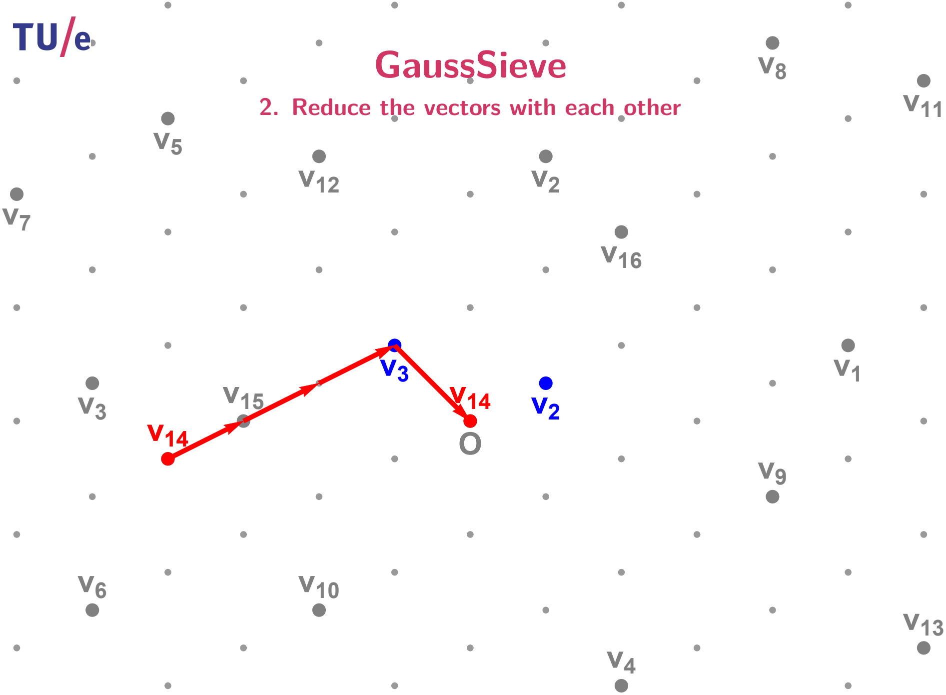

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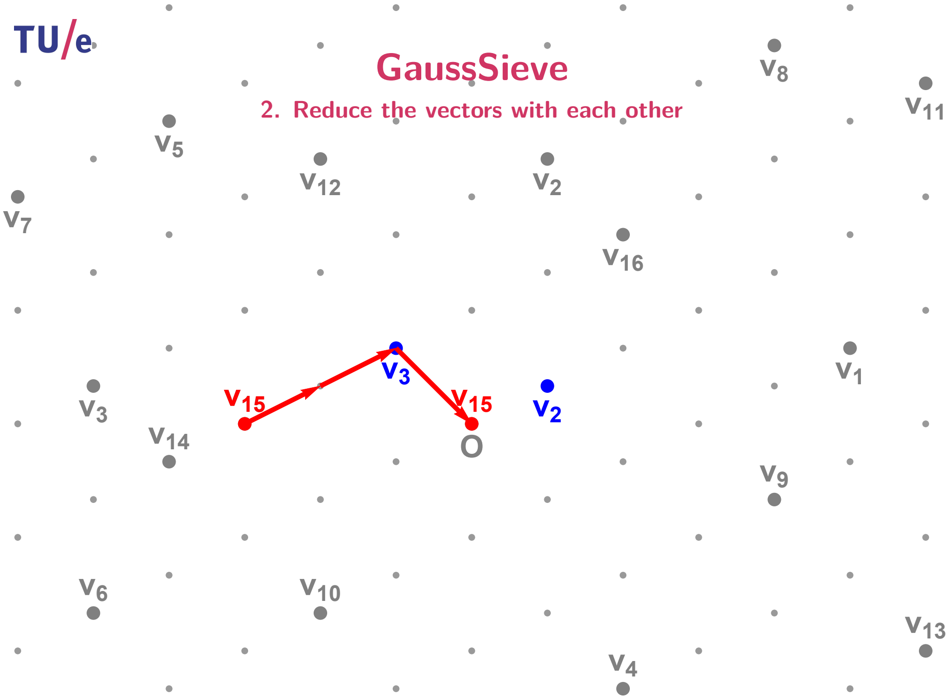

GaussSieve

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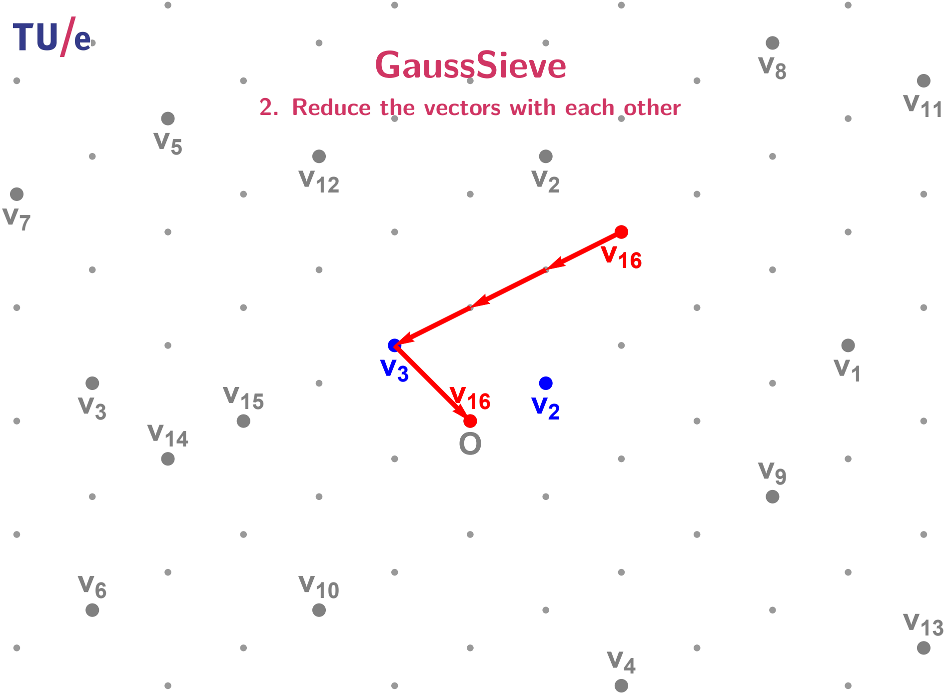

GaussSieve

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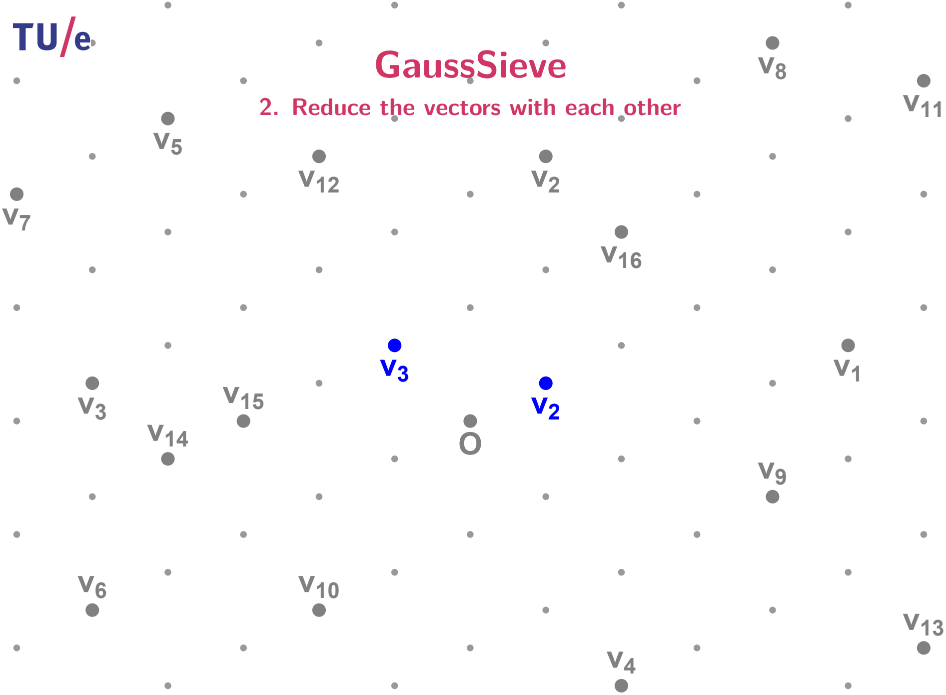
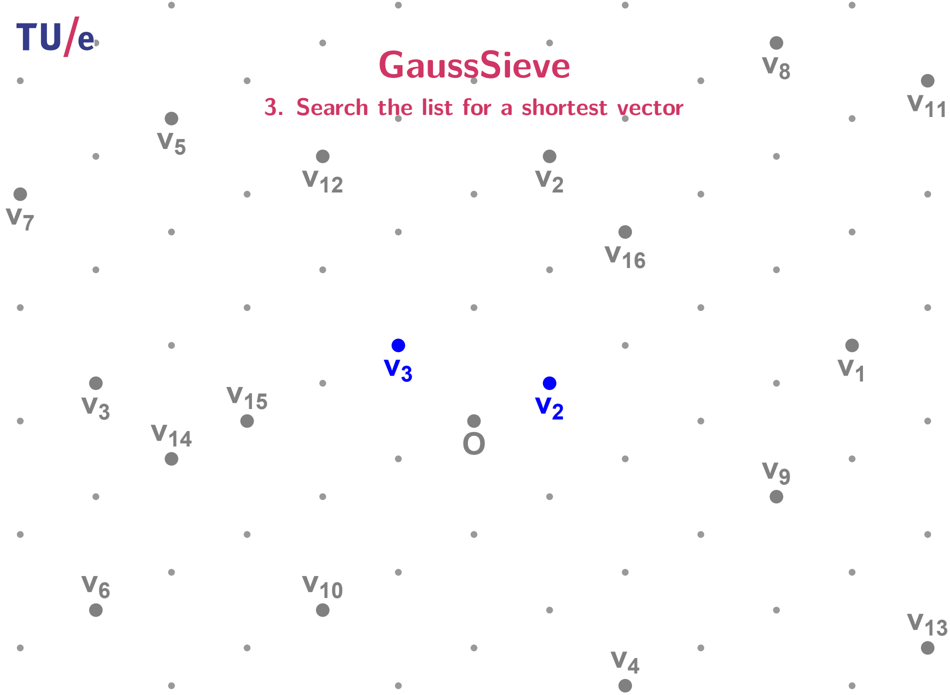


