Technische Universiteit Eindhoven University of Technology

Algorithms for hard lattice problems and lattice-based cryptography

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IBM Research, Rüschlikon, Switzerland
(November 26, 2015)

Lattices

What is a lattice?

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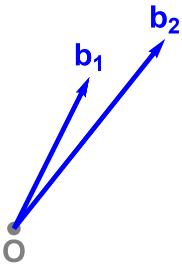

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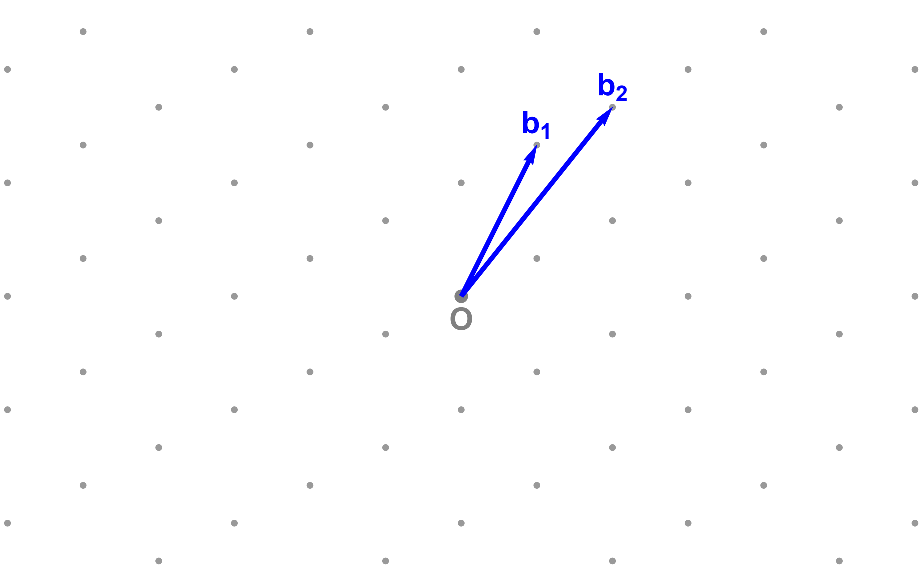
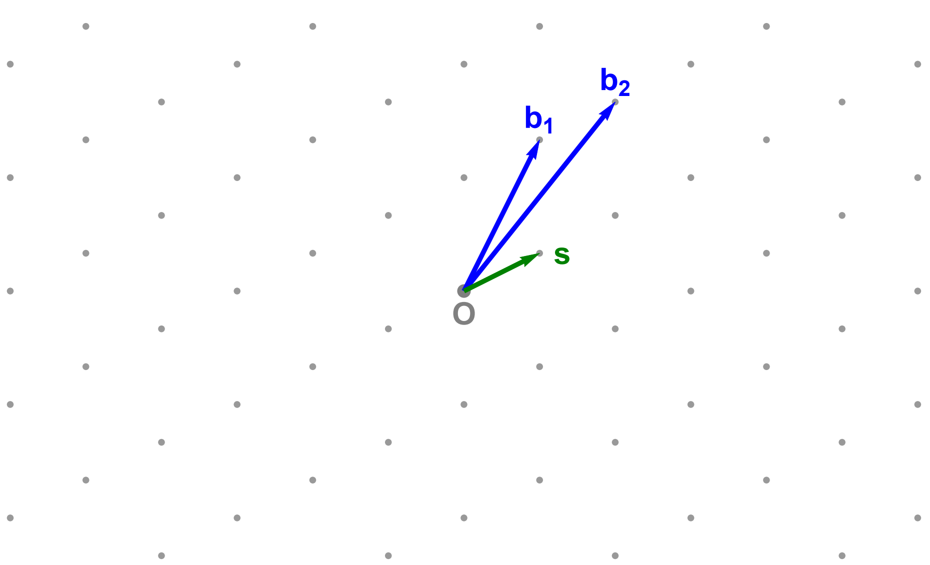


Lattices

Shortest Vector Problem (SVP)



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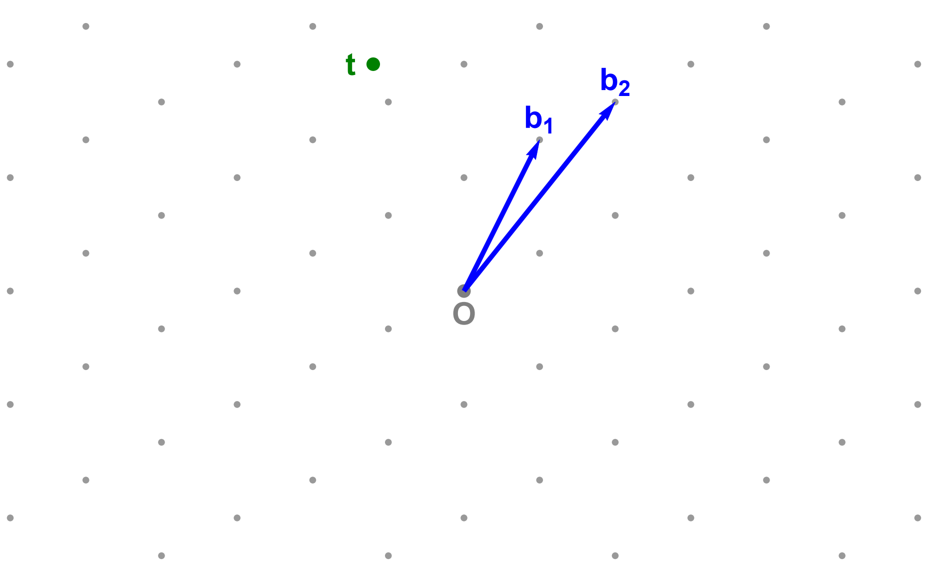

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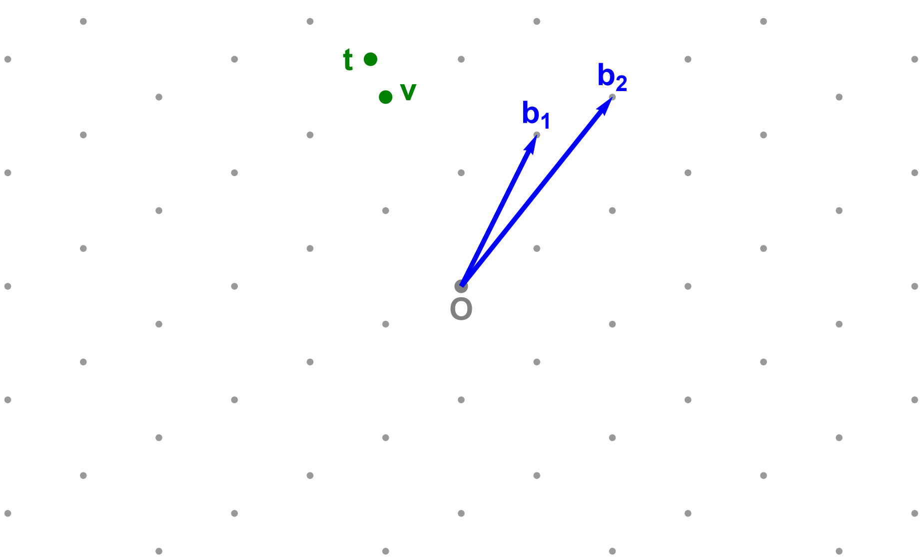
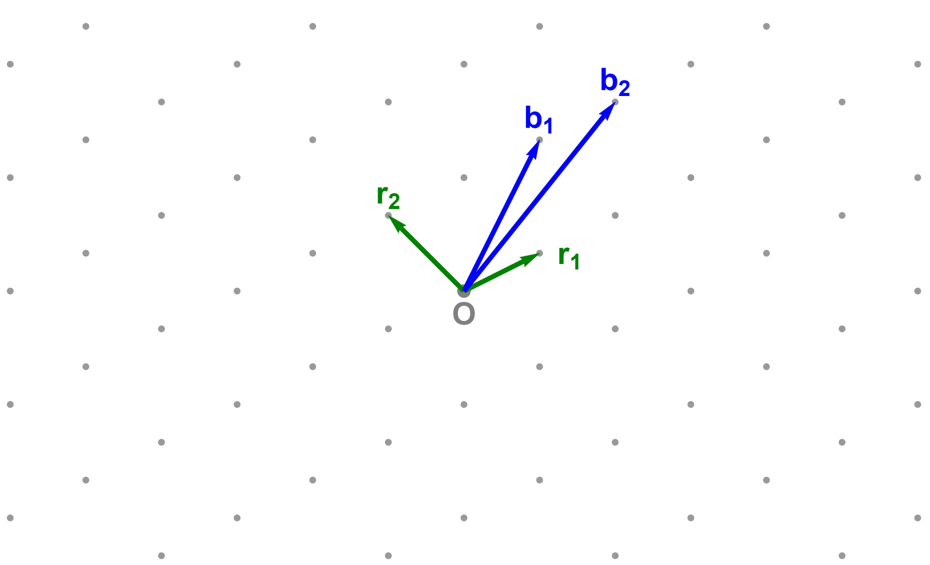


Lattices

Lattice basis reduction (e.g. LLL, BKZ)



Lattices
Applications

- "Constructive cryptography": Lattice-based cryptosystems
  - Based on hard lattice problems (SVP, CVP, LWE, SIS)
  - Worst-case to average-case reductions [Ajt96]
  - Candidate for "post-quantum cryptography"
  - NTRU cryptosystem [HPS98, ..., HPSSWZ15]
  - Fully homomorphic encryption [Gen09, ..., CM15]
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How does lattice-based cryptography work?

GGH cryptosystem

Overview [GGH97]

Private key: $R = \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \end{pmatrix}$

Public key: $B = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}$

Encrypt $\boldsymbol{m}$:

$$\boldsymbol{v} = \boldsymbol{m}B$$
$$\boldsymbol{c} = \boldsymbol{v} + \boldsymbol{e}$$

Decrypt $\boldsymbol{c}$:

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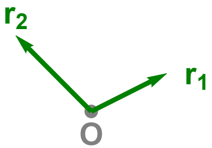

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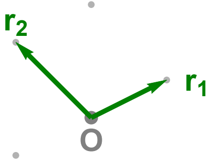

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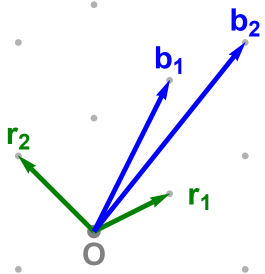

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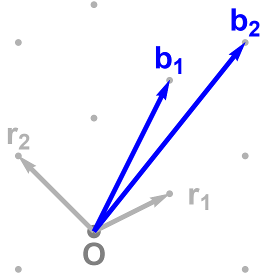

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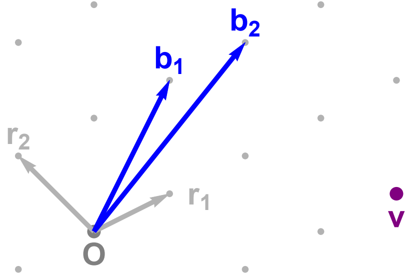

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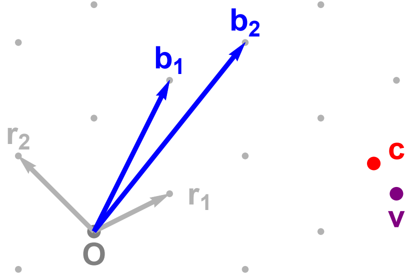

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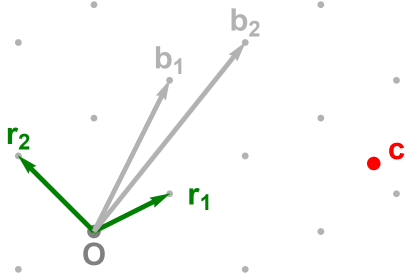

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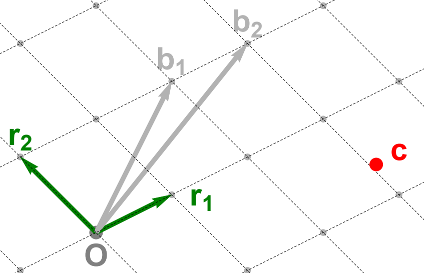

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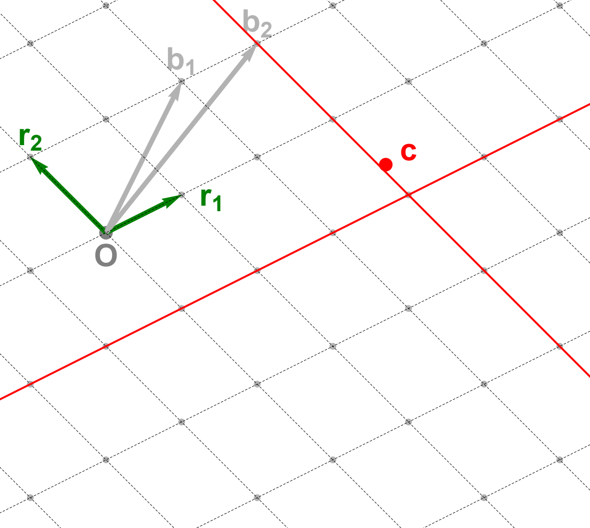

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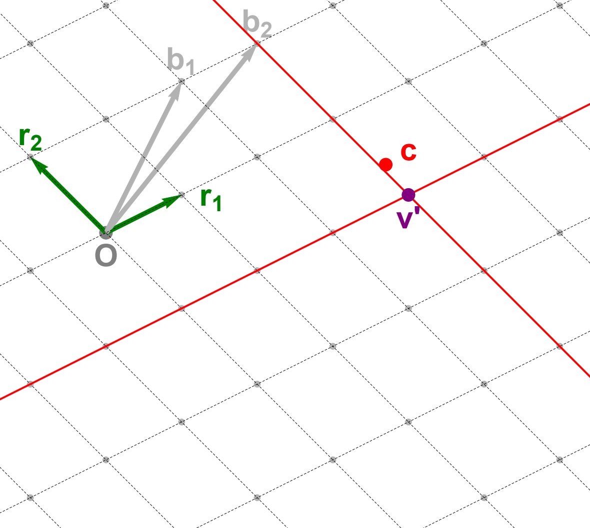

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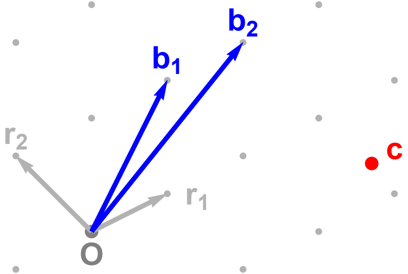

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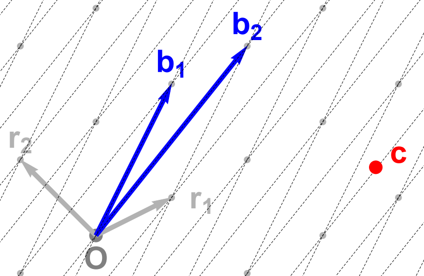

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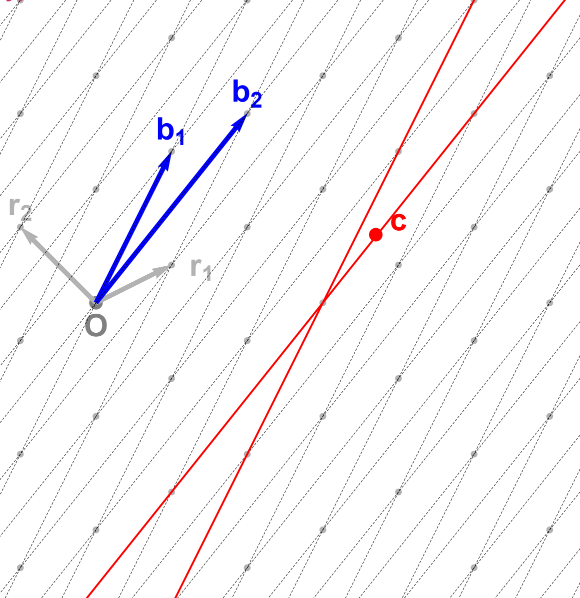

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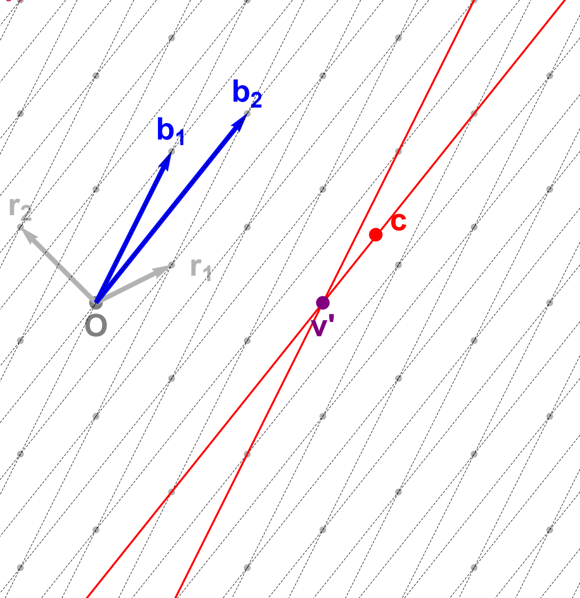

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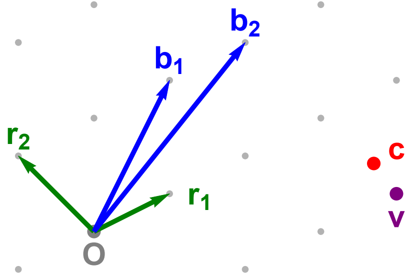

GGH signatures
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$\boldsymbol{c} = H(\boldsymbol{m})$

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Verify $(\boldsymbol{m}, \boldsymbol{s})$:

$\boldsymbol{s}$ lies on the lattice

$\|\boldsymbol{s} - H(\boldsymbol{m})\|$ is small

GGH signatures

Private and public keys

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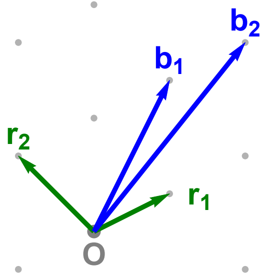

GGH signatures

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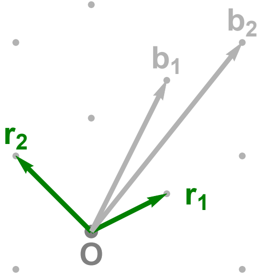

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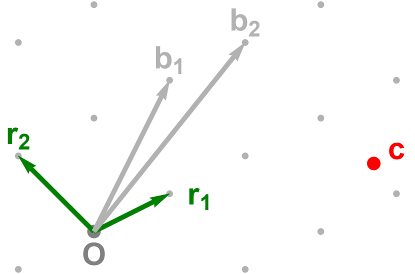

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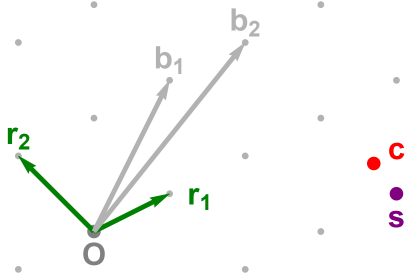

GGH signatures

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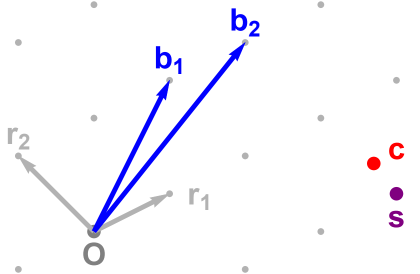

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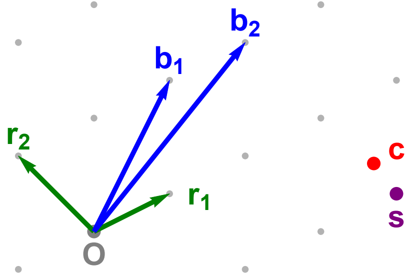

GGH signatures

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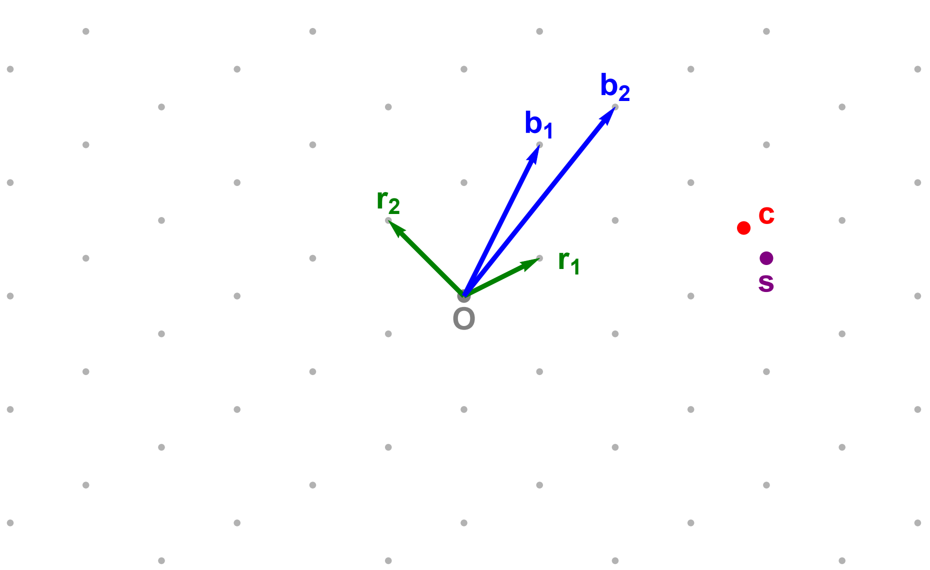

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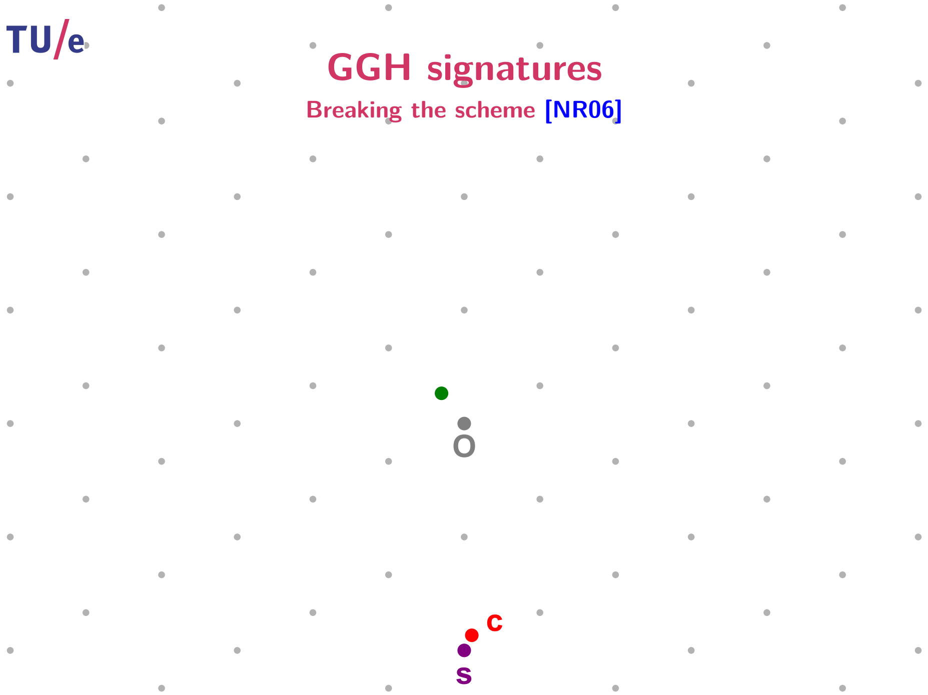

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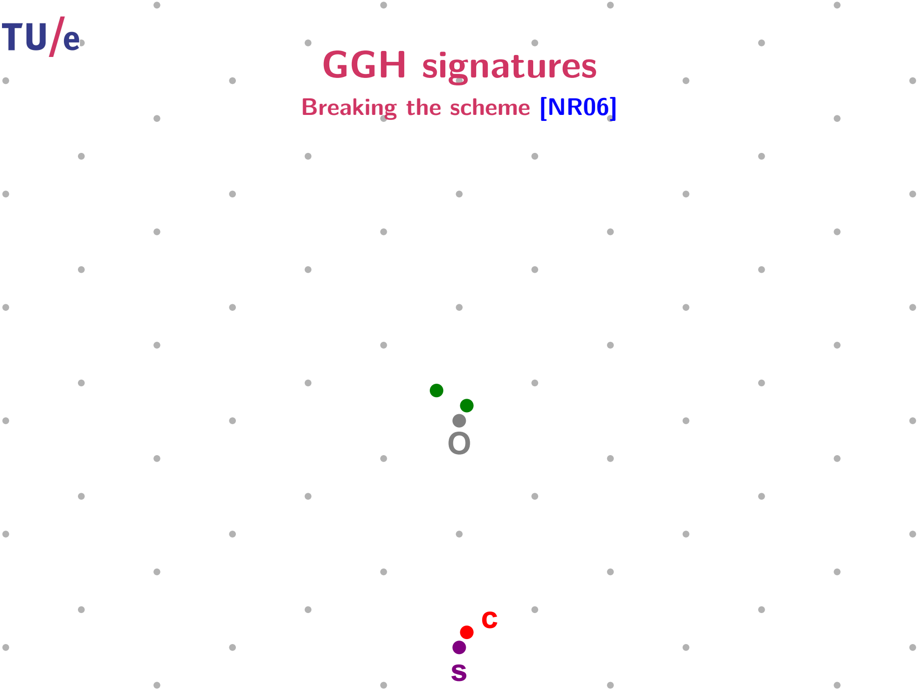

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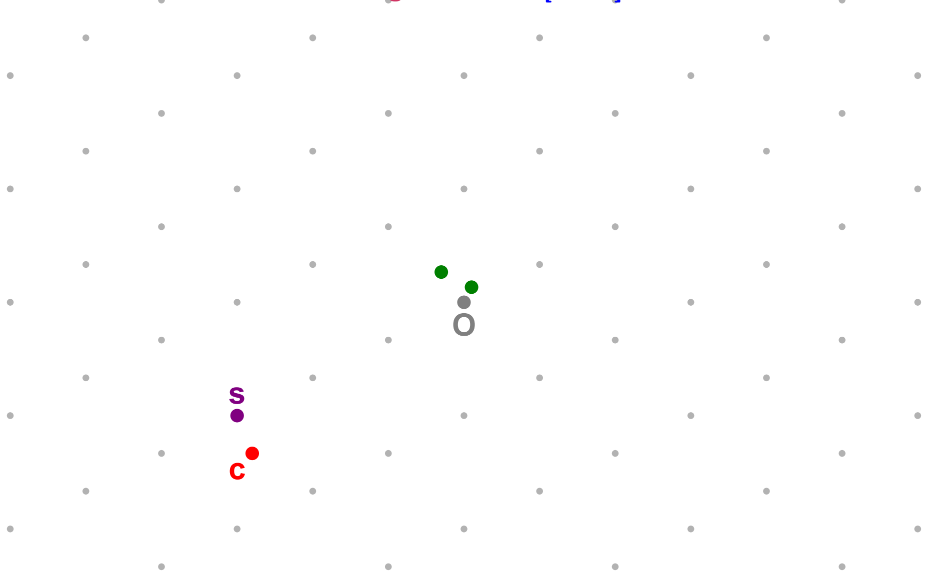

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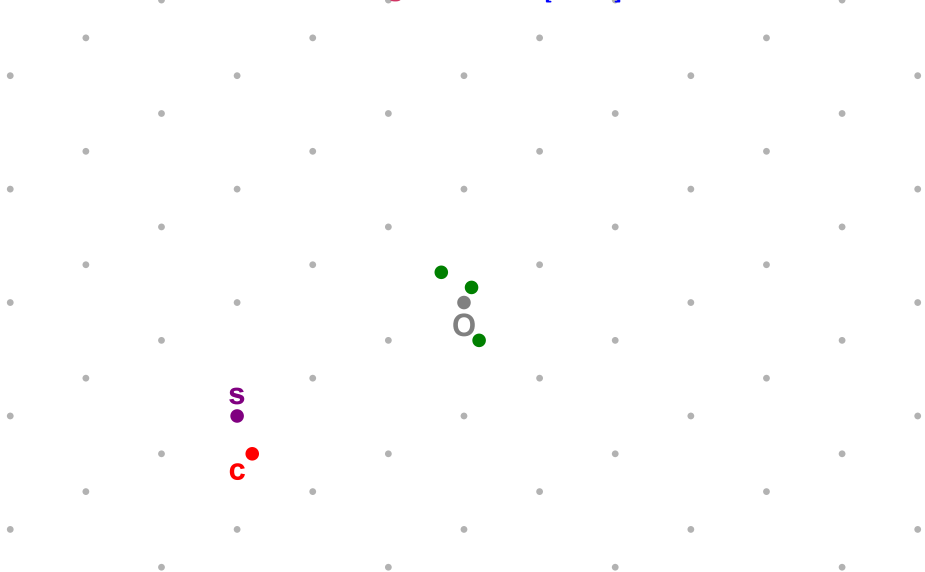

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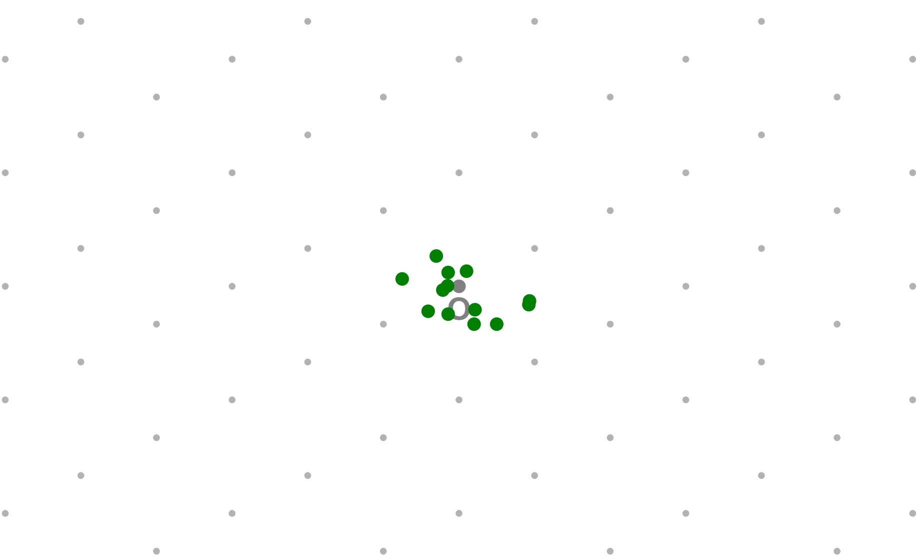

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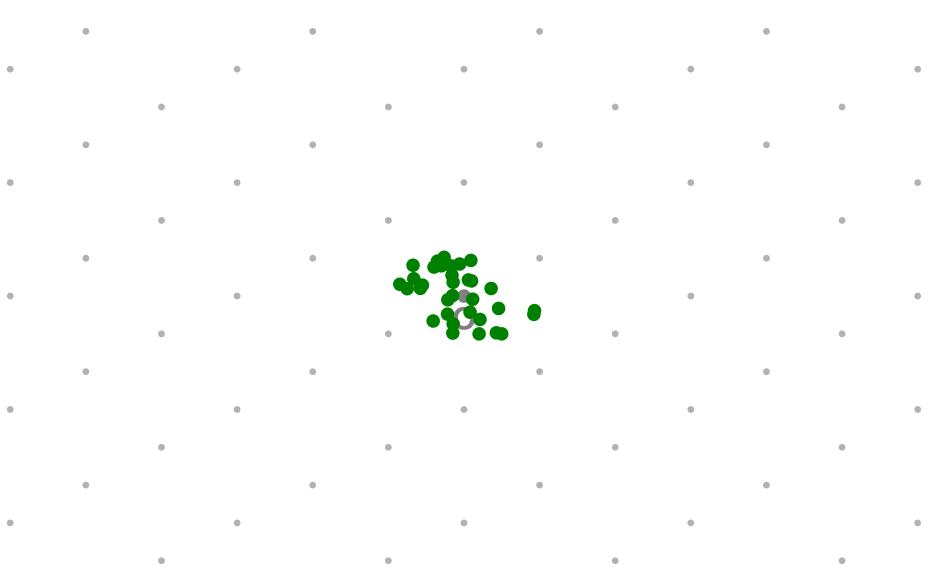

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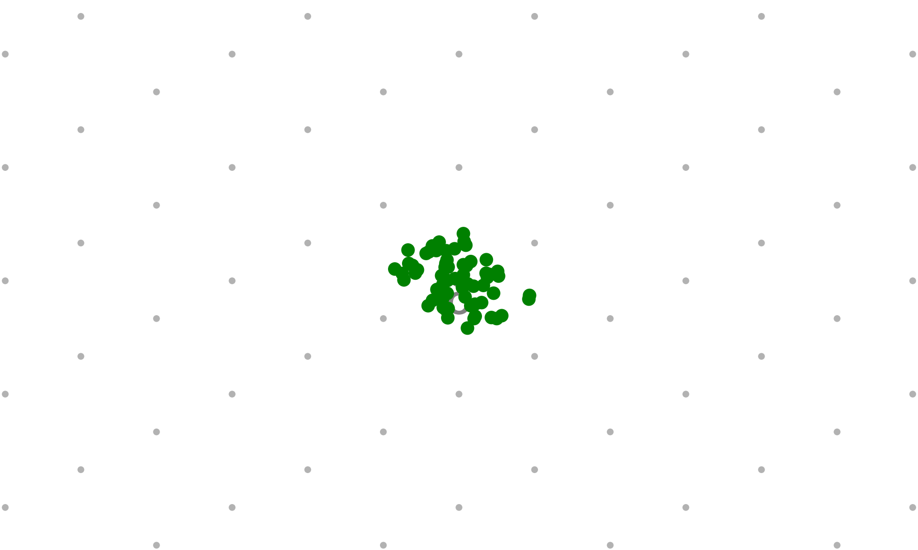

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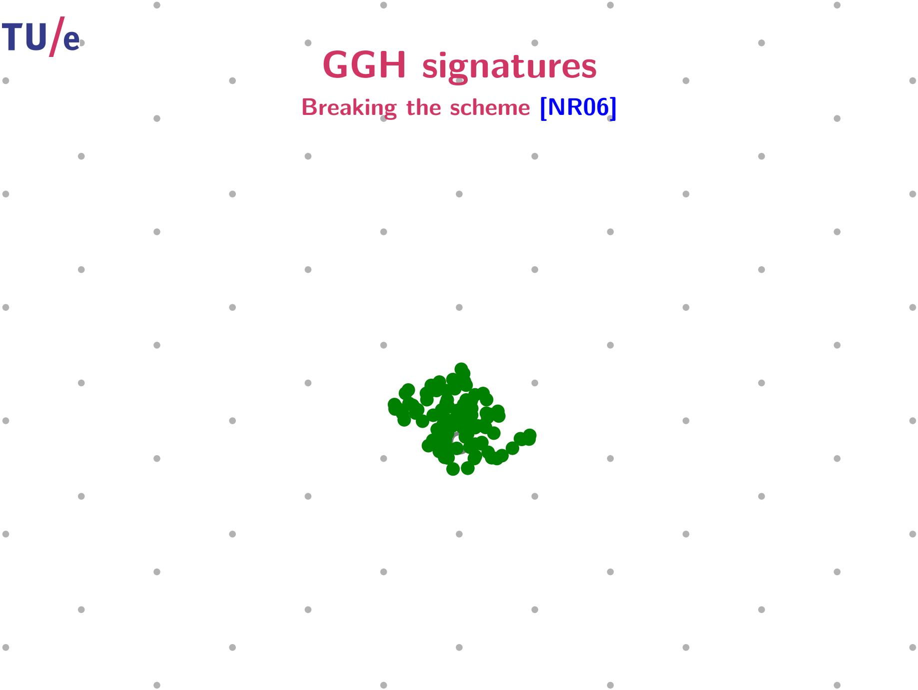

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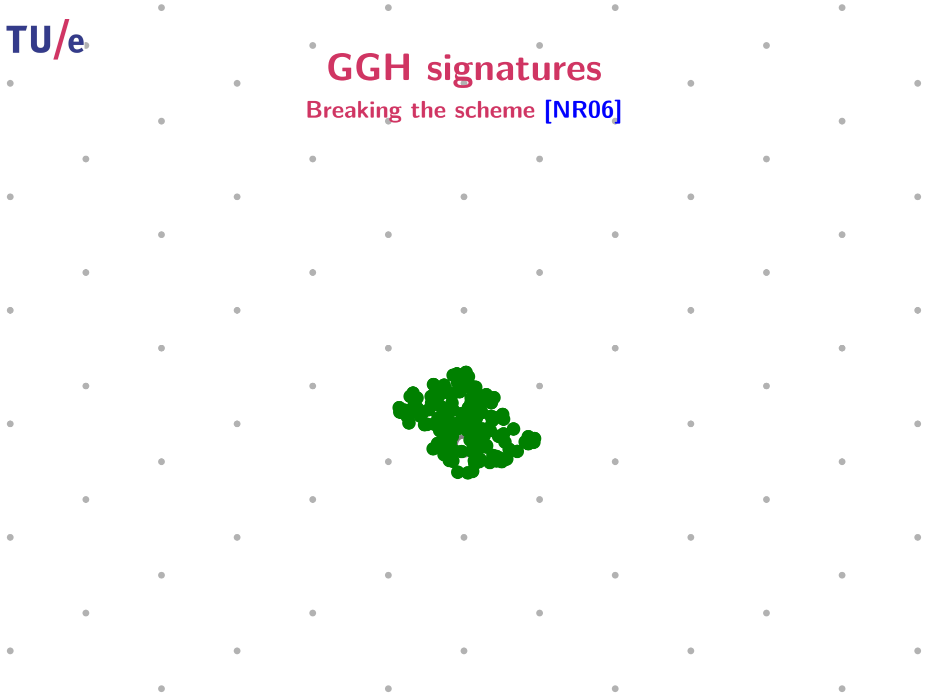

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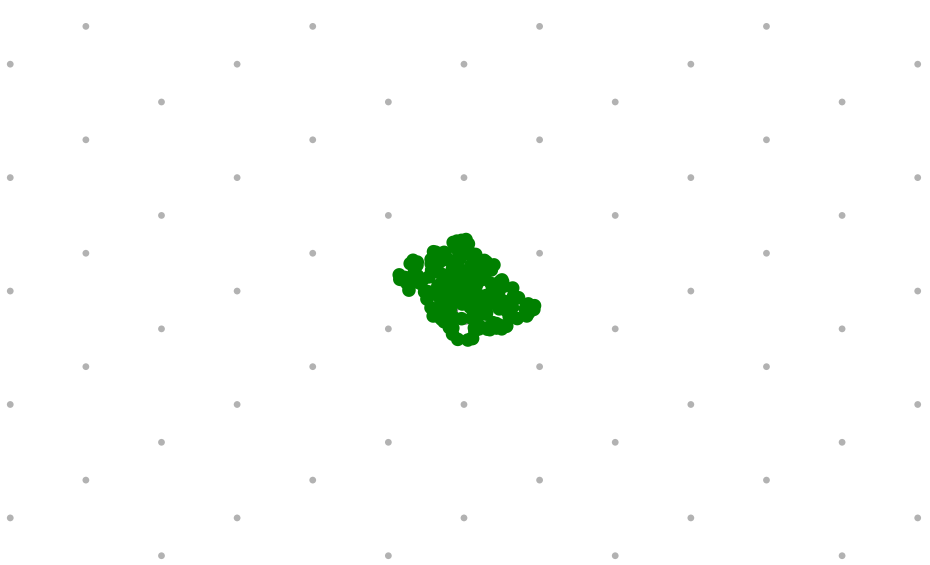

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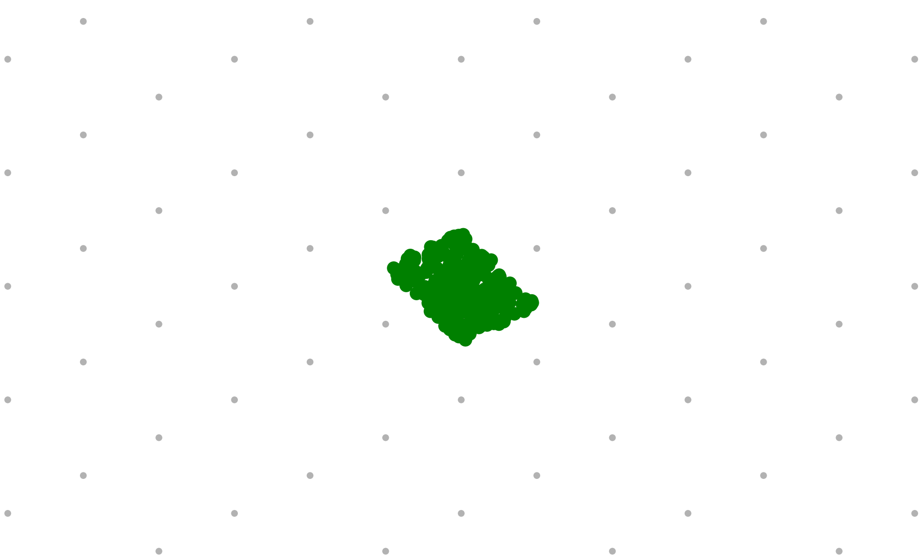

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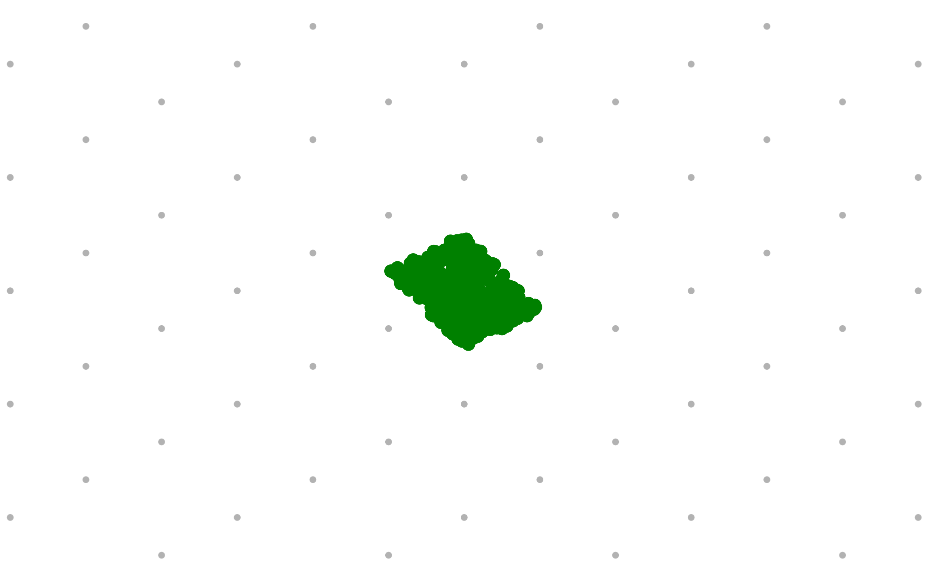

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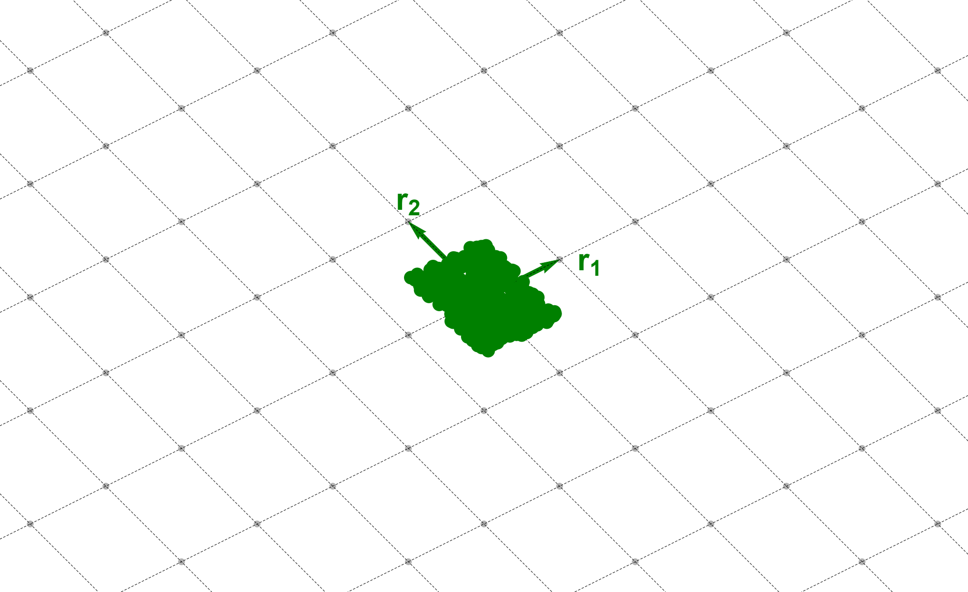

Lattices

Applications

- "Constructive cryptography": Lattice-based cryptosystems
  - Based on hard lattice problems (SVP, CVP, LWE, SIS)
  - Worst-case to average-case reductions [Ajt96]
  - Candidate for "post-quantum cryptography"
  - NTRU cryptosystem [HPS98, . . . , HPSSWZ15]
  - Fully homomorphic encryption [Gen09, . . . , CM15]
  - Multilinear maps(?) [GGH13, . . . , Cor15]
- "Destructive cryptography": Lattice cryptanalysis
  - Attack knapsack-based cryptosystems [Sha82, LO85, . . .]
  - Attack RSA with Coppersmith's method [Cop97, . . .]
  - Attack lattice-based cryptosystems [Ngu99, JJ00, . . .]

