Sieving for shortest vectors in lattices using (angular) locality-sensitive hashing

Thijs Laarhoven

mail@thijs.com
http://www.thijs.com/

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(August 17, 2015)

Lattices

What is a lattice?

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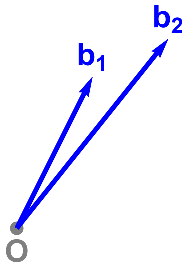

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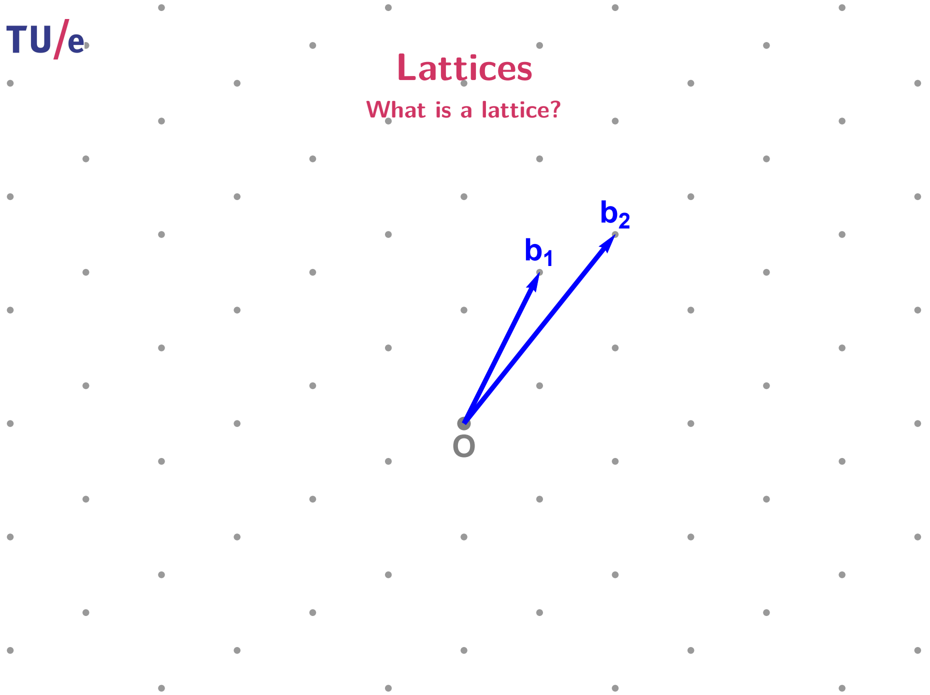
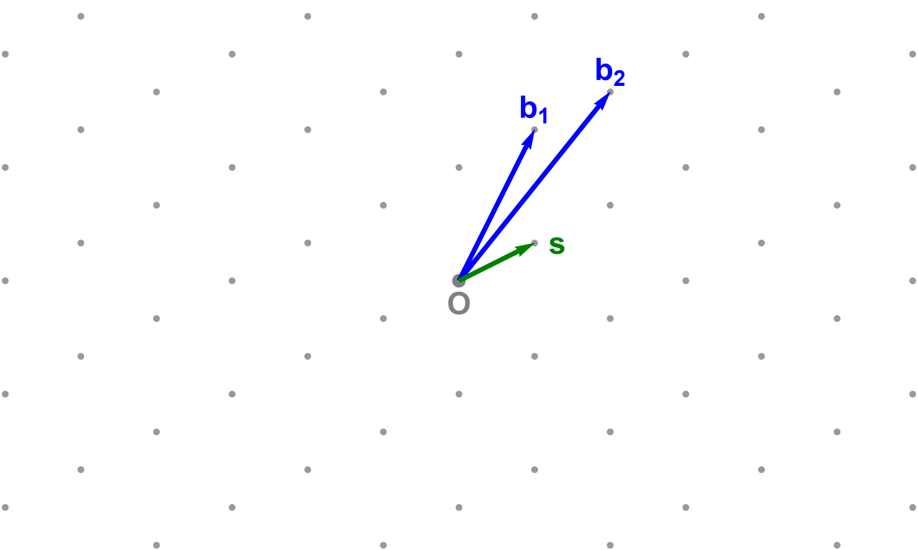


Lattices

Shortest Vector Problem (SVP)



Lattices

Exact SVP algorithms

	Algorithm	$\log_2(\text{Time})$	$\log_2(\text{Space})$
Provable SVP	Enumeration [Poh81, Kan83, . . . , GNR10]	$\Omega(n \log n)$	$O(\log n)$
	AKS-sieve [AKS01, NV08, MV10, HPS11]	$3.398n$	$1.985n$
	ListSieve [MV10, MDB14]	$3.199n$	$1.327n$
	AKS-sieve-birthday [PS09, HPS11]	$2.648n$	$1.324n$
	ListSieve-birthday [PS09]	$2.465n$	$1.233n$
	Voronoi cell algorithm [MV10b]	$2.000n$	$1.000n$
	Discrete Gaussian sampling [ADRS15]	$1.000n$	$1.000n$
Heuristic SVP	Nguyen-Vidick sieve [NV08]	$0.415n$	$0.208n$
	GaussSieve [MV10, . . . , IKMT14, BNvdP14]	$0.415n$?	$0.208n$
	Two-level sieve [WLTB11]	$0.384n$	$0.256n$
	Three-level sieve [ZPH13]	$0.3778n$	$0.283n$
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TU/e

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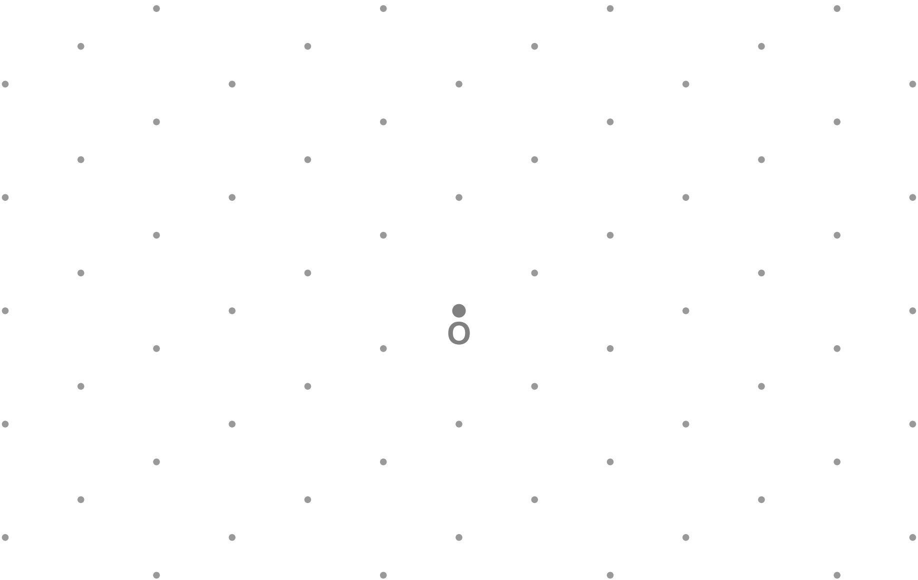
	Hyperplane LSH **[Laa15]**, [MLB15]	$0.337n$	$0.208n$
	May and Ozerov's NNS method [BGJ15]	$0.311n$	$0.208n$
	Spherical LSH [LdW15]	$0.298n$	$0.208n$
	Cross-polytope LSH [BL15]	$0.298n$	$0.208n$

Nguyen-Vidick sieve



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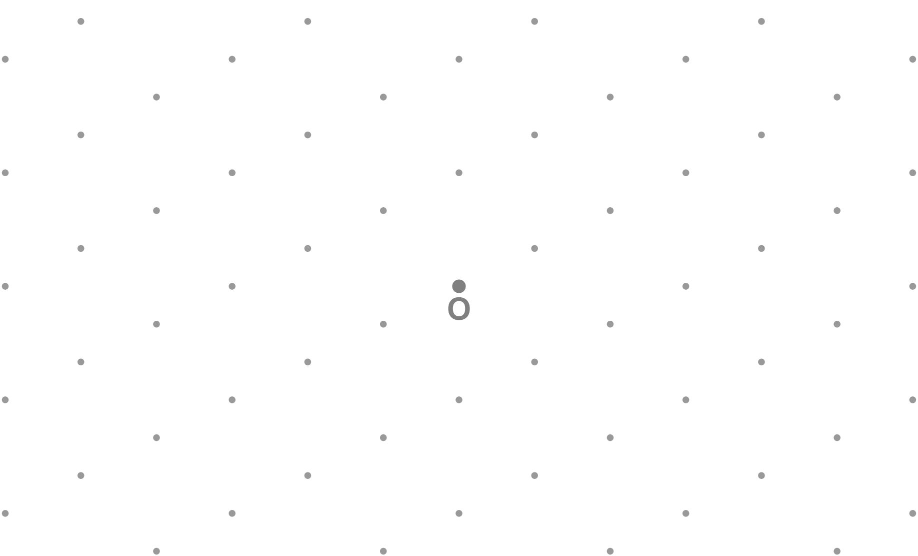