GaussSieve

3. Search the list for a shortest vector



GaussSieve

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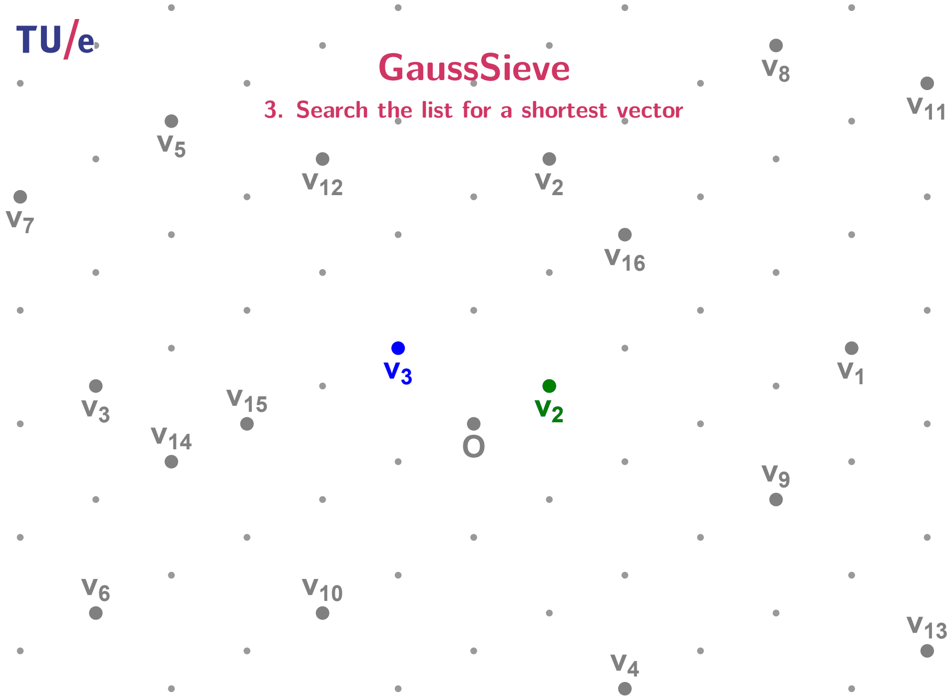
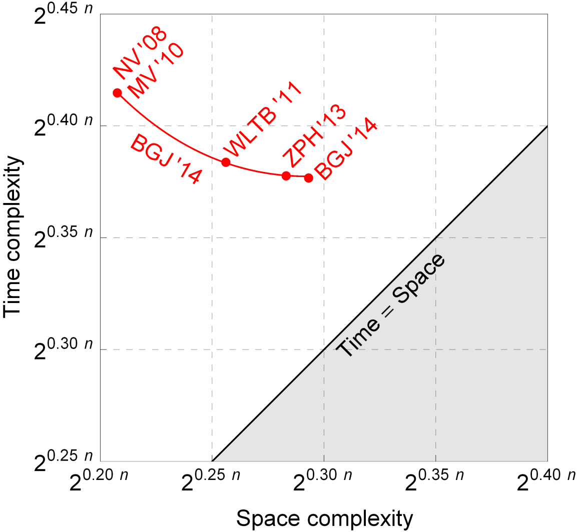


Sieving

Space/time trade-off



Locality-sensitive hashing

Introduction

"The key idea is to use hash functions such that the probability of collision is much higher for objects that are close to each other than for those that are far apart."

– Indyk and Motwani, *STOC'98*

Locality-sensitive hashing

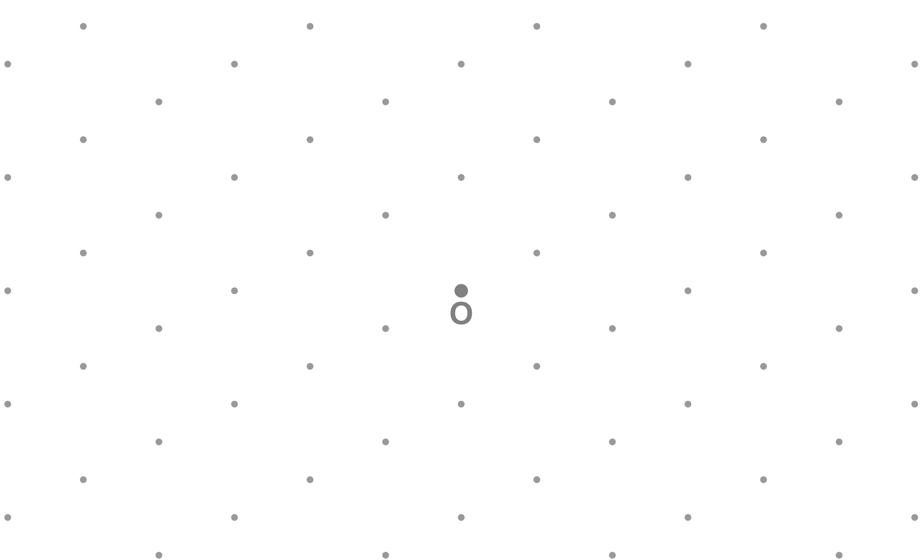
Introduction

> *"The key idea is to use hash functions such that the probability of collision is much higher for objects that are close to each other than for those that are far apart."*
>
> – Indyk and Motwani, *STOC'98*

(These hash functions are **not** cryptographic hash functions!)

Nguyen-Vidick sieve with LSH

1. Sample a list L of random lattice vectors



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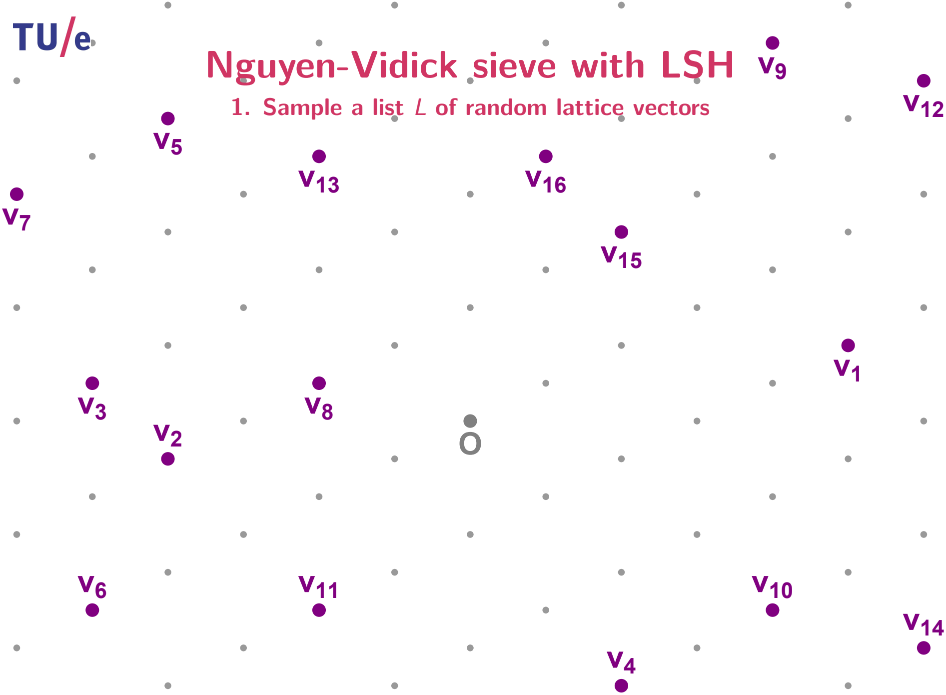
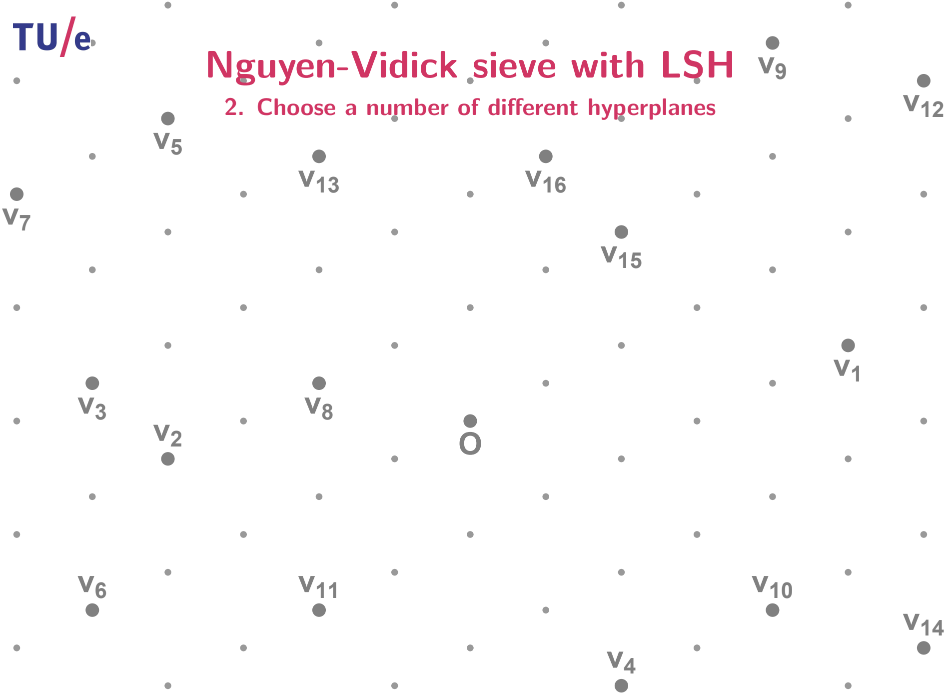