How does lattice-based cryptography work?

Lattices

Applications

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How does lattice-based cryptography work?
How hard are hard lattice problems such as SVP?

Lattices

Exact SVP algorithms

	Algorithm	$\log_2(\text{Time})$	$\log_2(\text{Space})$
Provable SVP	Enumeration [Poh81, Kan83, . . . , MW15]	$\Omega(n \log n)$	$O(\log n)$
	AKS-sieve [AKS01, NV08, MV10, HPS11]	$3.398n$	$1.985n$
	ListSieve [MV10, MDB14]	$3.199n$	$1.327n$
	AKS-sieve-birthday [PS09, HPS11]	$2.648n$	$1.324n$
	ListSieve-birthday [PS09]	$2.465n$	$1.233n$
	Voronoi cell algorithm [AEVZ02, MV10b]	$2.000n$	$1.000n$
	Discrete Gaussians [ADRS15, ADS15, Ste16]	$1.000n$	$1.000n$
Heuristic SVP	Nguyen–Vidick sieve [NV08]	$0.415n$	$0.208n$
	GaussSieve [MV10, . . . , IKMT14, BNvdP14]	$0.415n$	$0.208n$
	Two-level sieve [WLTB11]	$0.384n$	$0.256n$
	Three-level sieve [ZPH13]	$0.3778n$	$0.283n$
	Overlattice sieve [BGJ14]	$0.3774n$	$0.293n$
	Hyperplane LSH [Laa15, MLB15, Mar15]	$0.337n$	$0.208n$
	May and Ozerov's NNS method [BGJ15]	$0.311n$	$0.208n$
	Spherical LSH [LdW15]	$0.298n$	$0.208n$
	Cross-polytope LSH [BL15]	$0.298n$	$0.208n$
	Spherical filtering [BDGL16, Laa15, ML15]	$0.293n$	$0.208n$

Nguyen–Vidick sieve

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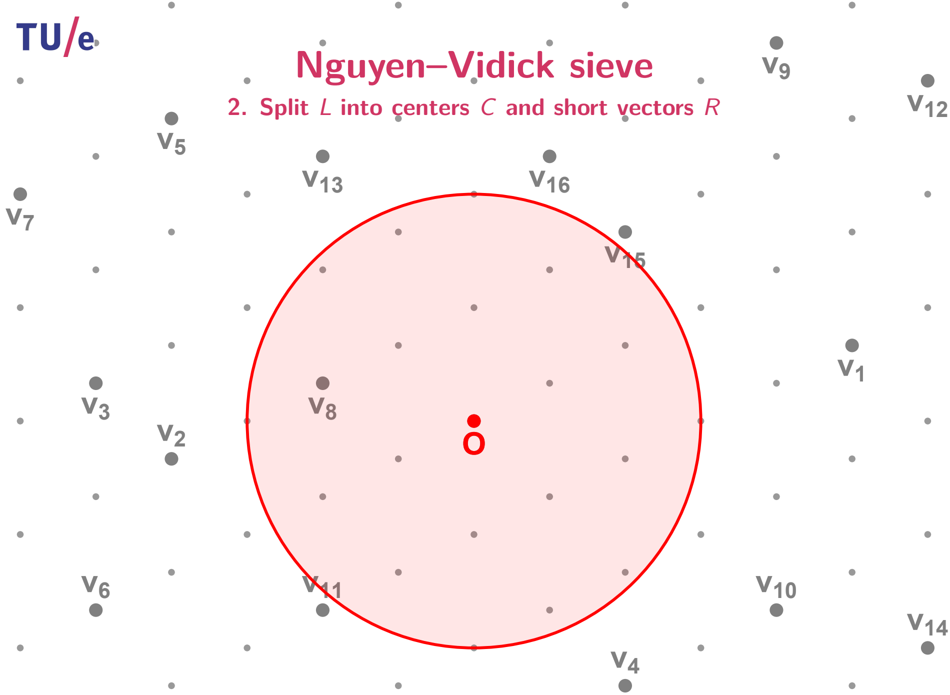

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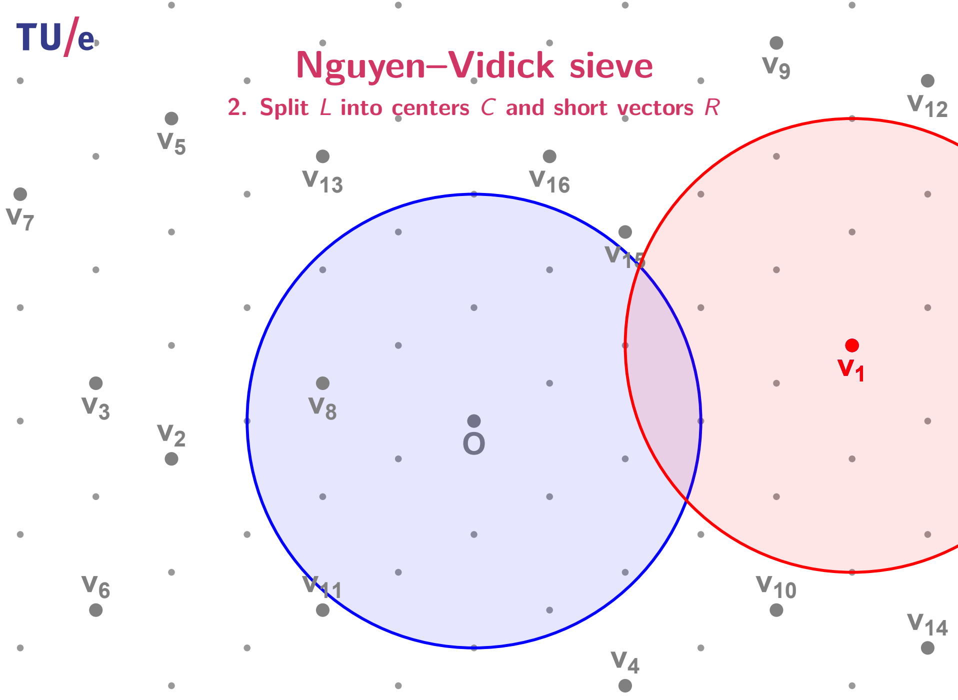

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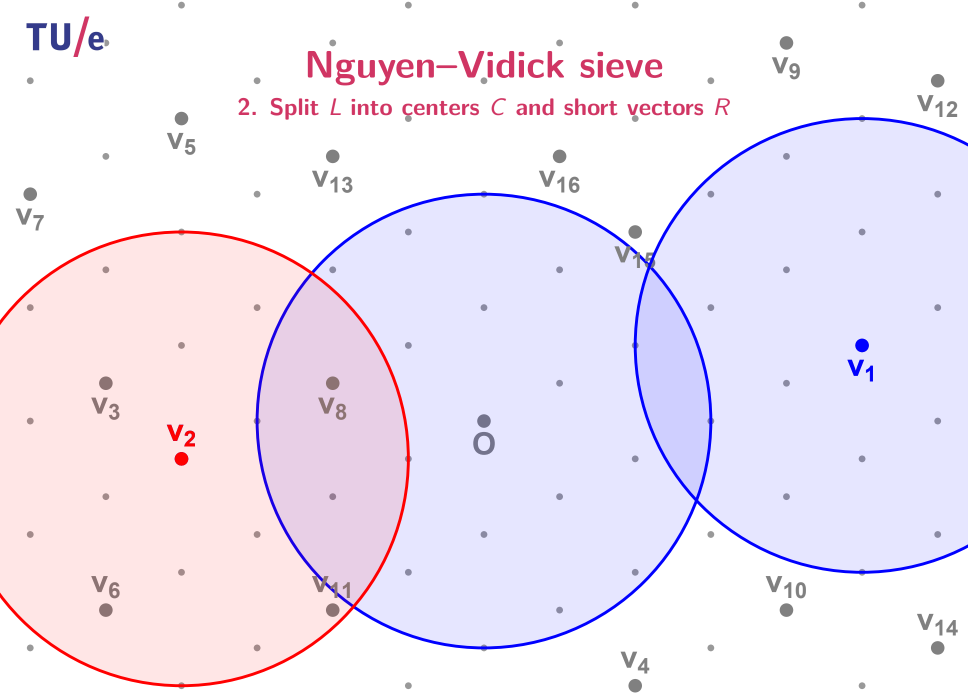

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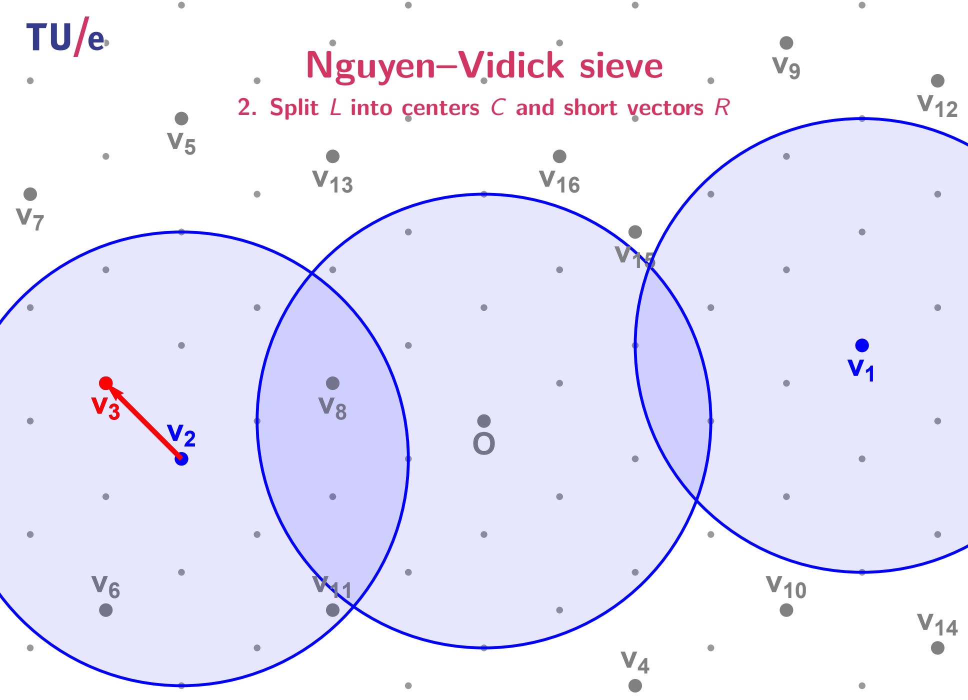

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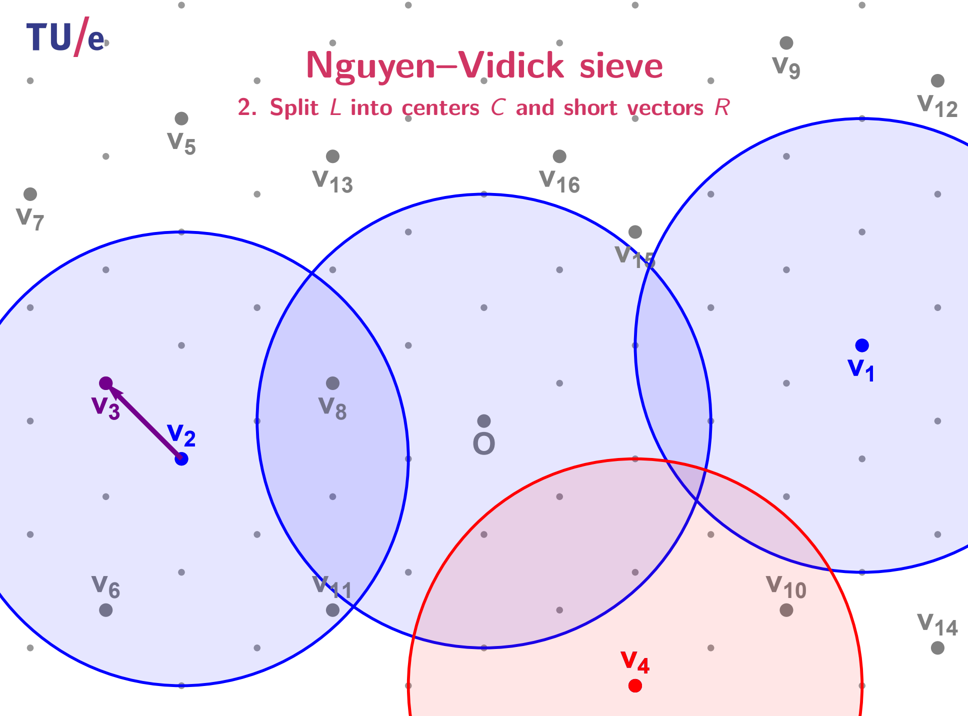

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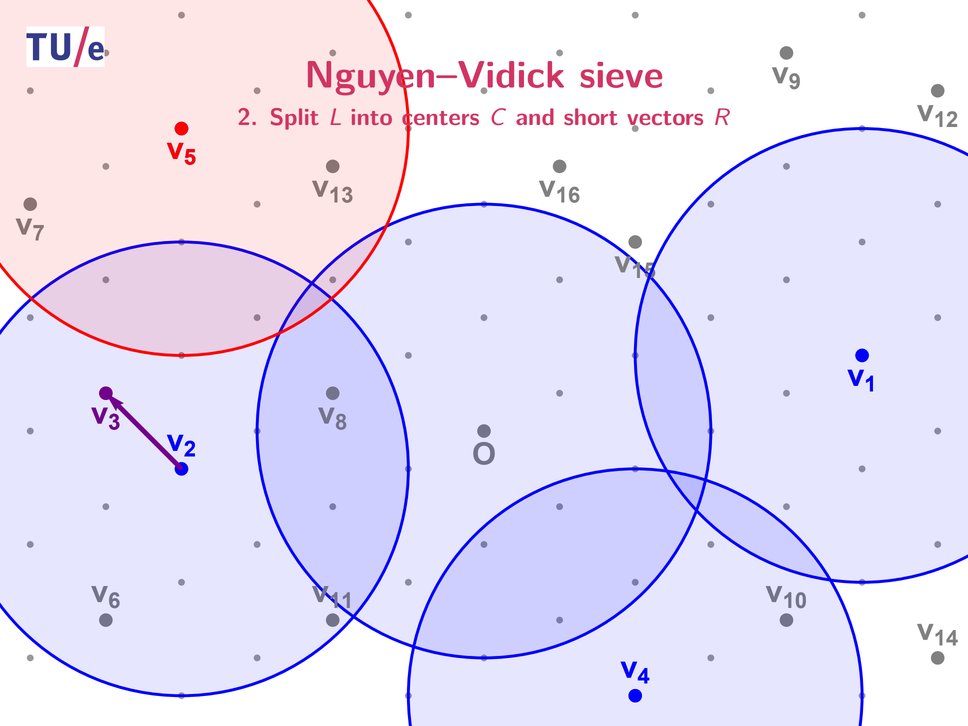

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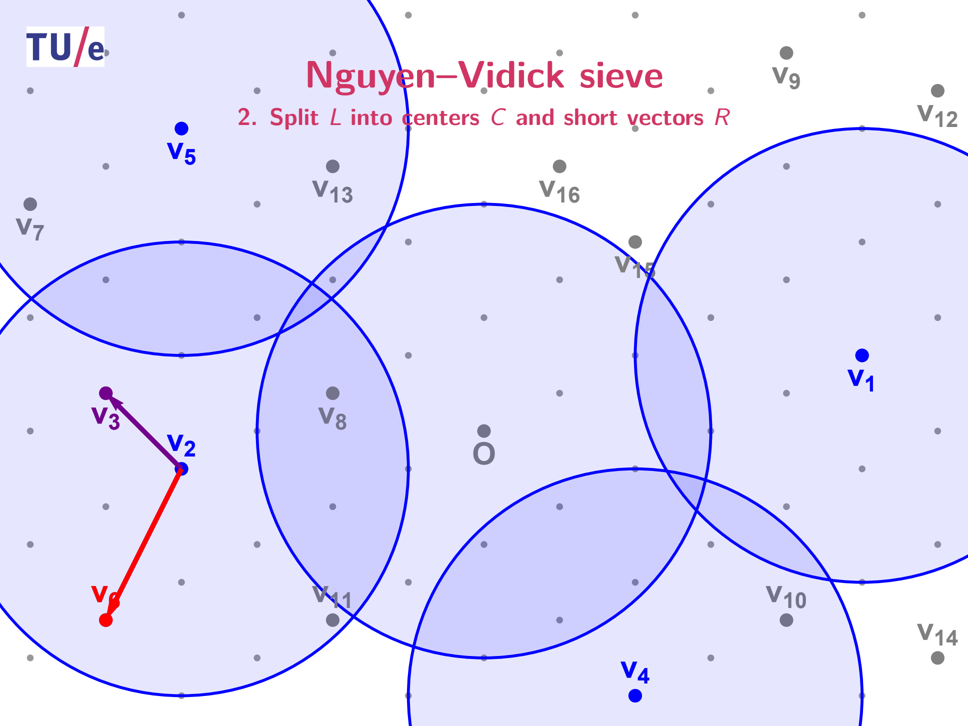

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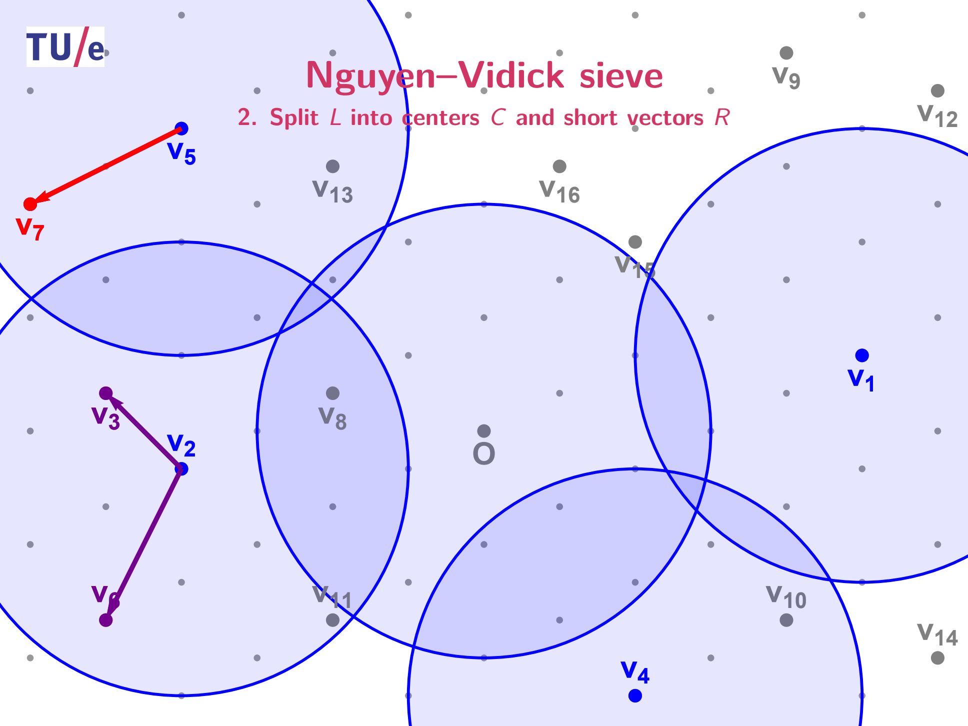

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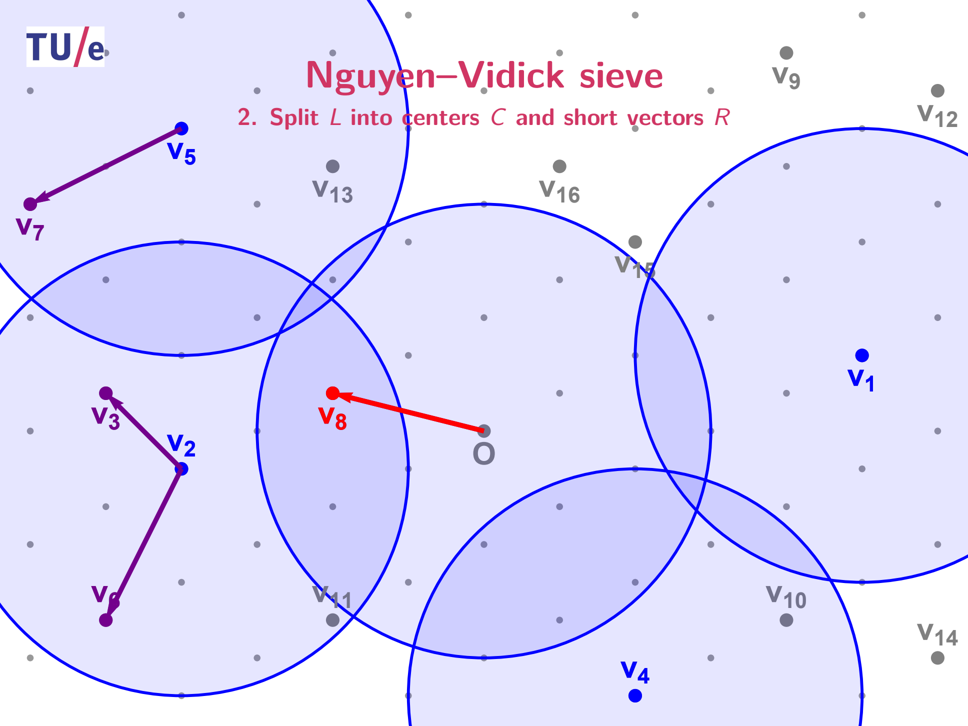

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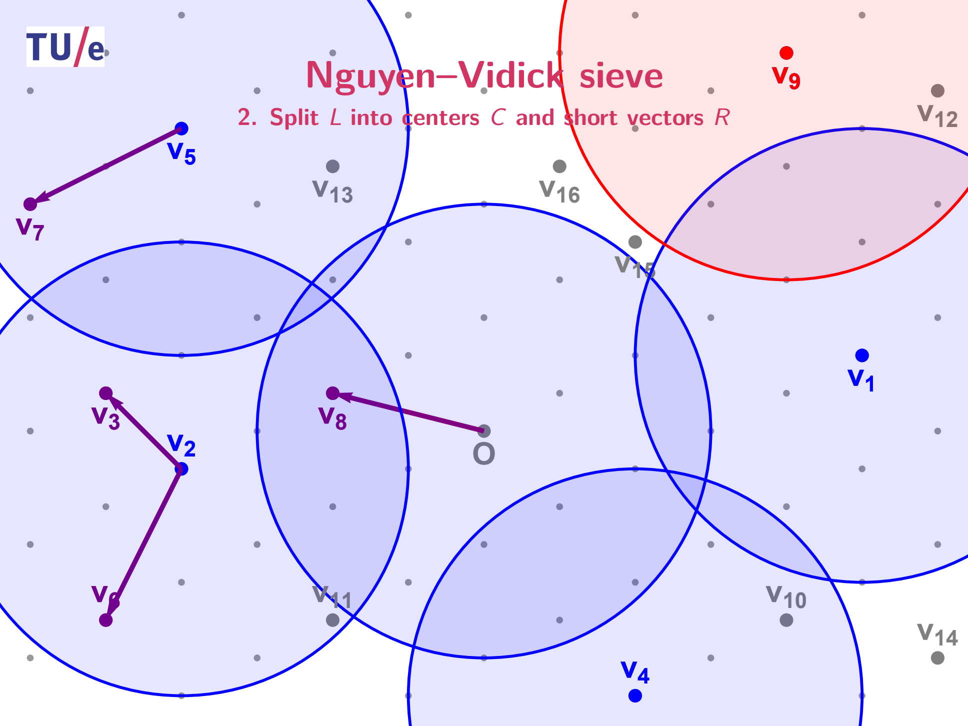

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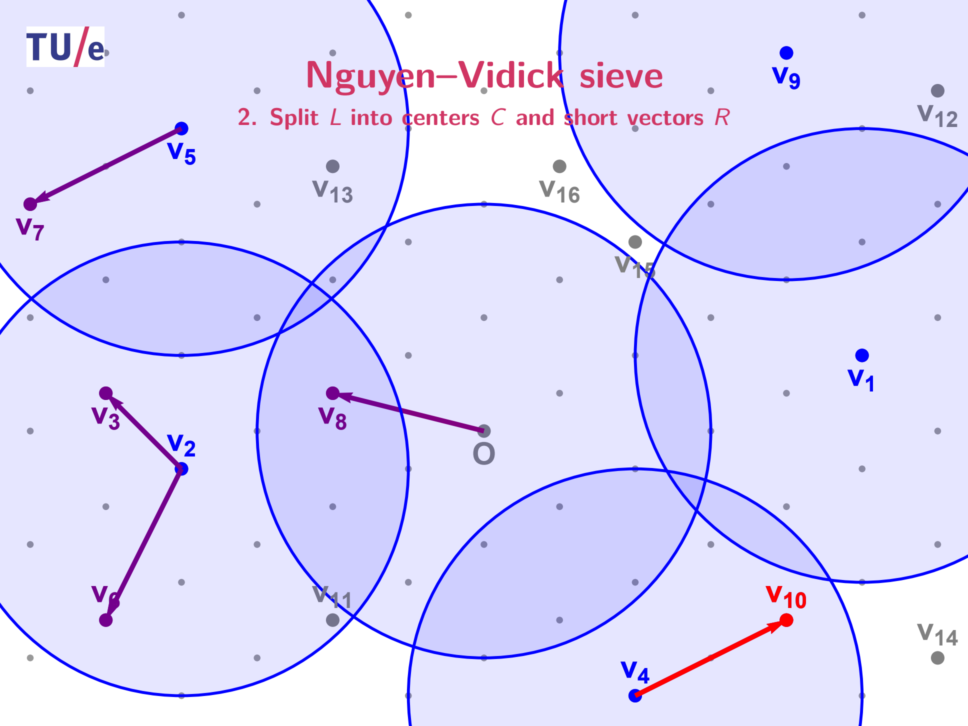

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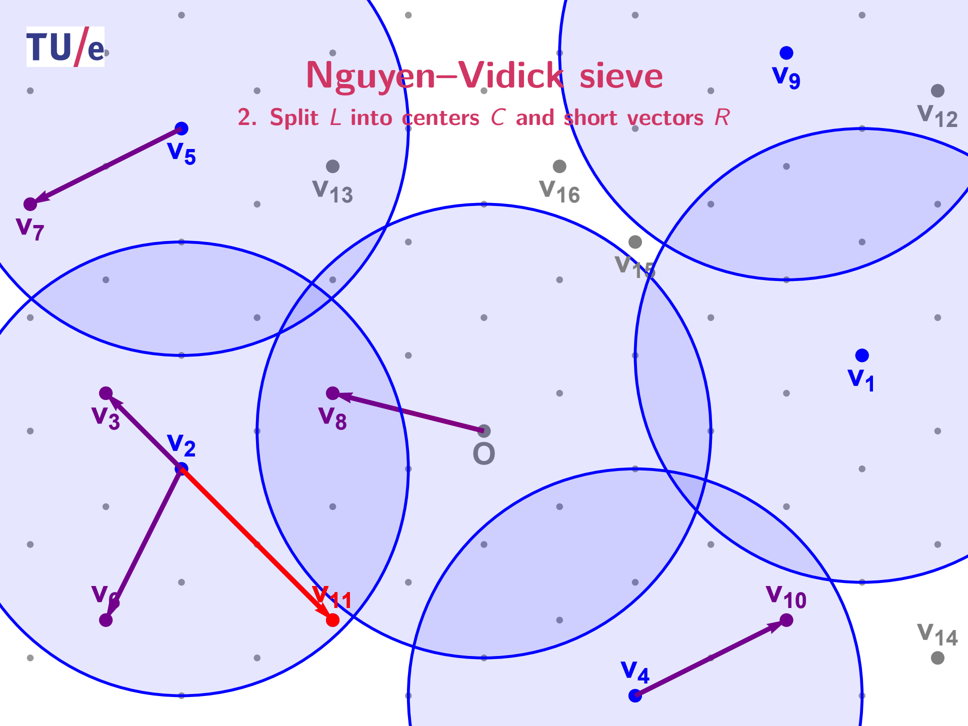

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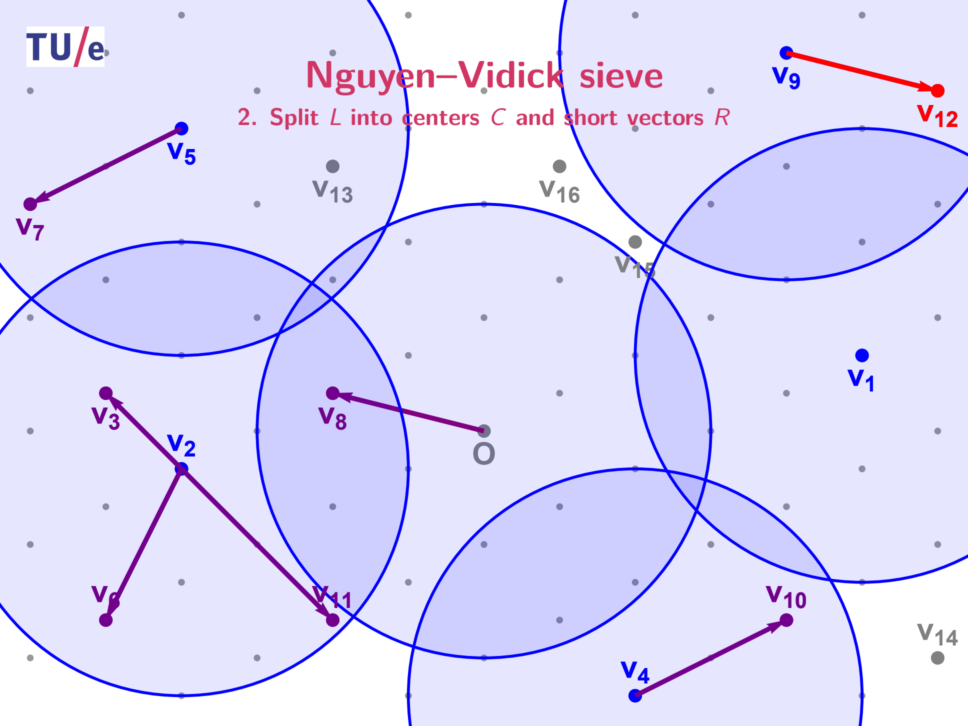

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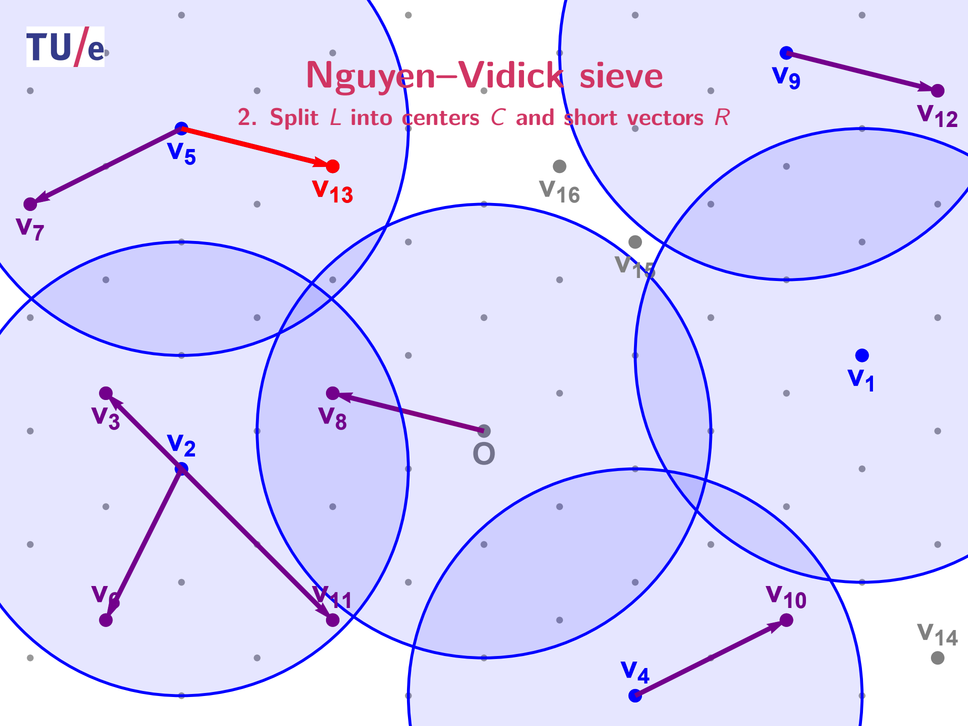

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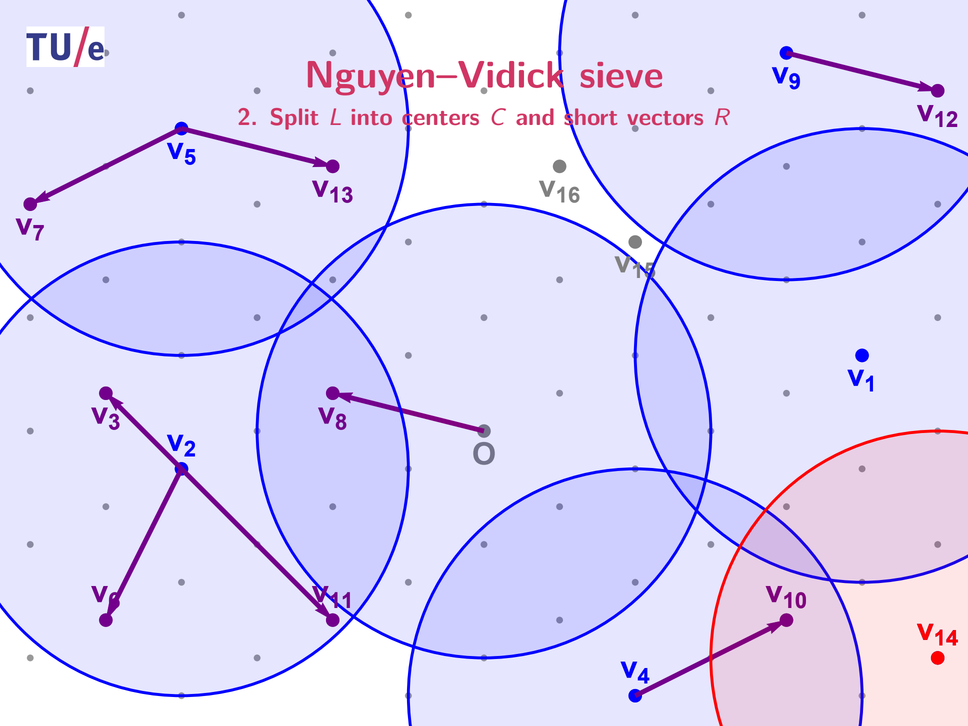