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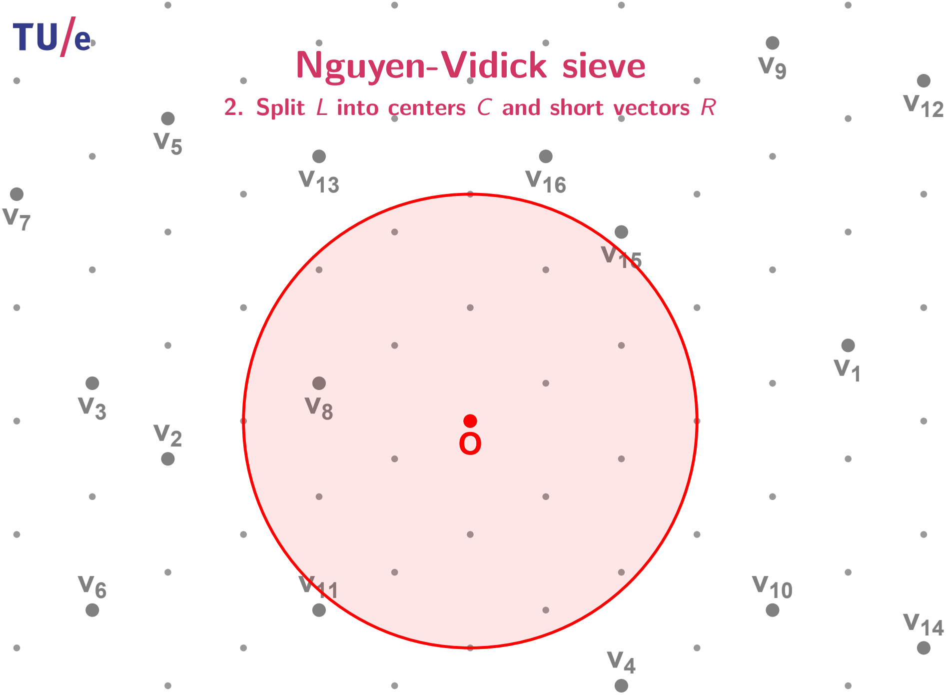

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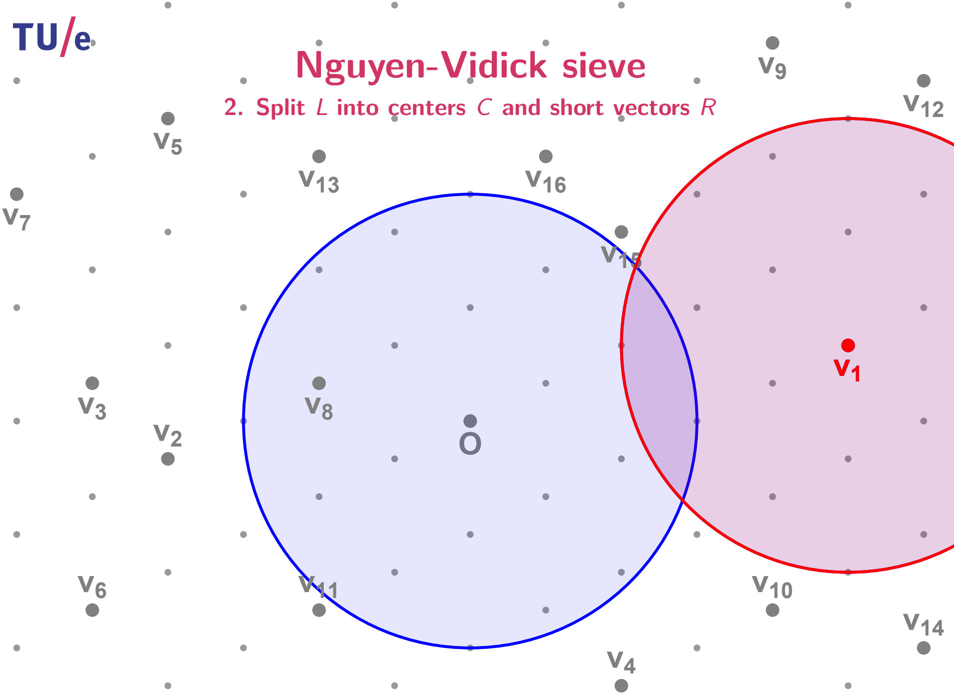

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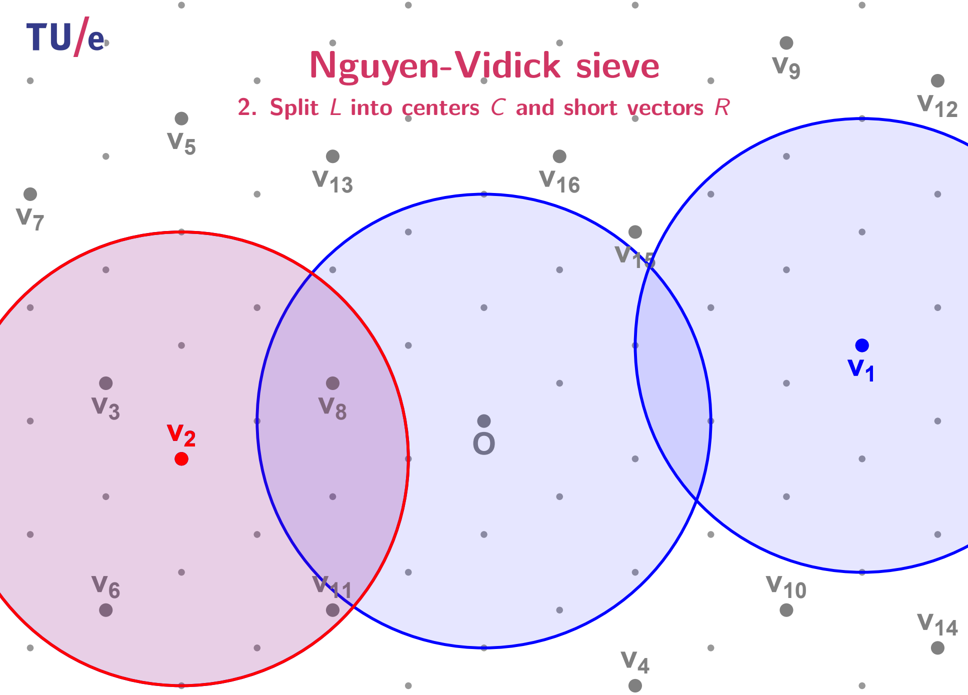

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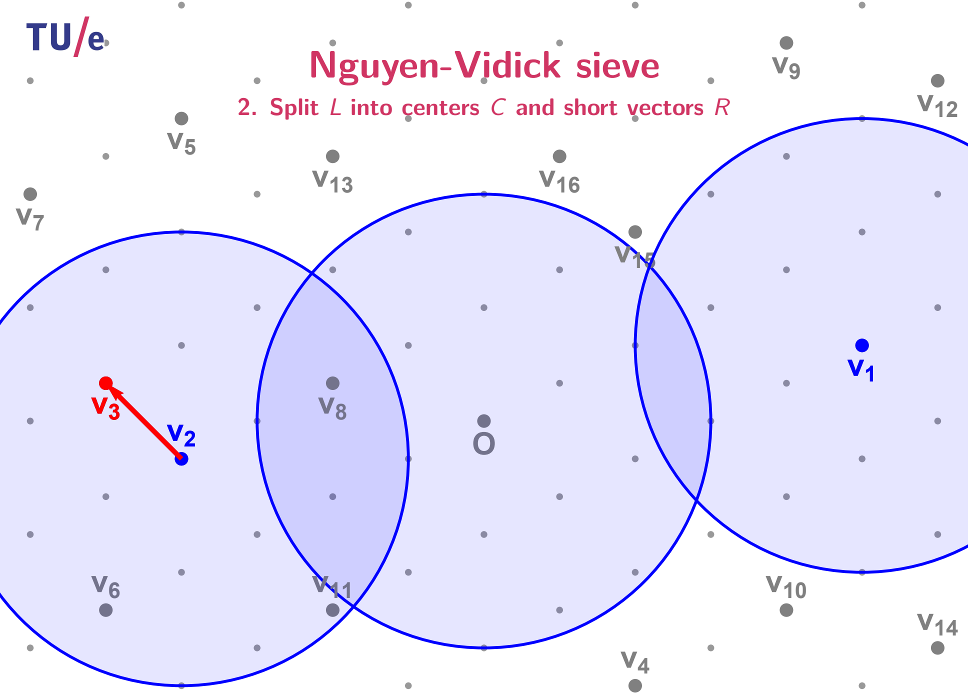

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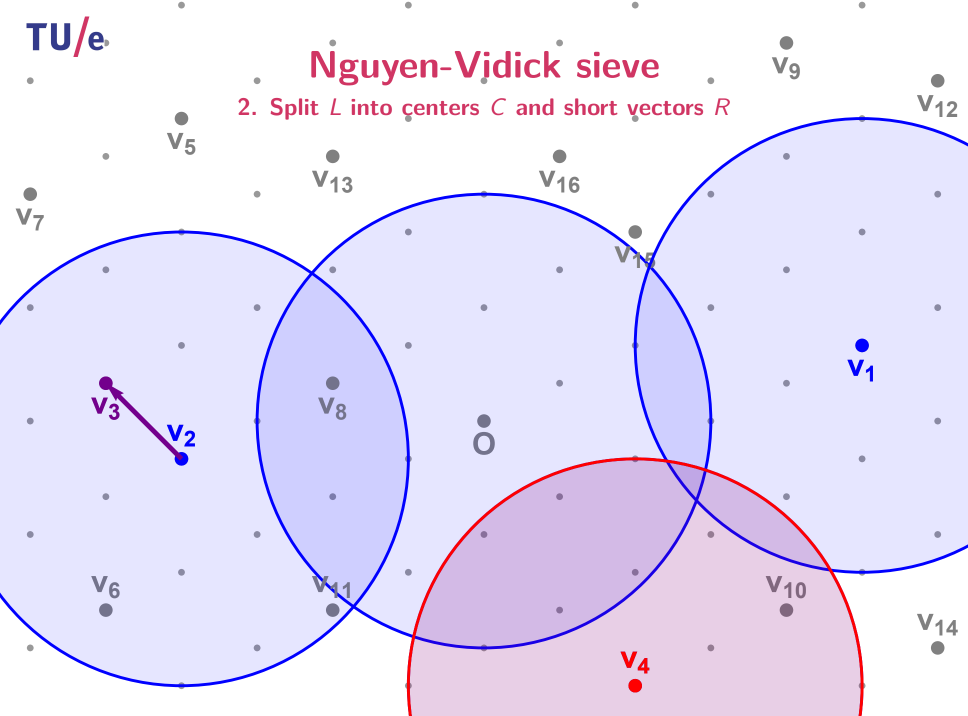