Nguyen-Vidick sieve with LSH

2. Choose a number of different hyperplanes



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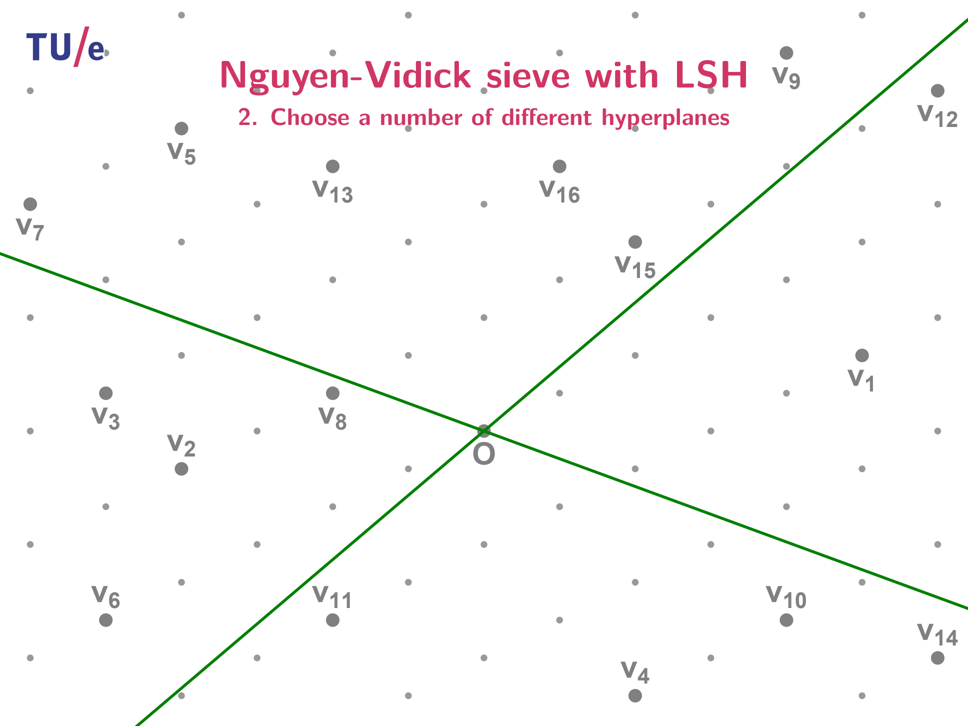
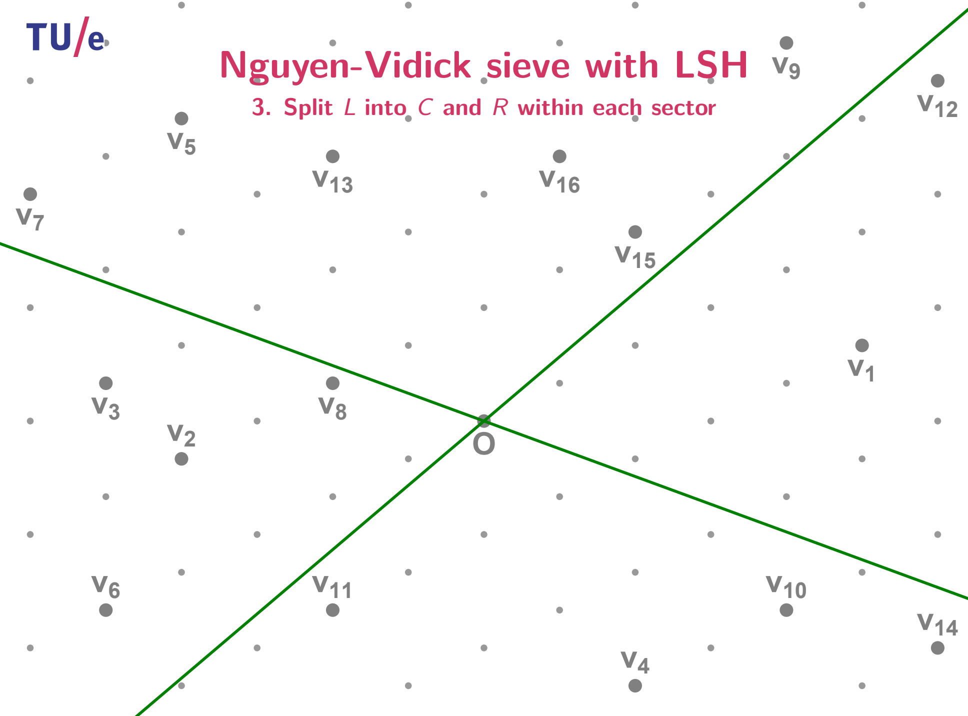


Nguyen-Vidick sieve with LSH

3. Split L into C and R within each sector



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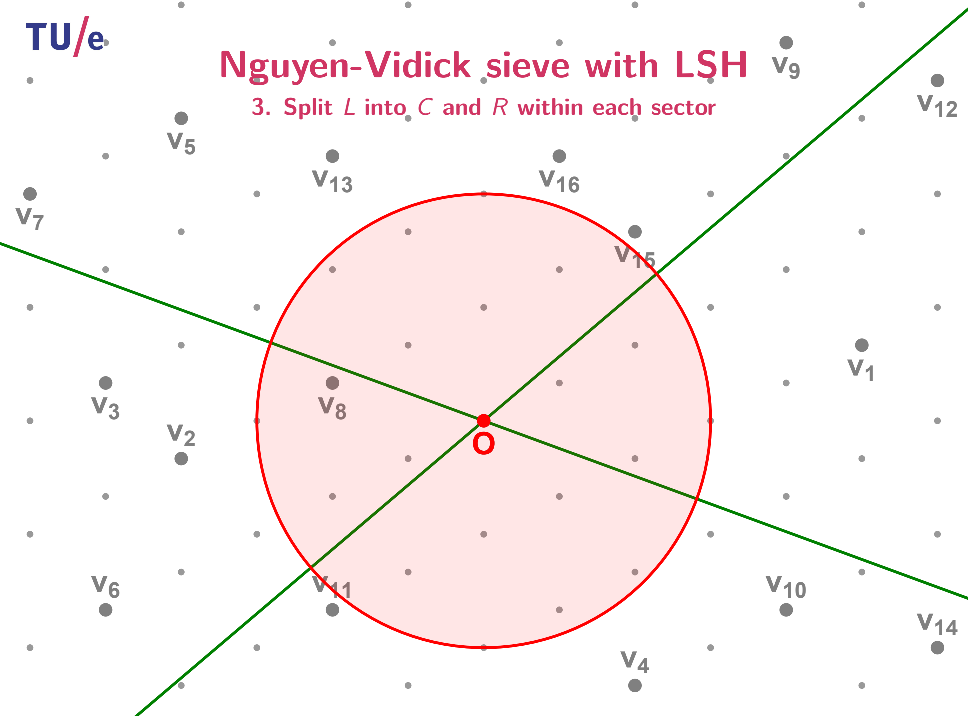

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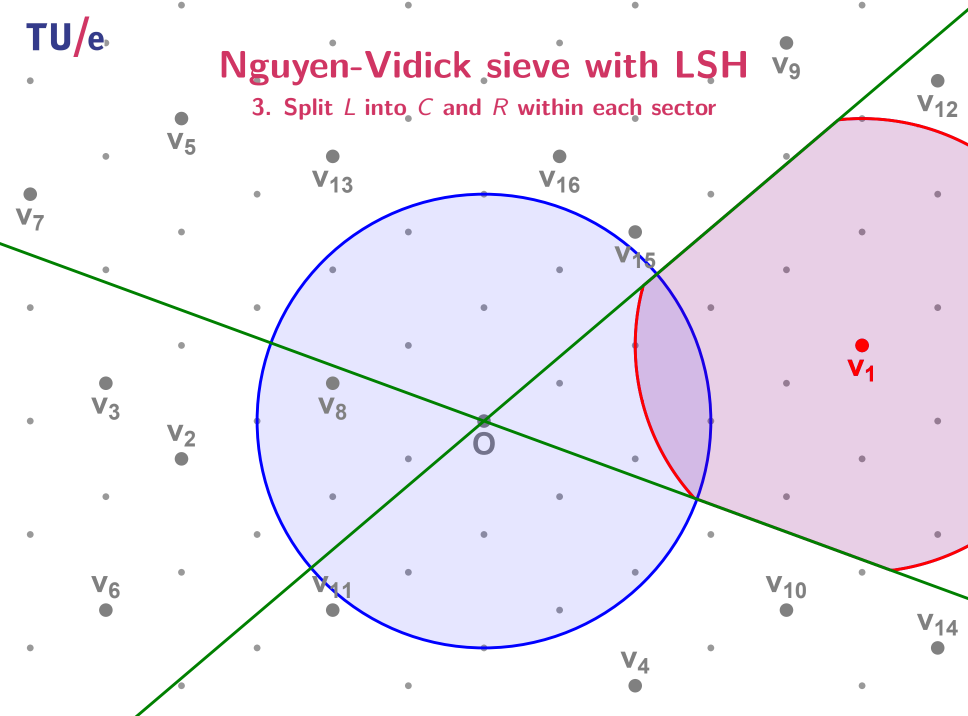