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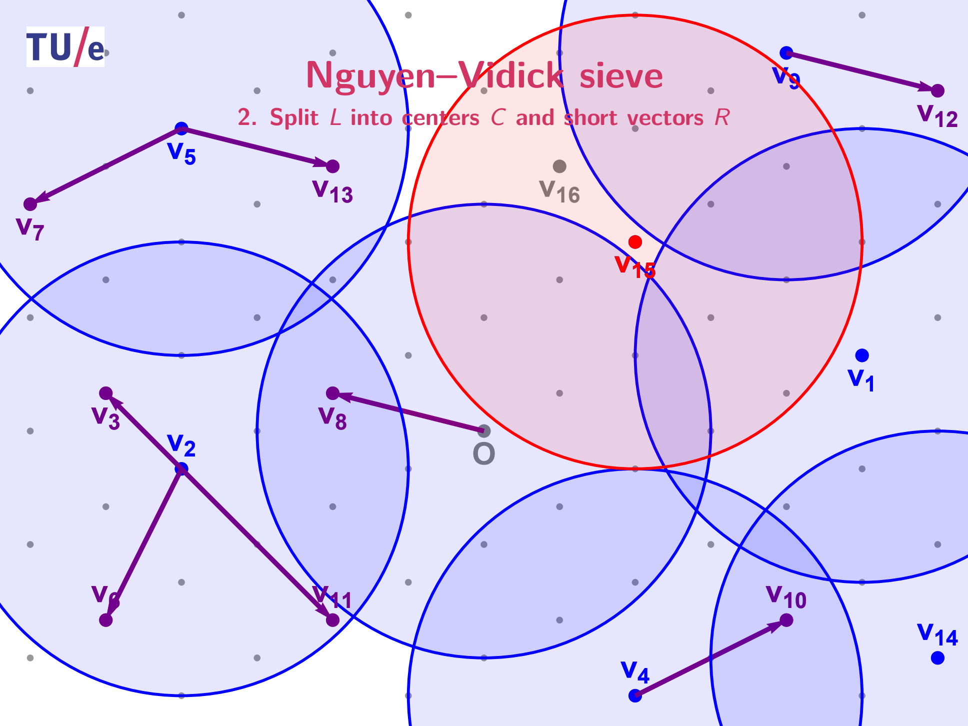

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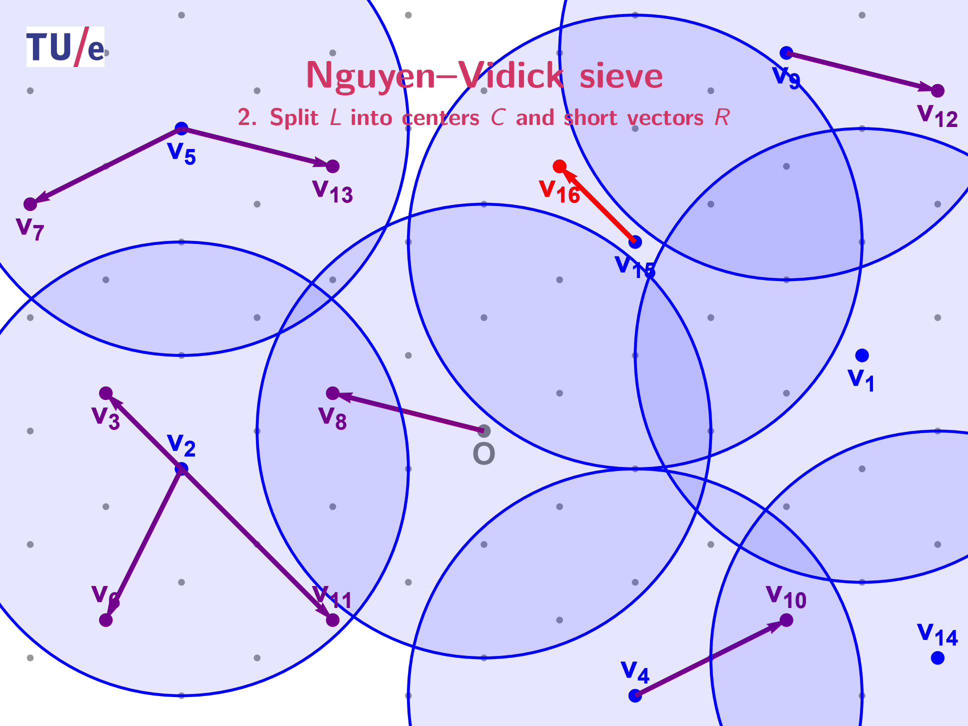

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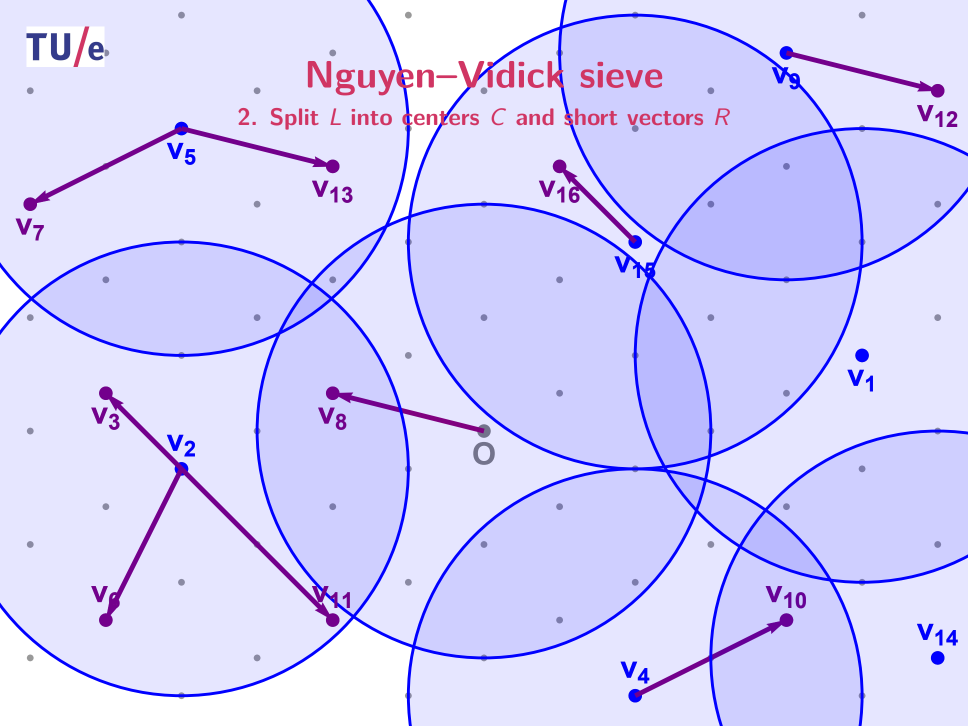

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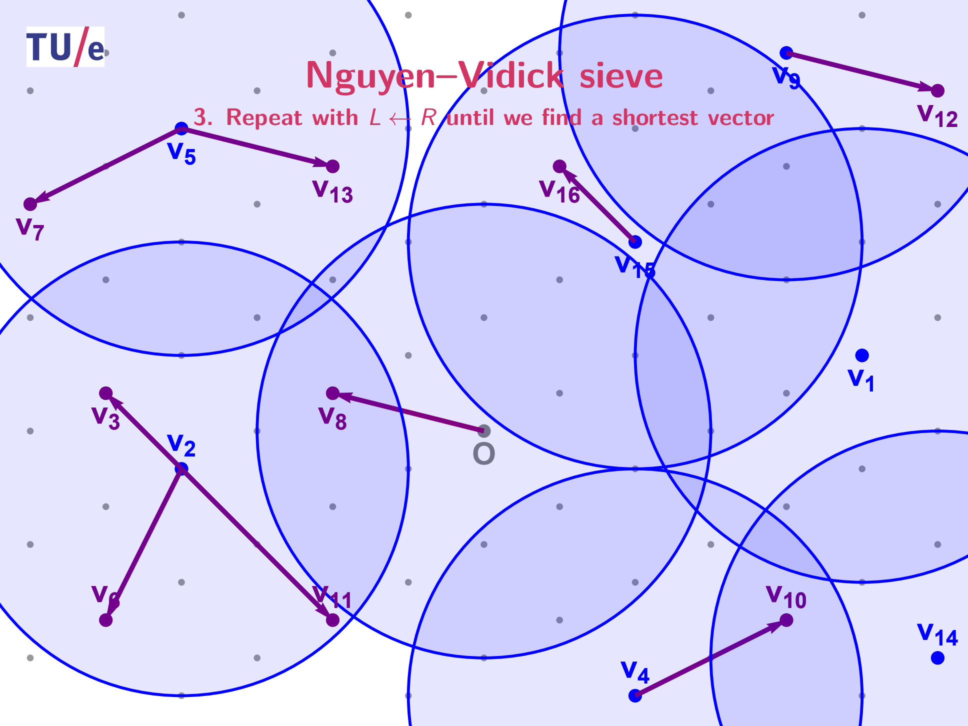

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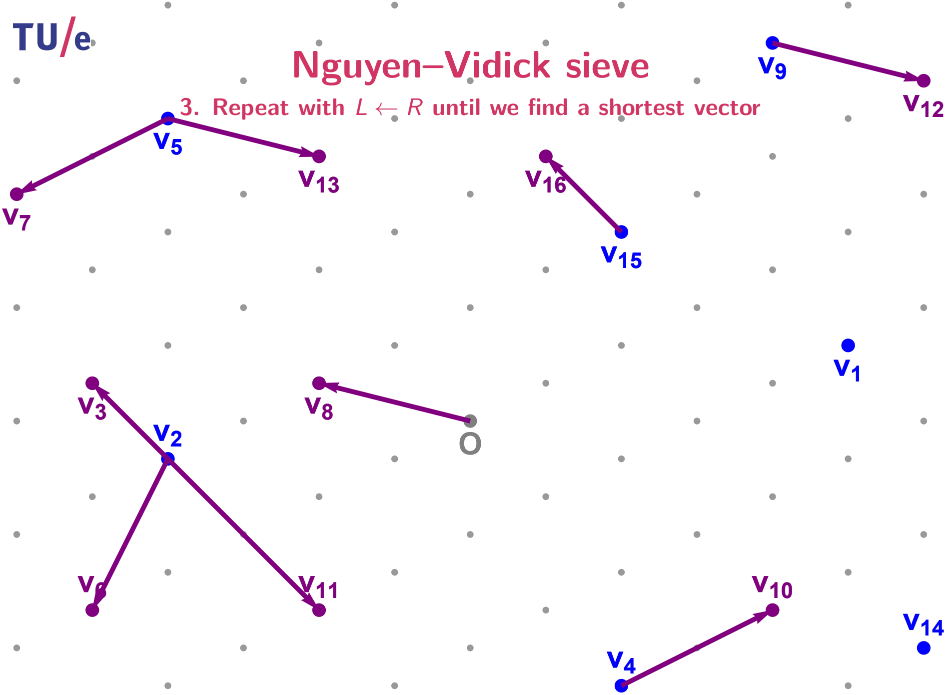

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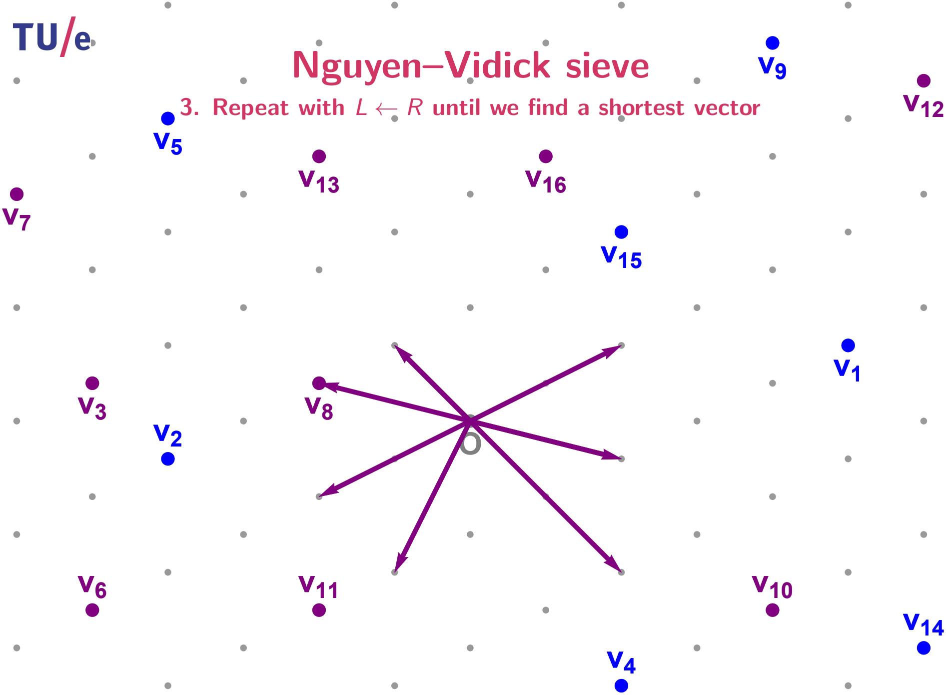

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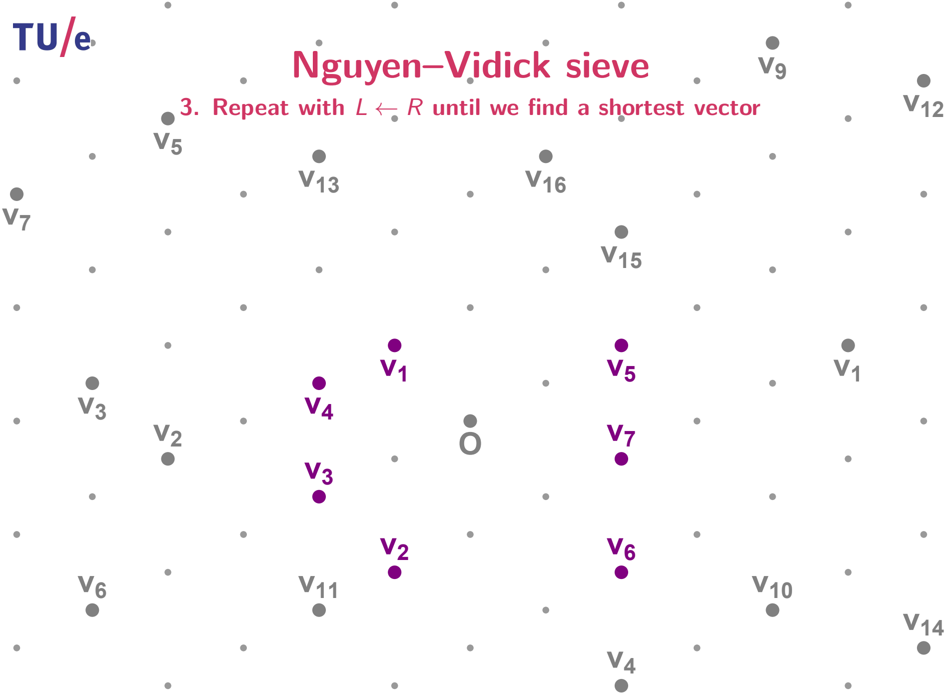

Nguyen–Vidick sieve

Overview

Nguyen–Vidick sieve

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- Space complexity: $\sqrt{4/3}^n \approx 2^{0.21n+o(n)}$ vectors
  - Need $\sqrt{4/3}^n$ vectors to cover all corners of $\mathbb{R}^n$

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  - Need $\sqrt{4/3}^n$ vectors to cover all corners of $\mathbb{R}^n$
- Time complexity: $(4/3)^n \approx 2^{0.42n+o(n)}$
  - Comparing a target vector to all centers: $2^{0.21n+o(n)}$
  - Repeating this for each list vector: $2^{0.21n+o(n)}$
  - Repeating the whole sieving procedure: $\mathrm{poly}(n)$

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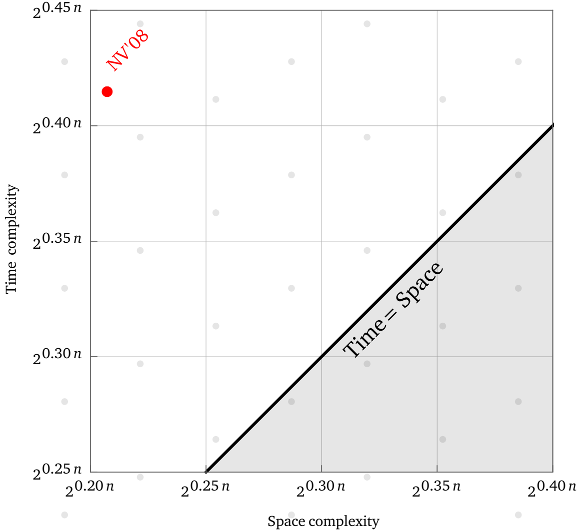
Heuristic (Nguyen–Vidick, J. Math. Crypt. '08)

The NV-sieve runs in time $2^{0.42n+o(n)}$ and space $2^{0.21n+o(n)}$.

Nguyen–Vidick sieve

Space/time trade-off



GaussSieve

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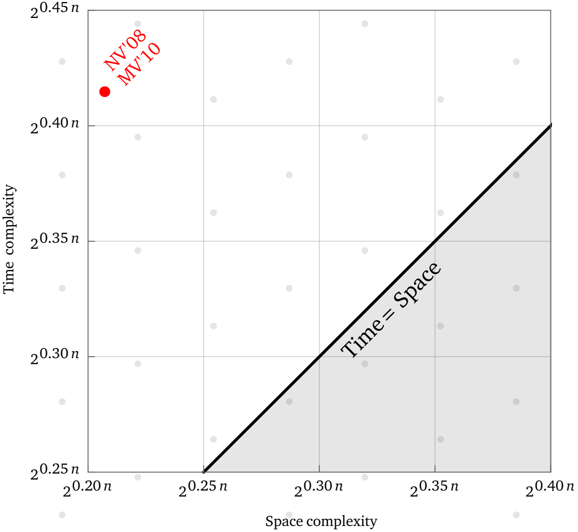

Two-level sieve

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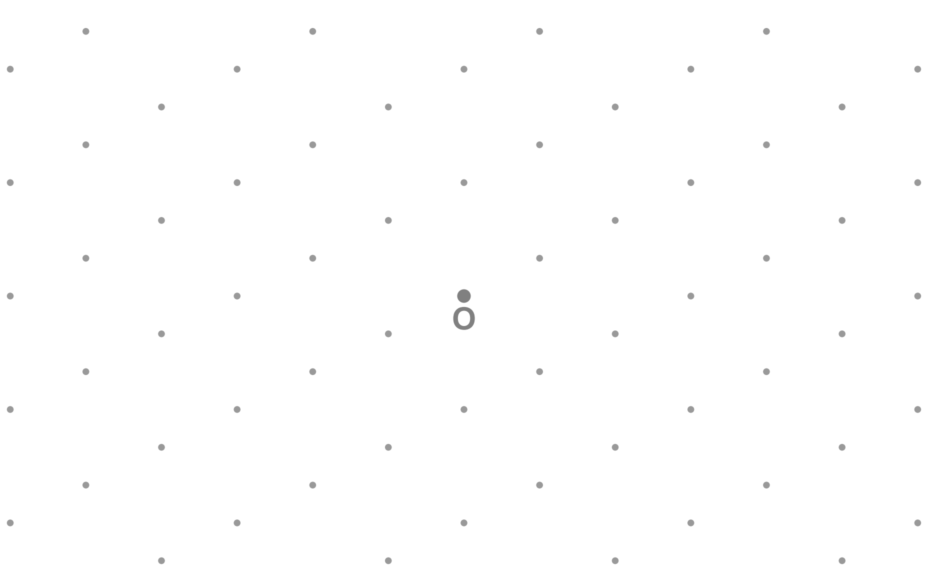

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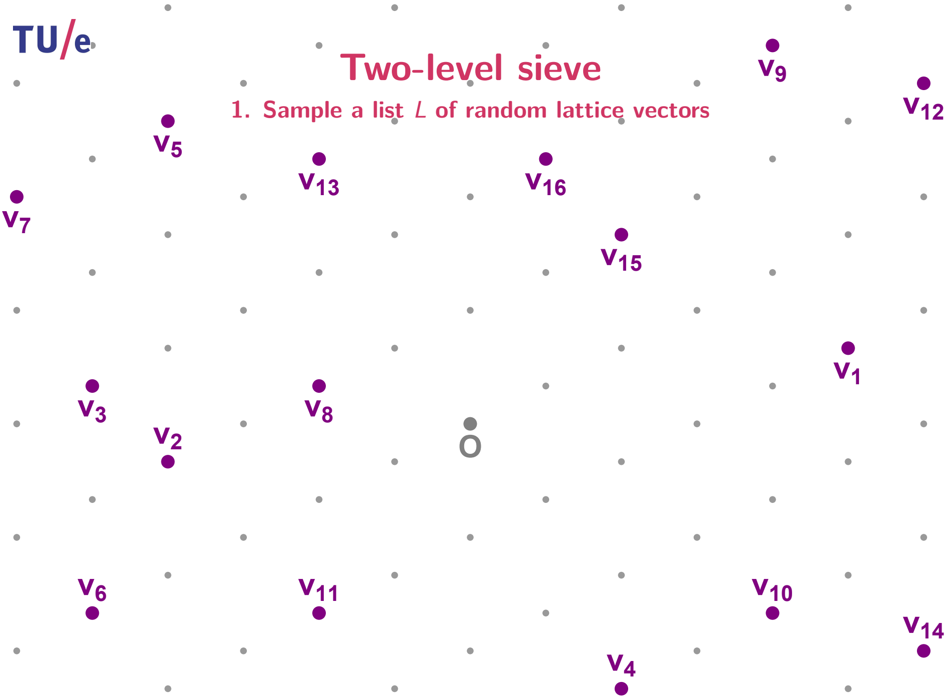

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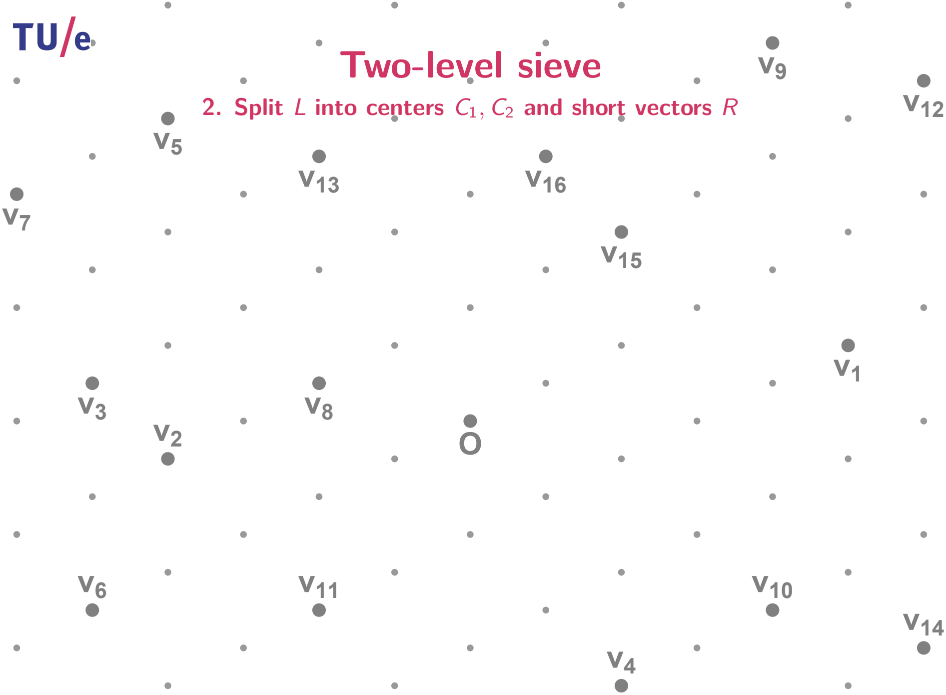

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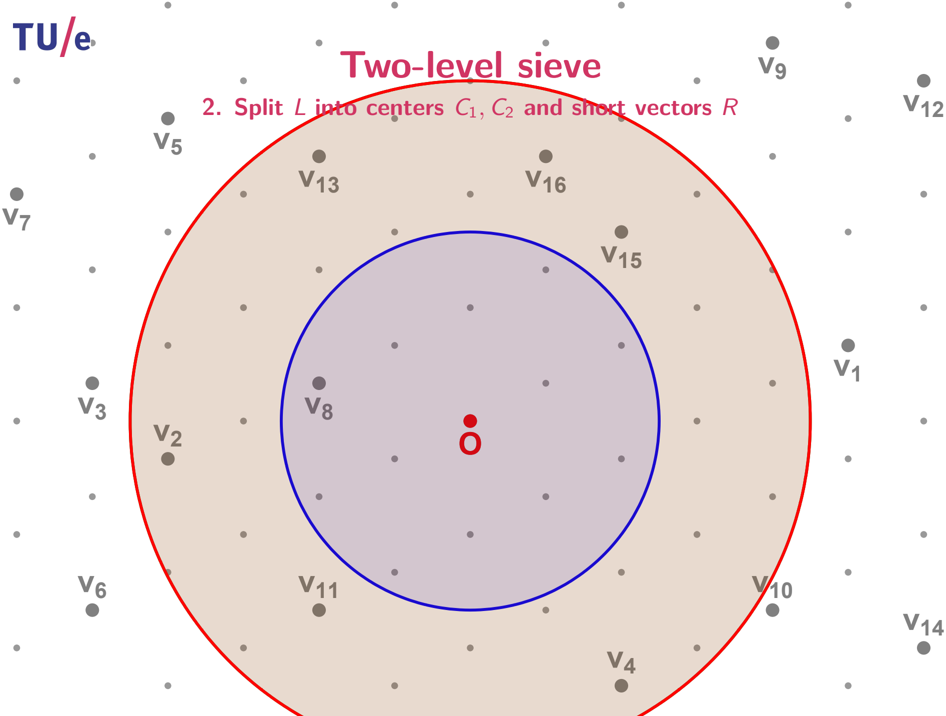

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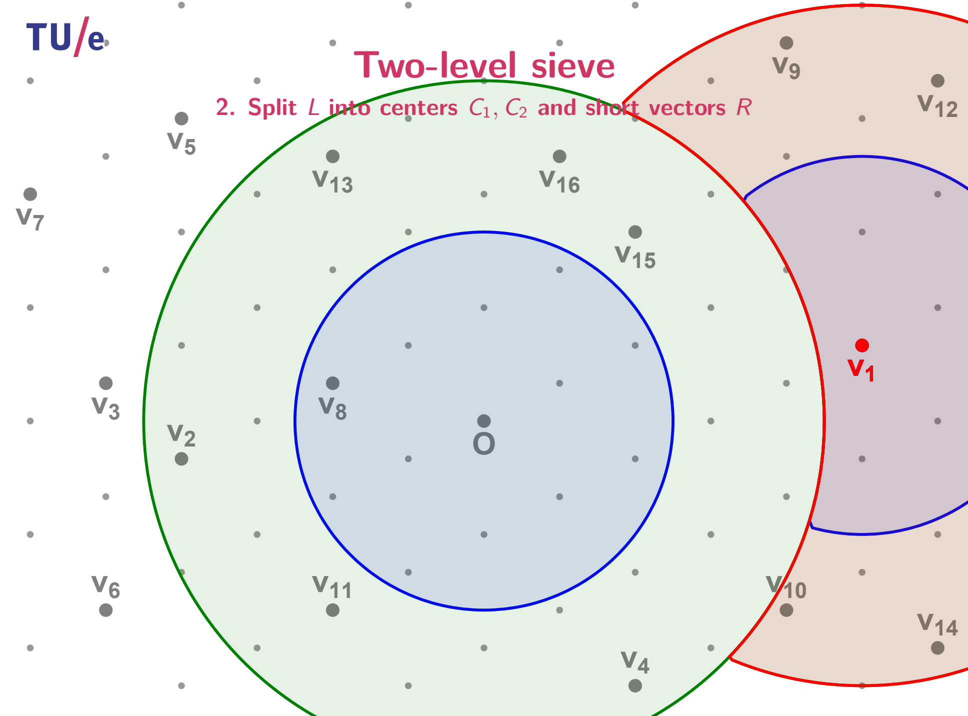

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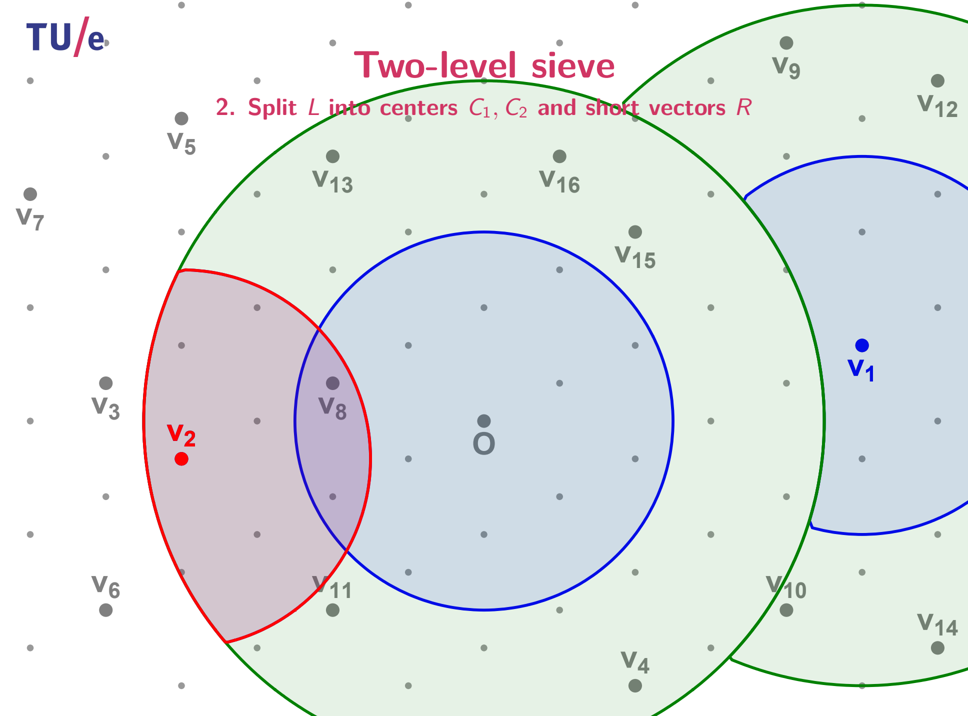

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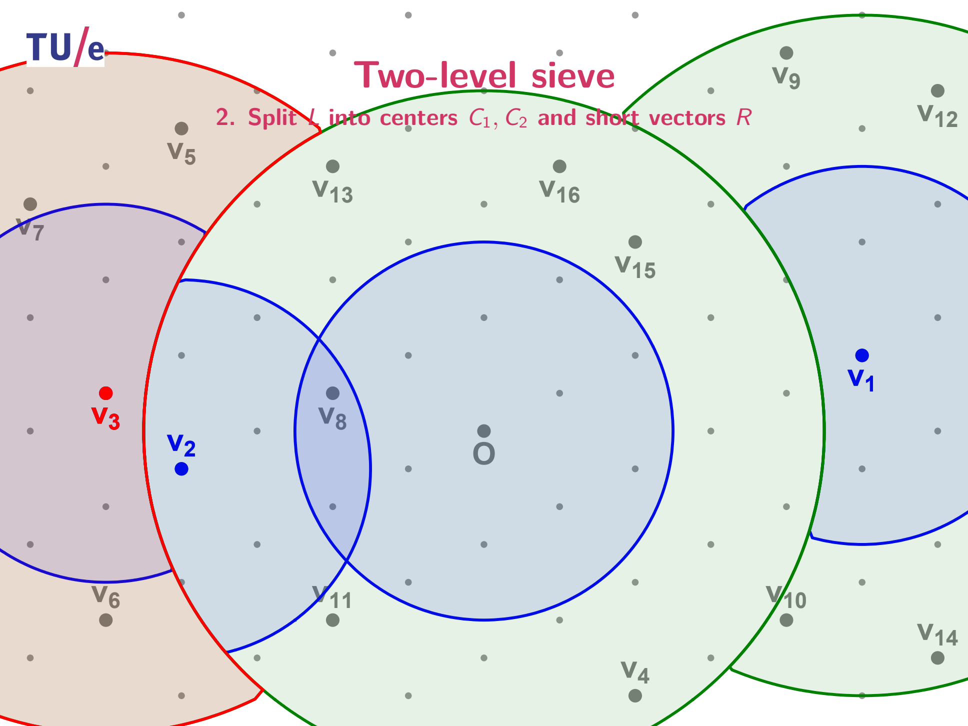

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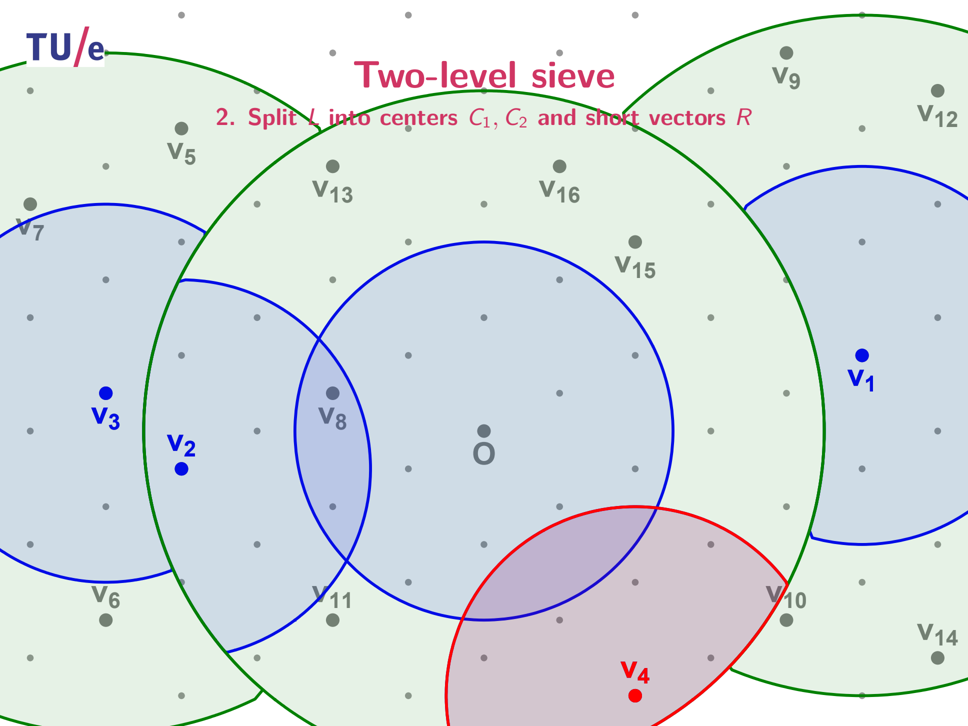

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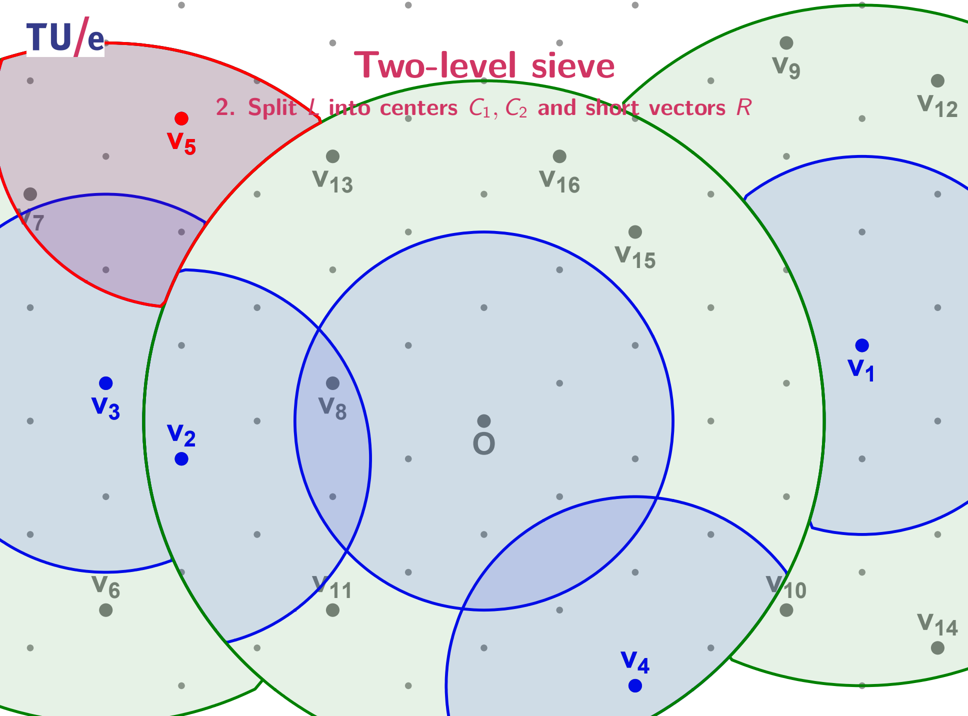

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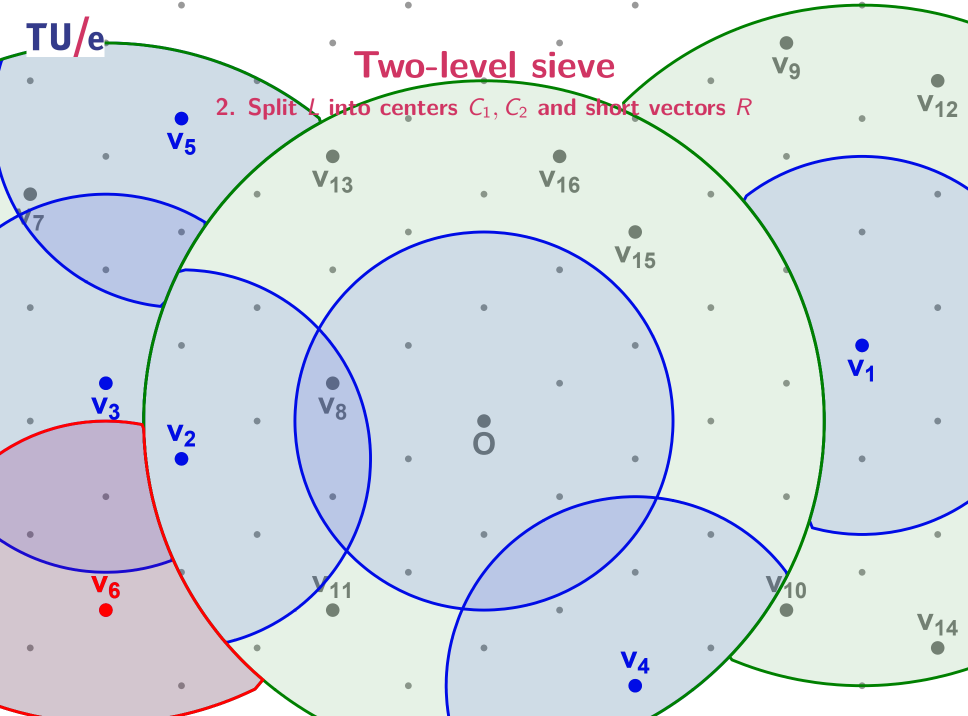

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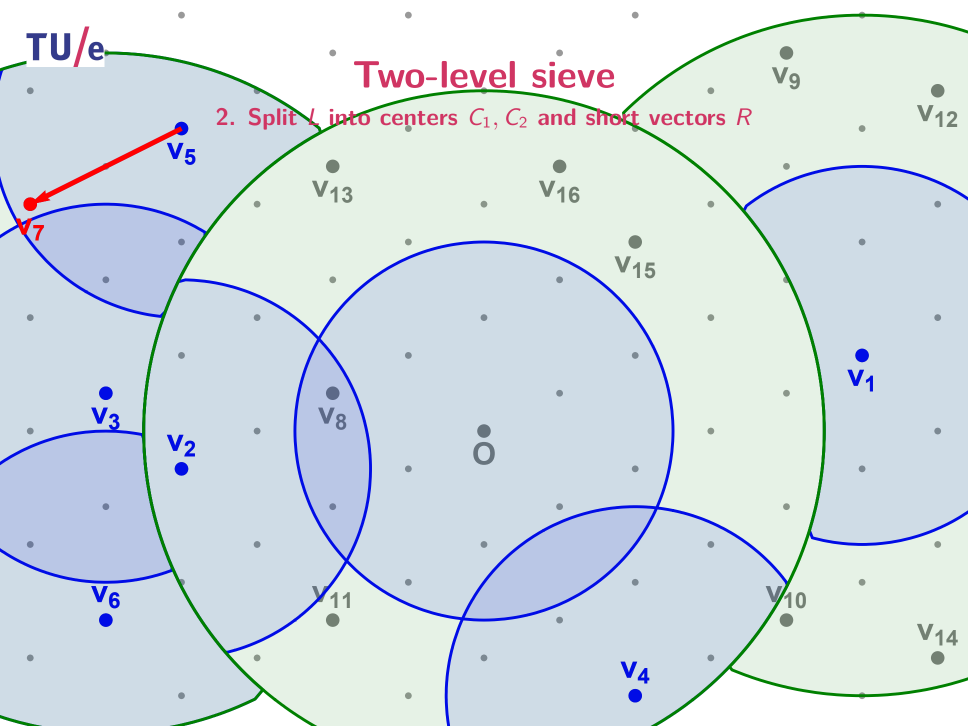