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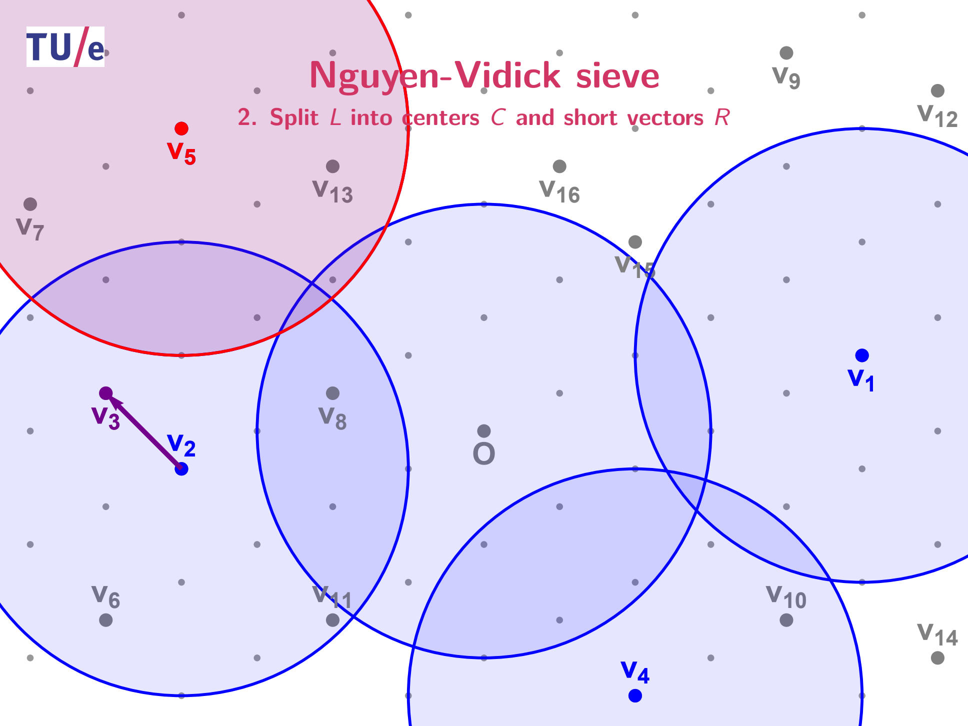

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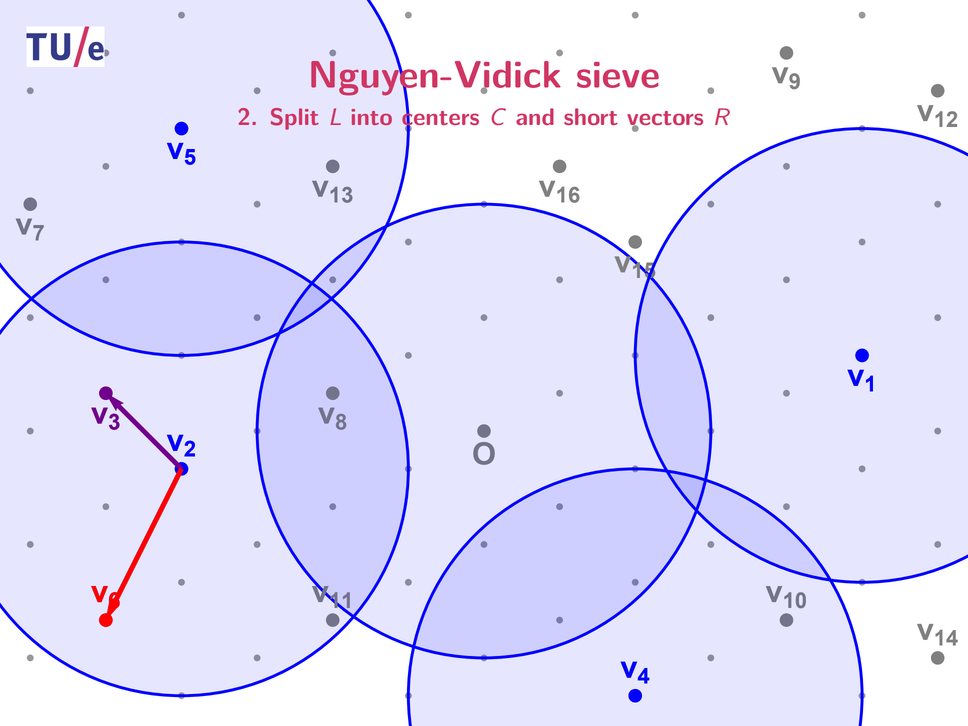

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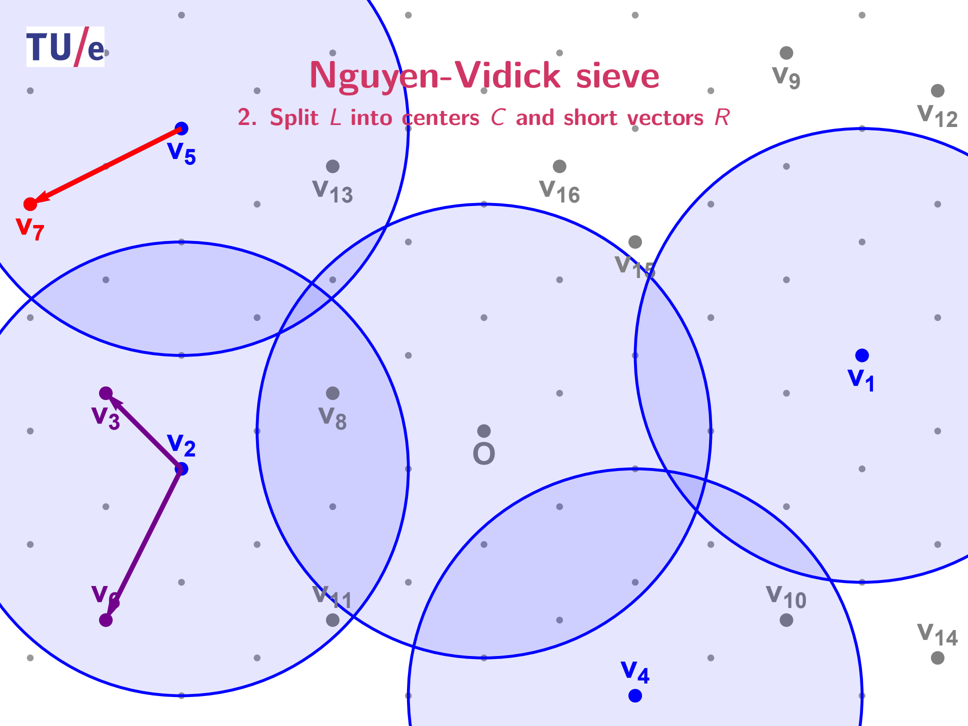

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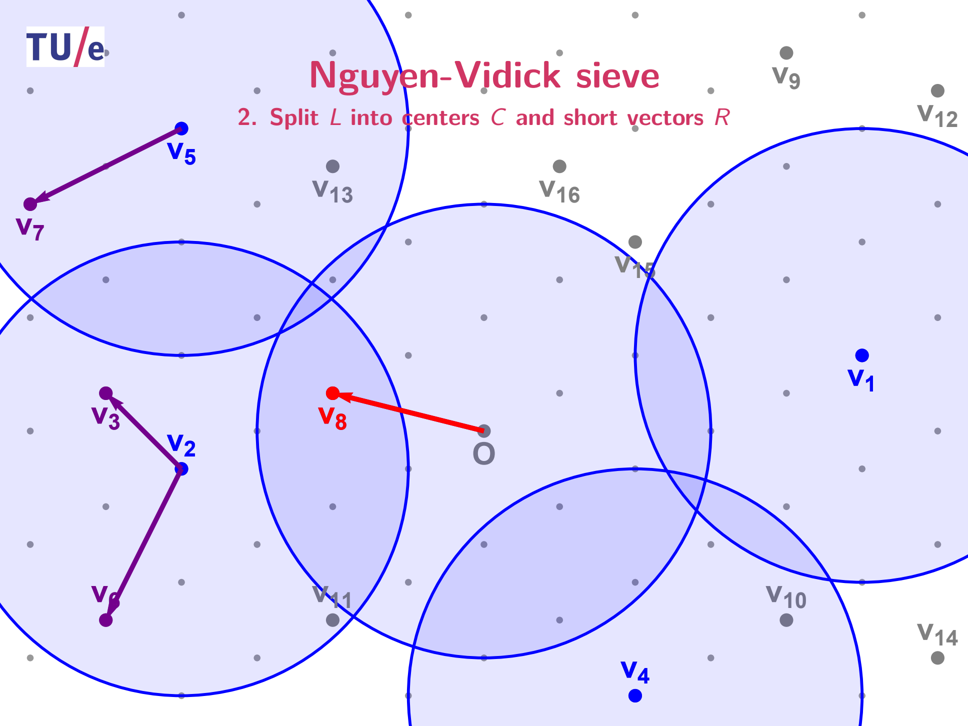

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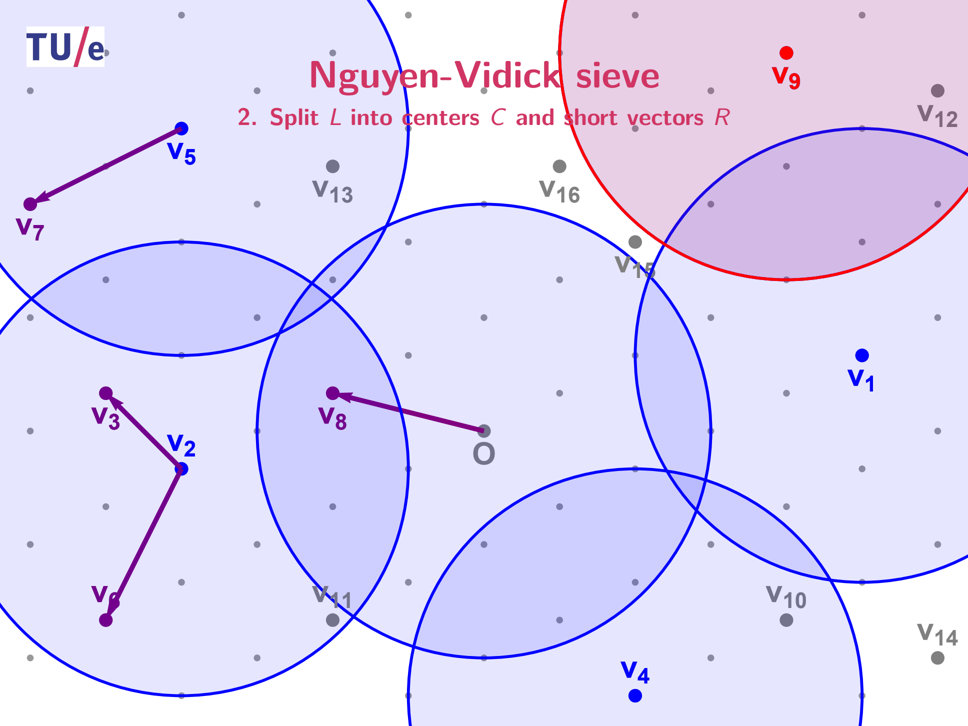