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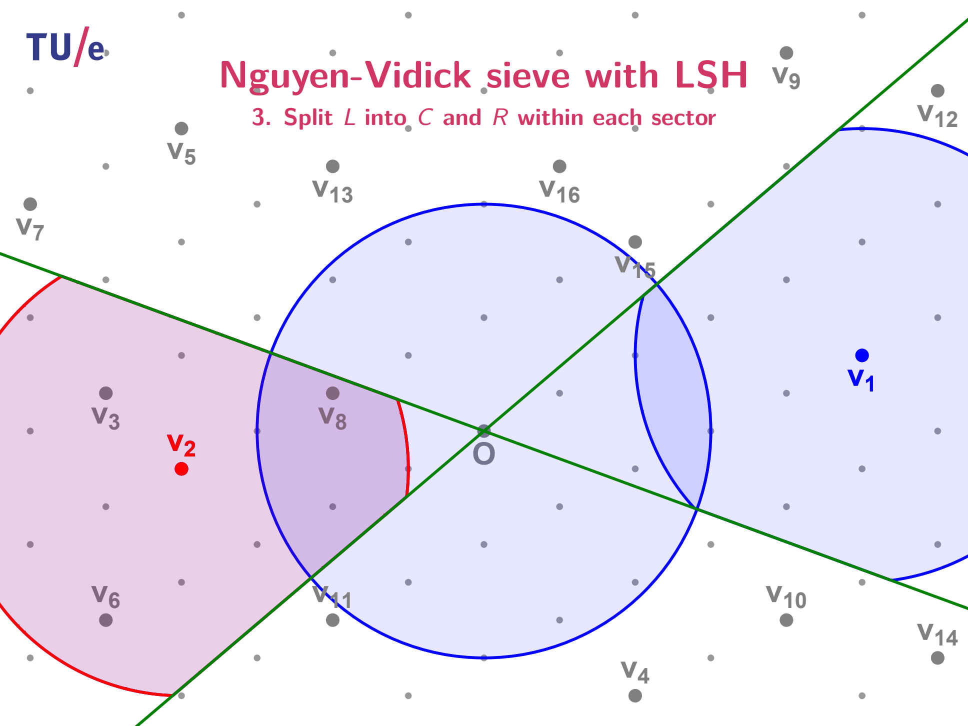

TU/e

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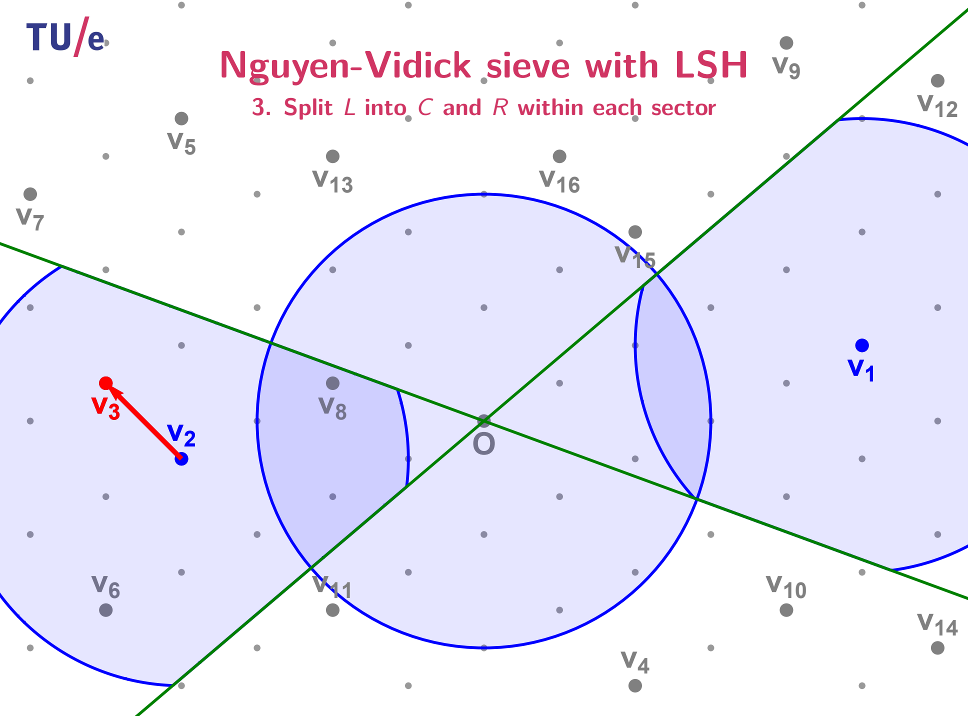

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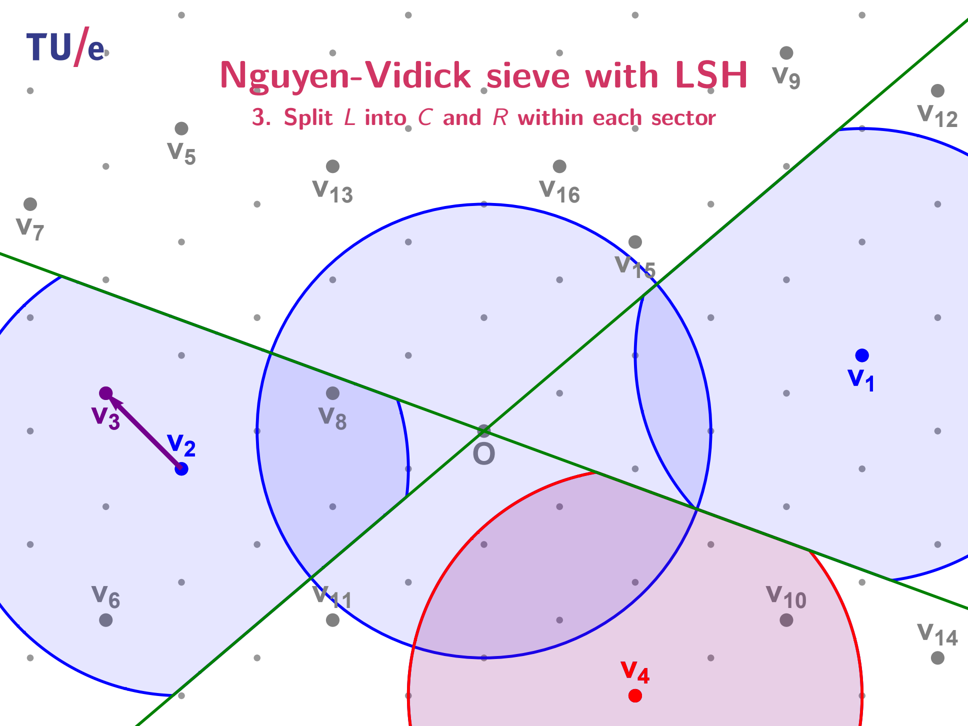

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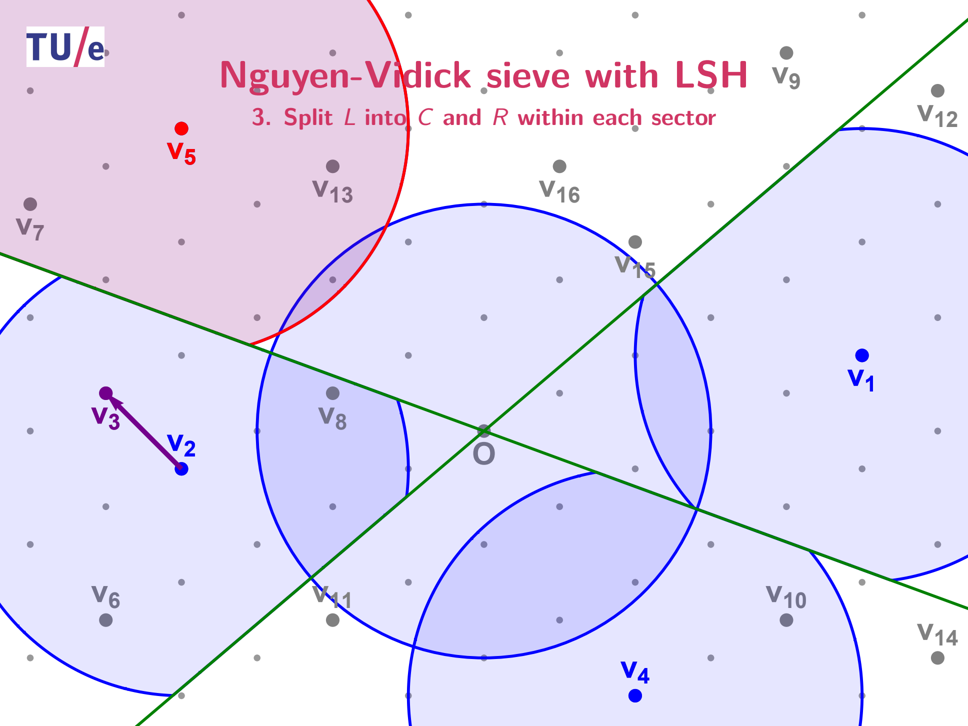