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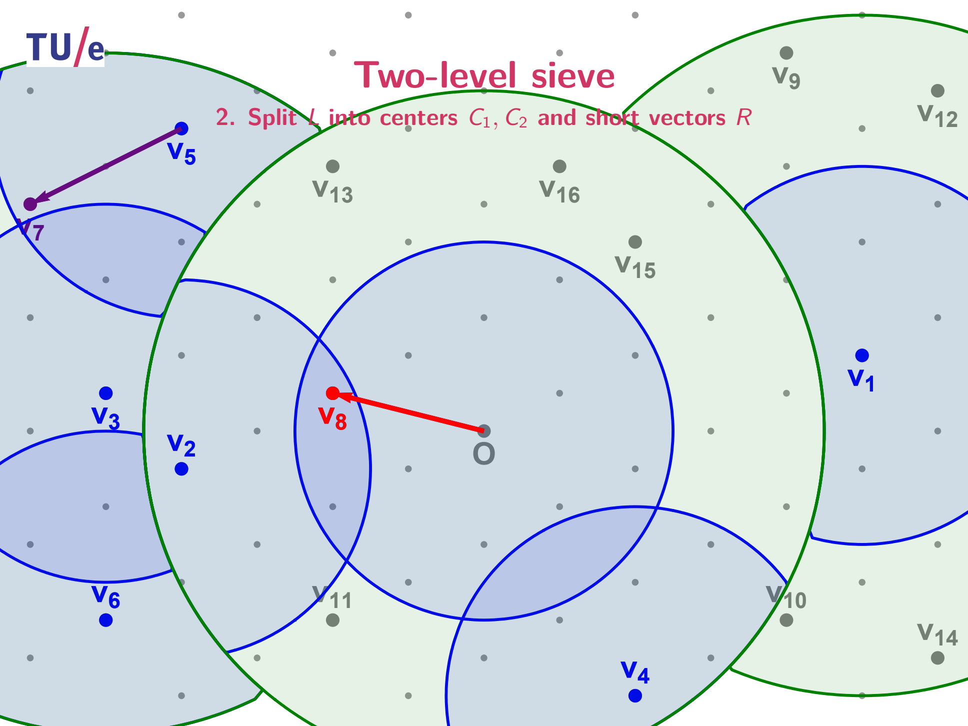

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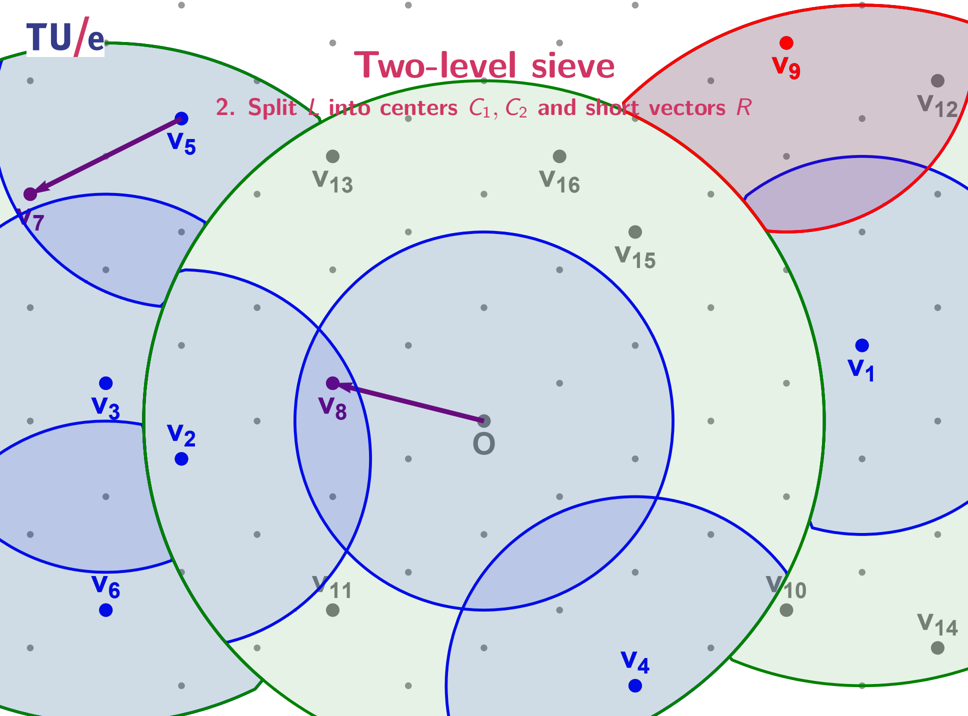

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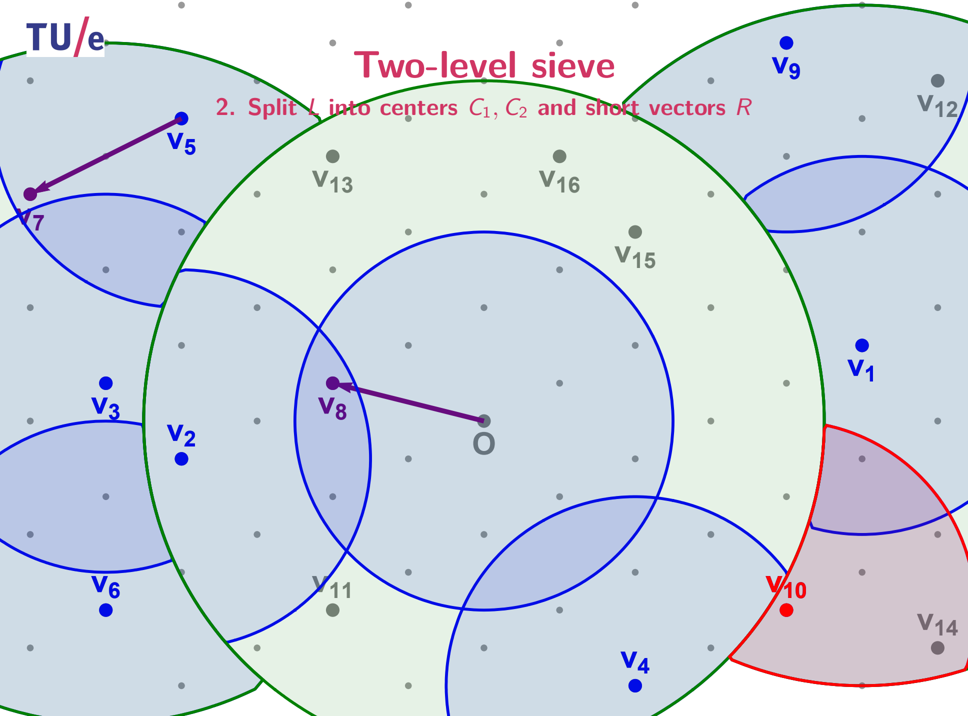

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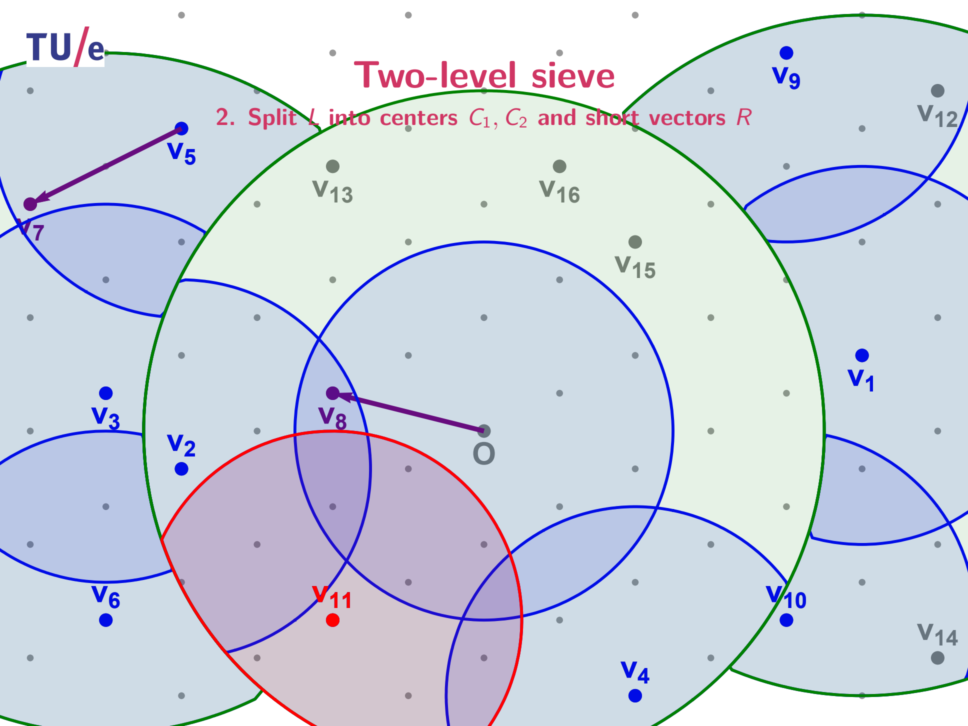

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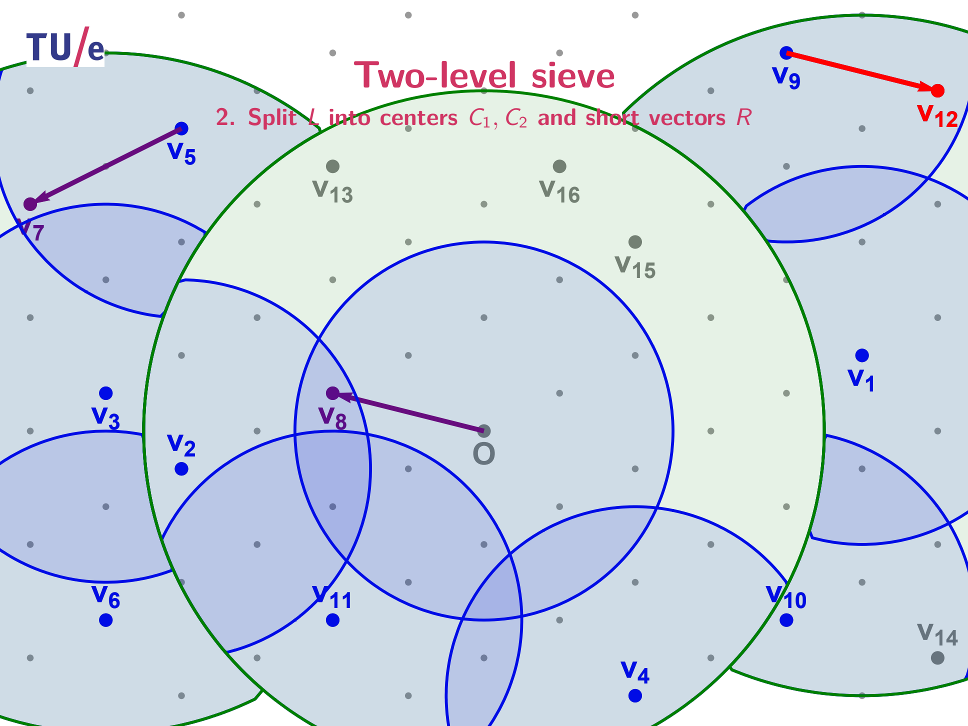

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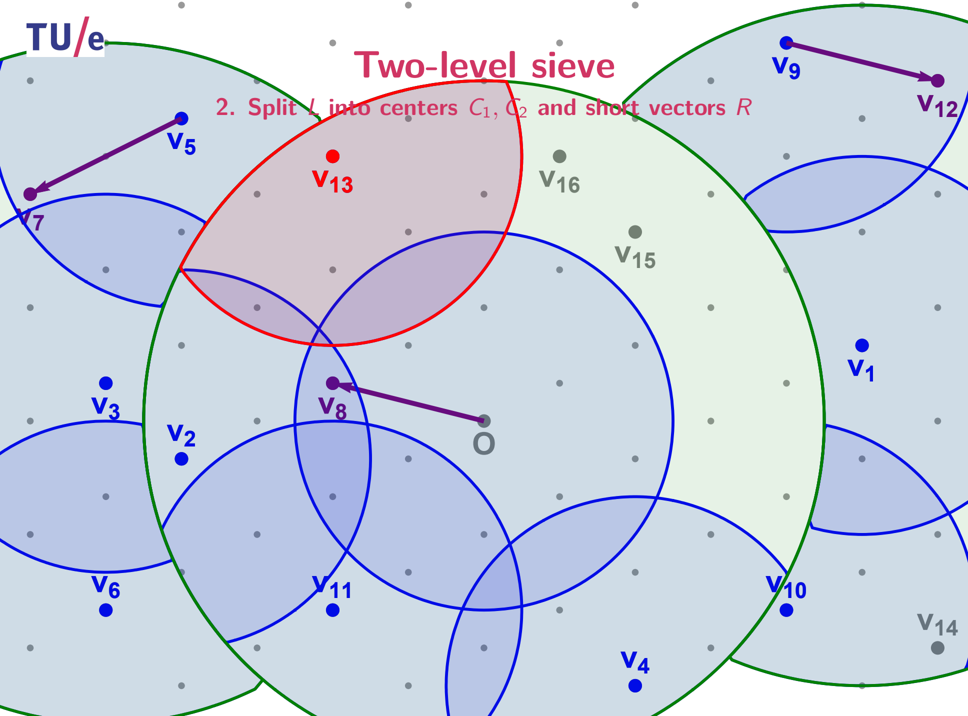

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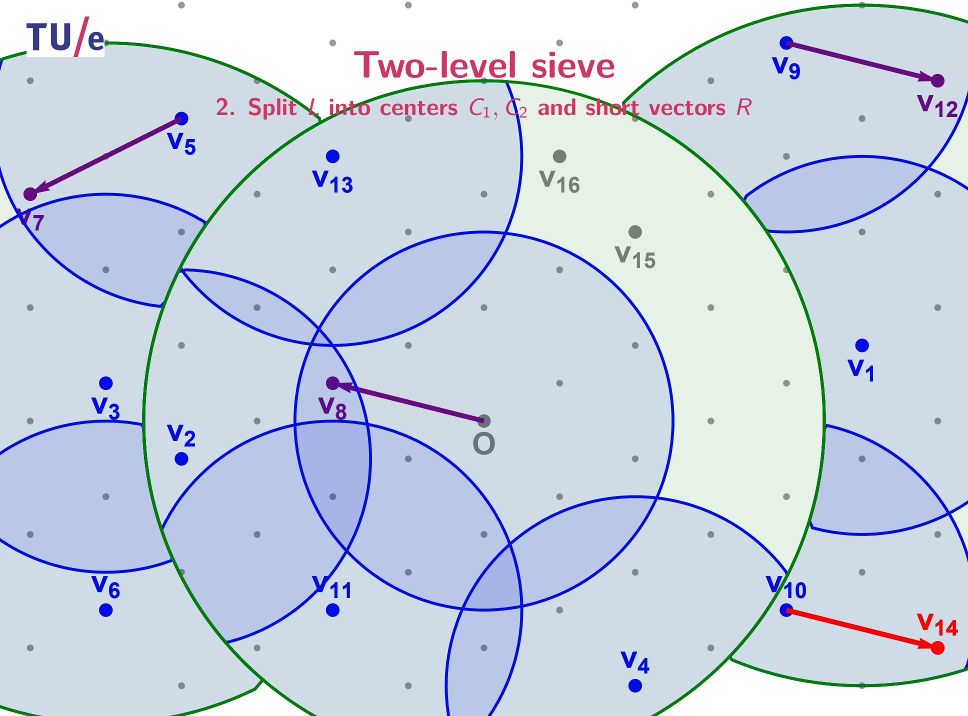

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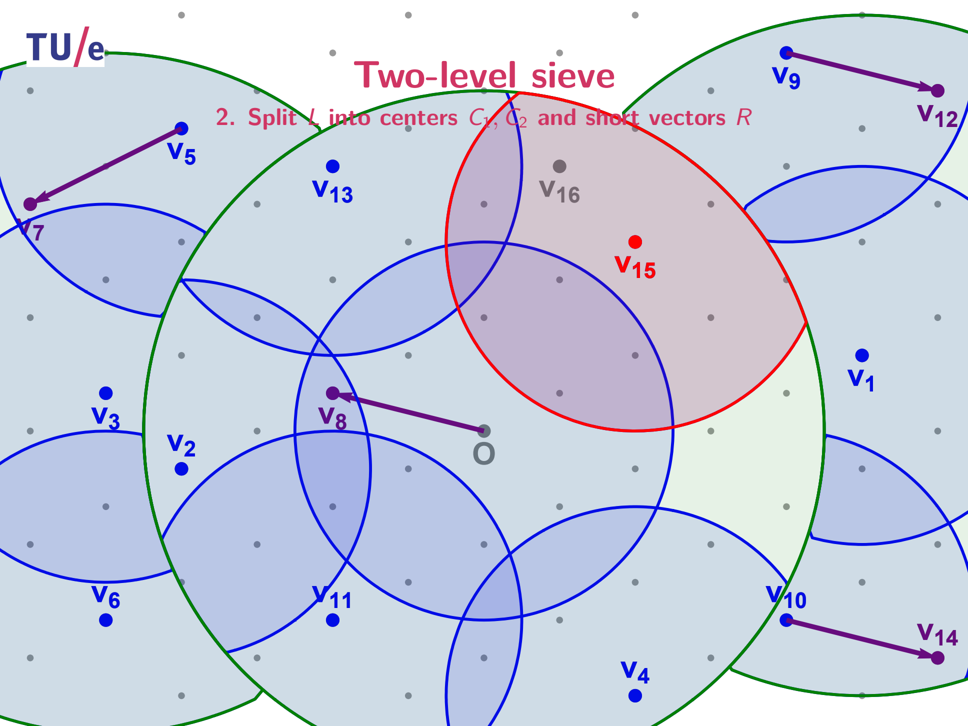
2. Split L into centers C_1, C_2 and short vectors R



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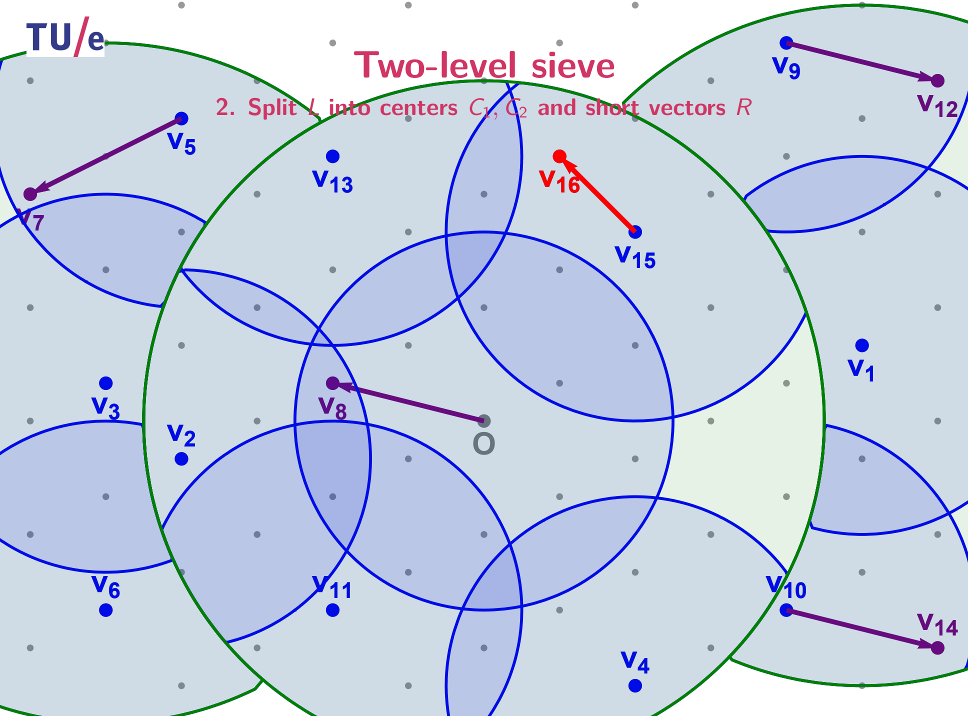

TU/e

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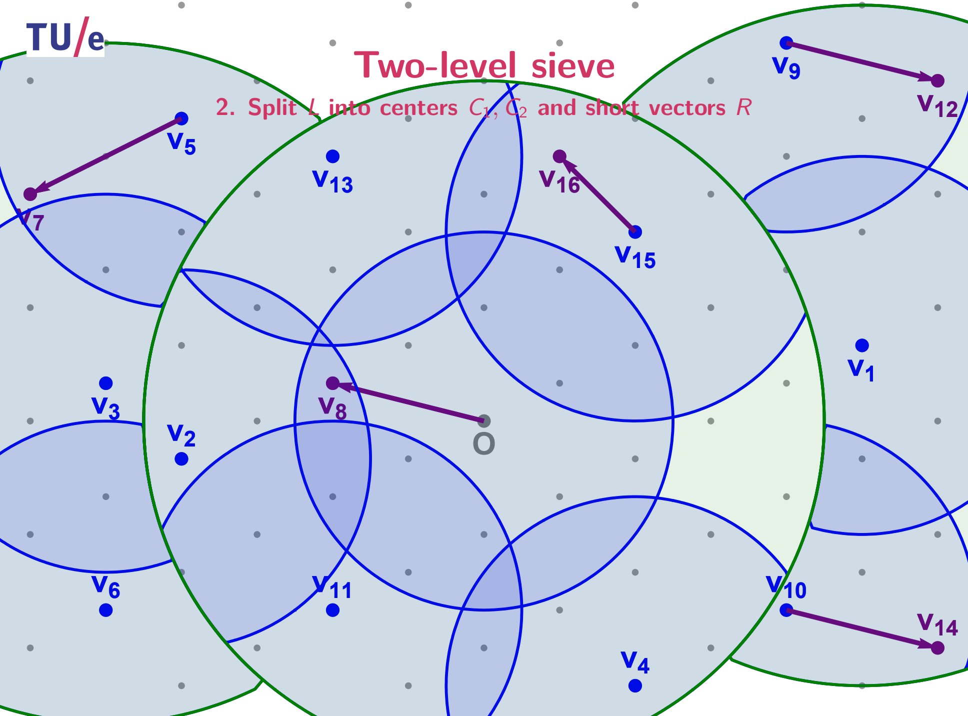

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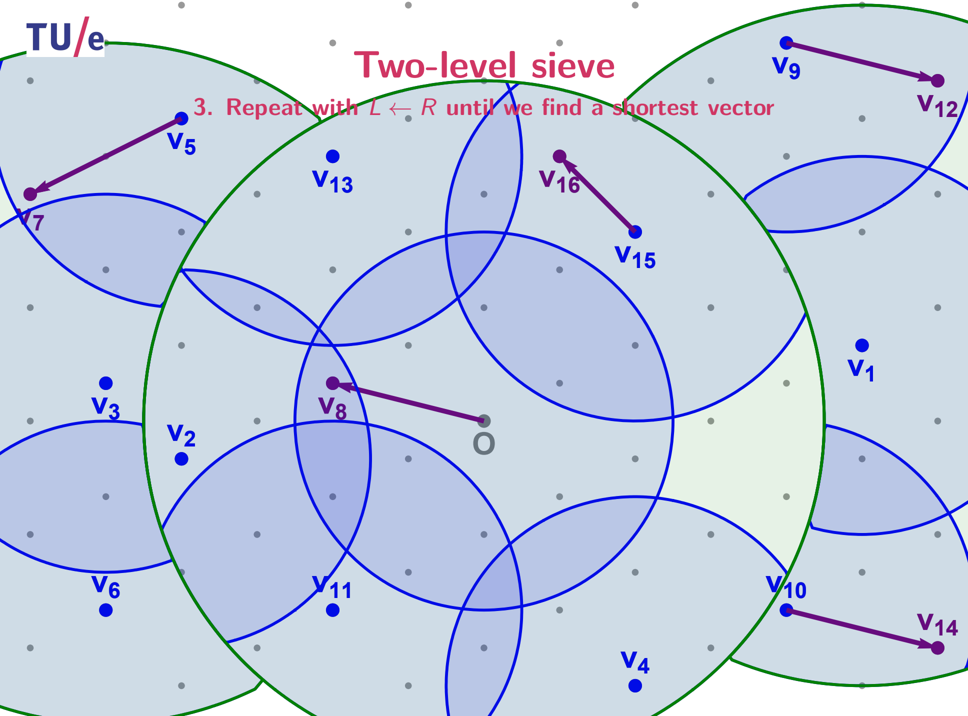

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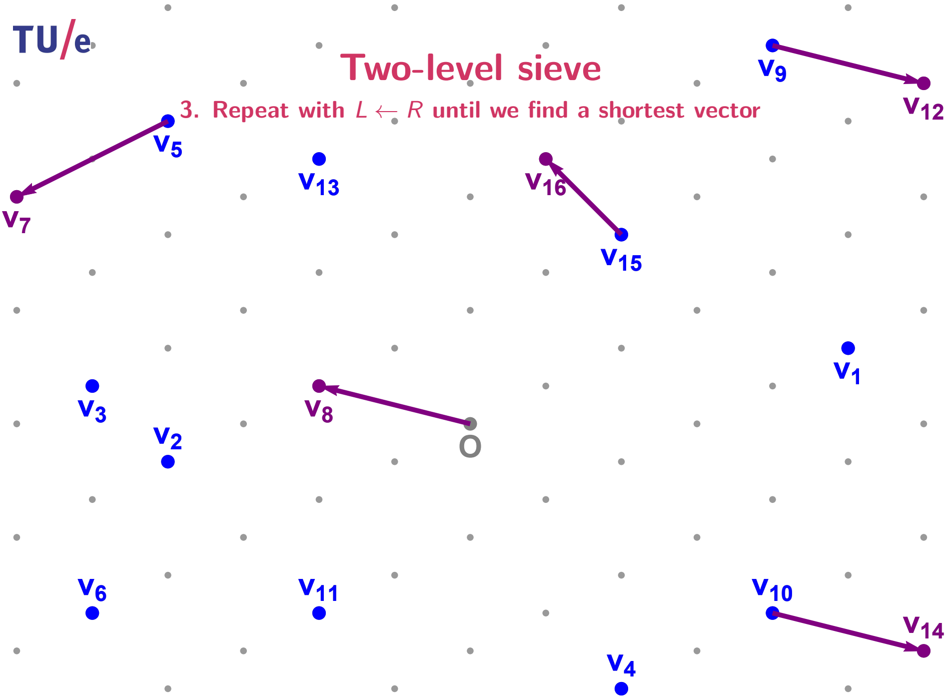

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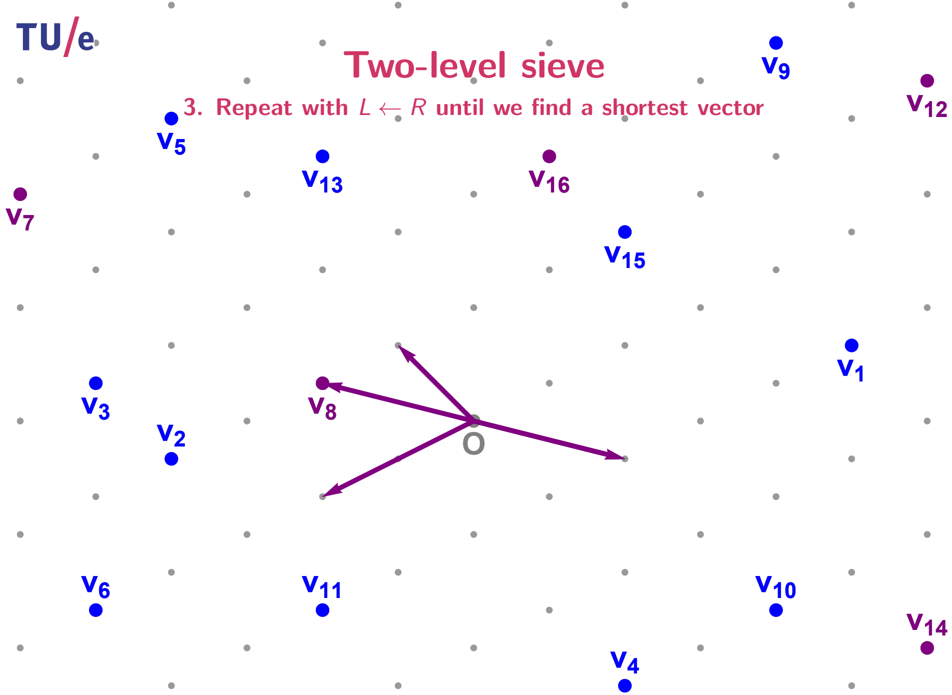

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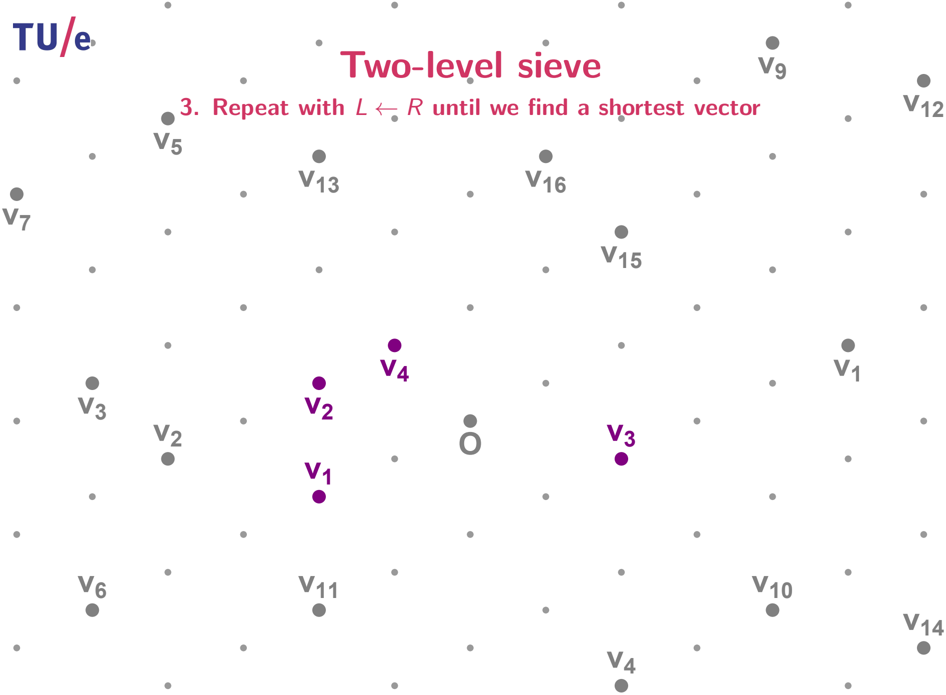
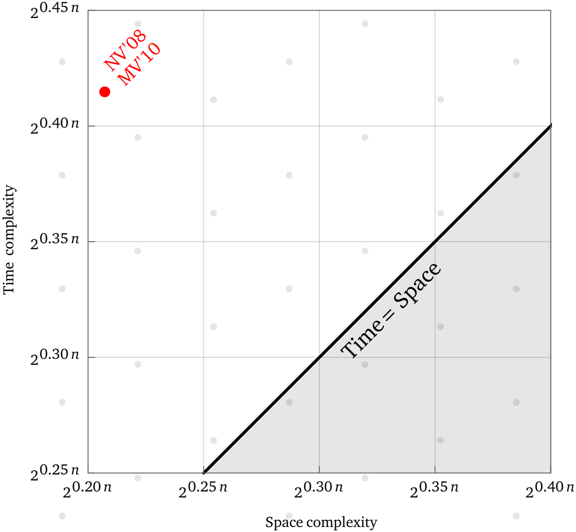


Two-level sieve

Space/time trade-off



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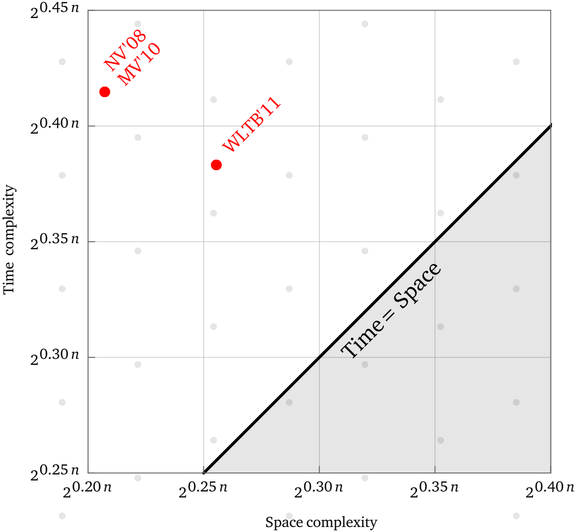

Three-level sieve

Overview

Theorem (Nguyen–Vidick, J. Math. Crypt. '08)

The one-level sieve runs in time $2^{0.4150n}$ *and space* $2^{0.2075n}$.

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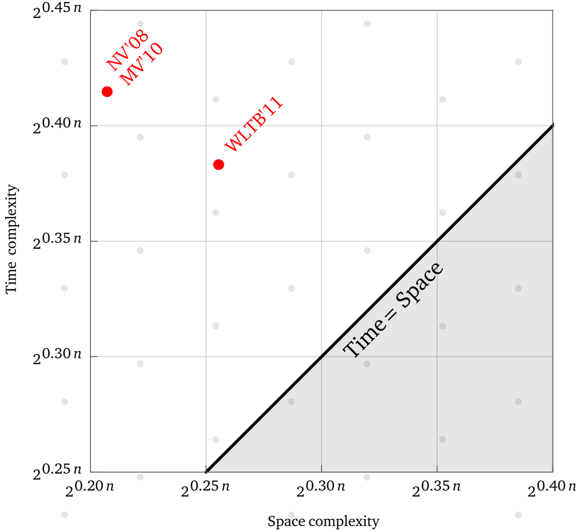
Conjecture (L, PhD thesis)

The four-level sieve runs in time $2^{0.3774n}$ and space $2^{0.2925n}$, and higher-level sieves are not faster than this.