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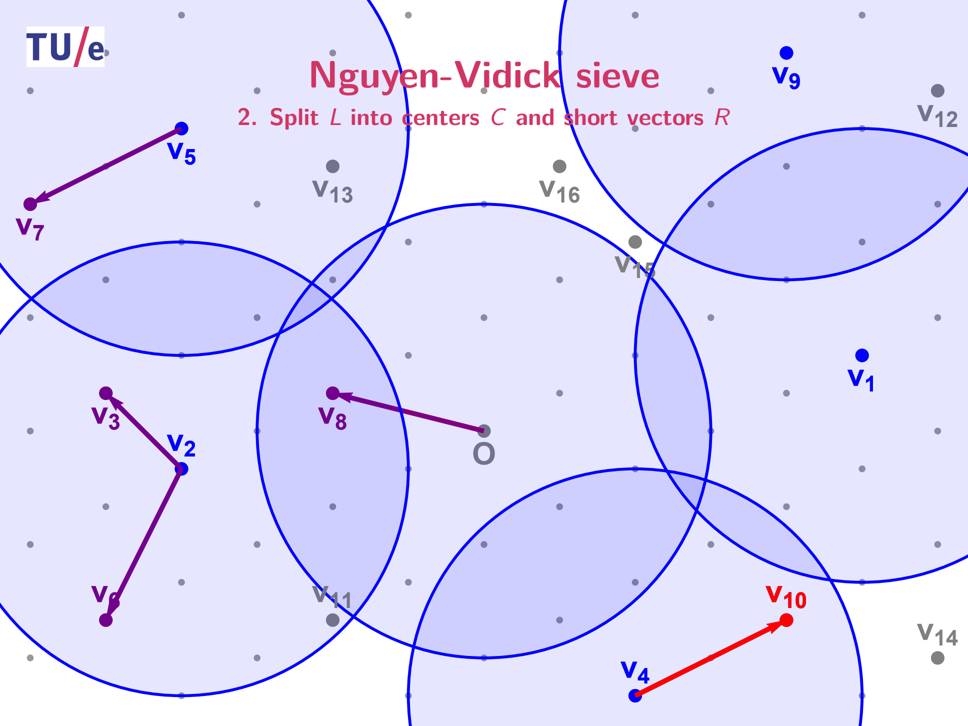

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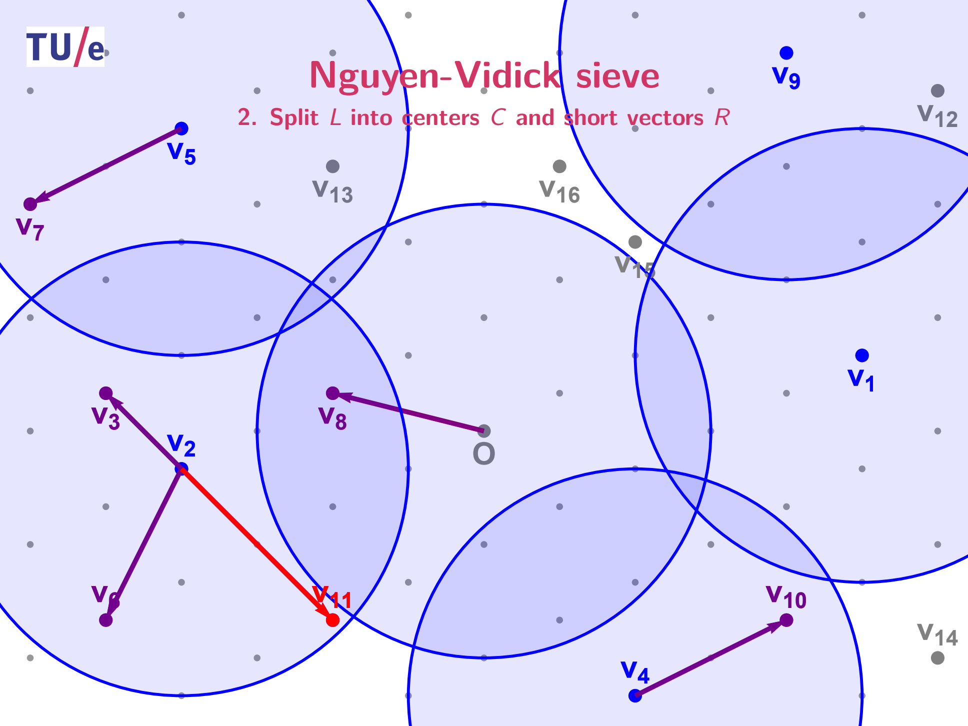

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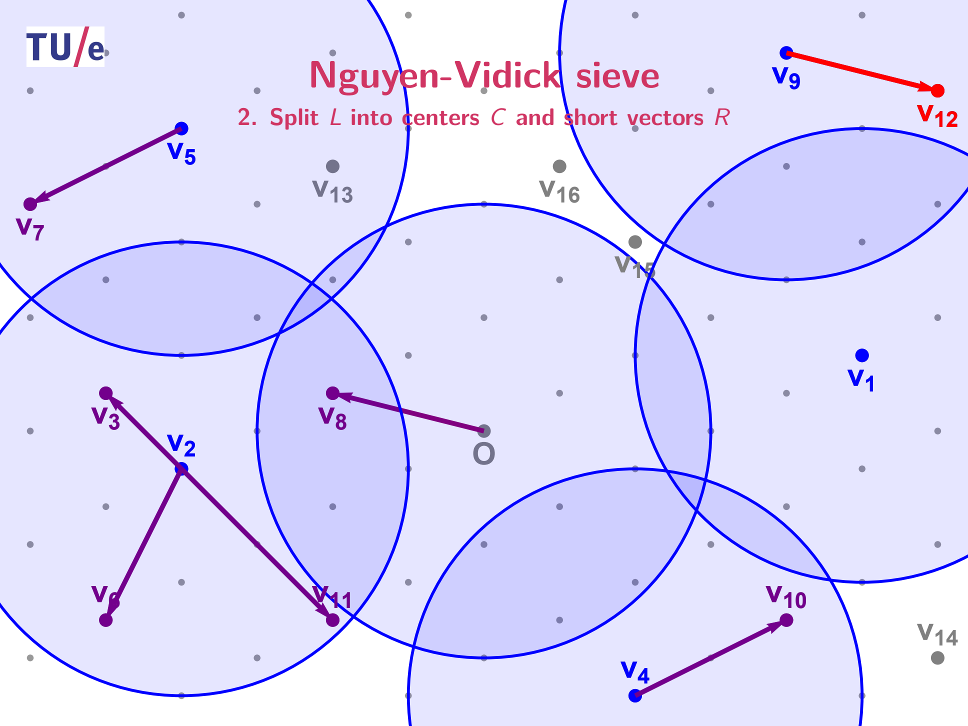

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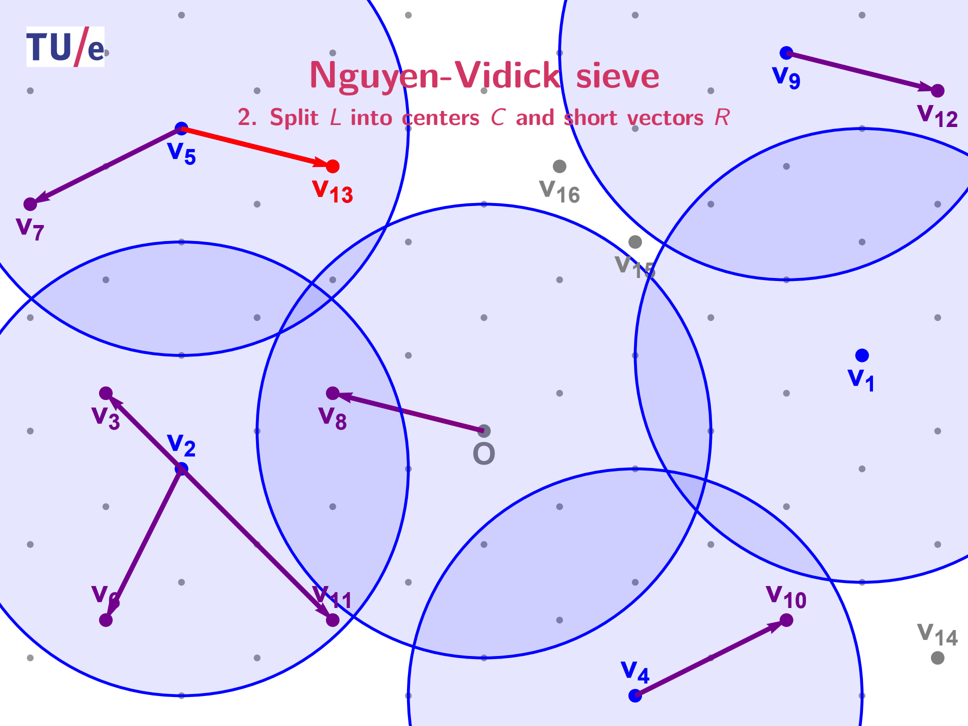