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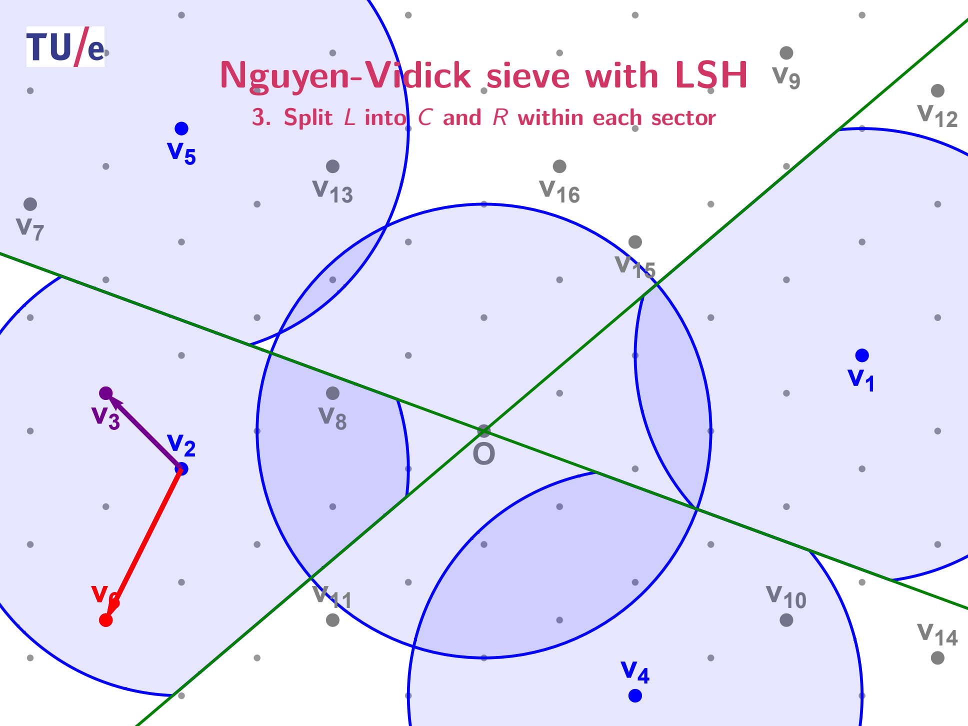

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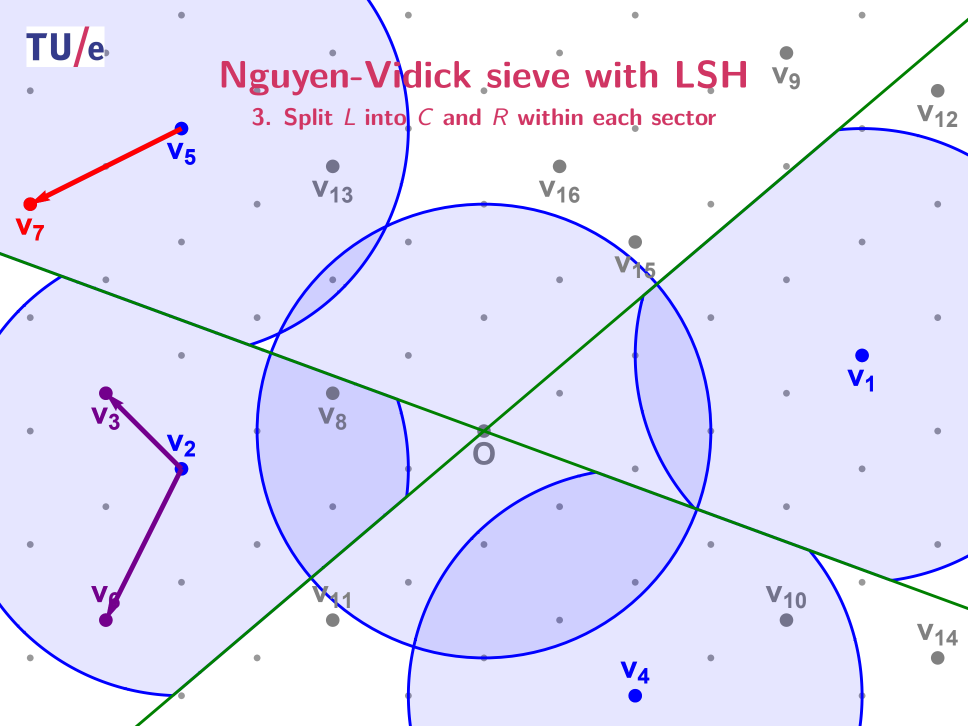

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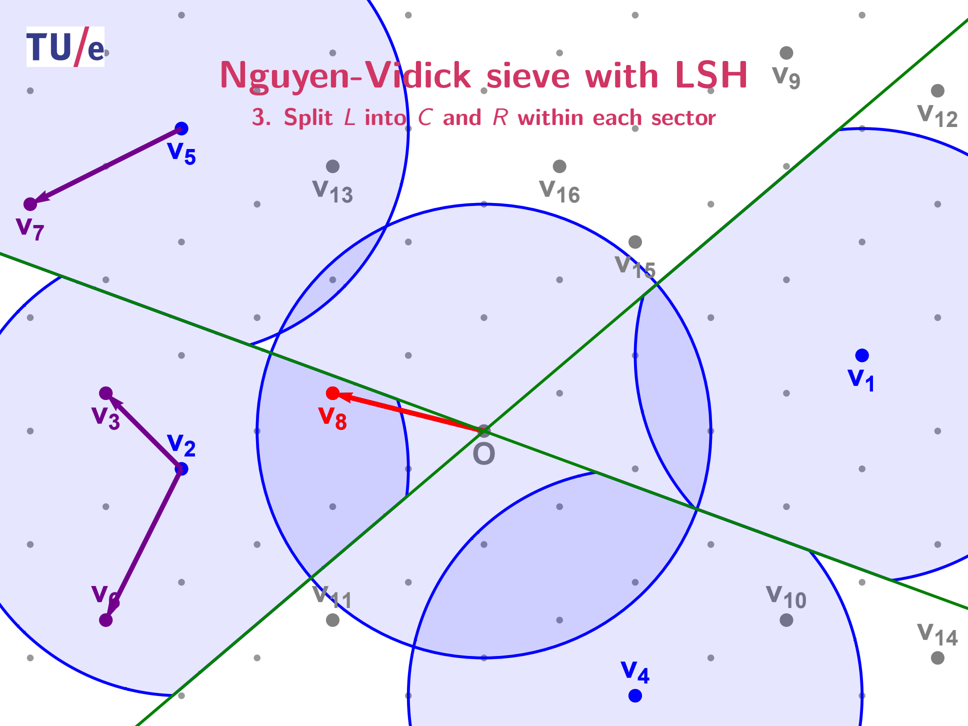

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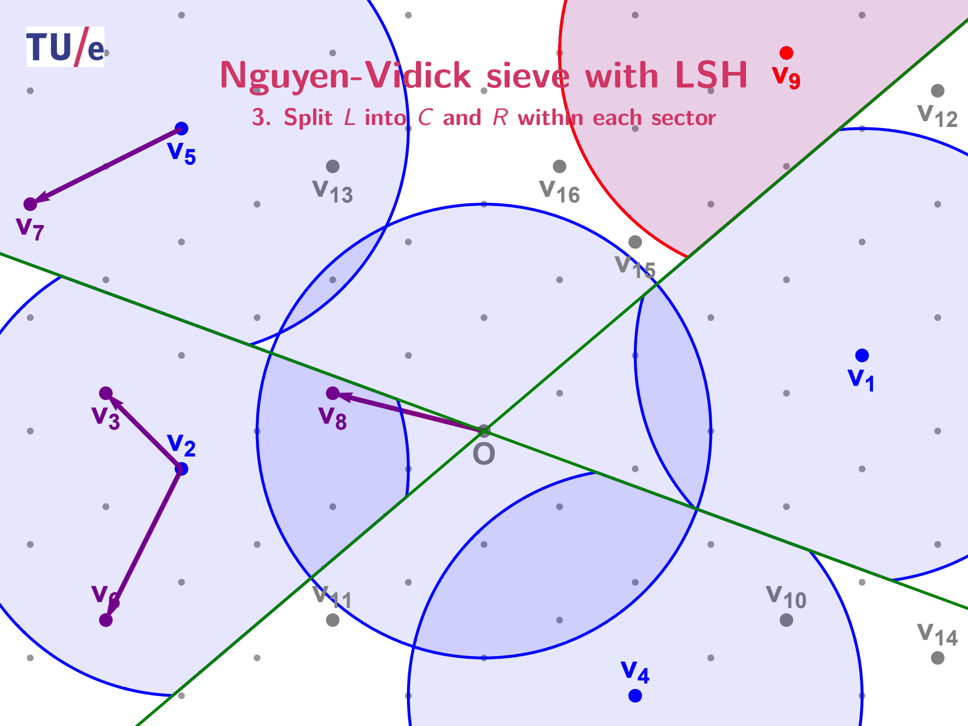