Three-level sieve

Space/time trade-off



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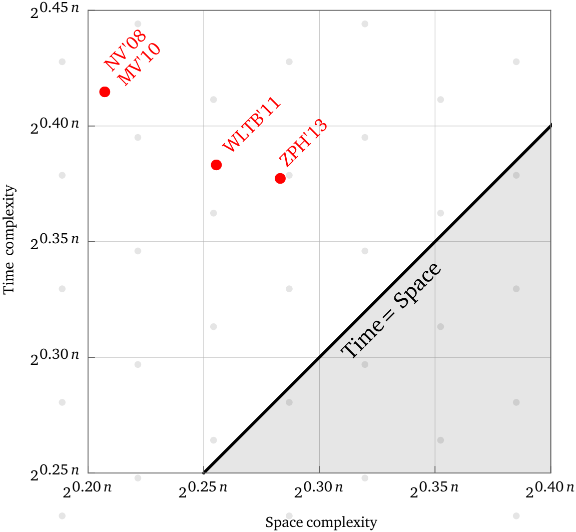
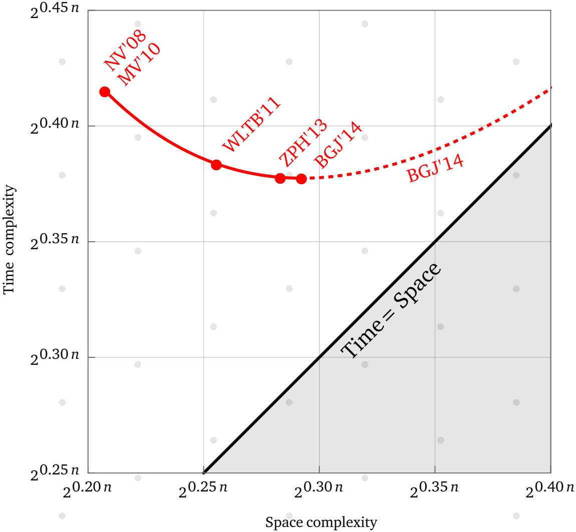


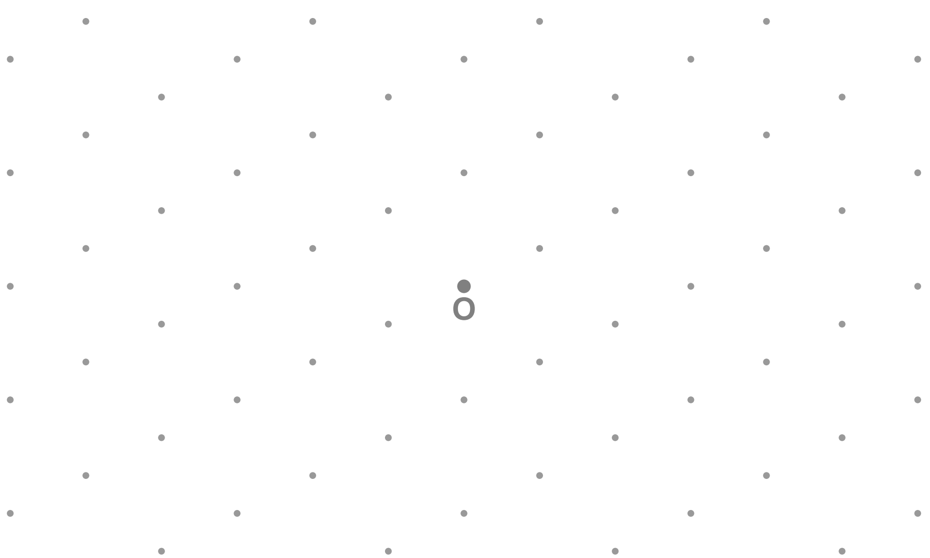
Decomposition approach

Space/time trade-off



Hyperplane LSH

1. Sample a list L of random lattice vectors



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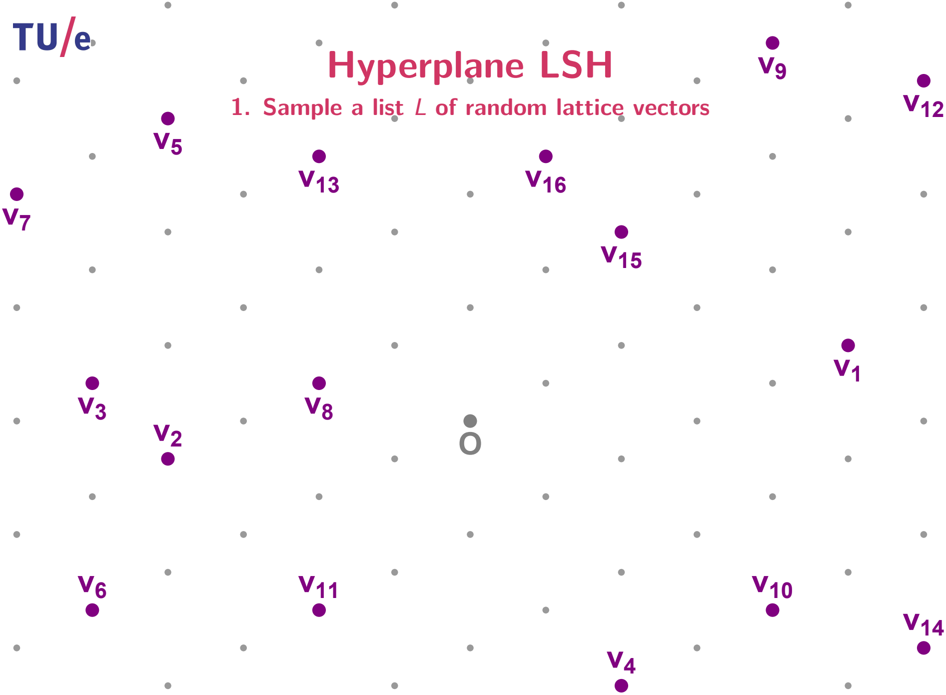

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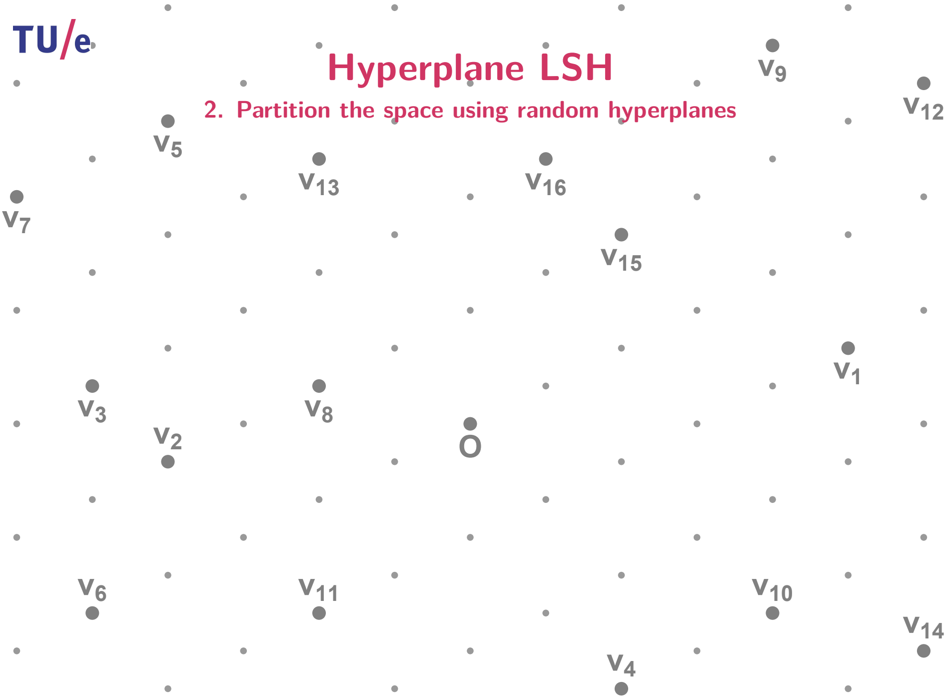
2. Partition the space using random hyperplanes



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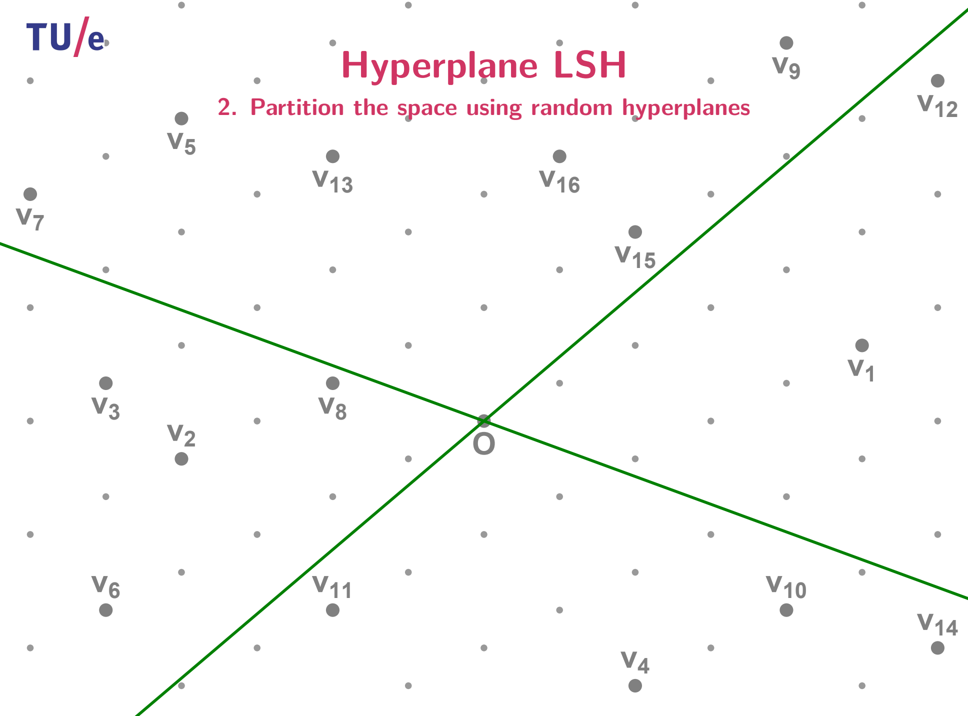

Hyperplane LSH

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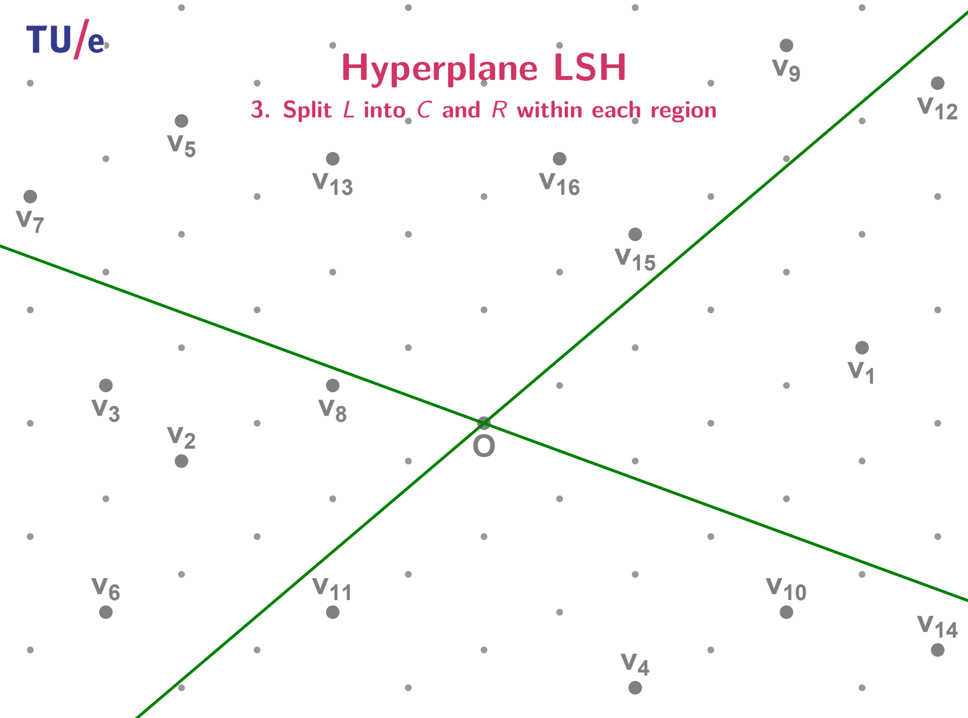

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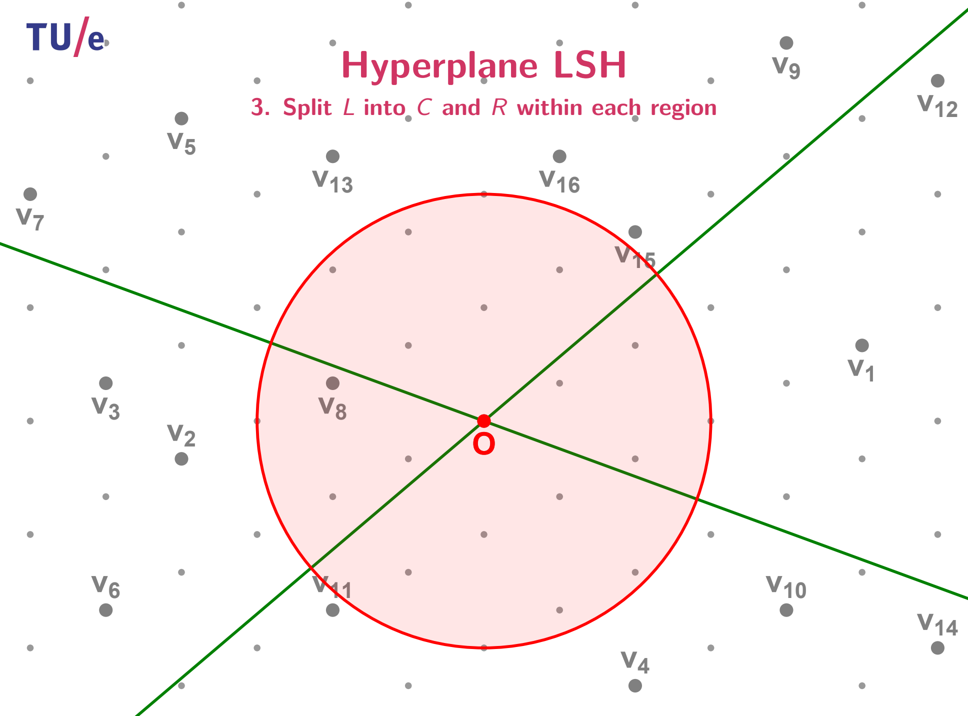

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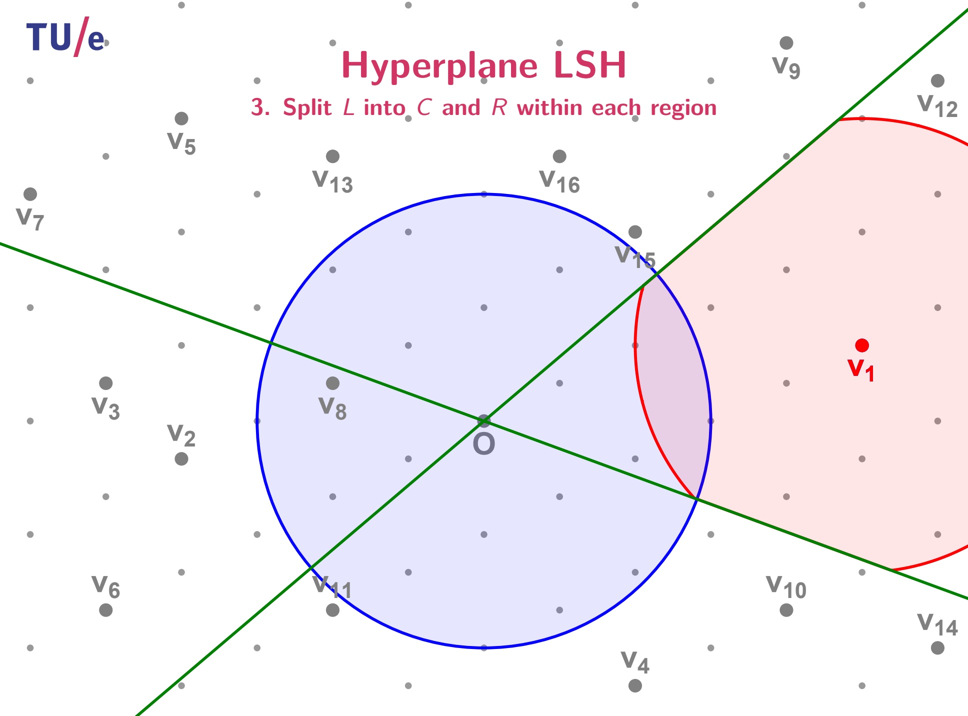

TU/e

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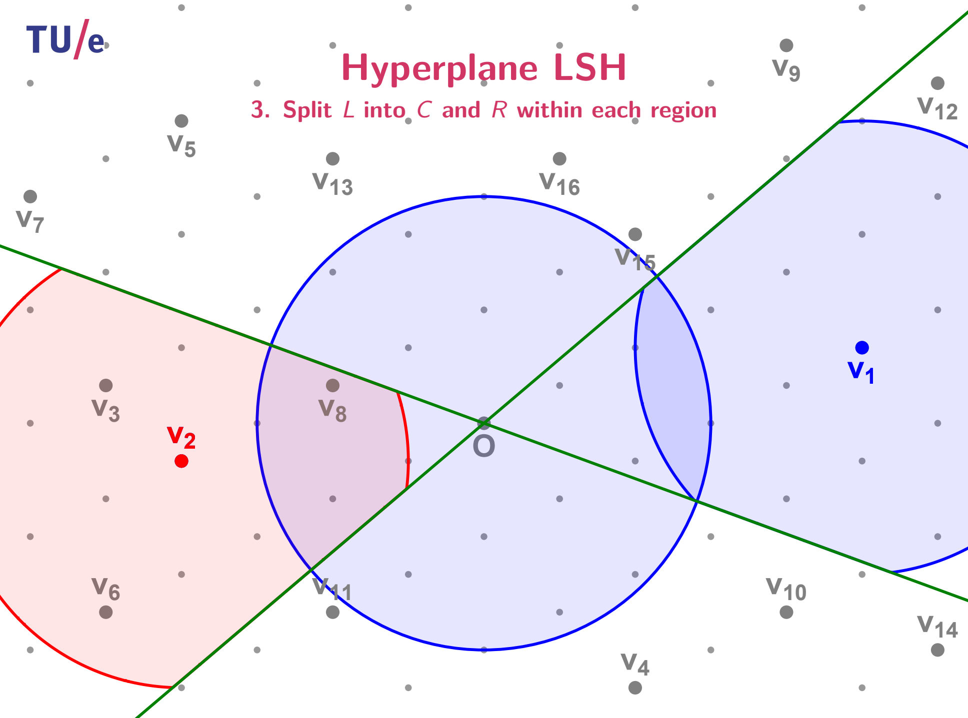

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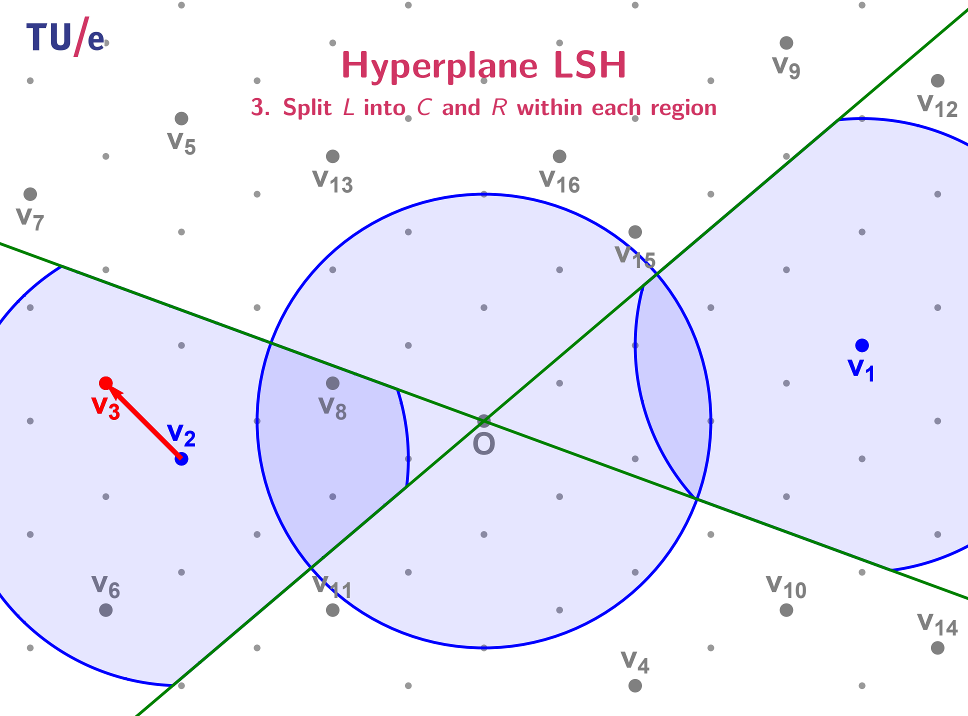

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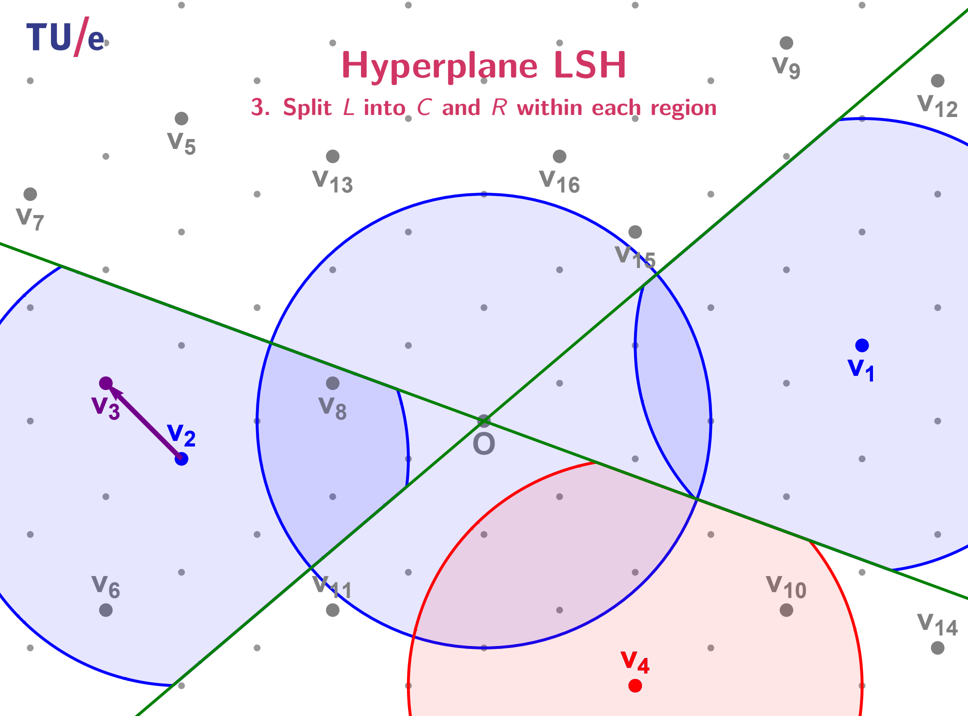

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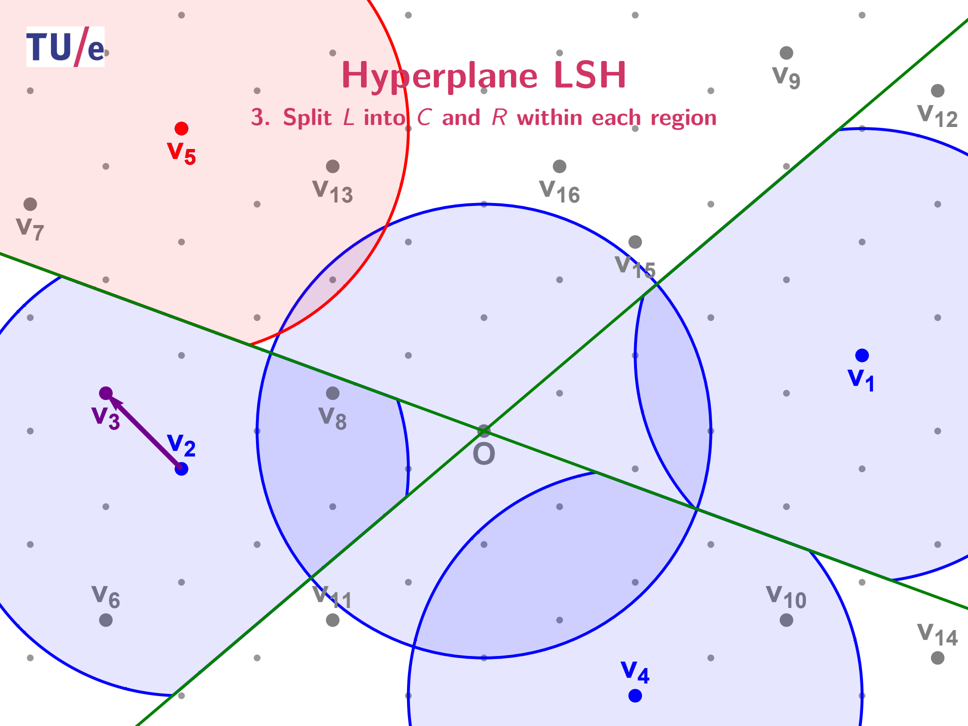

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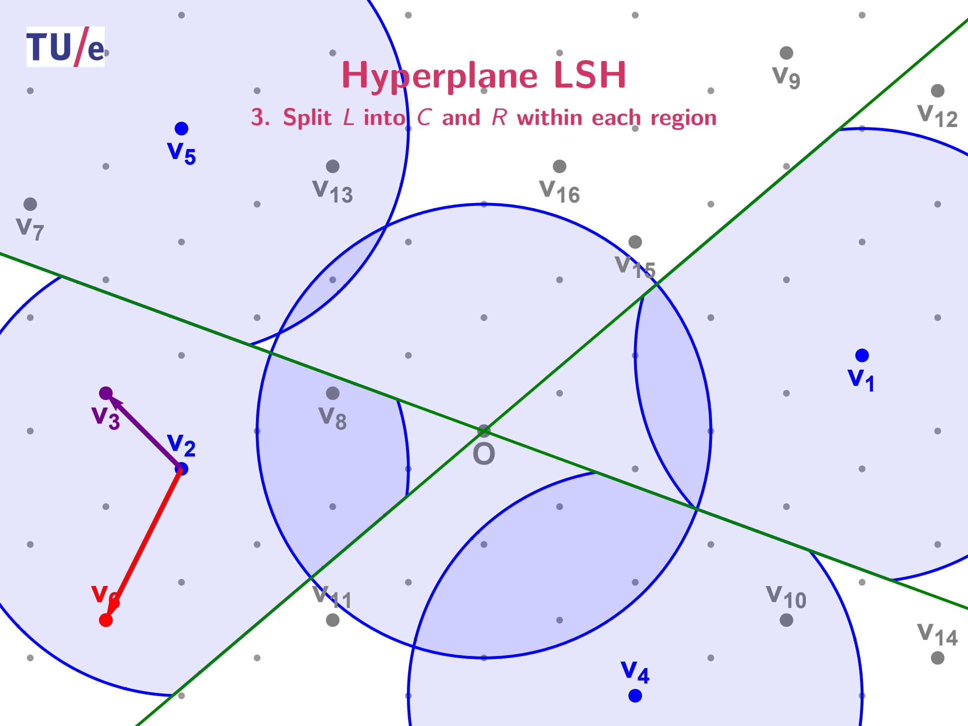

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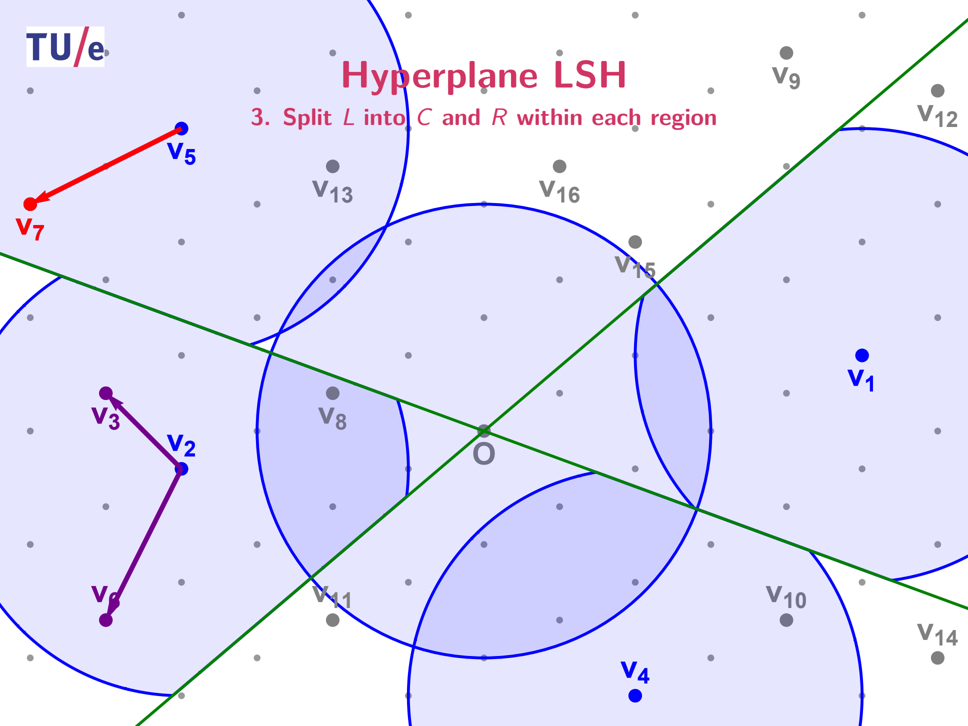

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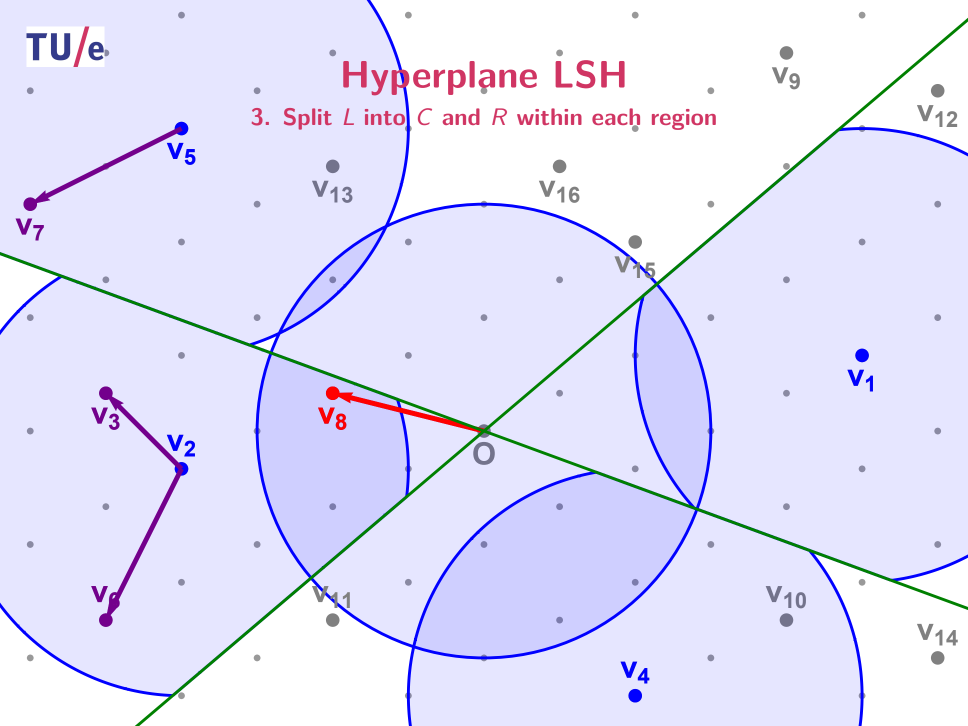

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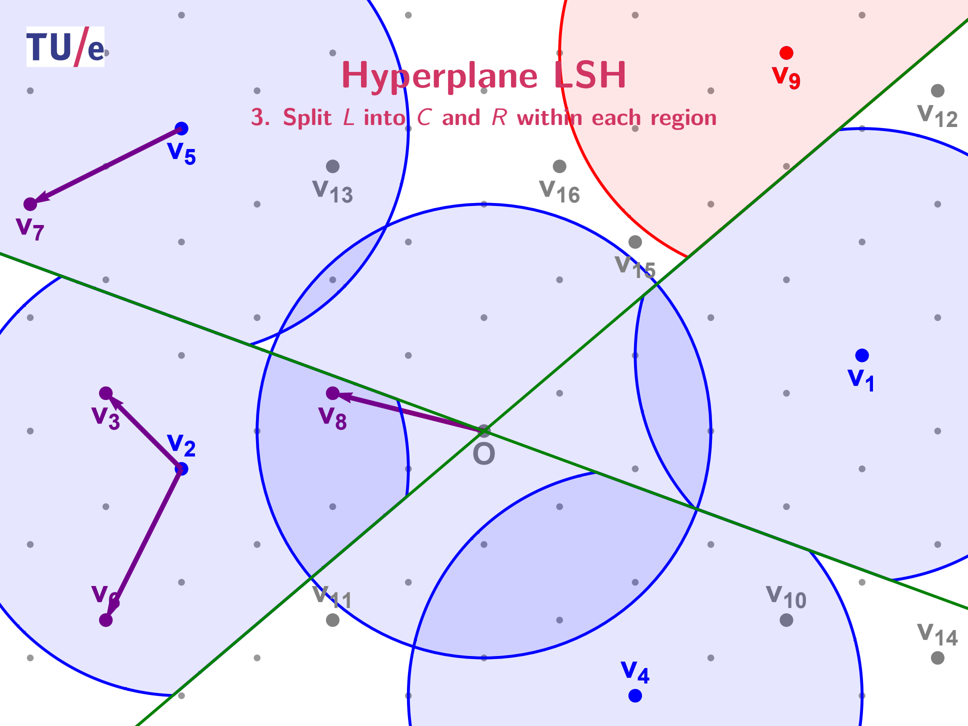

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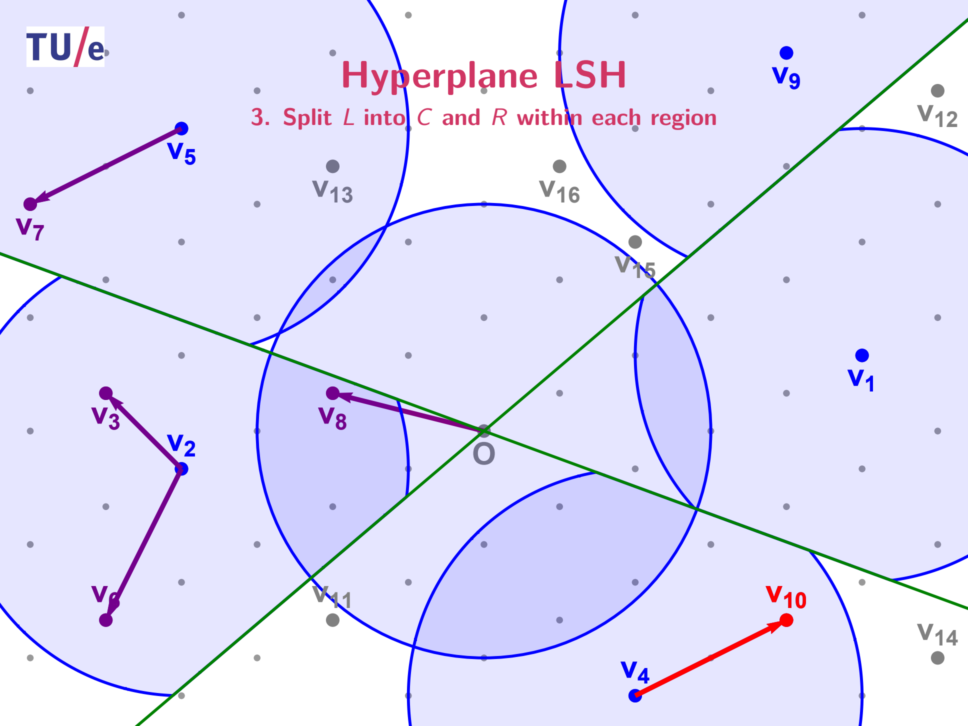

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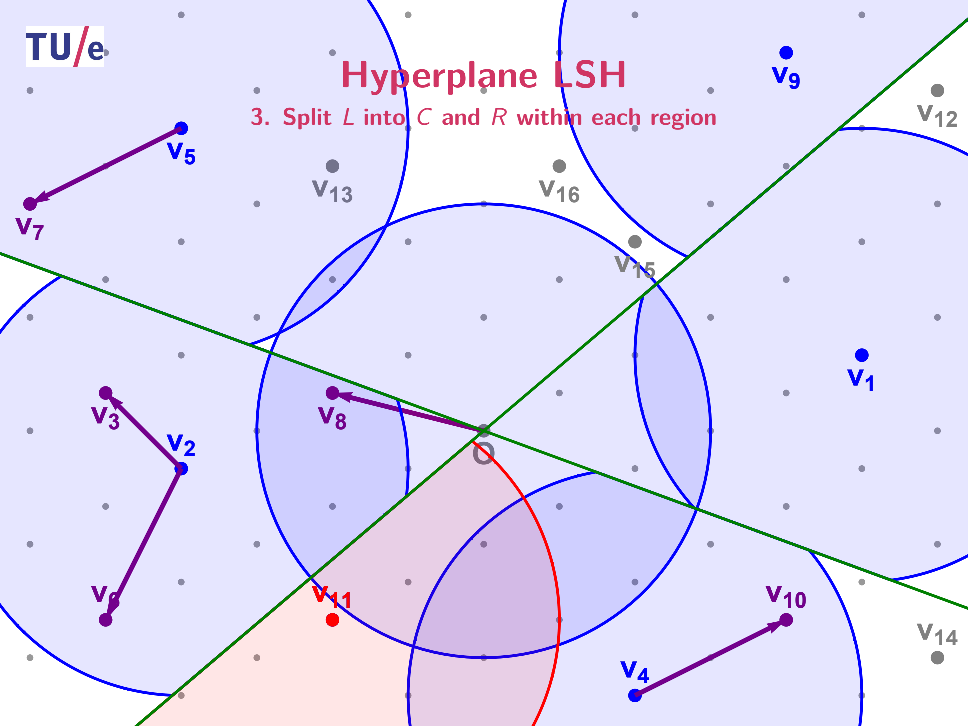

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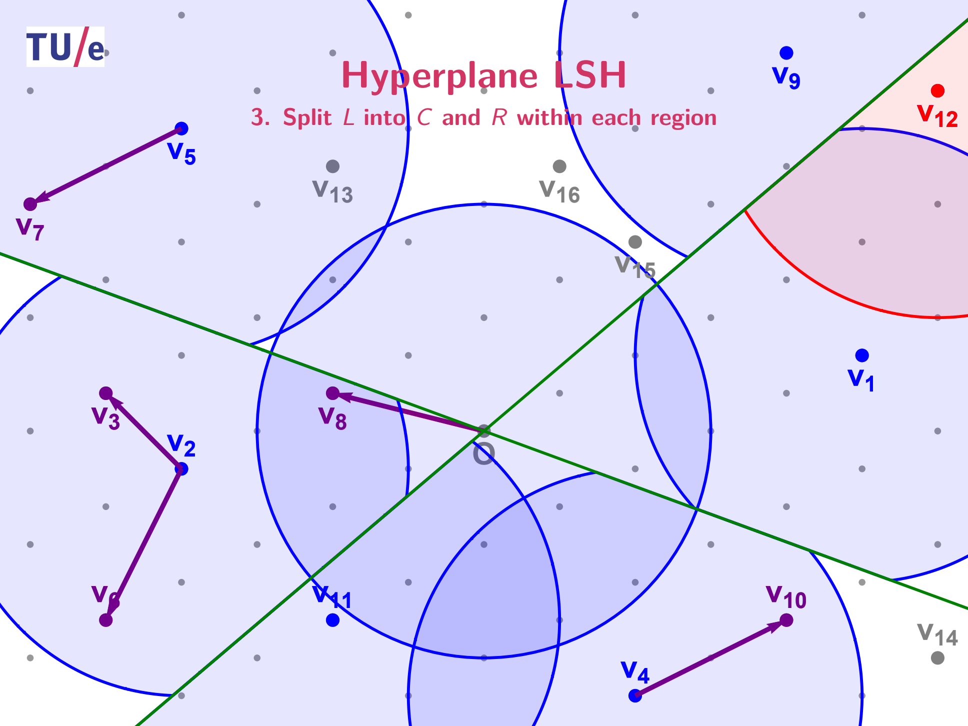

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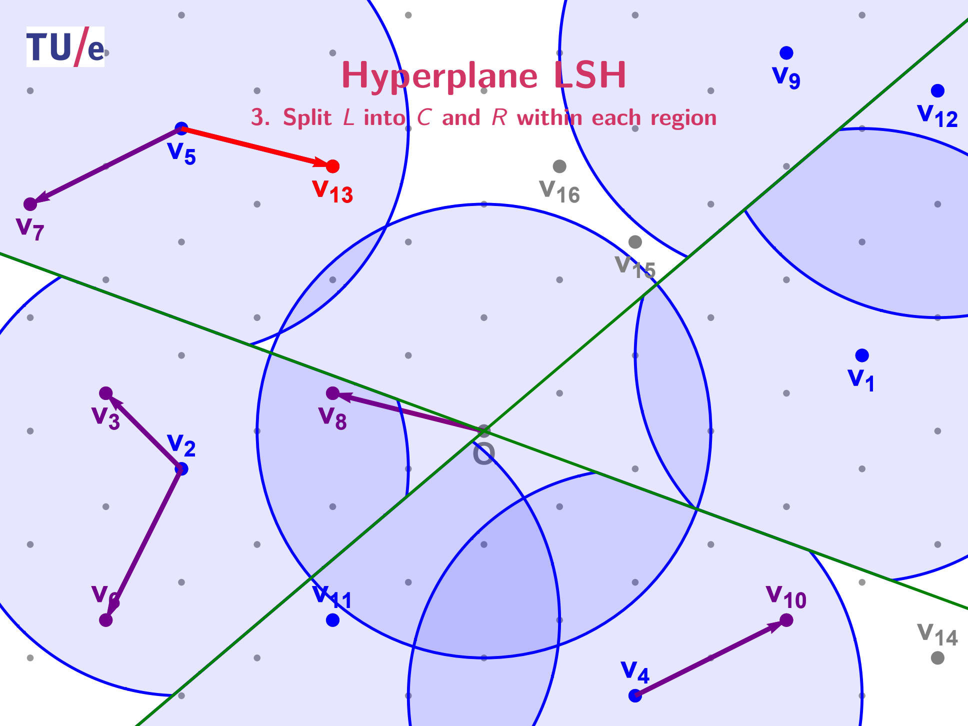

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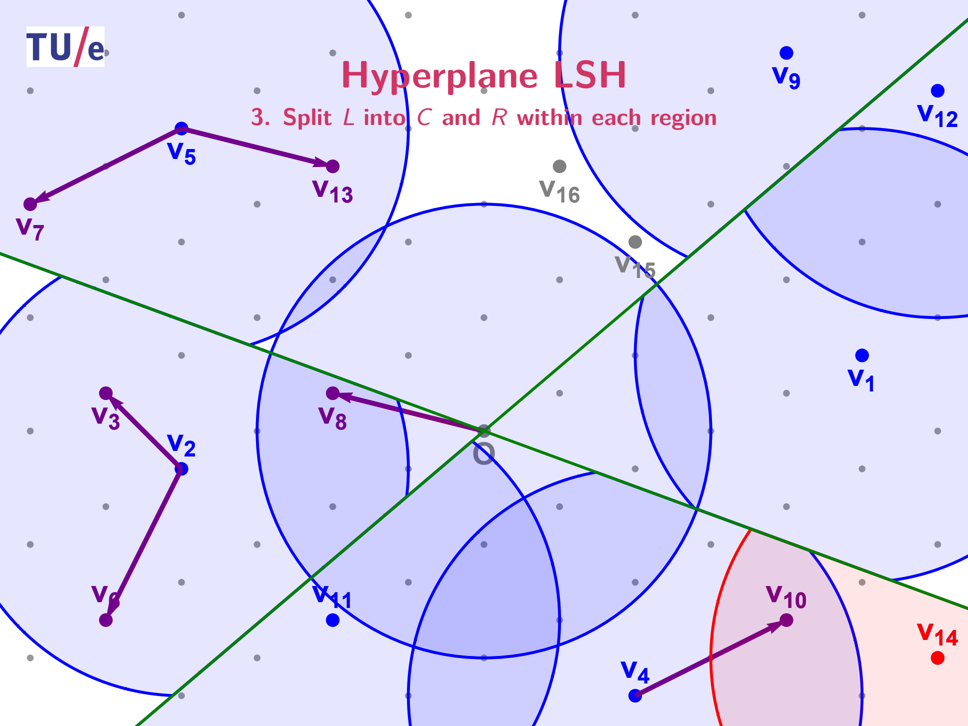