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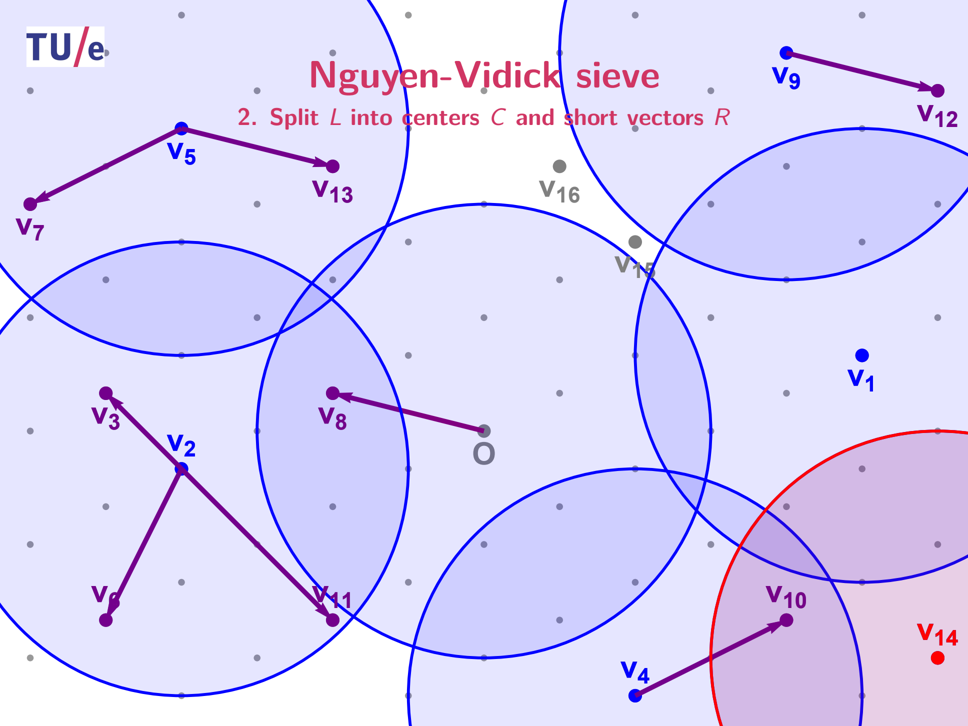

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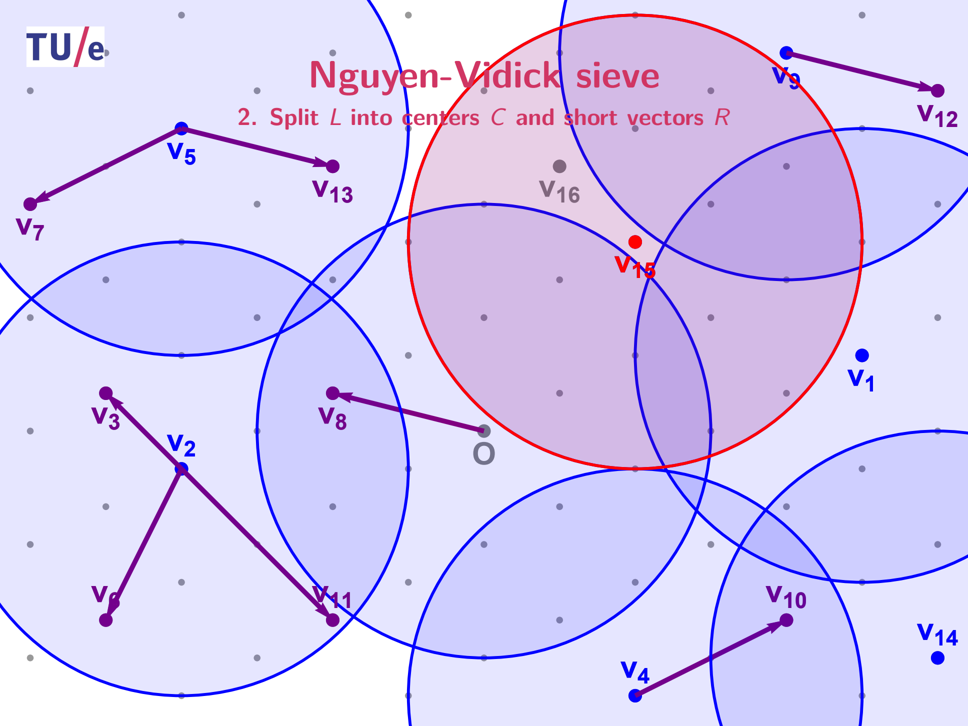

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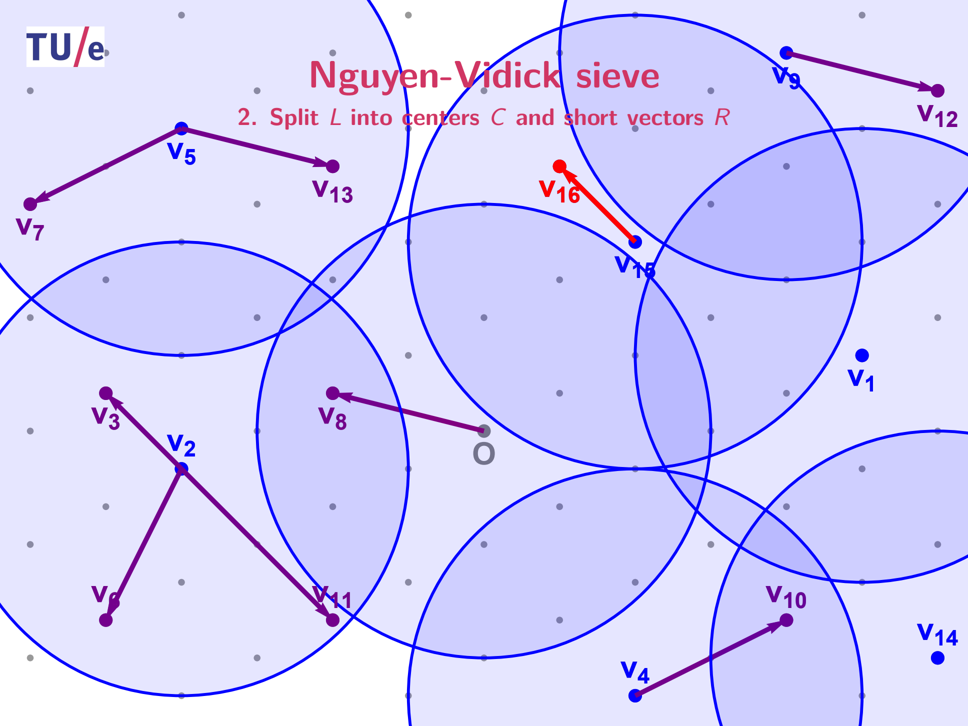

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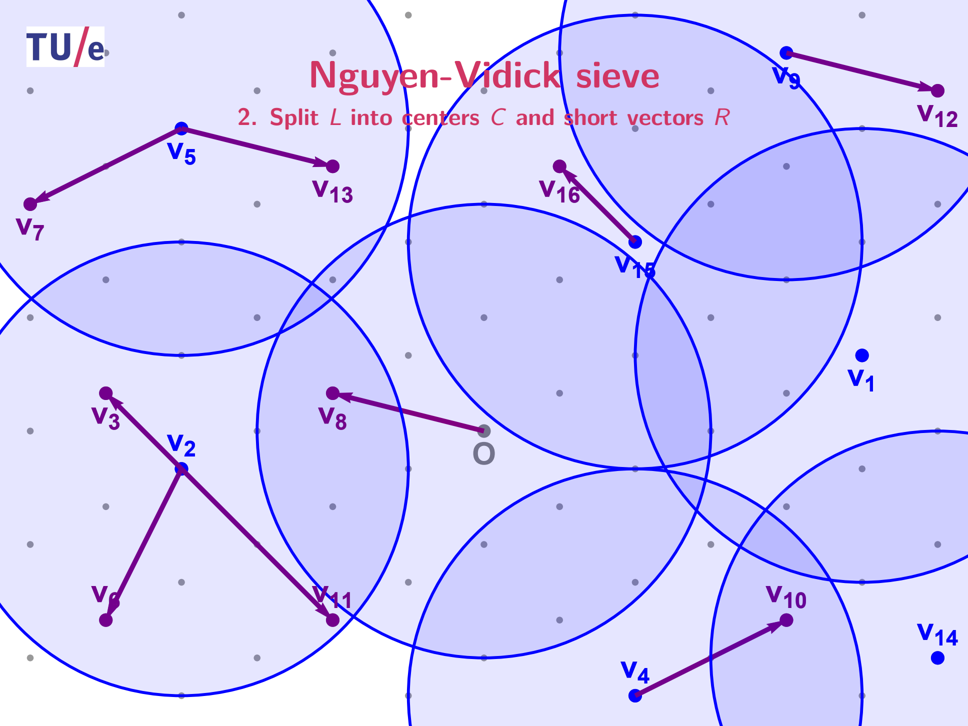