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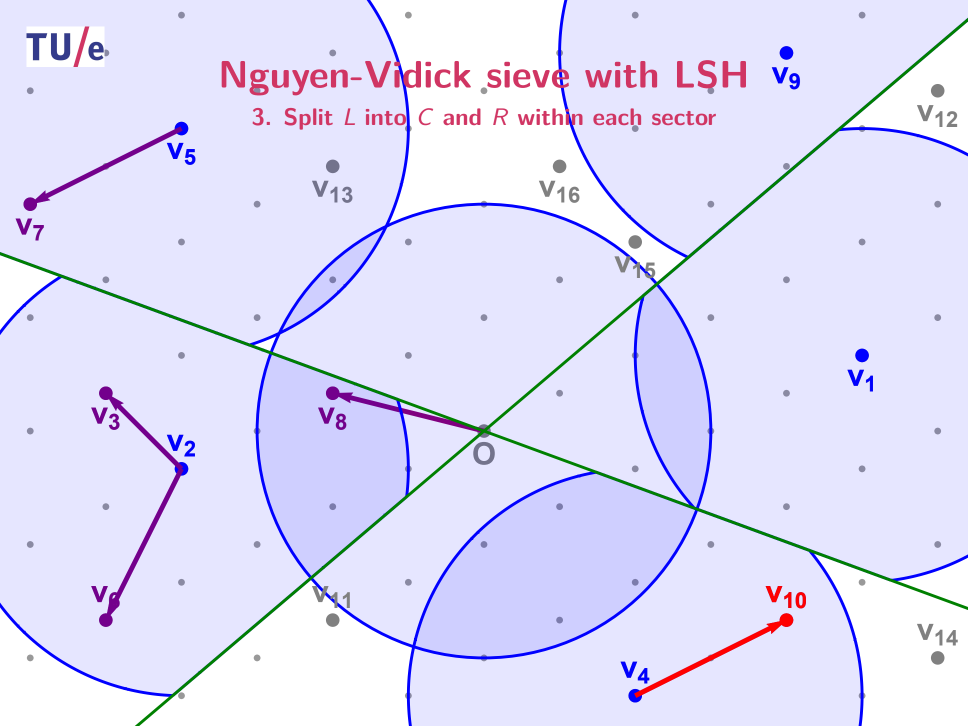

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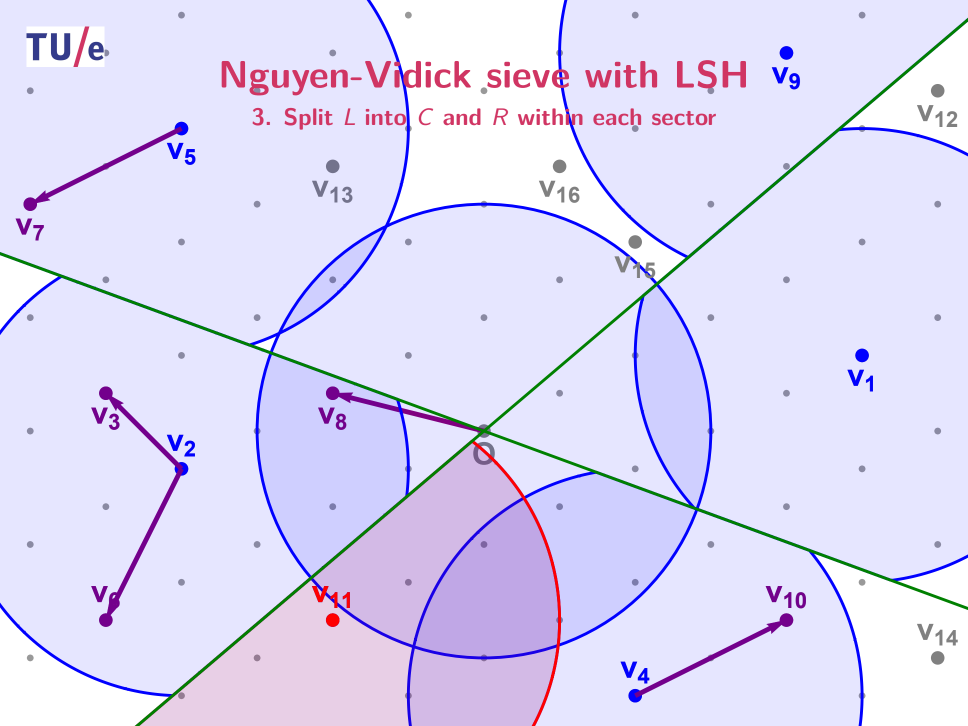

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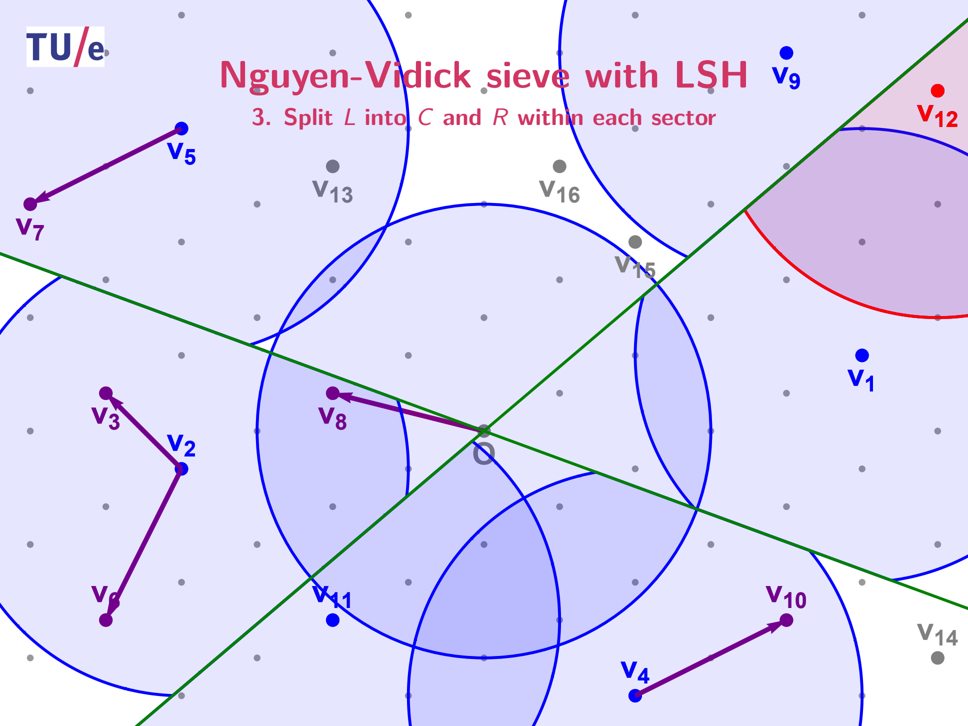

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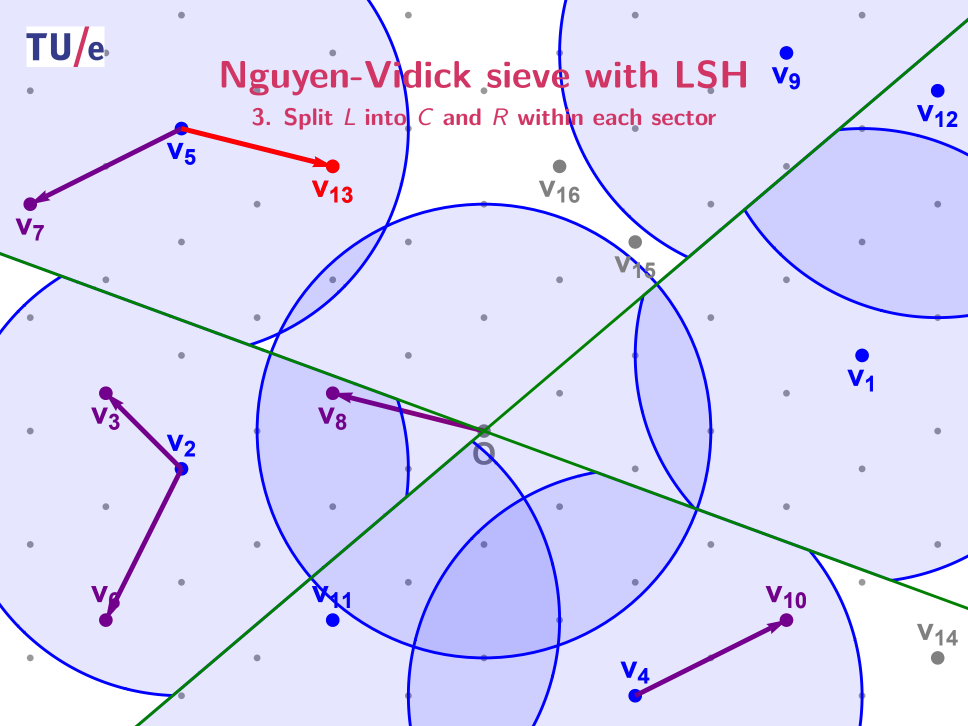