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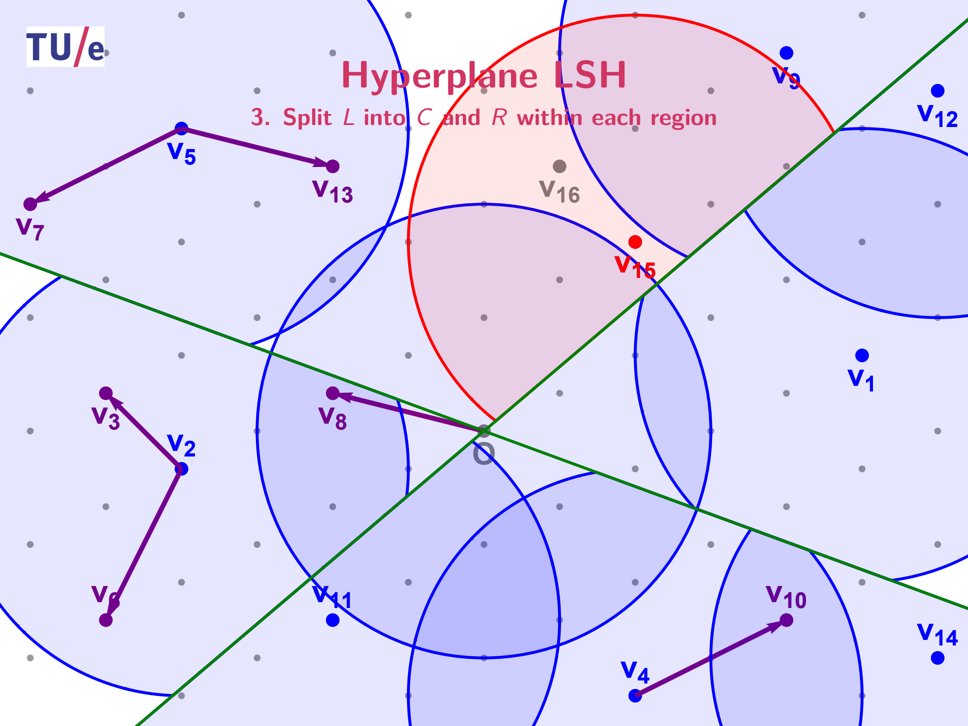

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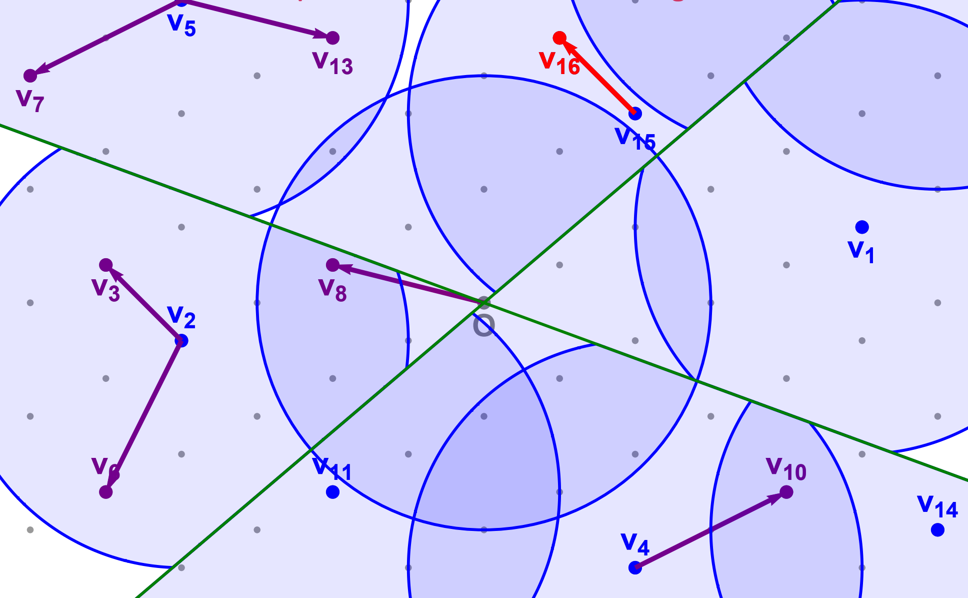

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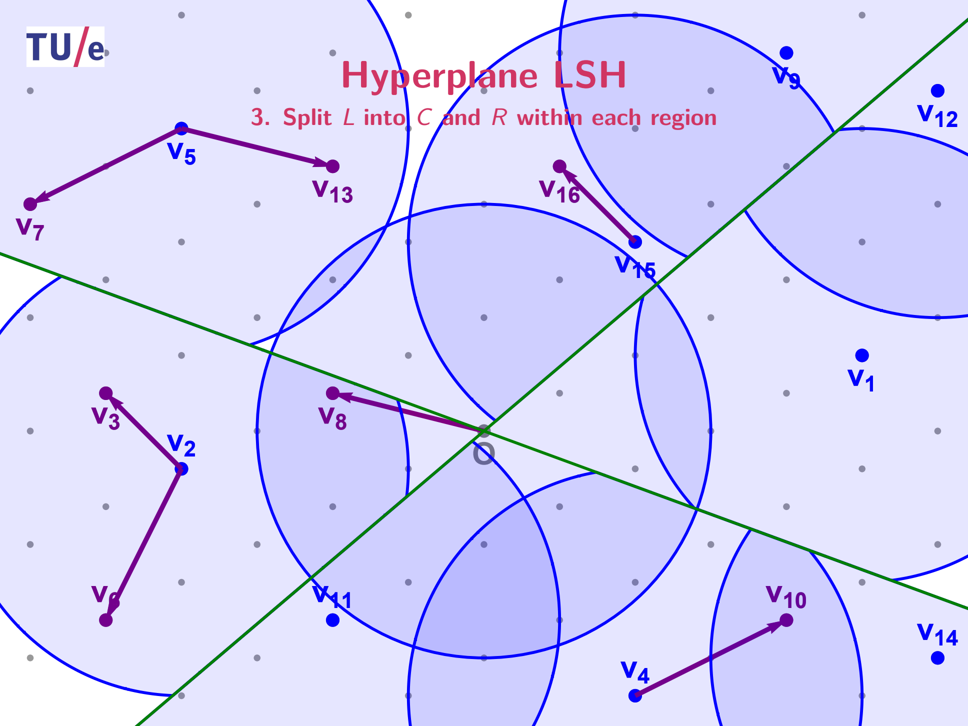

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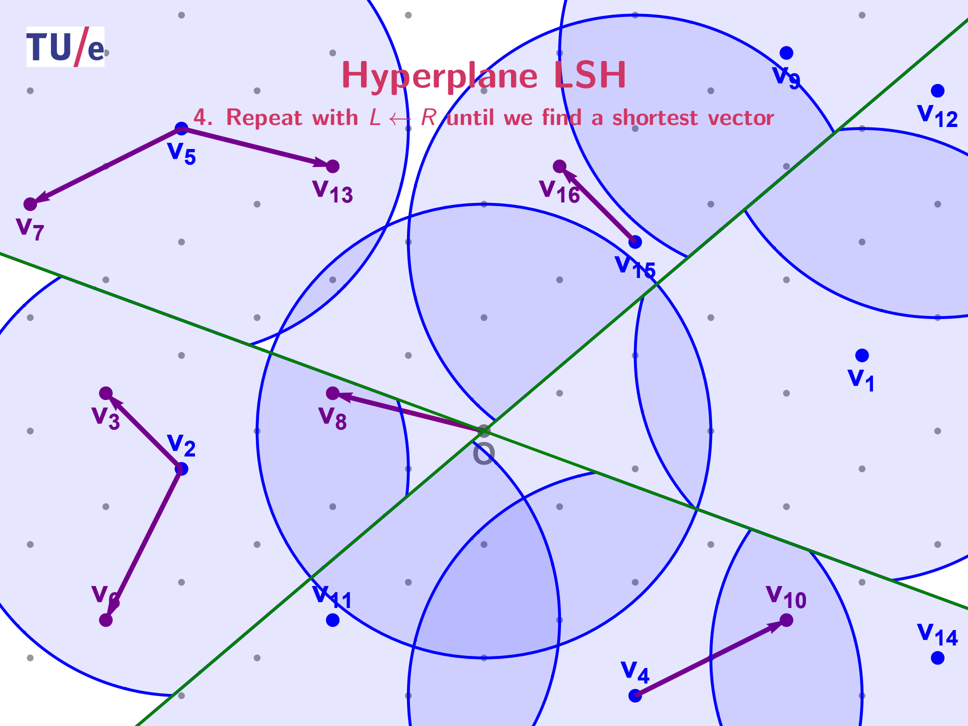

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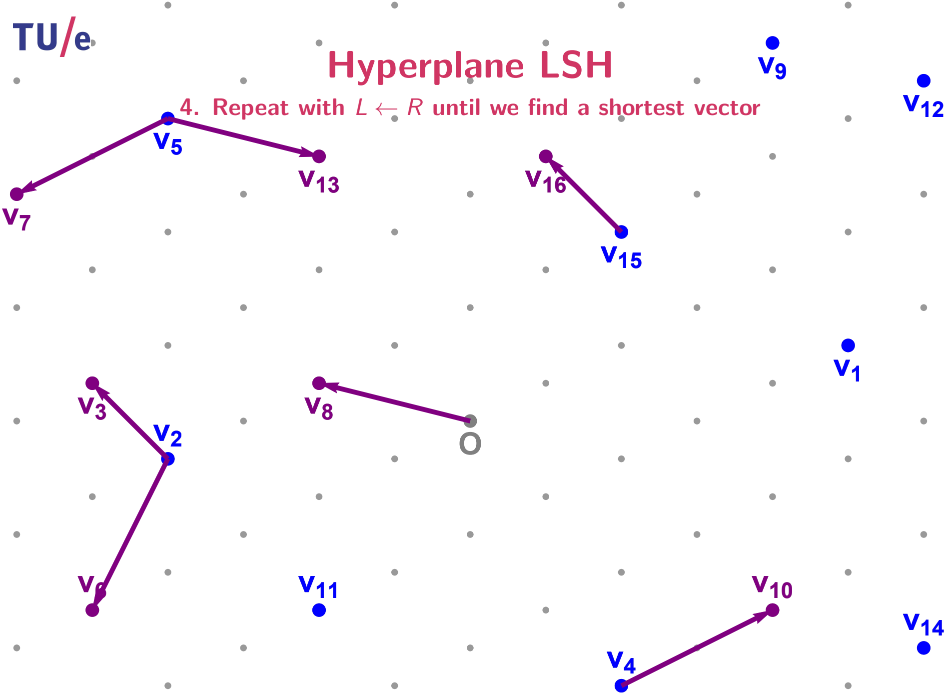

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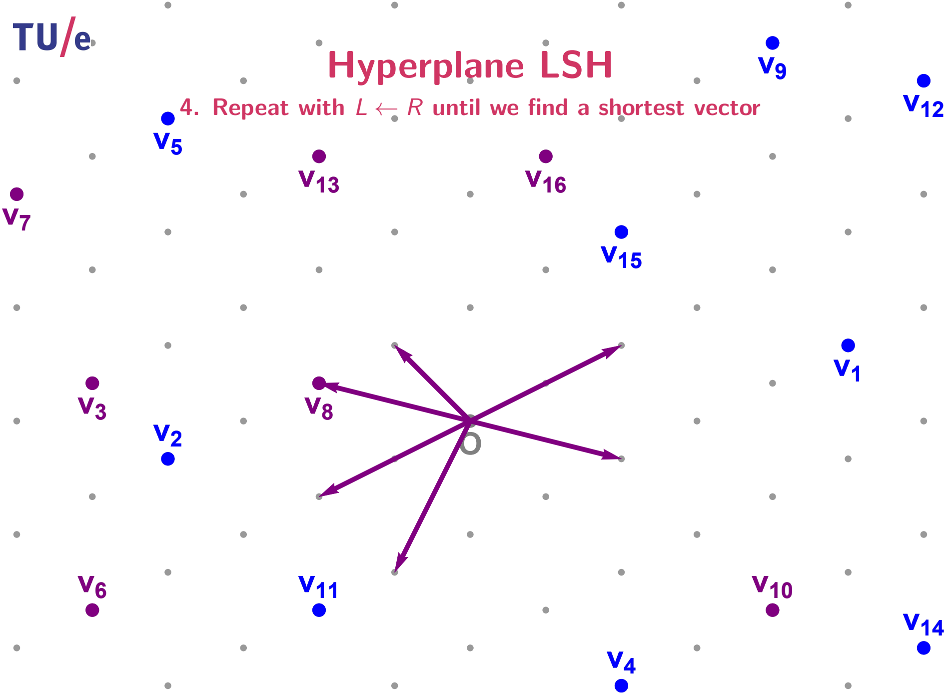

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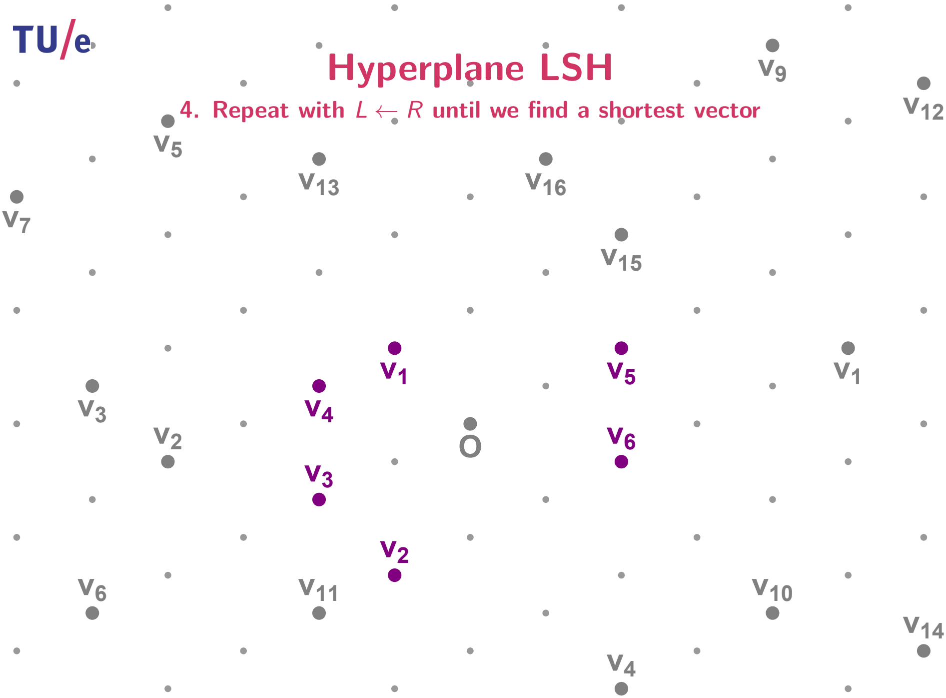

Hyperplane LSH

Overview

Hyperplane LSH

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- Two parameters to tune
  - $k = O(n)$: Number of hyperplanes, leading to 2^k regions
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  - Number of vectors: $2^{0.208n+o(n)}$
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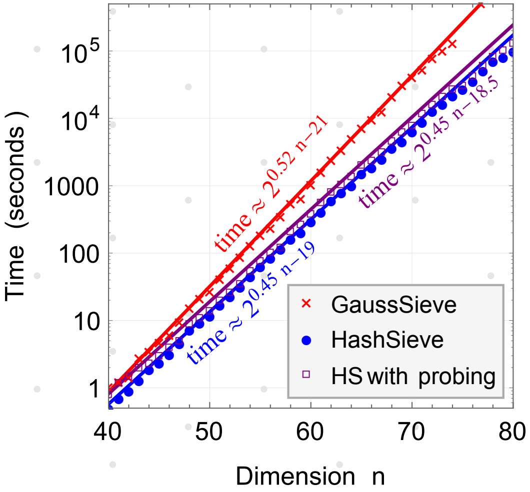
Theorem (L, CRYPTO'15)

Sieving with hyperplane LSH heuristically solves SVP in time and space $2^{0.337n+o(n)}$.

Hyperplane LSH

Experimental results



Hyperplane LSH

Experimental results [MLB15]

Dimension	90	92	94	96	98	100
Hyperplanes (k)	20	20	21	21	22	22
Hash tables (t)	3126	3738	4470	465	533	637
Probing?	N	N	N	Y	Y	Y
BKZ-β preproc.	34	34	34	34	40	40
Used vectors ($\cdot 10^6$)	2.43	3.00	4.50	5.57	7.05	10.05
Time (hours)	0.86	1.72	3.74	6.52	10.03	18.19
Memory (GB)	310	380	872	95	113	256

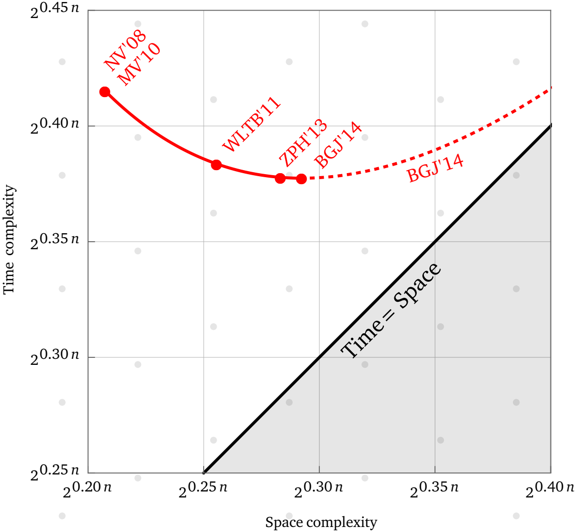
Hyperplane LSH

Experimental results [MLB15]

22	118	2782	0	Kenji Kashiwabara and Masaharu Fukase	vec	Other	2013-08-12	1.00811
23	118	2868	8	Yuanmi Chen and Phong Nguyen	vec	ENUM	2013-02-13	1.04441
24	116	2743	0	Thorsten Kleinjung	vec	Sieving	2014-05-2	1.00492
25	116	2764	0	Kenji Kashiwabara and Masaharu Fukase	vec	Other	2014-03-21	1.01287
26	116	2786	0	Kenji Kashiwabara and Masaharu Fukase	vec	Other	2013-08-3	1.02075
40	108	2508	0	Yuanmi Chen and Phong Nguyen	vec	Other	2010-06-16	0.95162
41	108	2755	0	Yuanmi Chen and Phong Nguyen	vec	Other	2010-05-30	1.04519
42	107	2626	0	Artur Mariano	vec	Sieving	2015-05-22	1.00056
43	107	2713	0	Artur Mariano	vec	Sieving	2015-05-22	1.03379
44	107	2716	0	Artur Mariano	vec	Sieving	2015-05-22	1.03490
45	107	2719	0	Artur Mariano	vec	Sieving	2015-05-22	1.03609
46	107	2720	0	Artur Mariano	vec	Sieving	2015-05-22	1.03657
47	107	2721	0	Artur Mariano	vec	Sieving	2015-05-22	1.03688
48	107	2722	0	Artur Mariano	vec	Sieving	2015-05-22	1.03721
49	107	2724	8	Po-Chun Kuo, Michael Schneider	vec	ENUM,BKZ	2011-03-12	1.03655

Hyperplane LSH

Space/time trade-off



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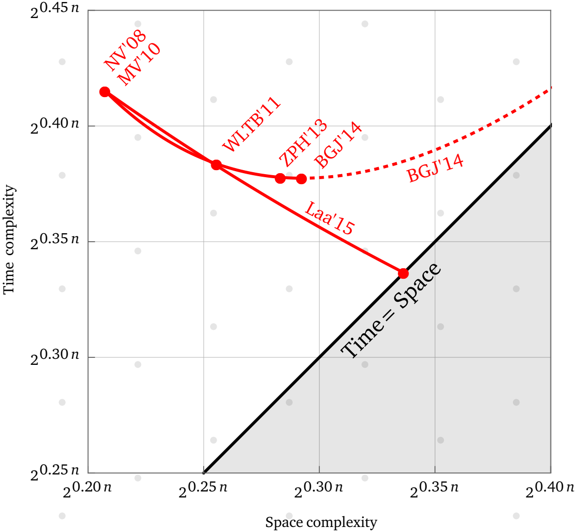

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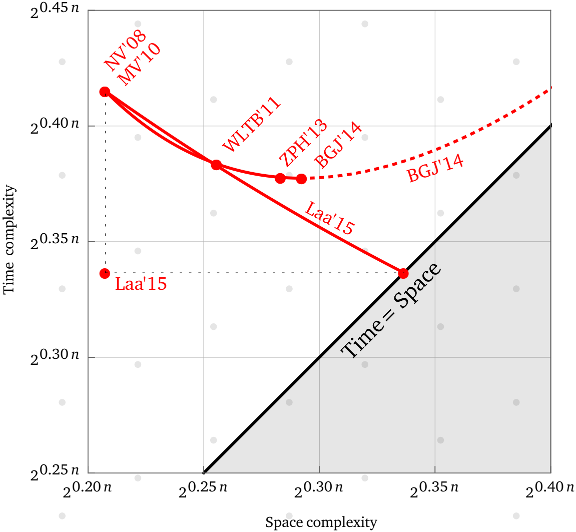
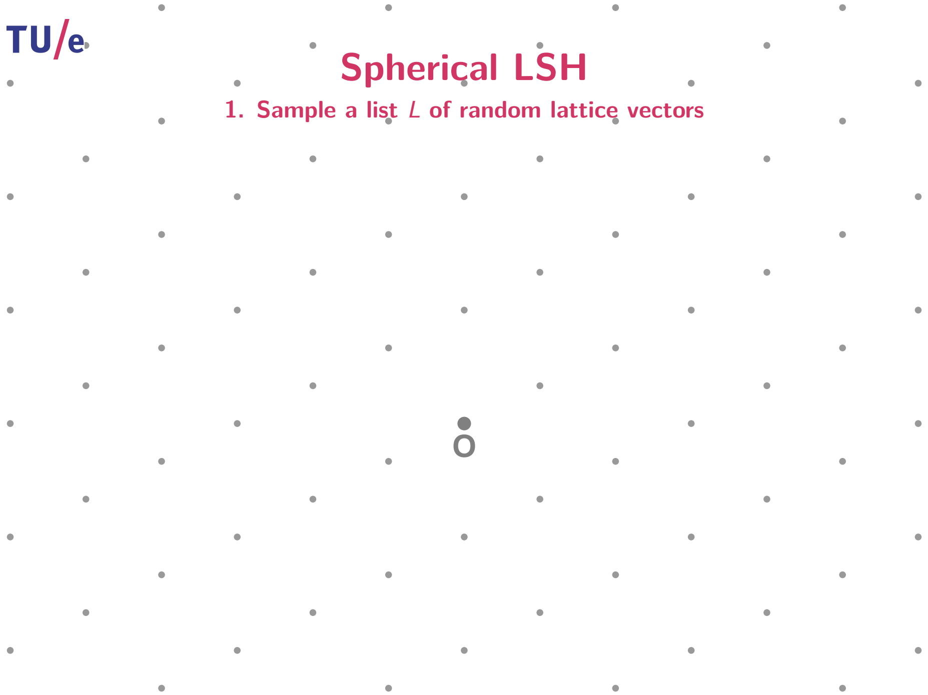


Spherical LSH

1. Sample a list L of random lattice vectors



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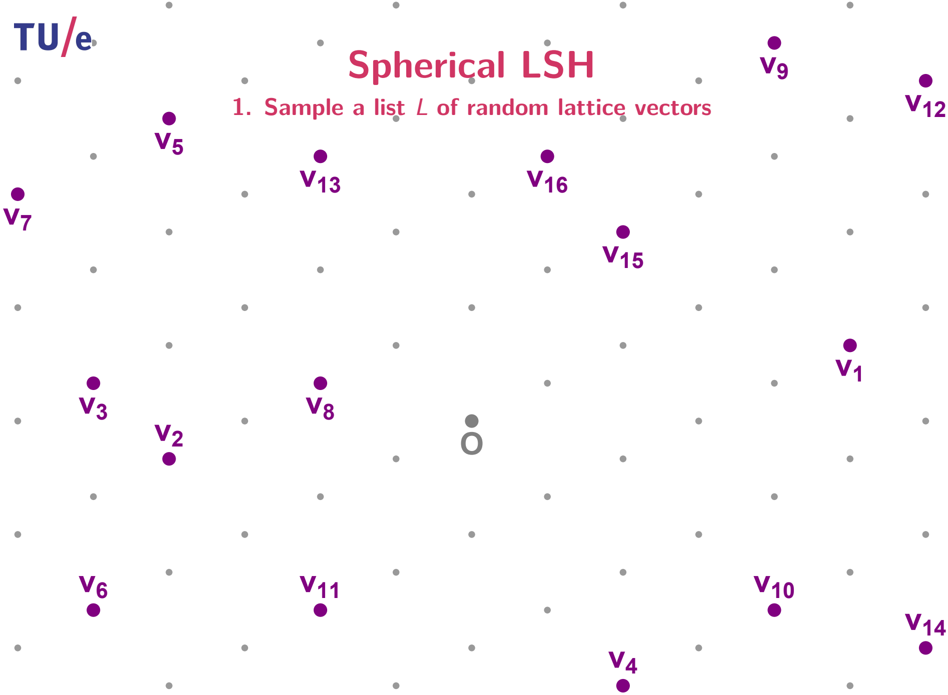

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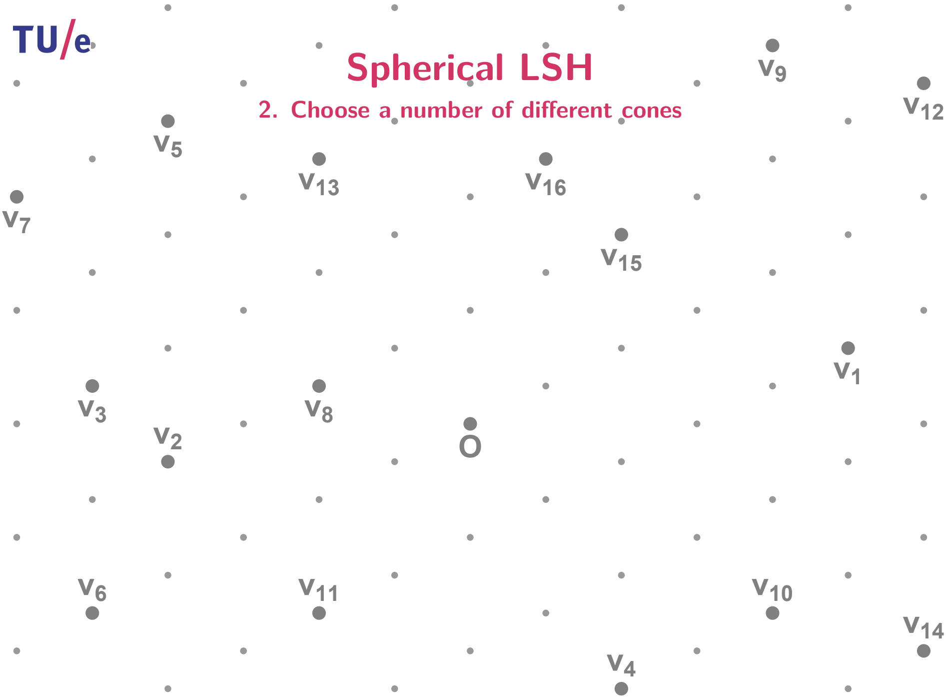
2. Choose a number of different cones



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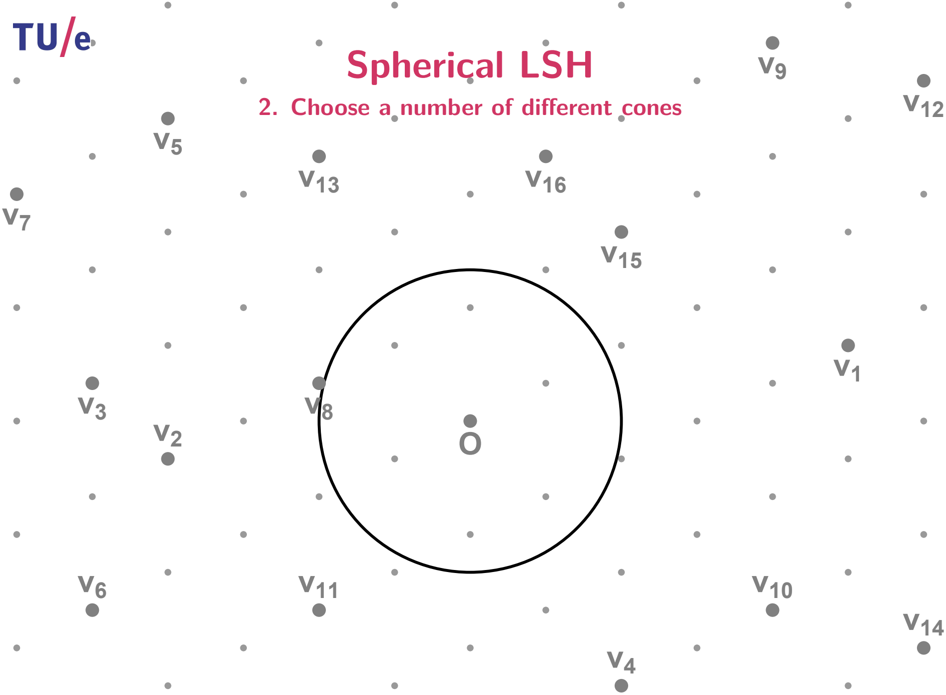

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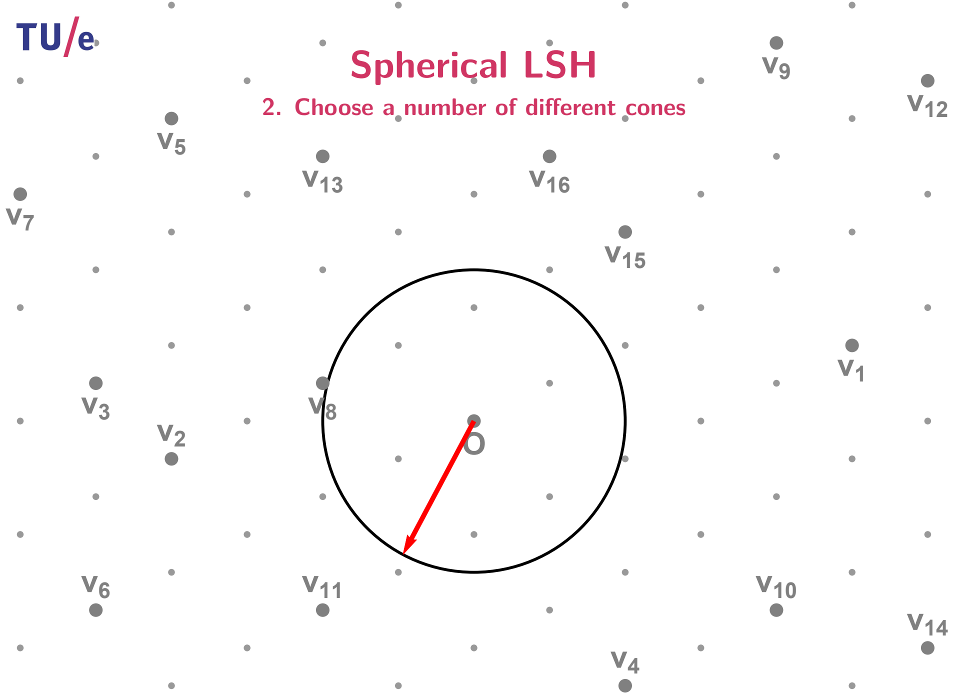

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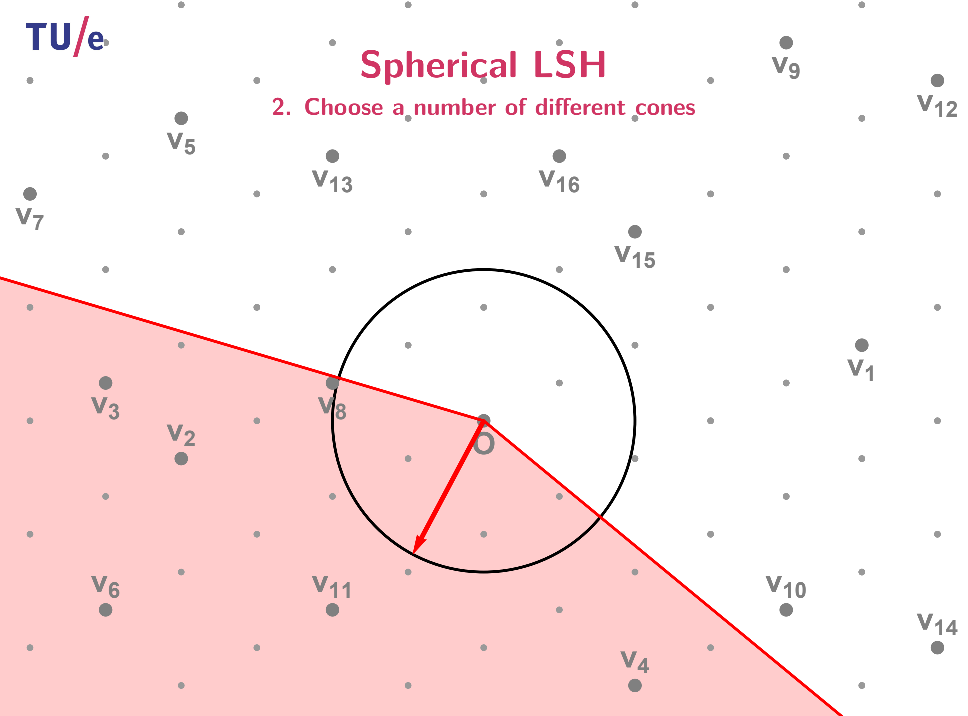

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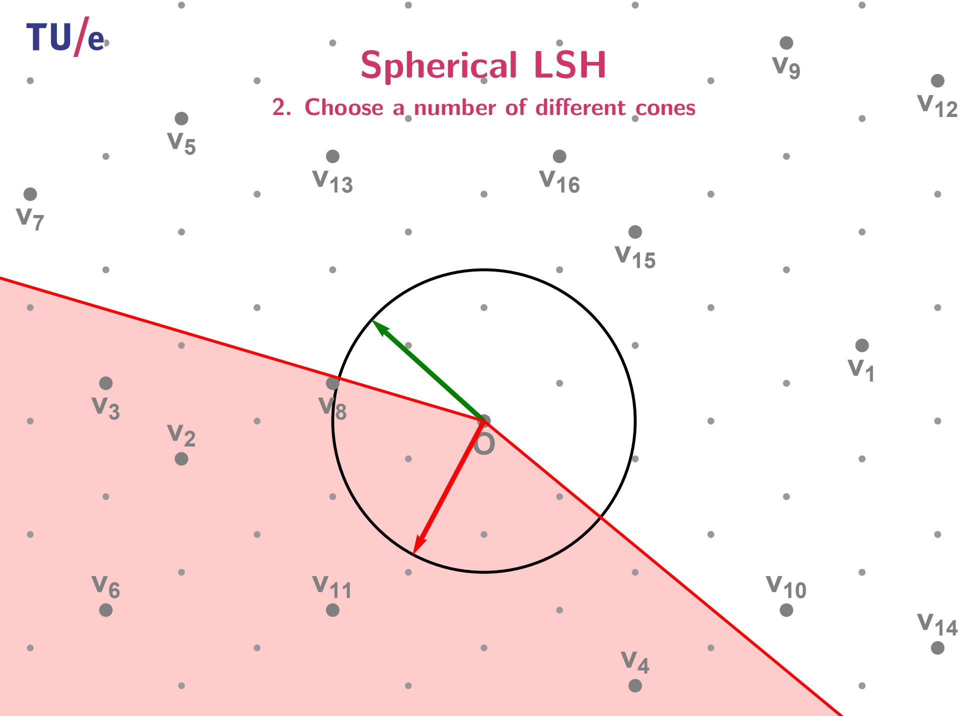

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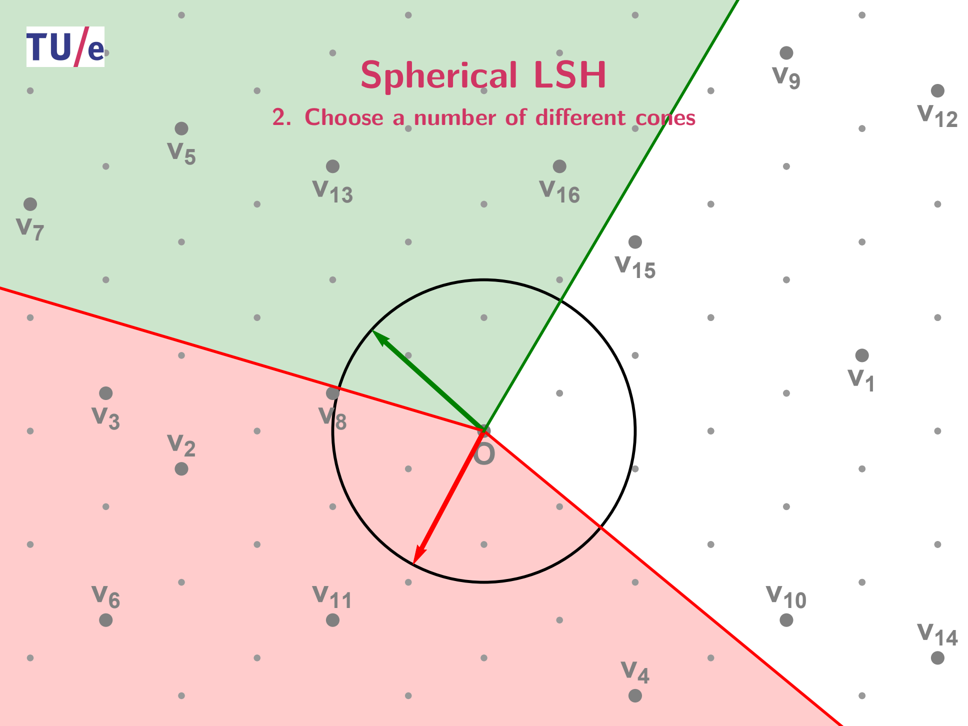

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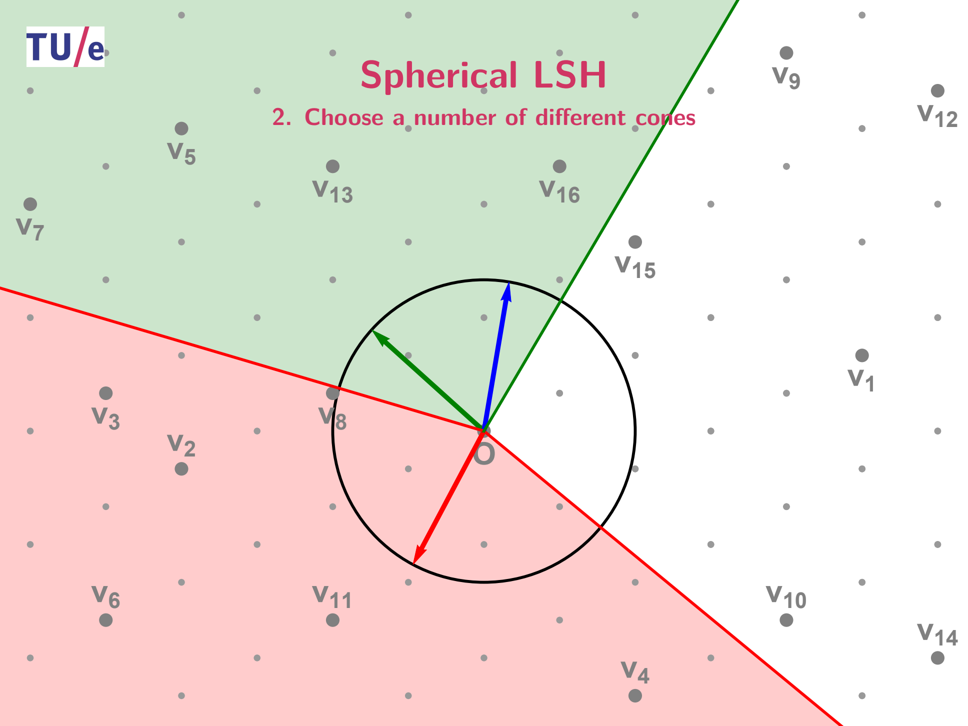