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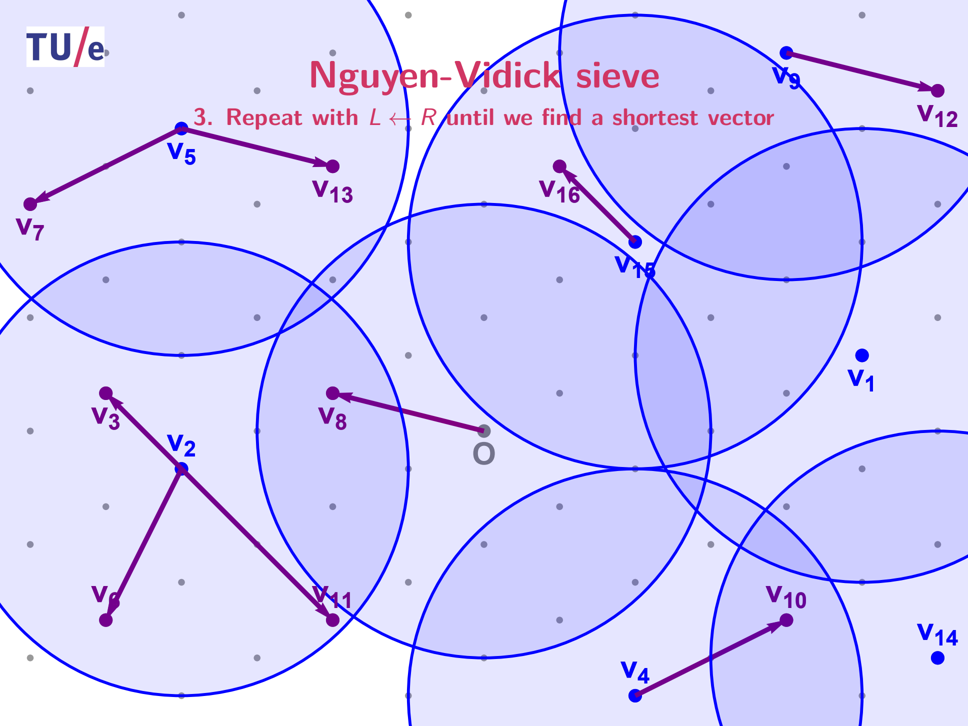

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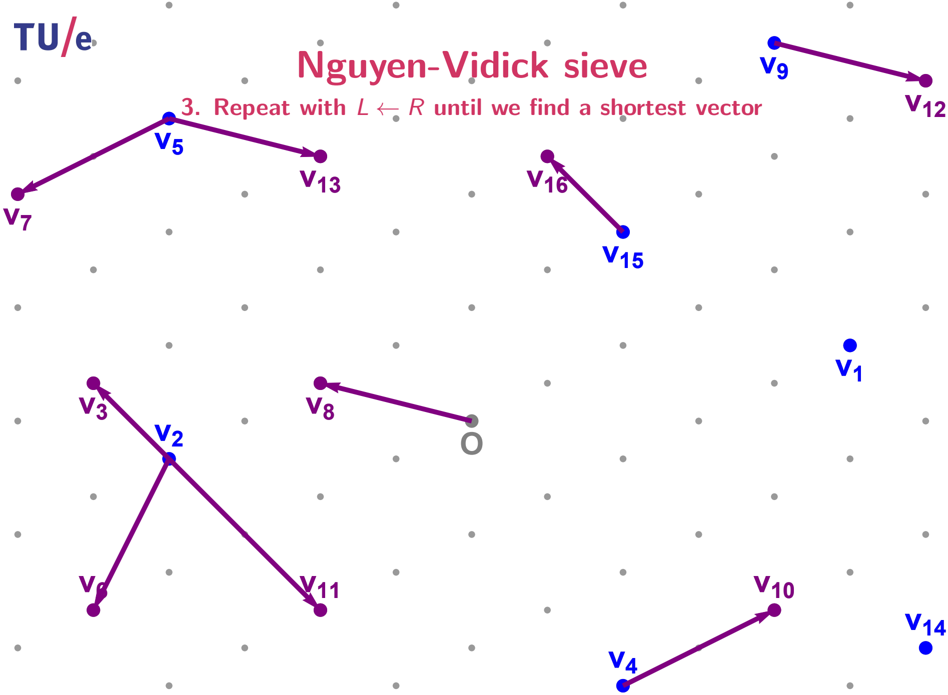

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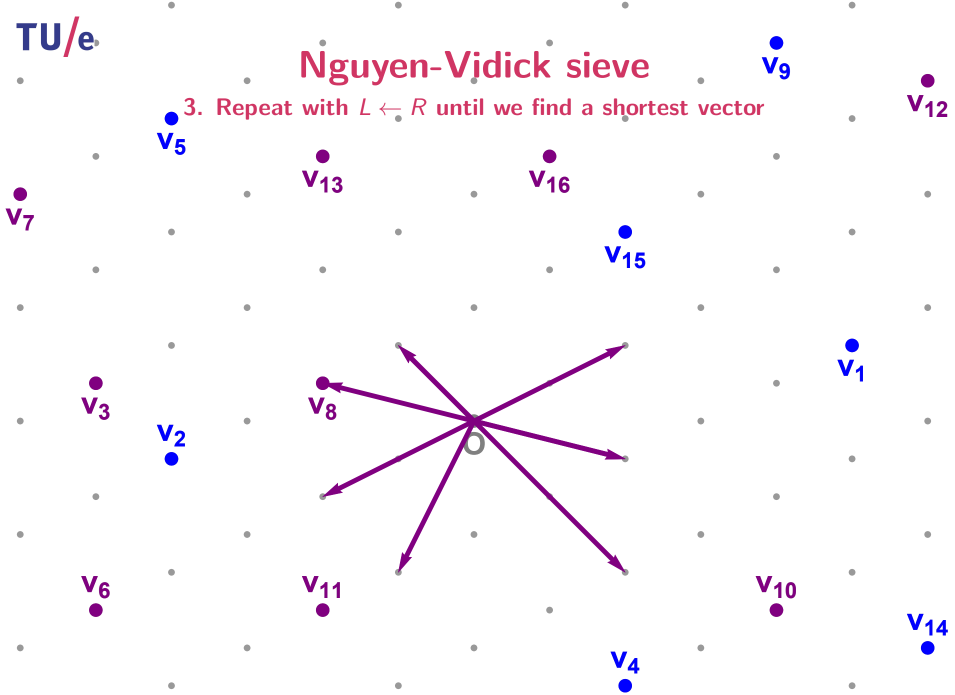

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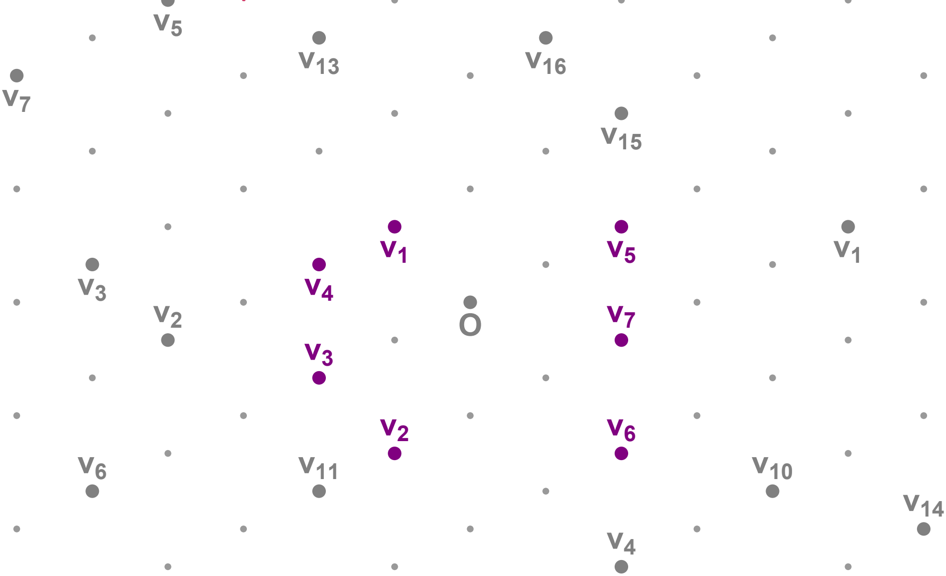

Nguyen-Vidick sieve

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- Space complexity: $(\sqrt{4/3})^n \approx 2^{0.208n+o(n)}$ vectors
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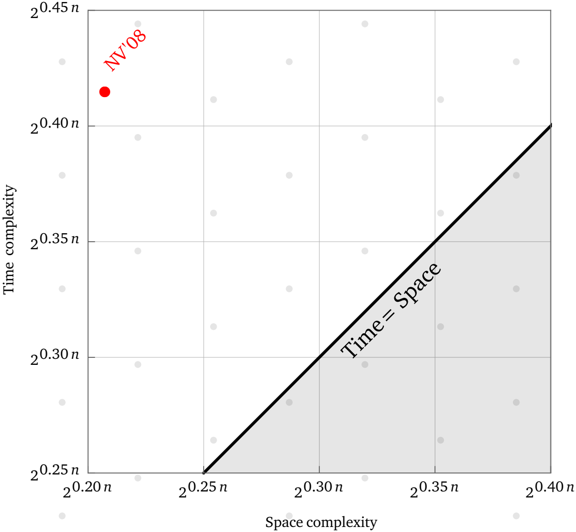
Theorem (Nguyen and Vidick, J. Math. Crypt. '08)

The Nguyen-Vidick sieve heuristically solves SVP in time $2^{0.415n+o(n)}$ and space $2^{0.208n+o(n)}$.