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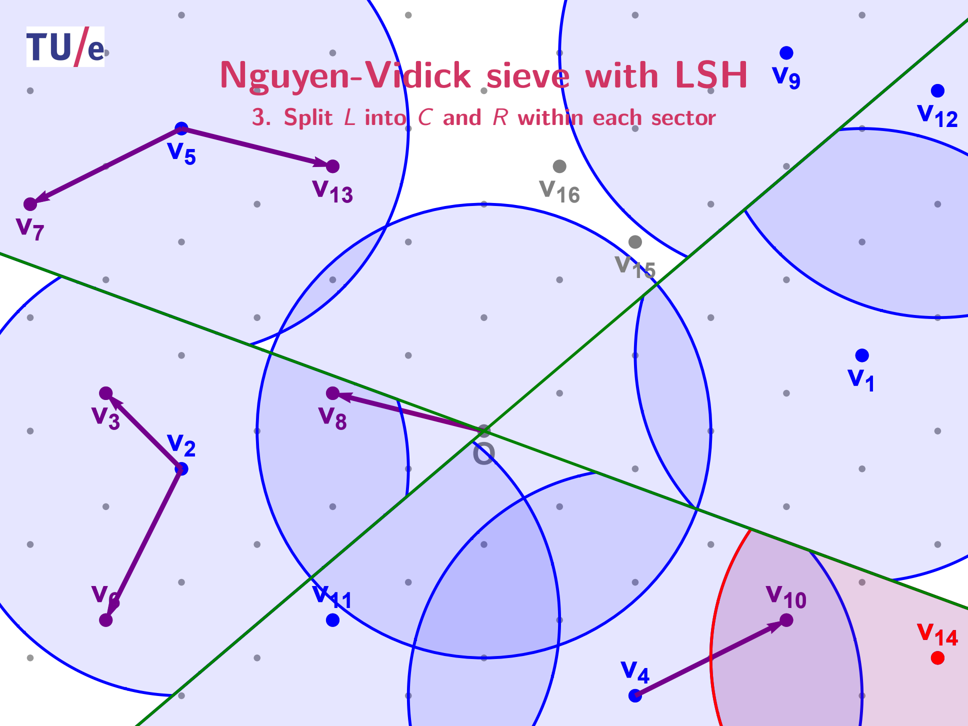

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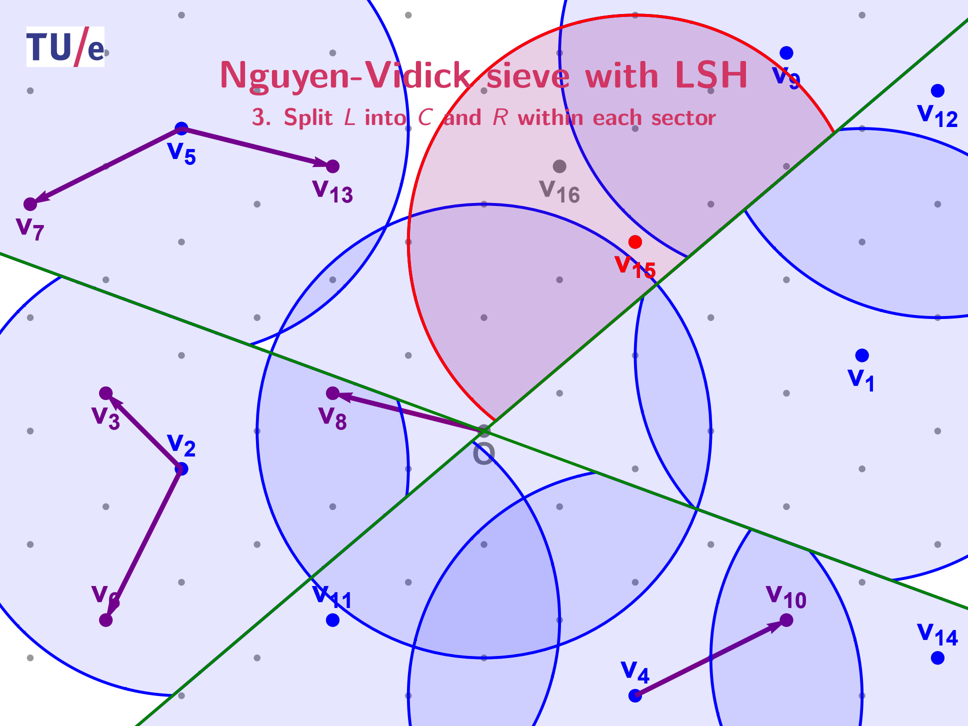

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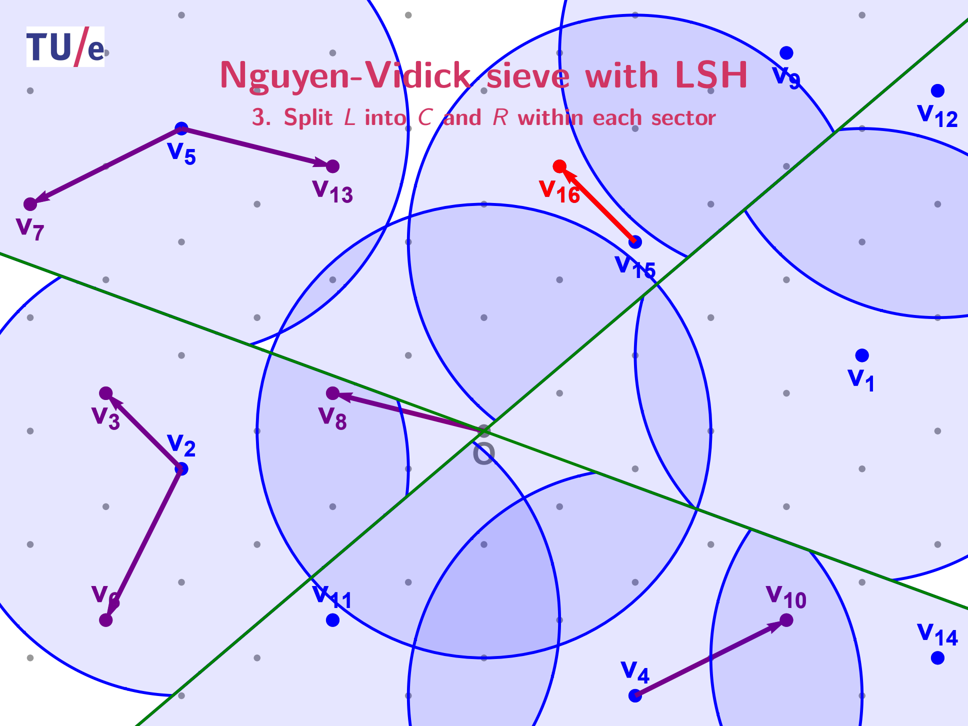

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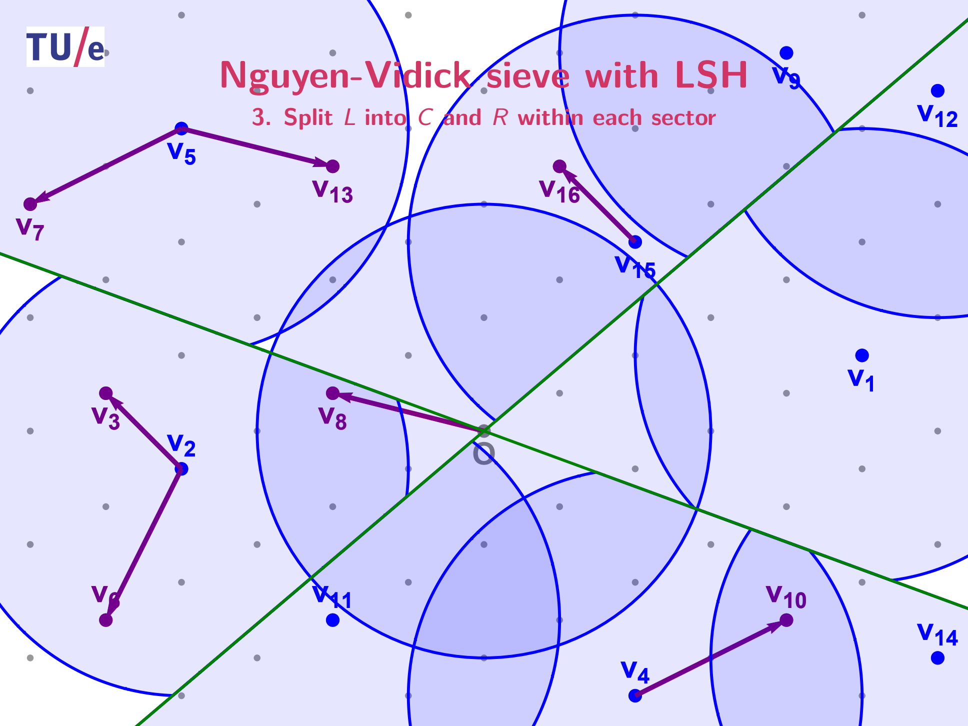
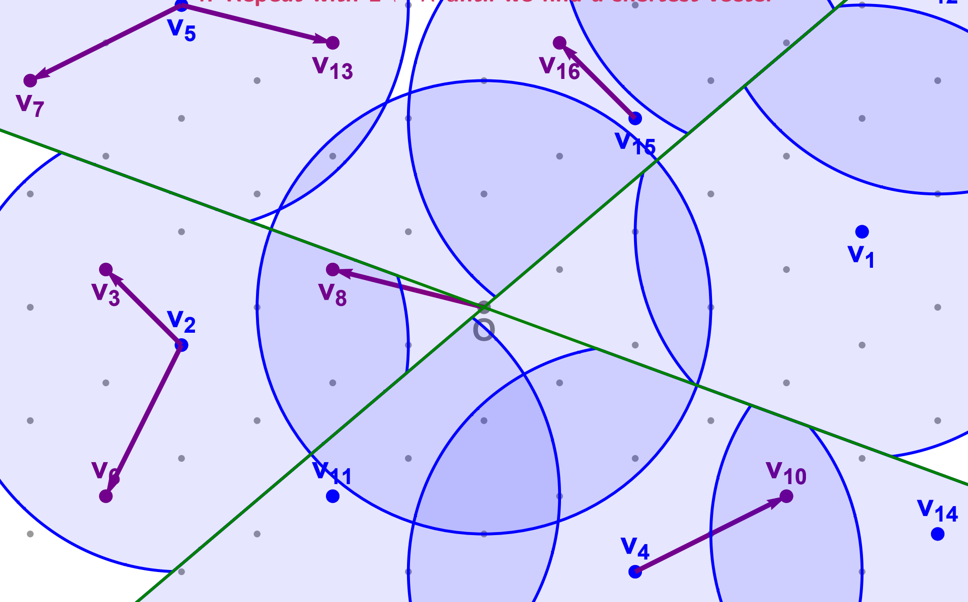