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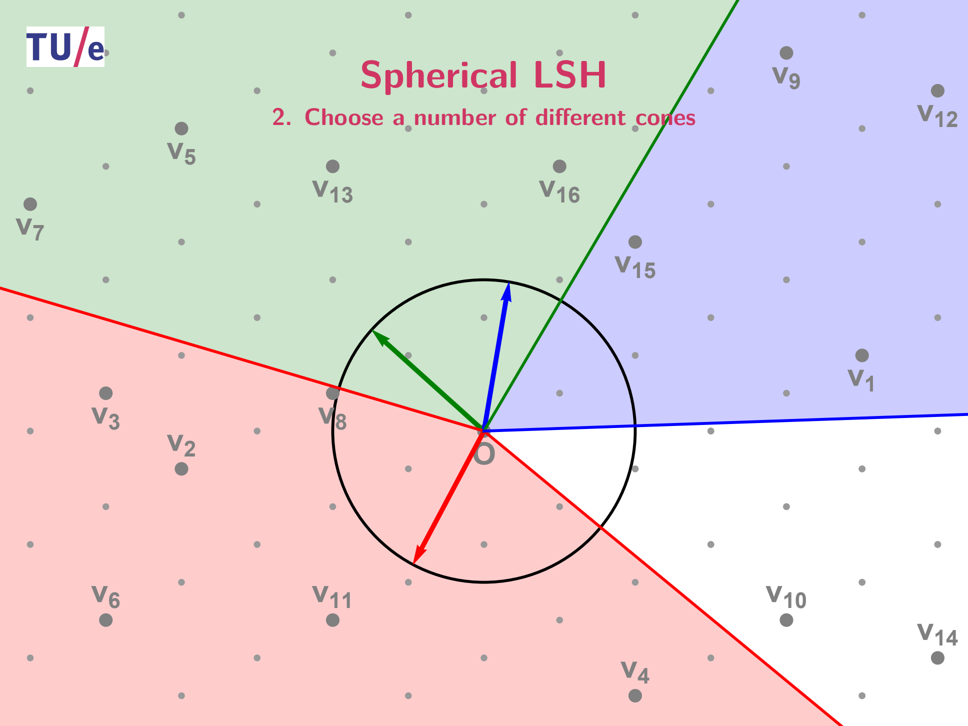

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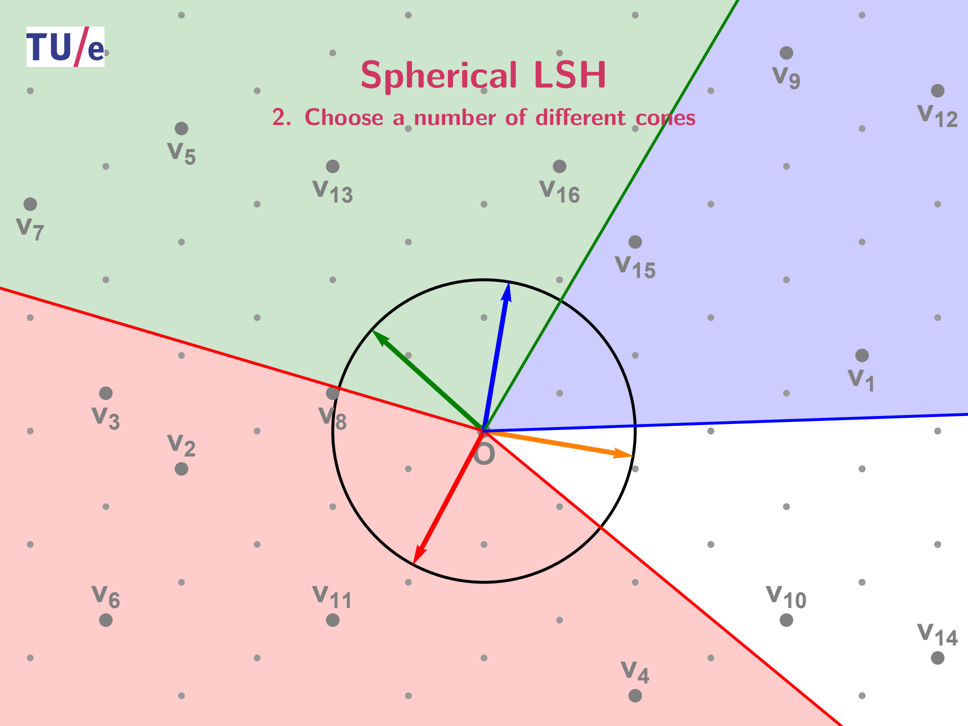

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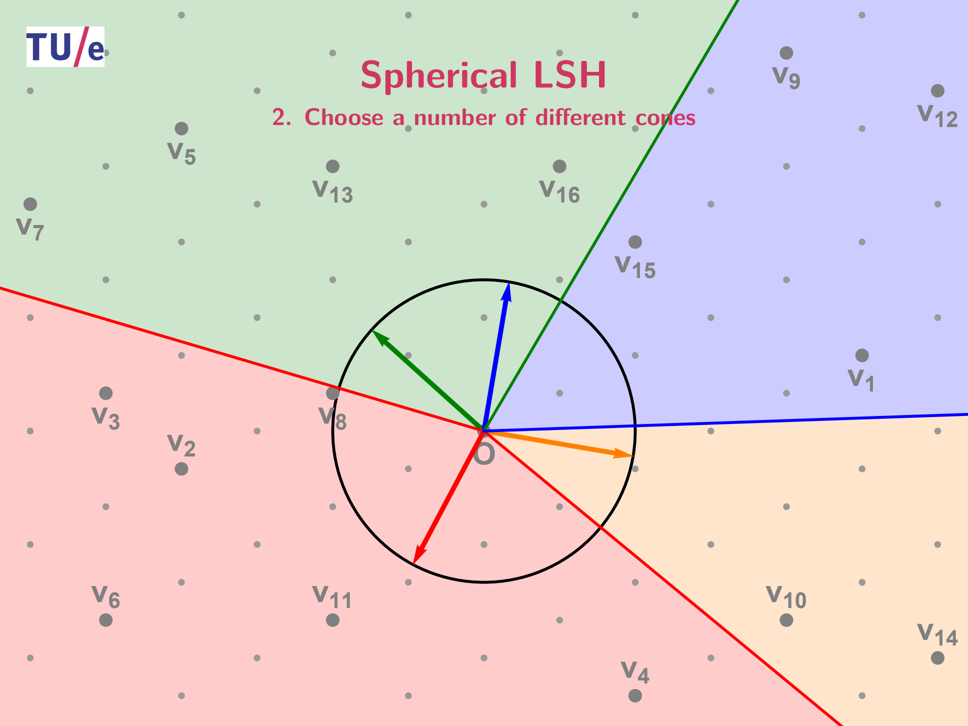

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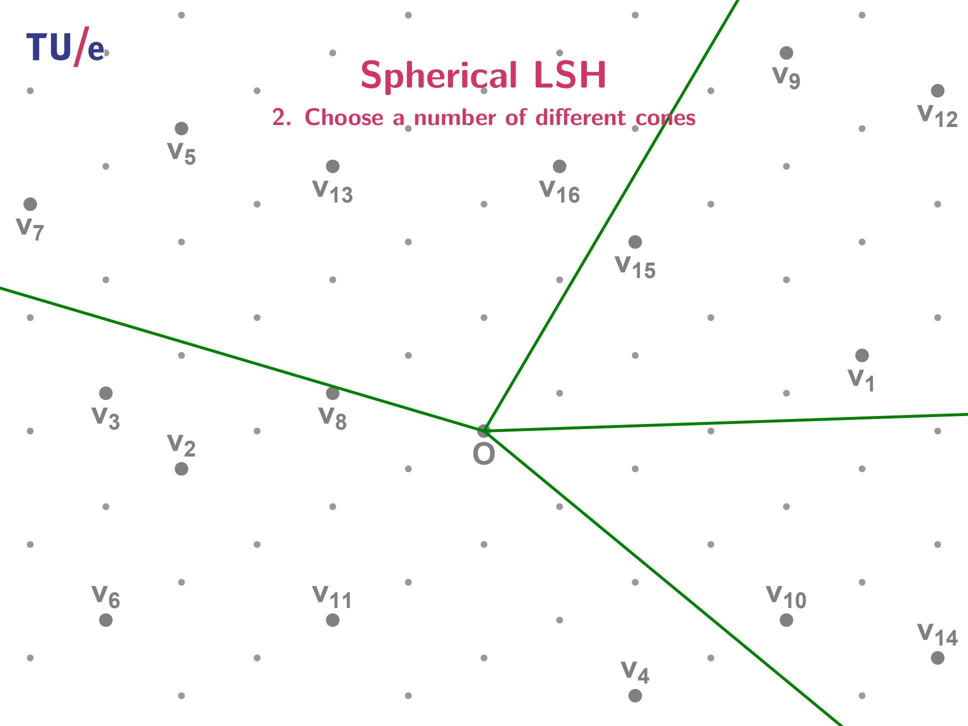

Spherical LSH

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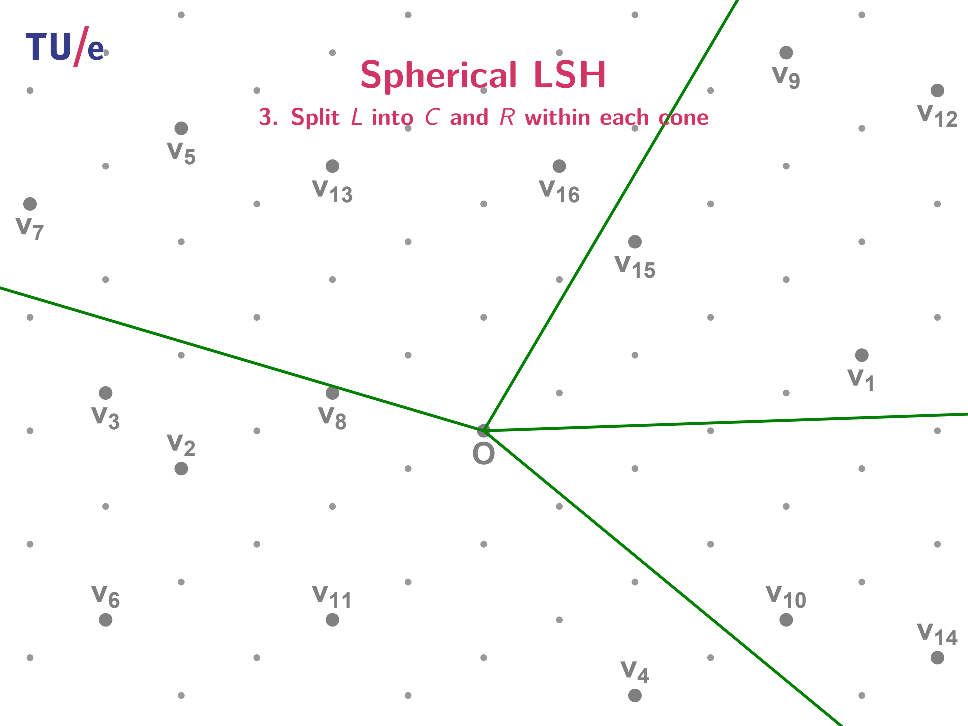

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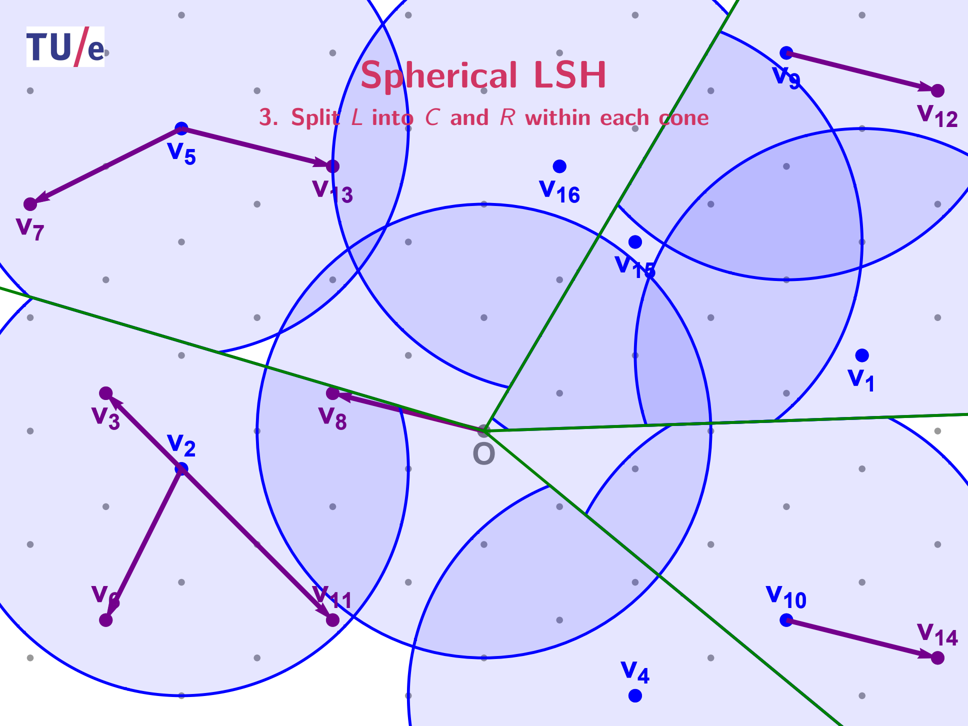
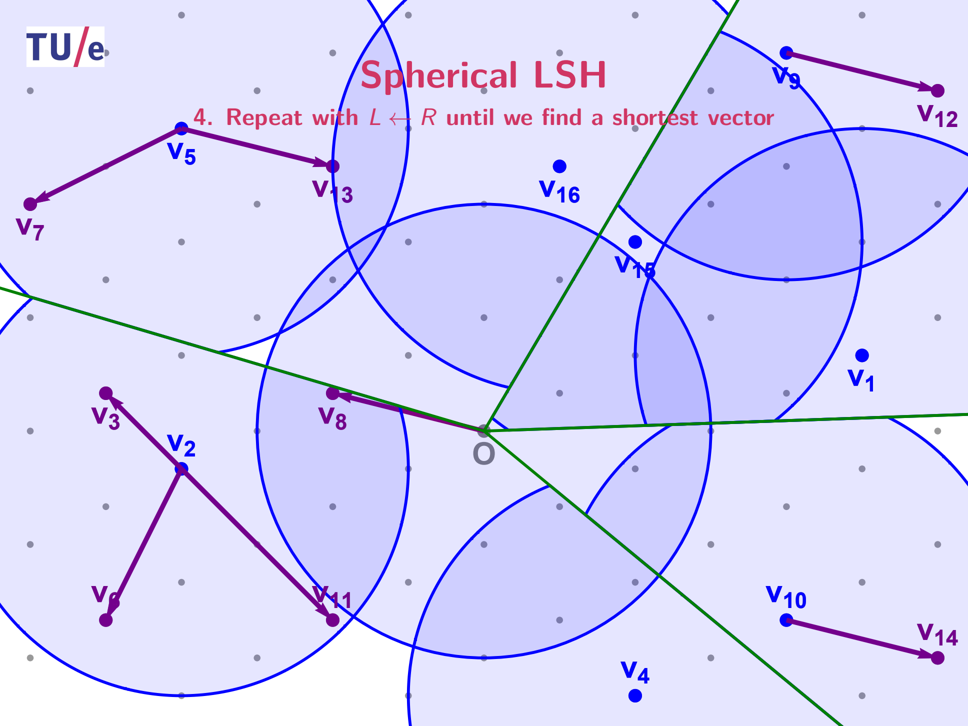


Spherical LSH

4. Repeat with $L \leftarrow R$ until we find a shortest vector



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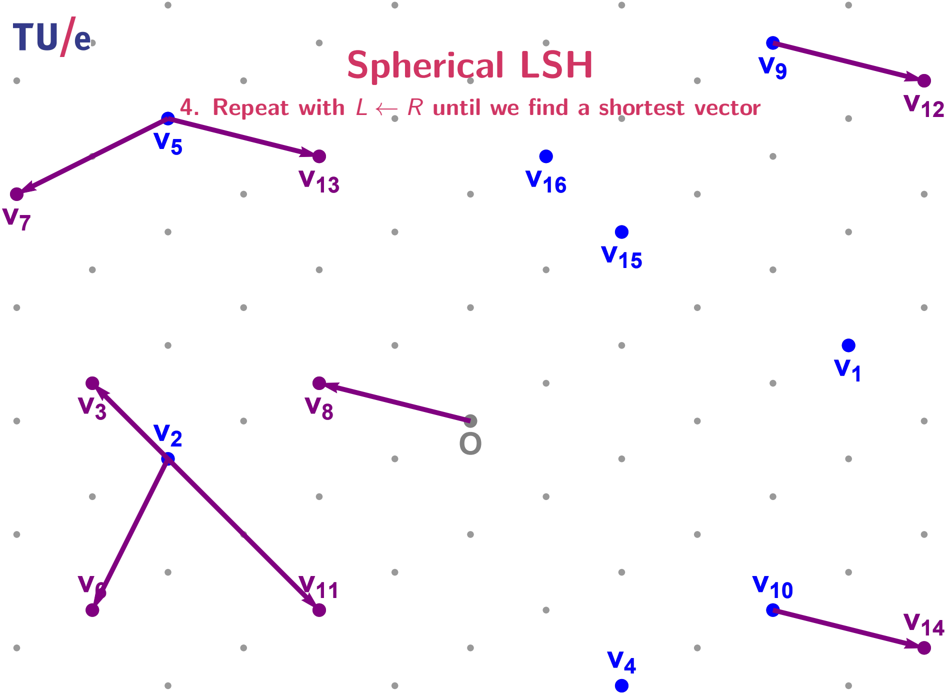

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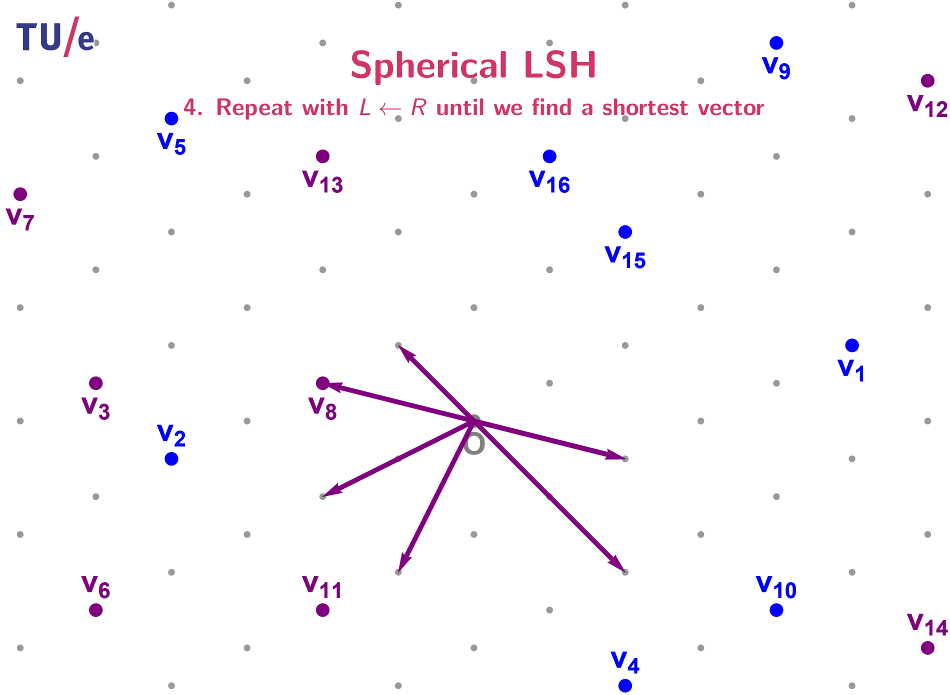

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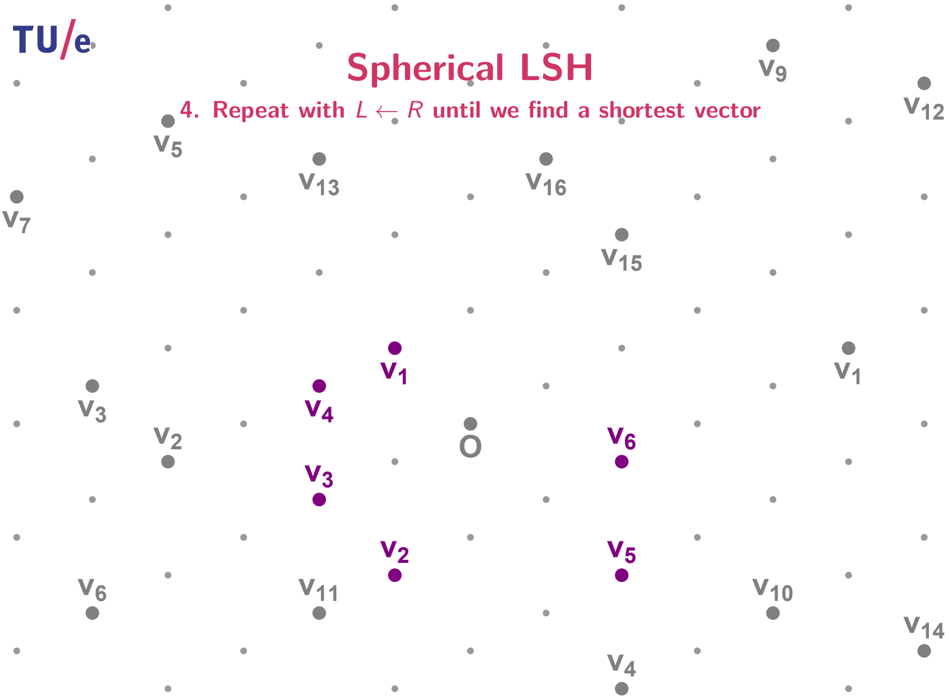
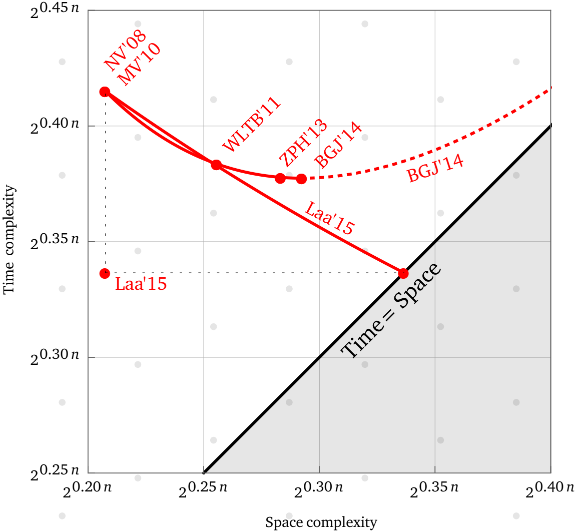


TU/e

Spherical LSH

Space/time trade-off



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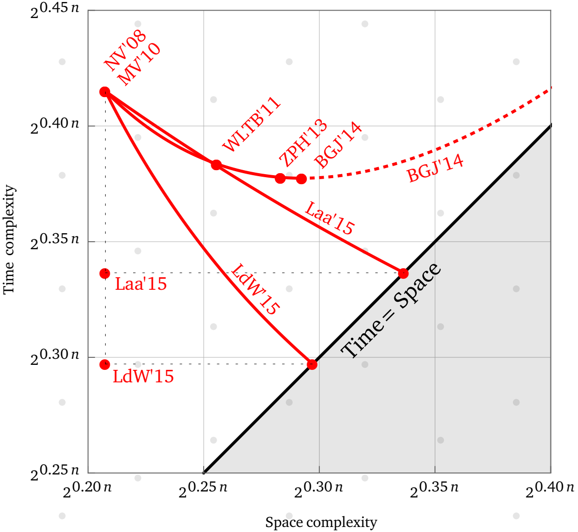

Cross-Polytope LSH

1. Sample a list L of random lattice vectors

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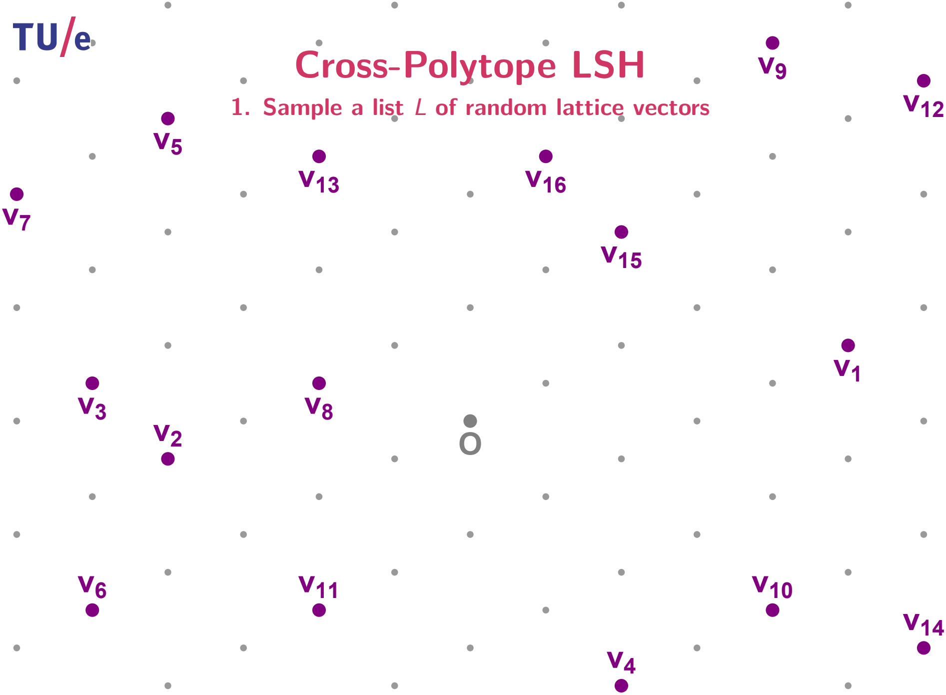
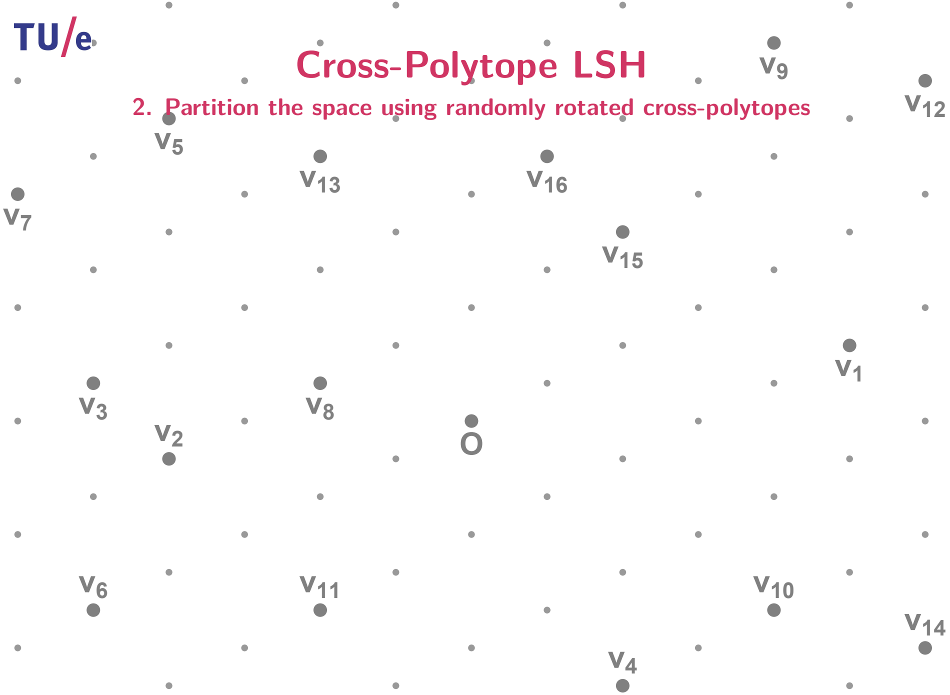


Cross-Polytope LSH

2. Partition the space using randomly rotated cross-polytopes



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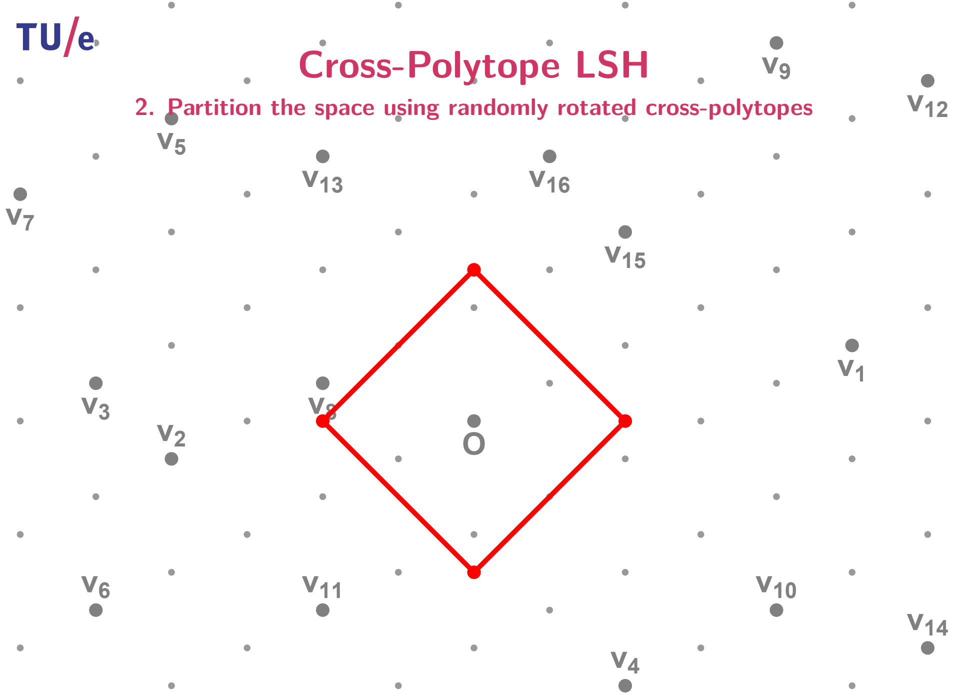

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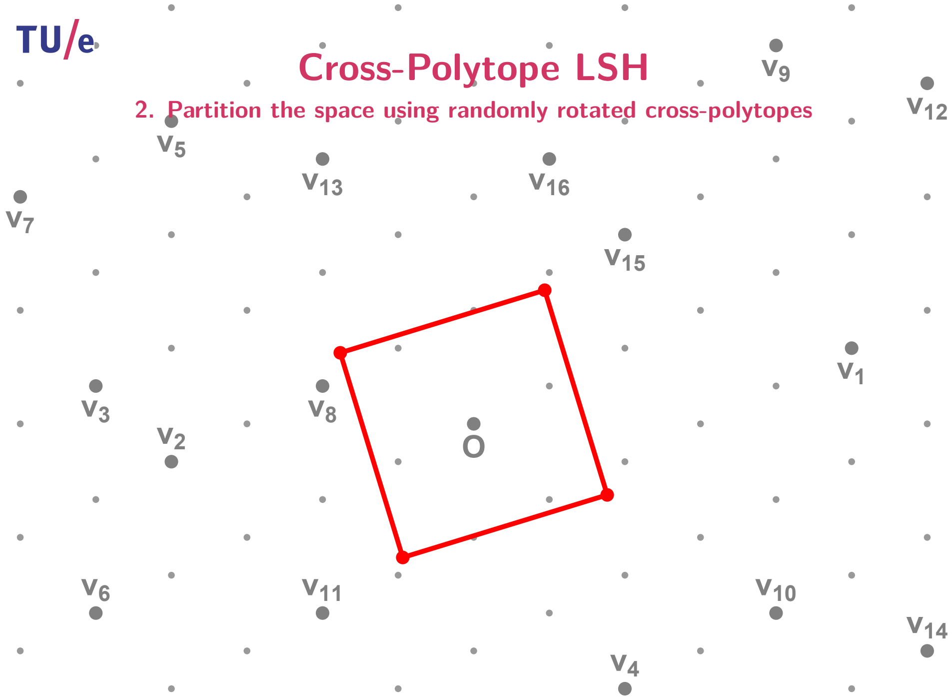

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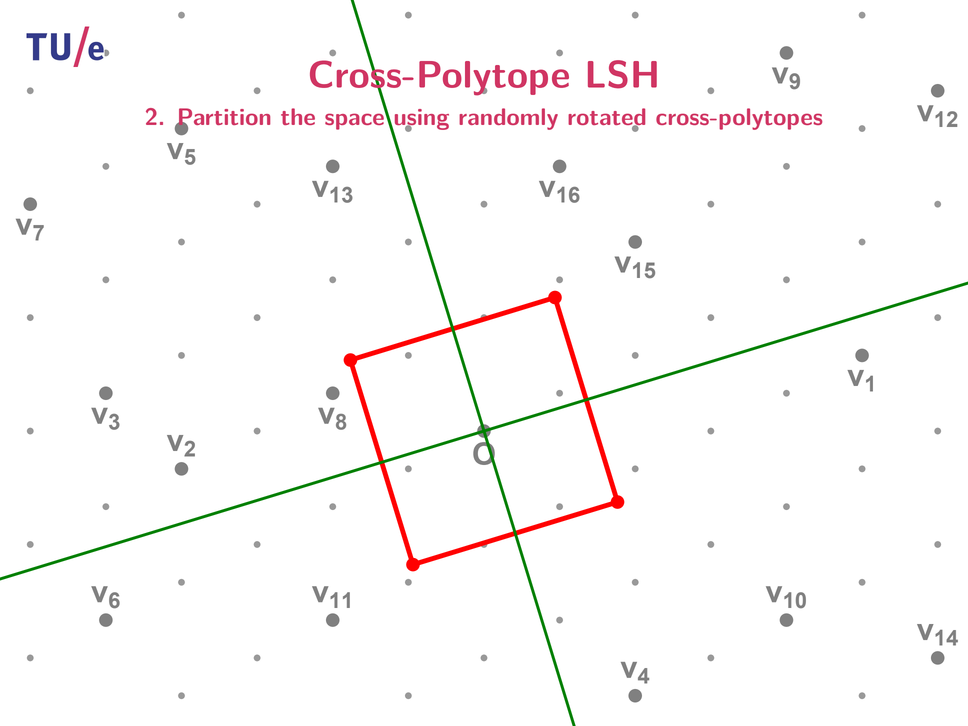

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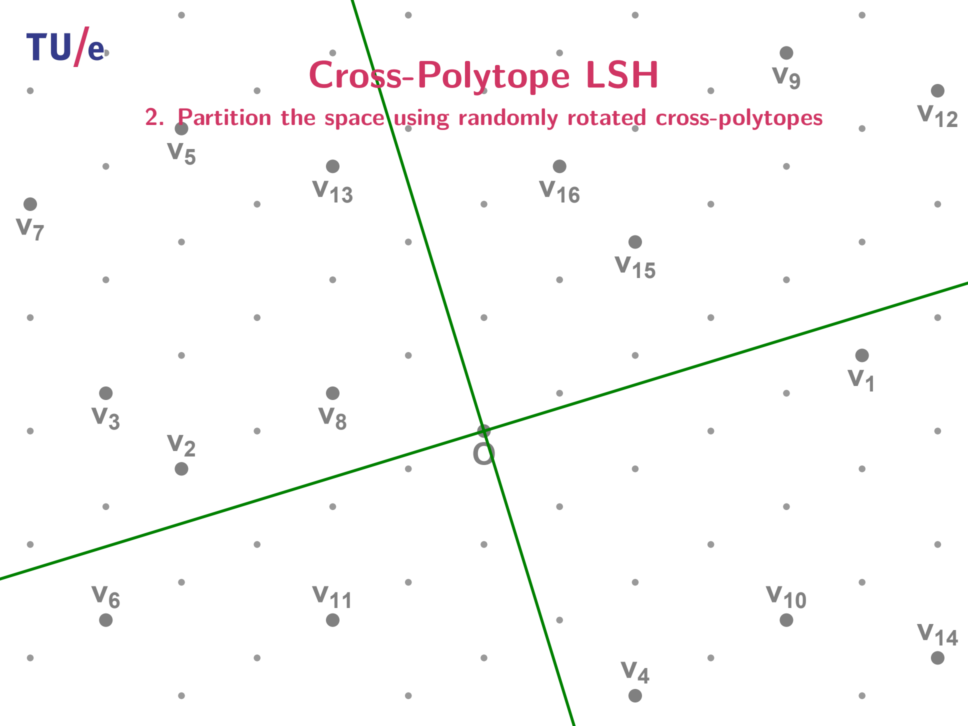

TU/e

Cross-Polytope LSH

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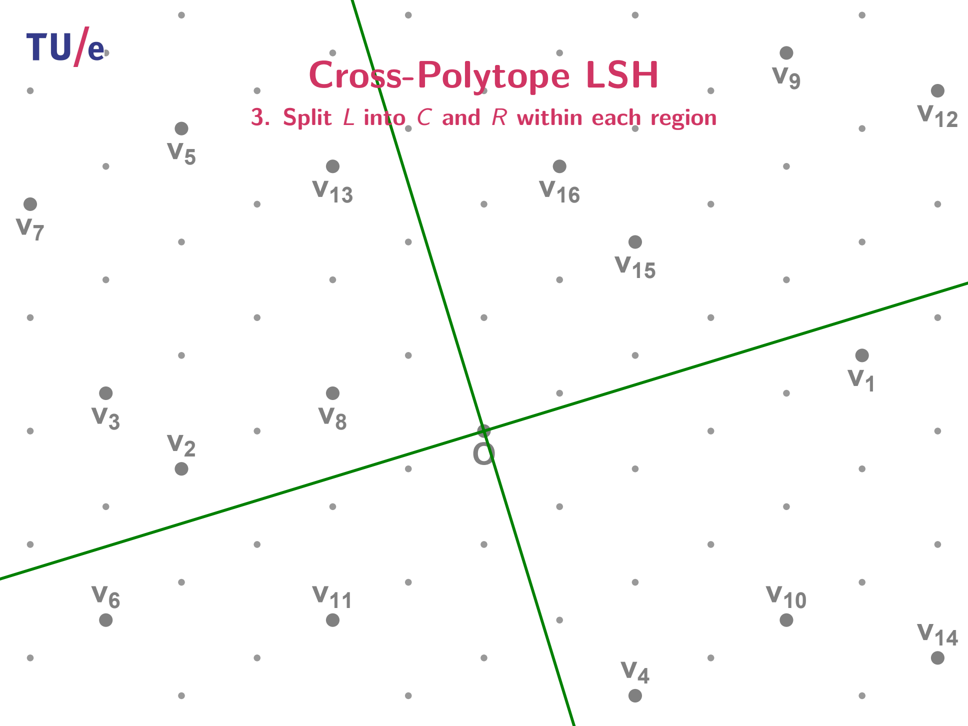

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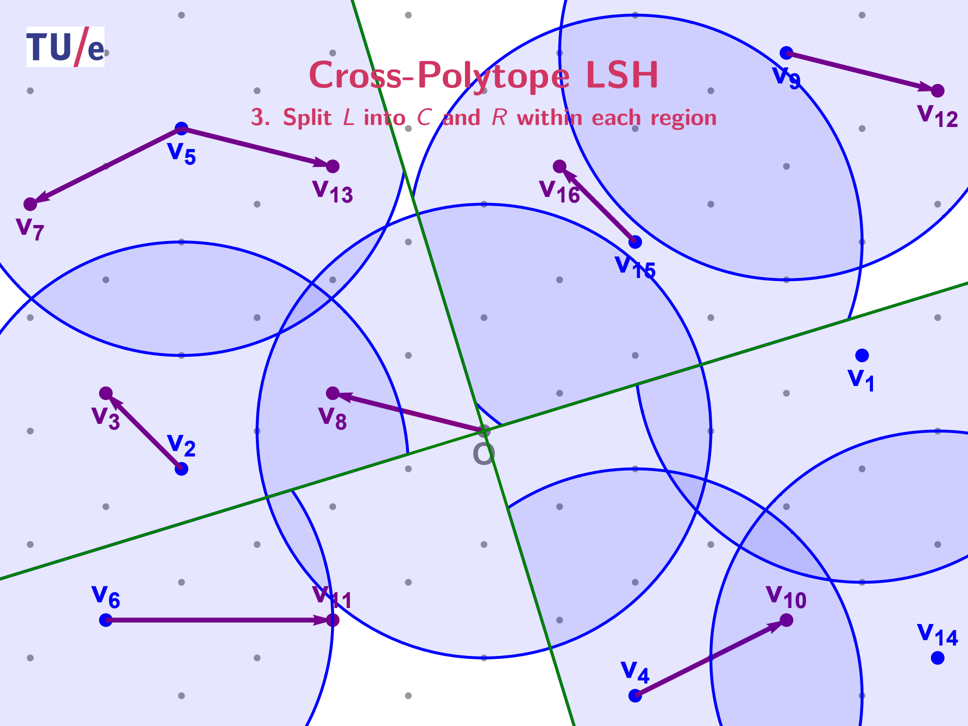