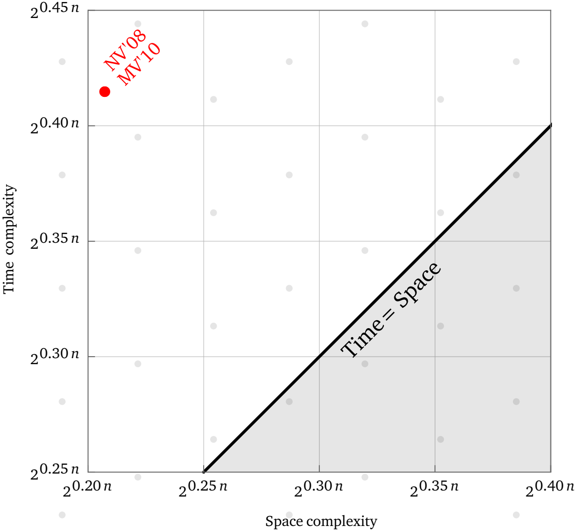
Nguyen-Vidick sieve

Space/time trade-off



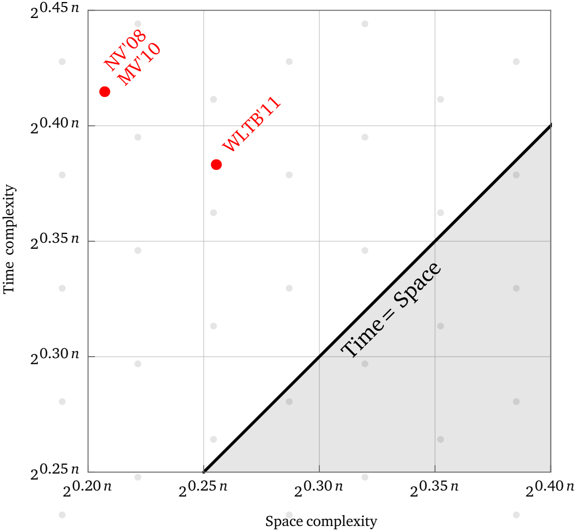
GaussSieve

Space/time trade-off



Two-level sieve

Space/time trade-off



Three-level sieve

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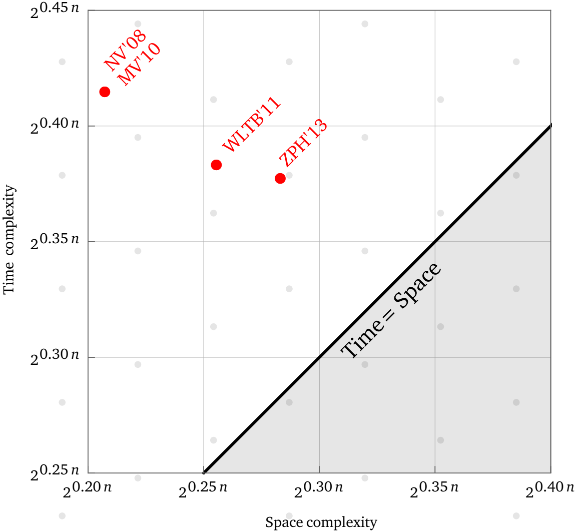
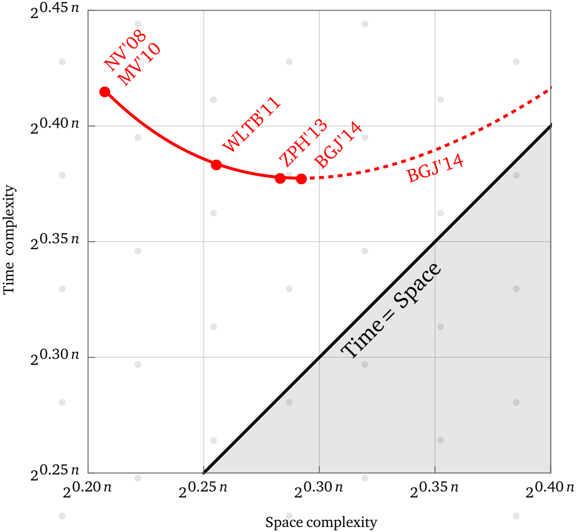


Overlattice sieving

Space/time trade-off



Hyperplane LSH

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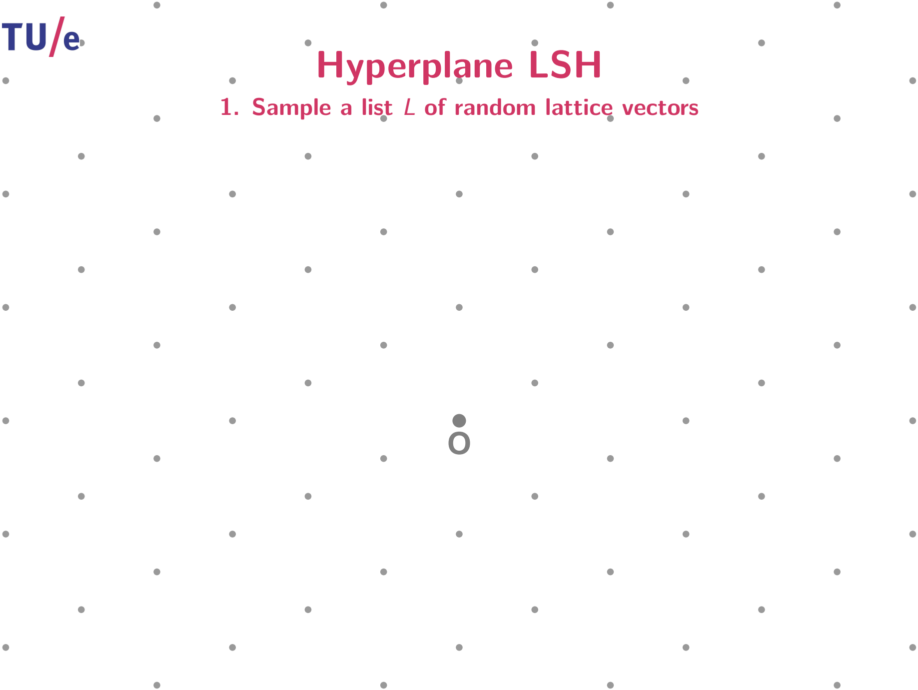

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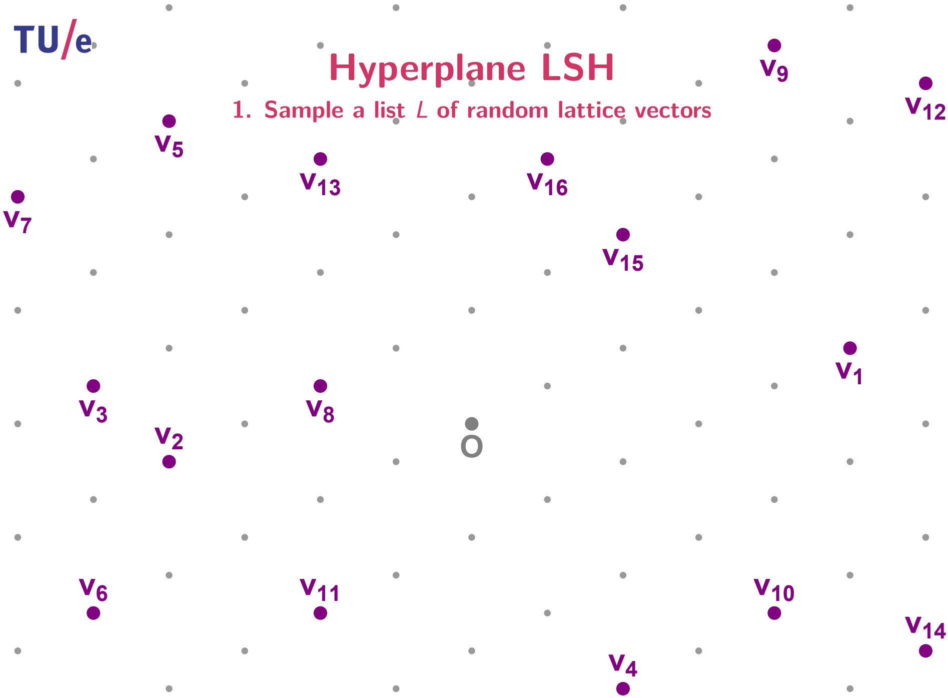

Hyperplane LSH

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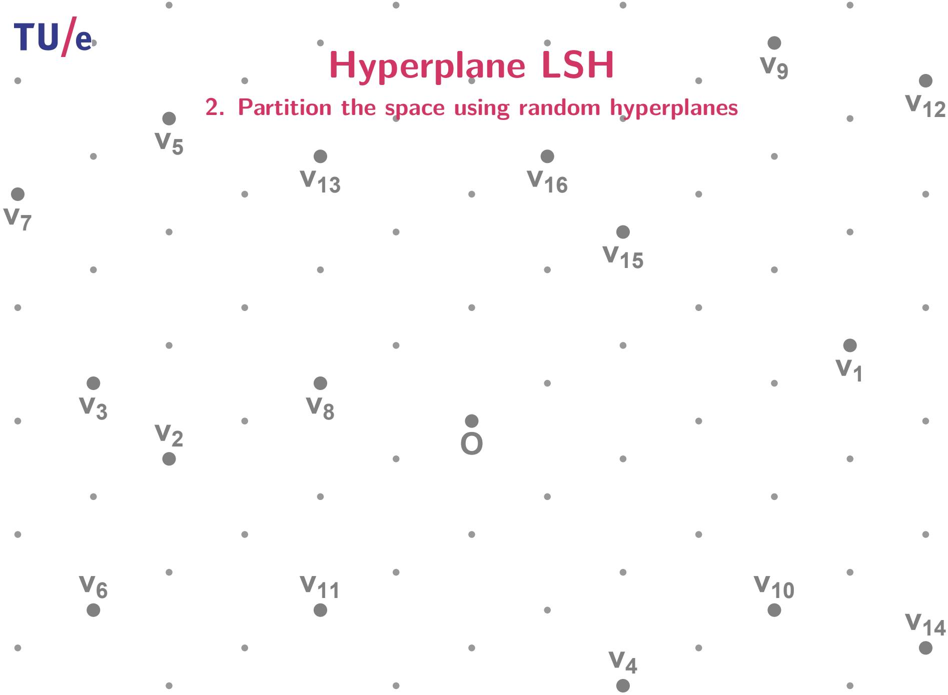