Nguyen-Vidick sieve with LSH

4. Repeat with $L \leftarrow R$ until we find a shortest vector



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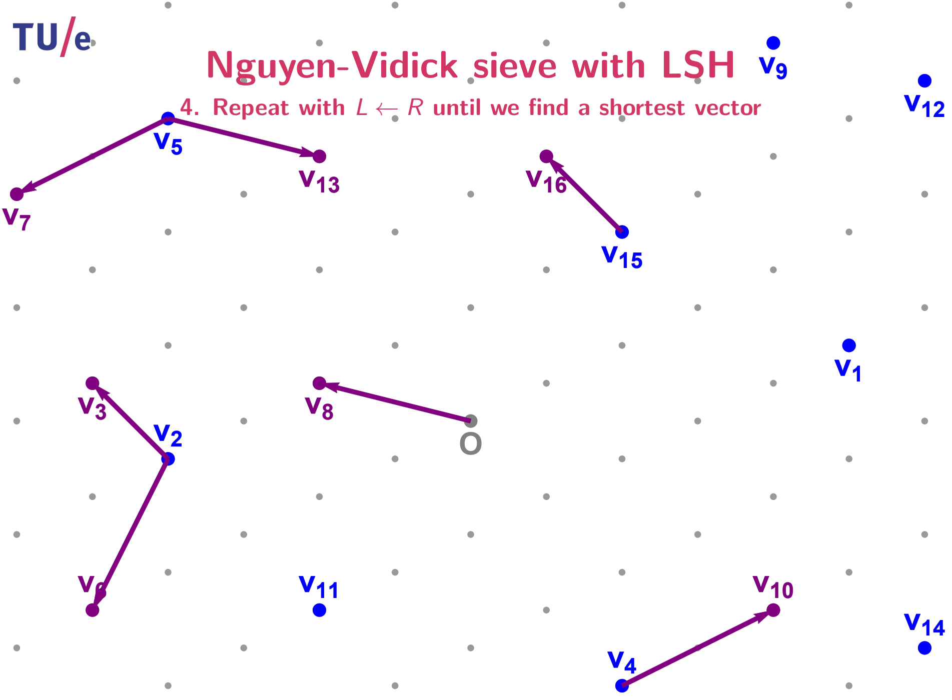

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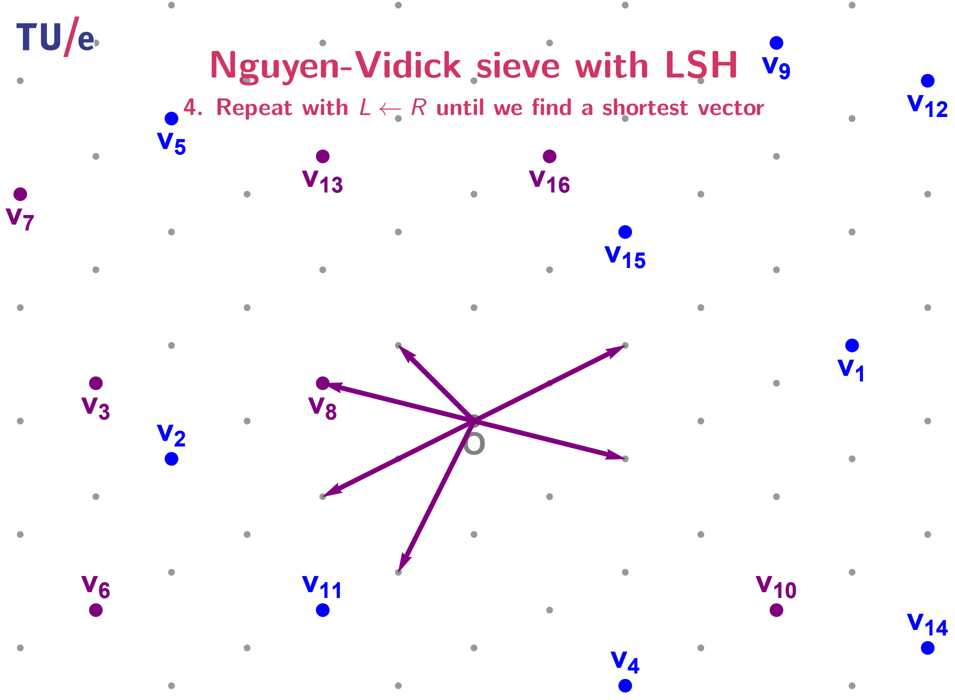

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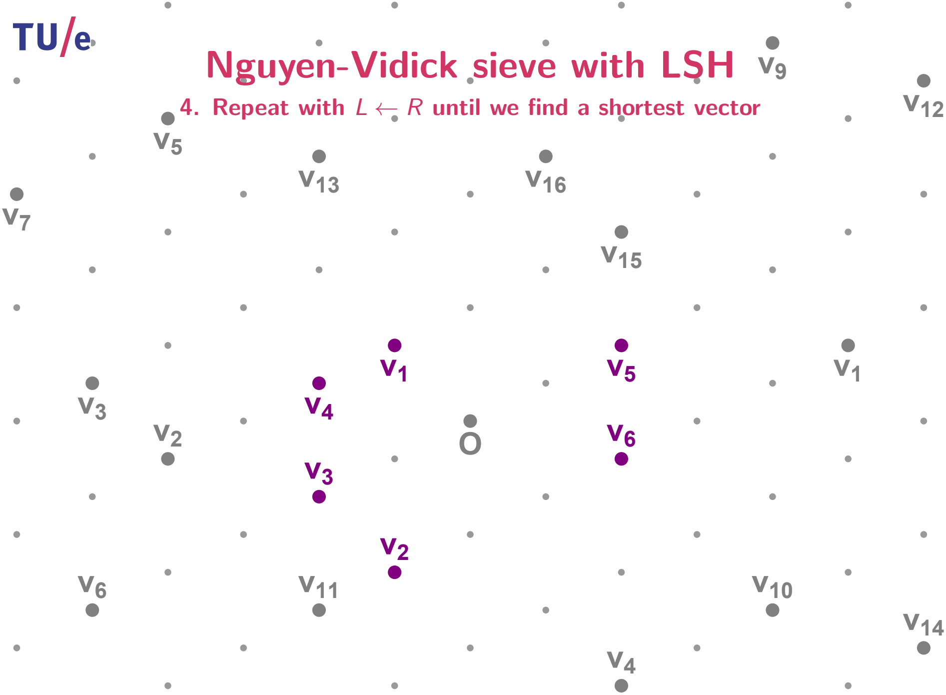
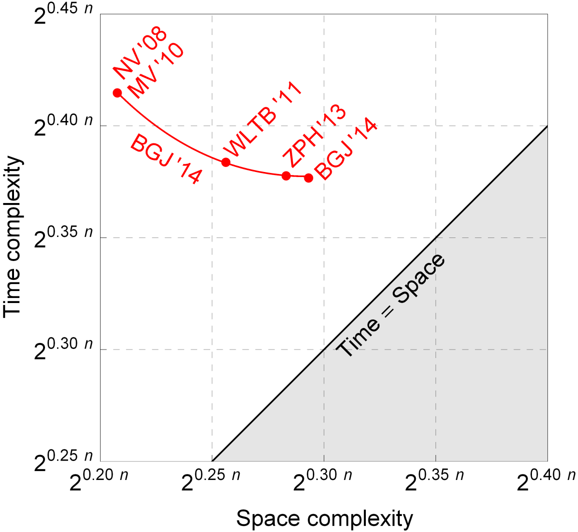


Sieving

Space/time trade-off



Sieving with LSH

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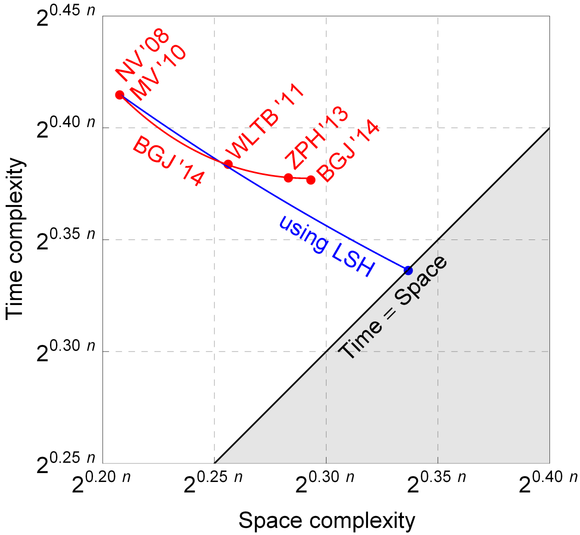

The End

Do not forget to register for the exam before **January 11, 2015**!