Cross-Polytope LSH

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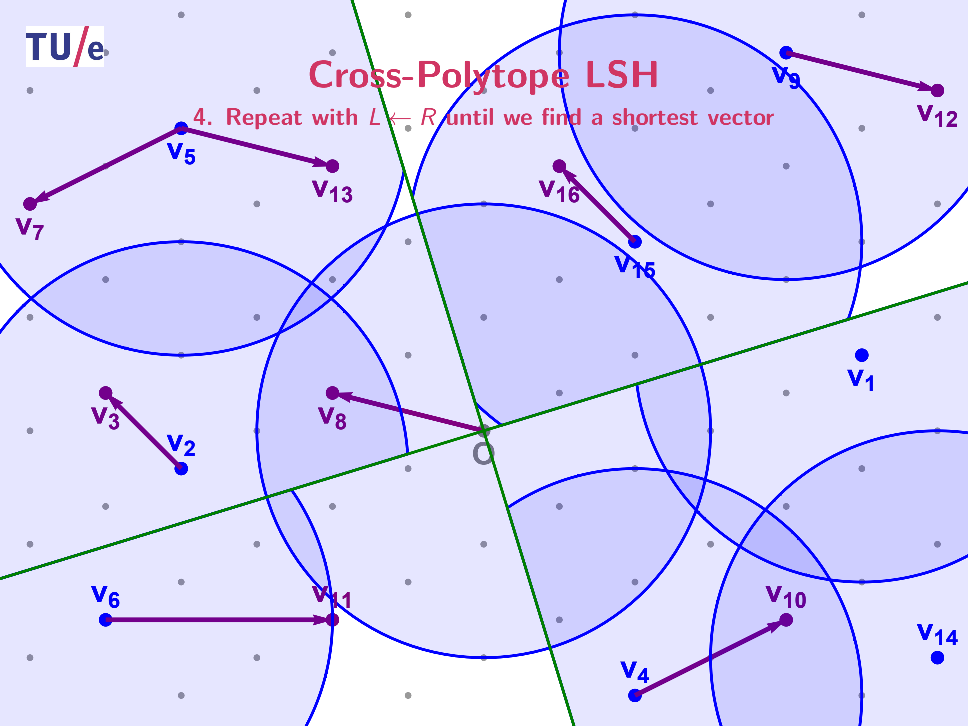

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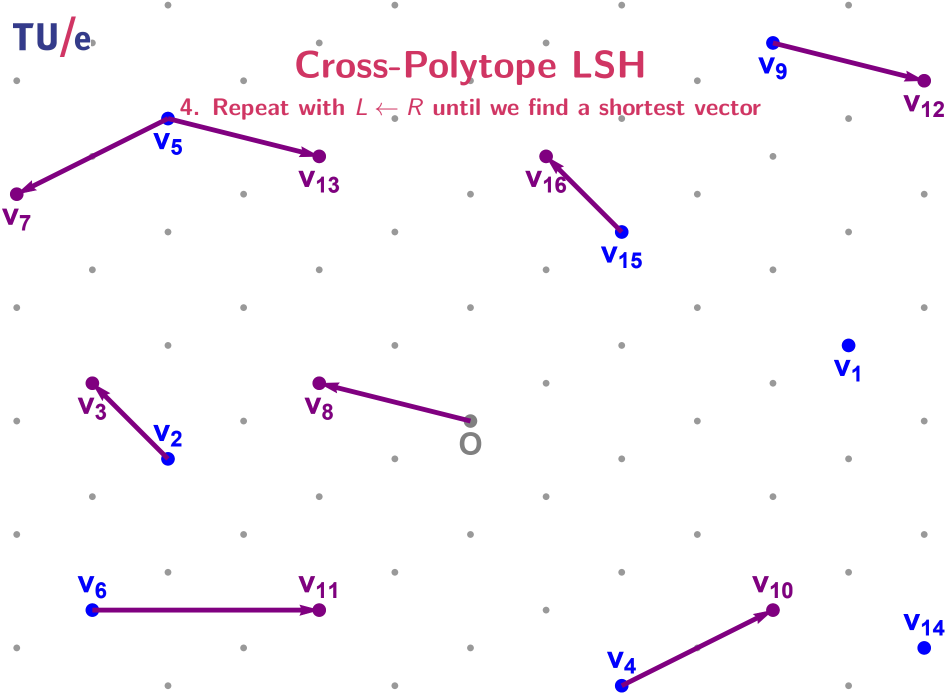

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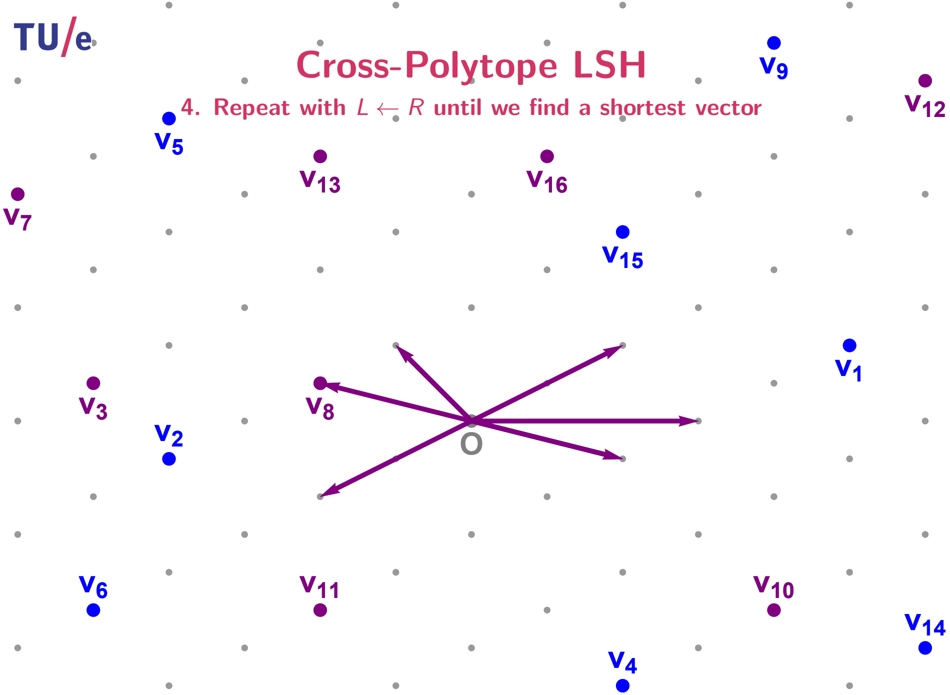

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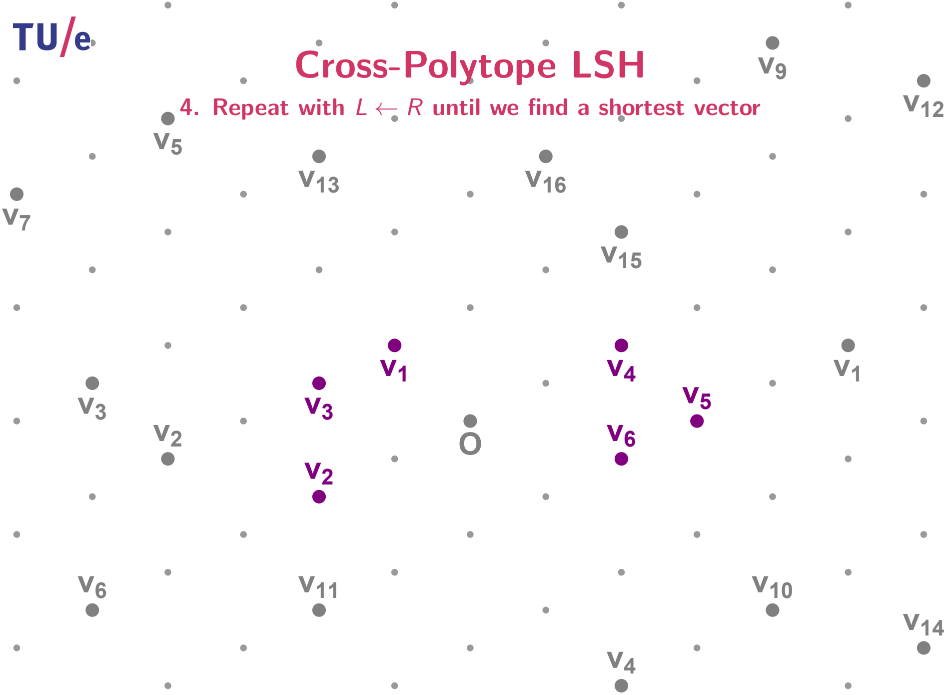
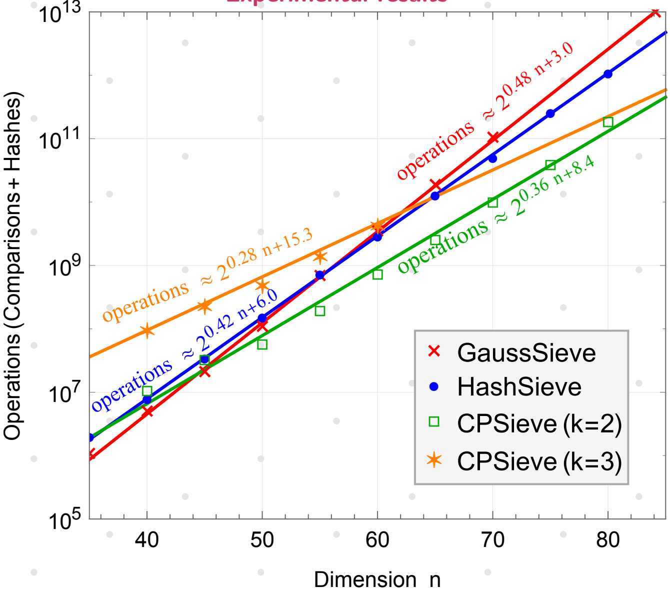


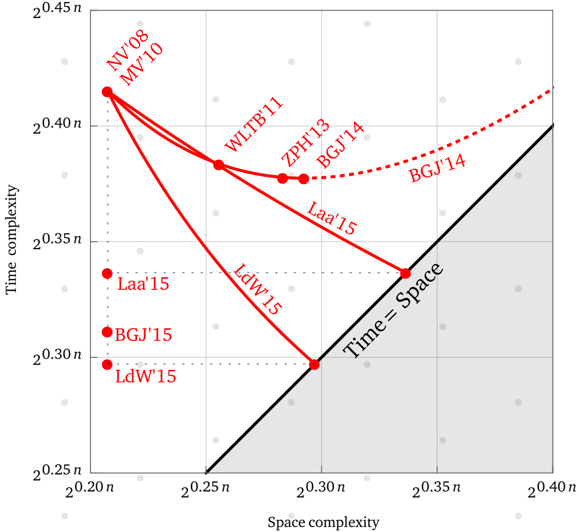
Cross-Polytope LSH

Experimental results



Cross-Polytope LSH

Space/time trade-off



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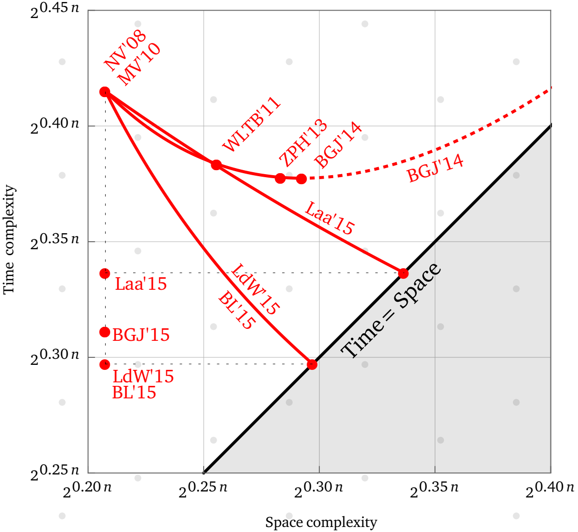

Spherical filtering

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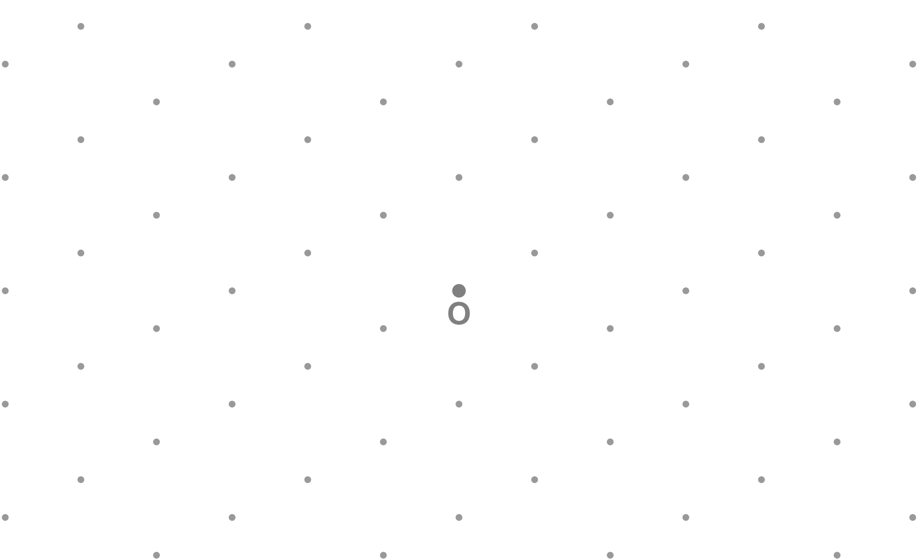

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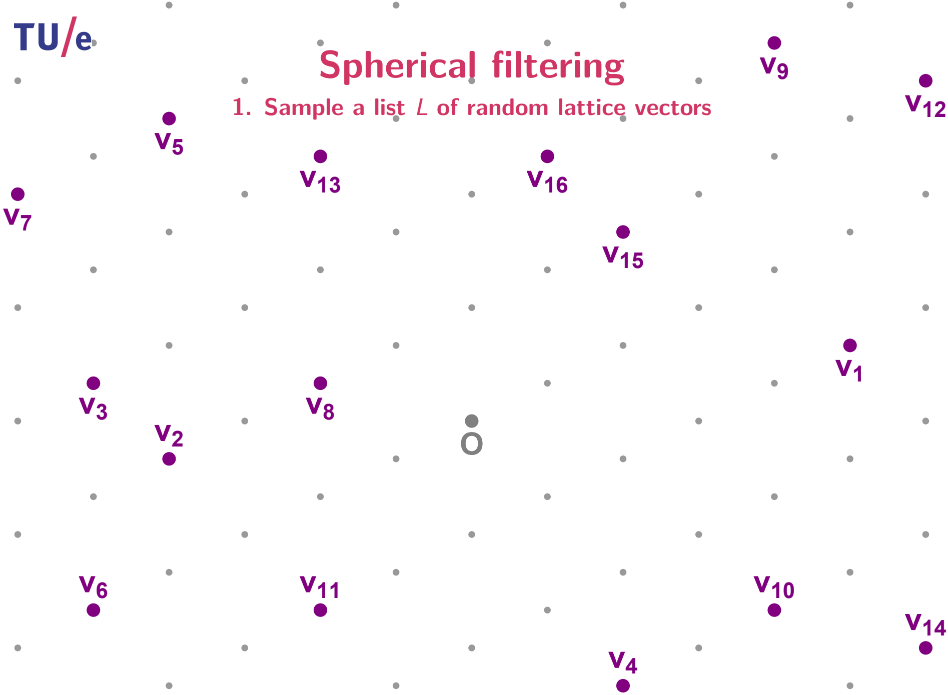
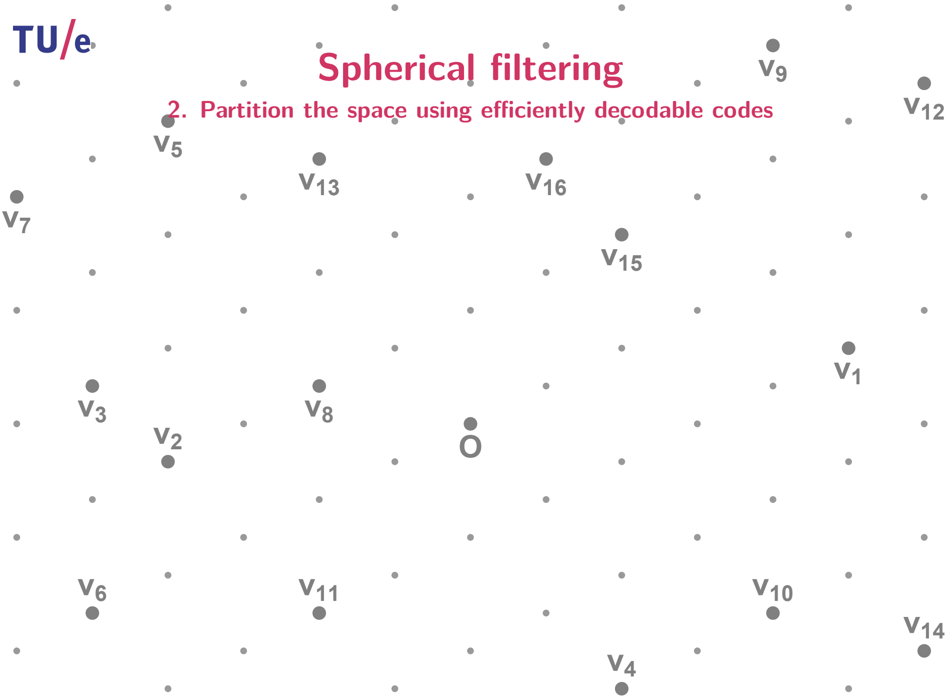


Spherical filtering

2. Partition the space using efficiently decodable codes



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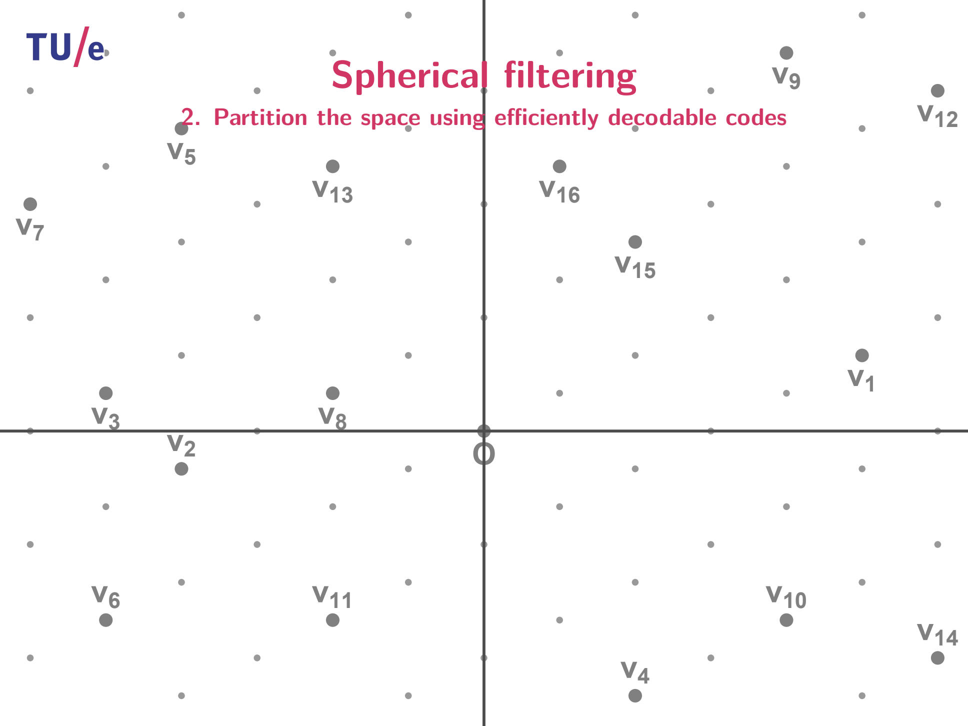

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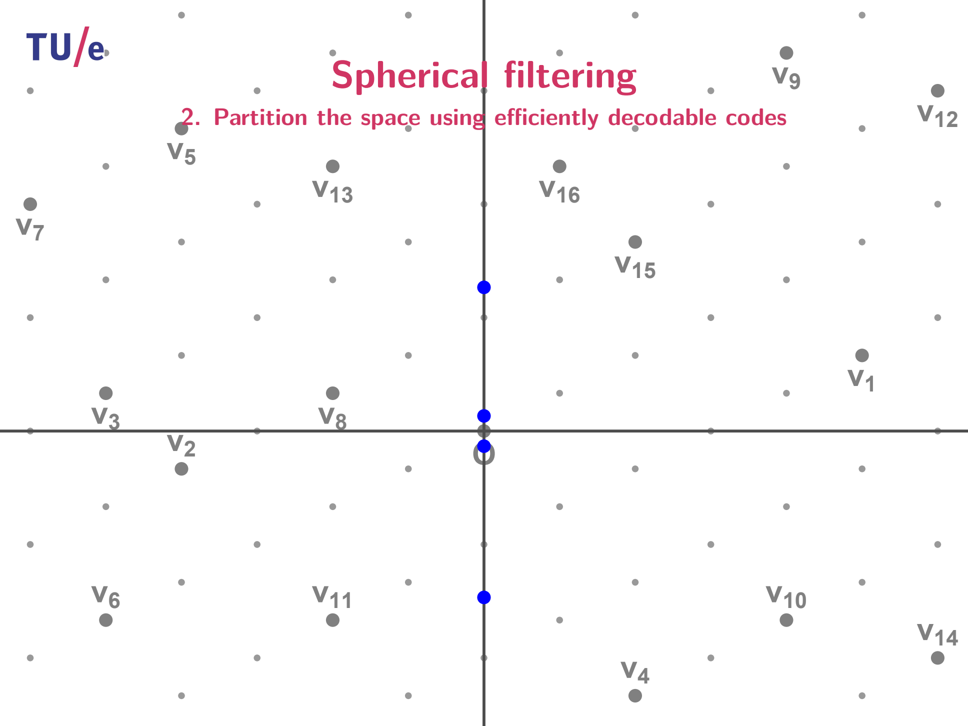

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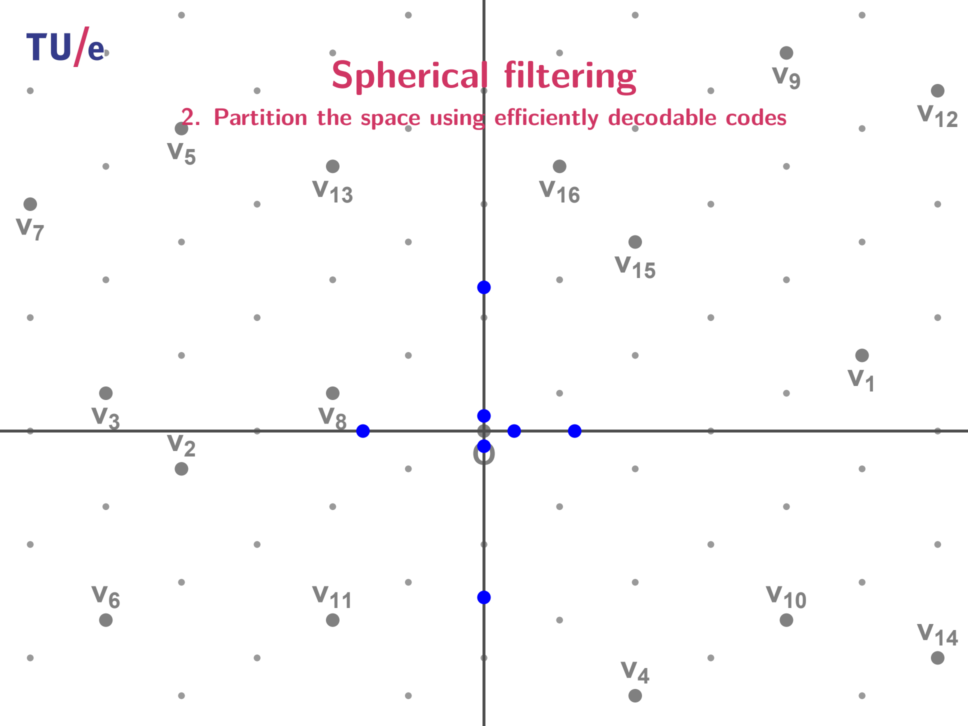

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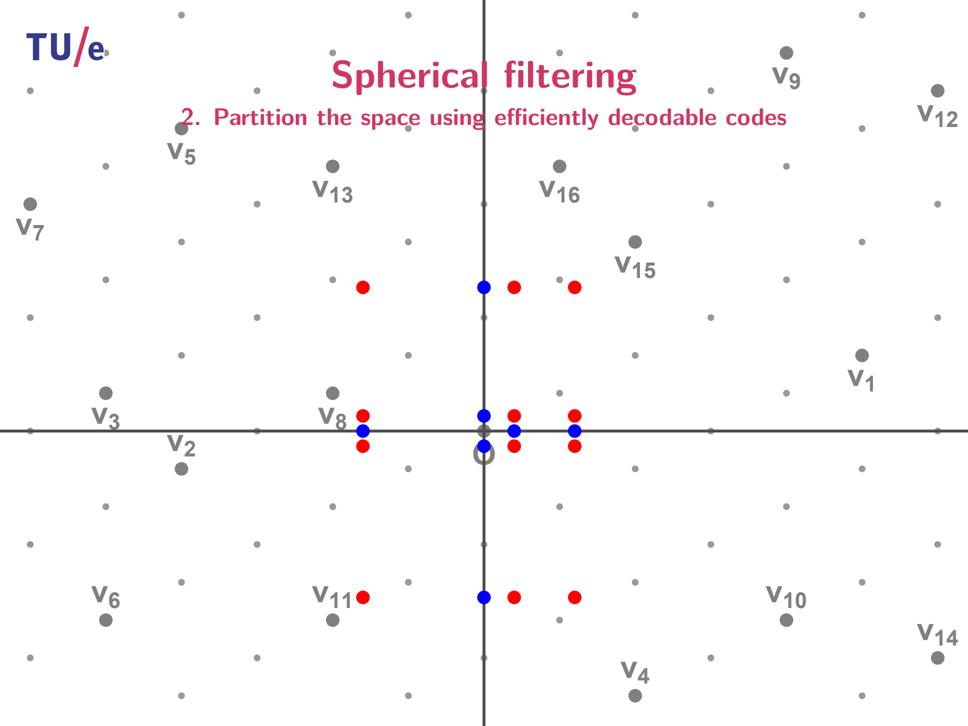

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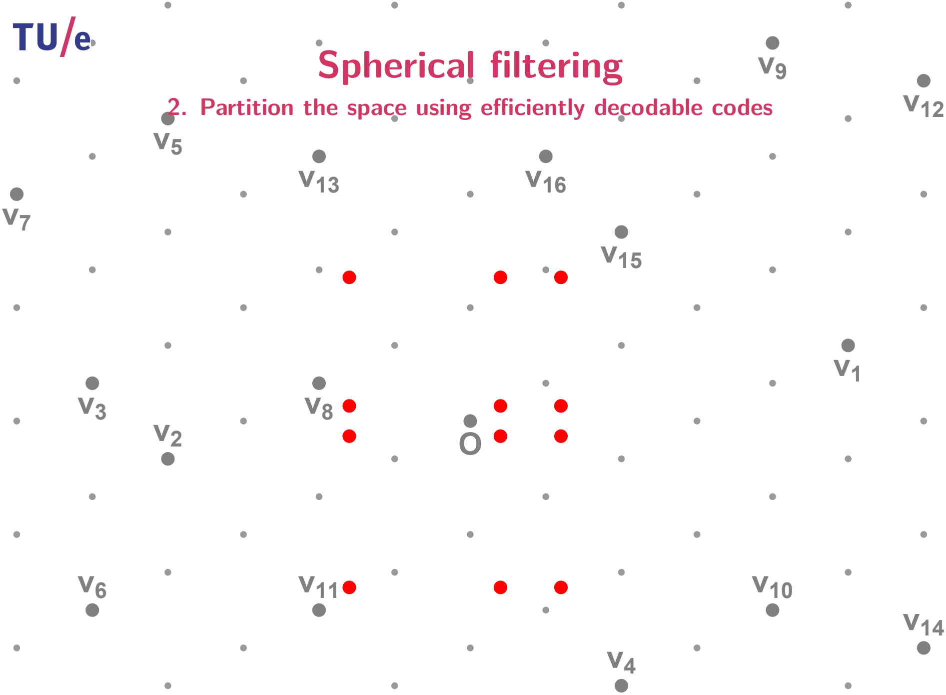

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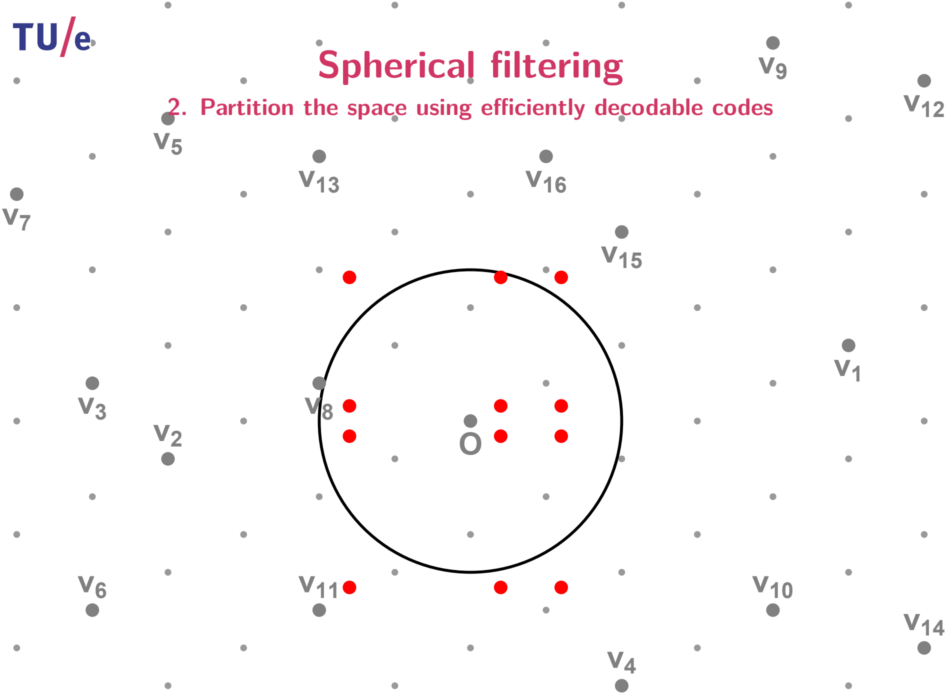

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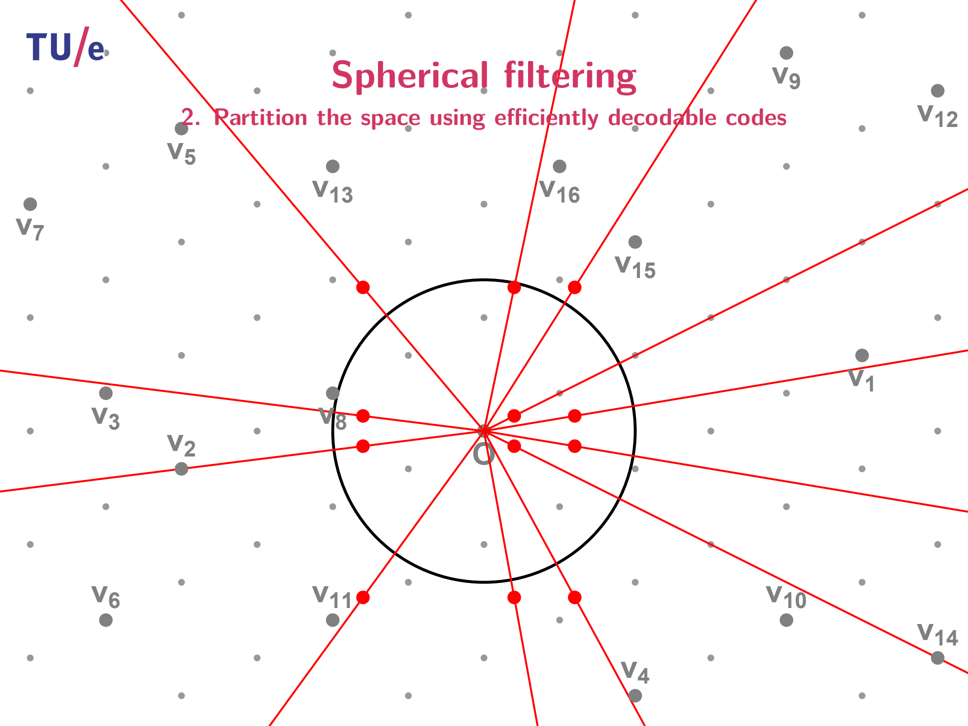

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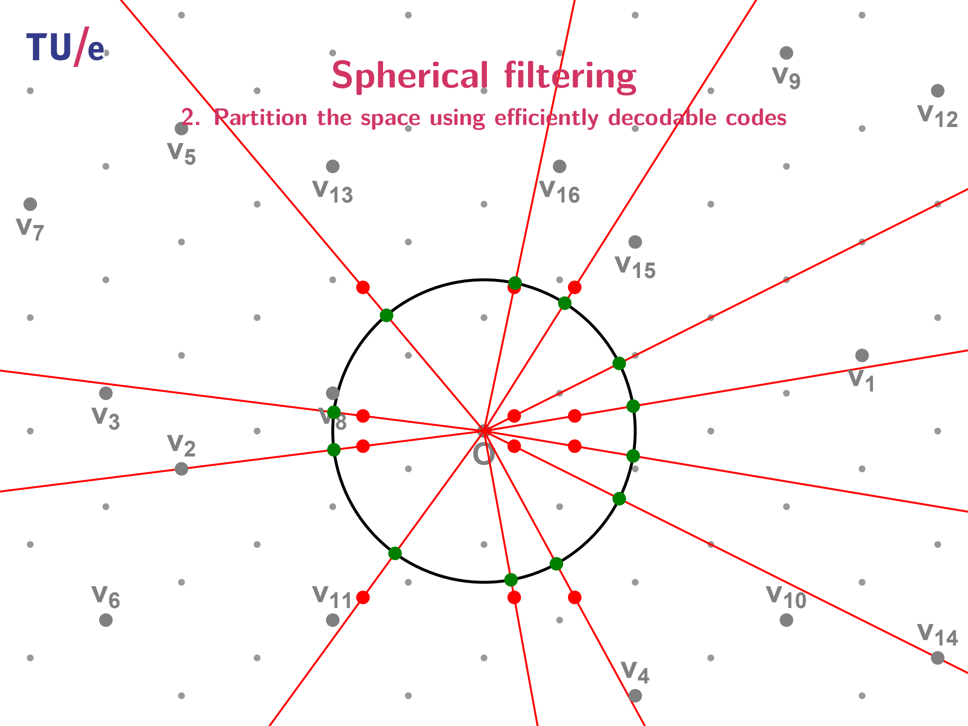

TU/e

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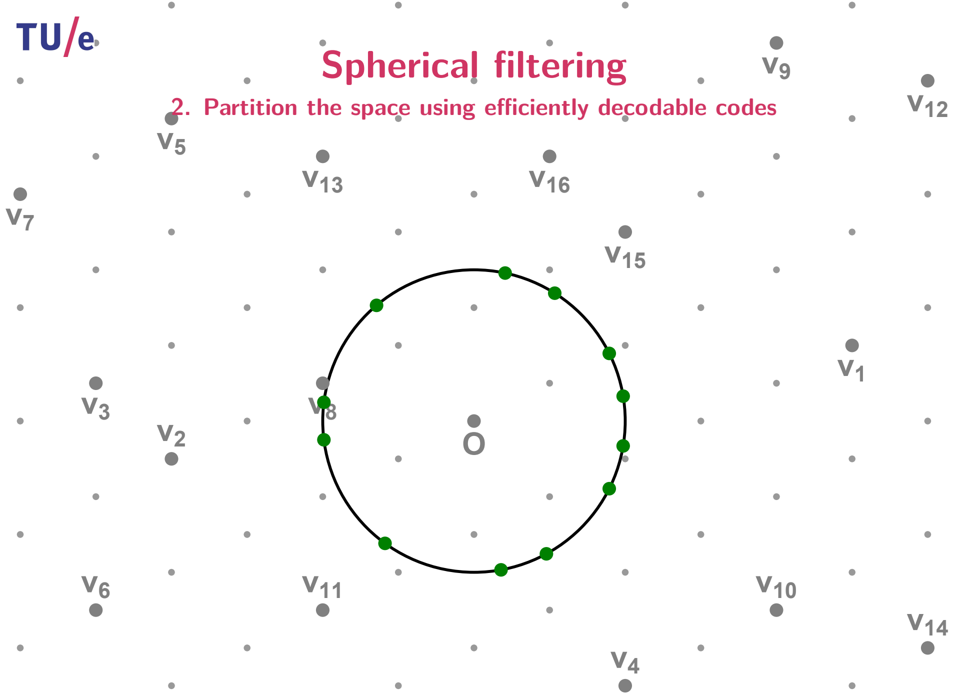

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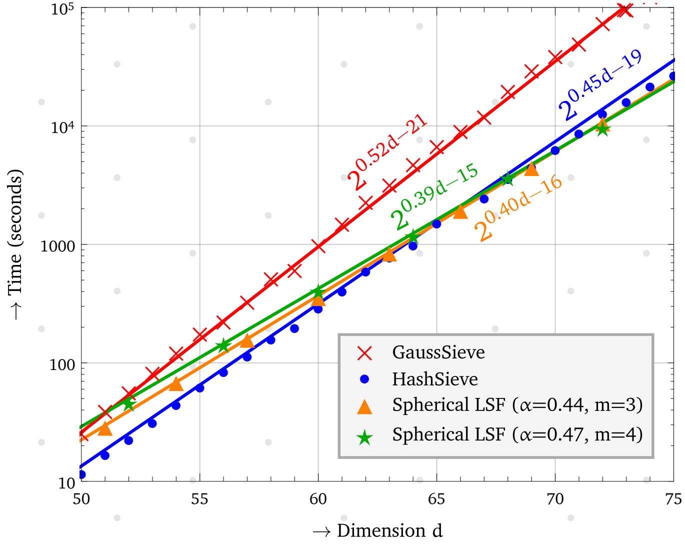

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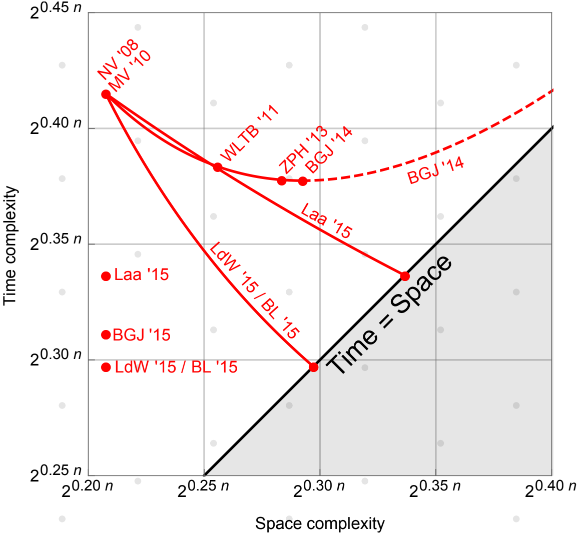

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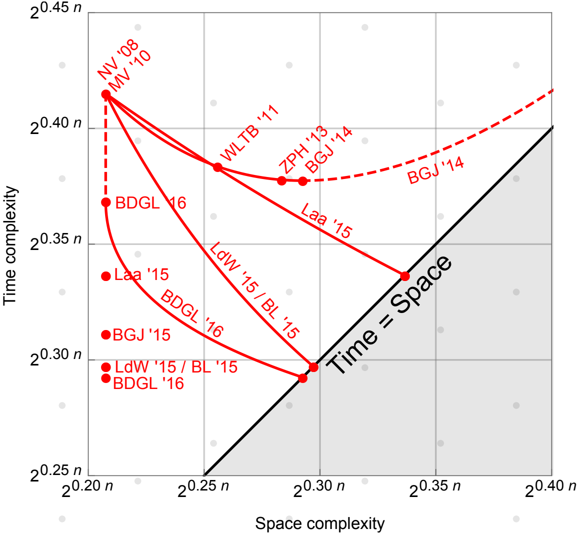

Implications for cryptography

Hardness of SVP in general

- Classically, dimension n costs $\approx 2^{0.29n}$ time and space
- Quantumly, dimension n costs $\approx 2^{0.25n}$ time and space [LMP15]

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- Sieving with cross-polytope LSH improves by factor $\approx n$
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Many challenges still remain!

Questions

[vdP'12]