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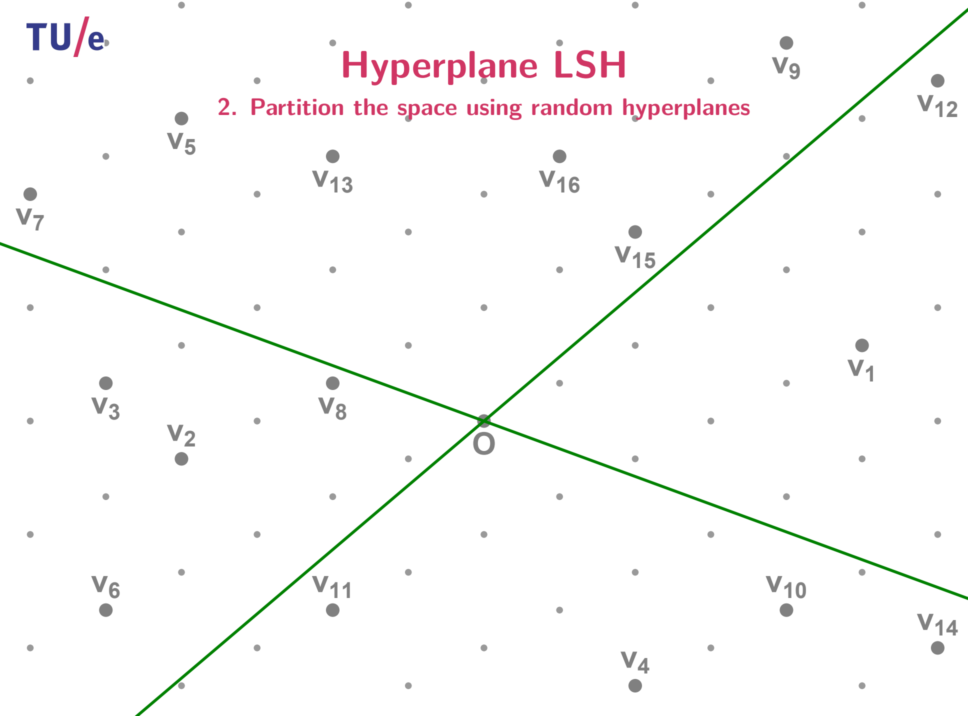
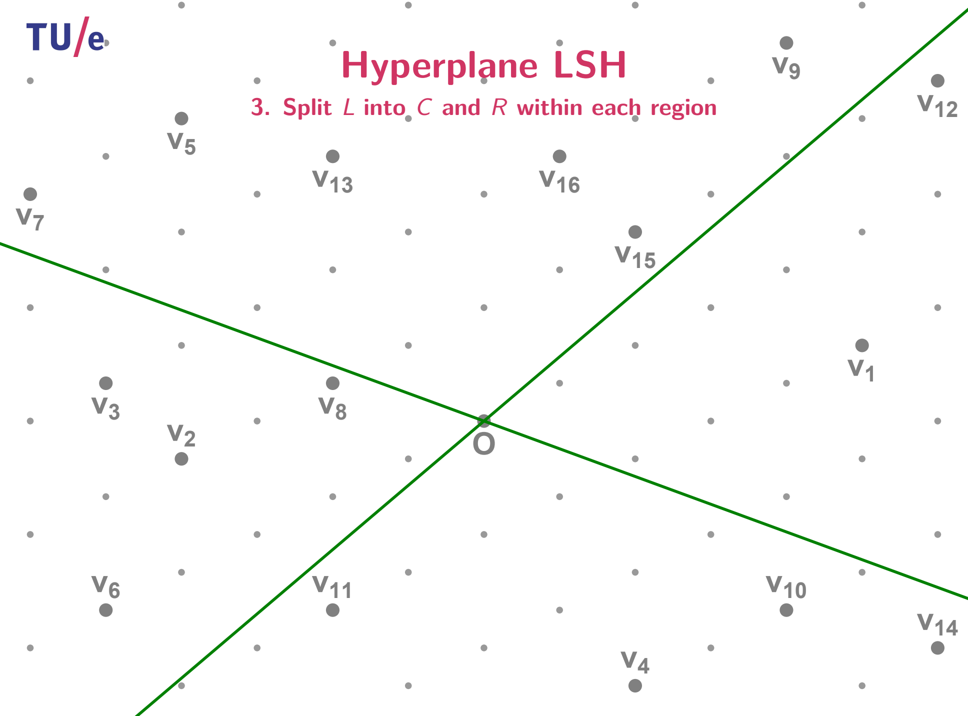


Hyperplane LSH

3. Split L into C and R within each region



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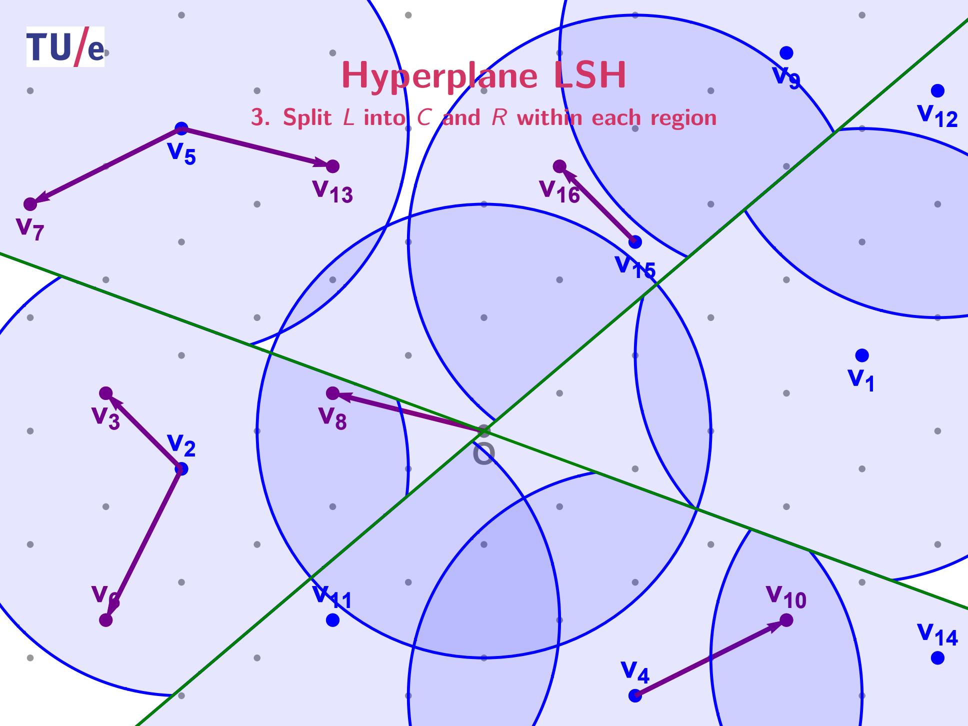

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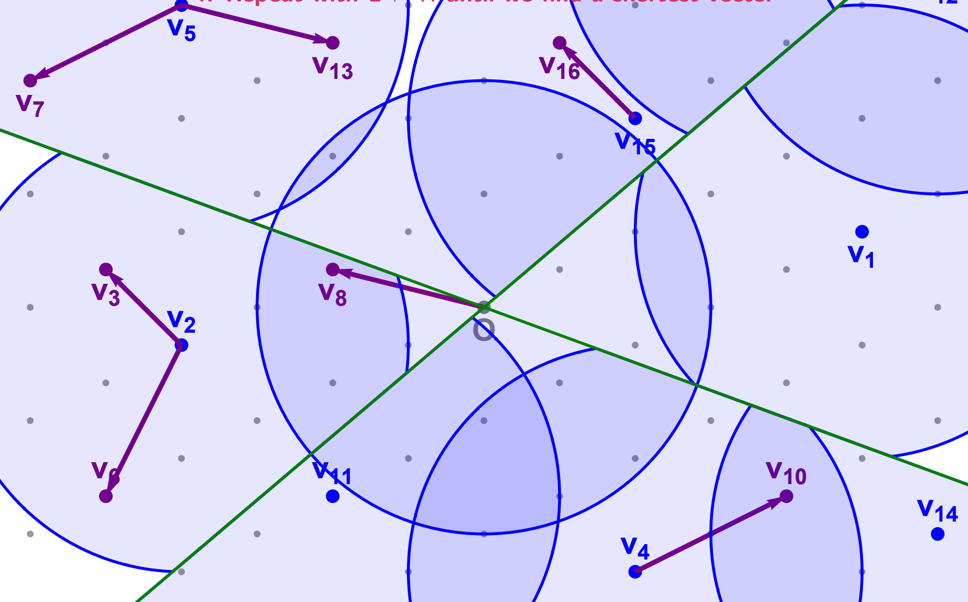

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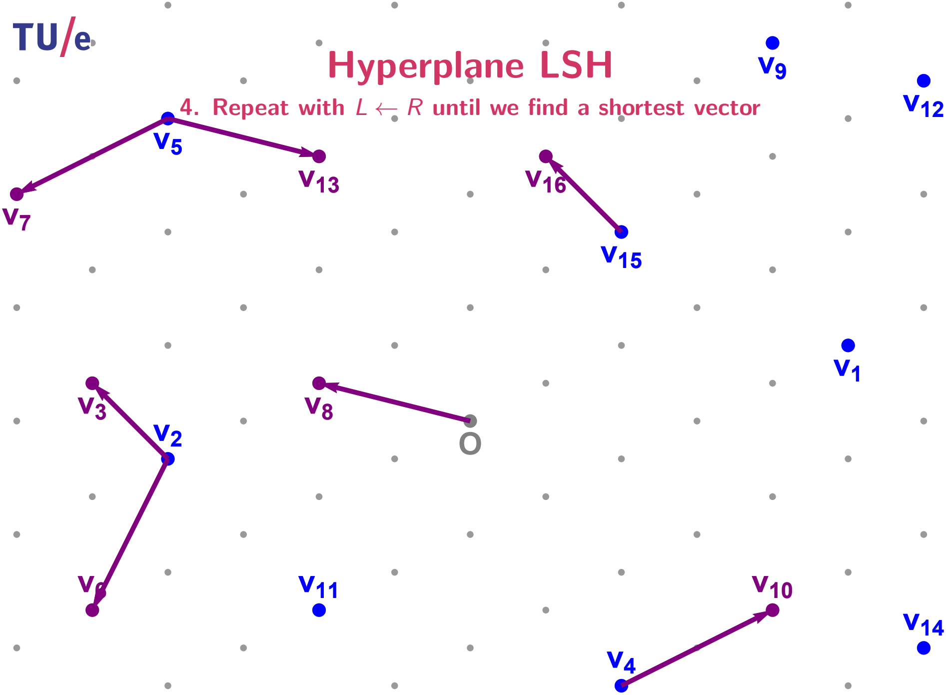

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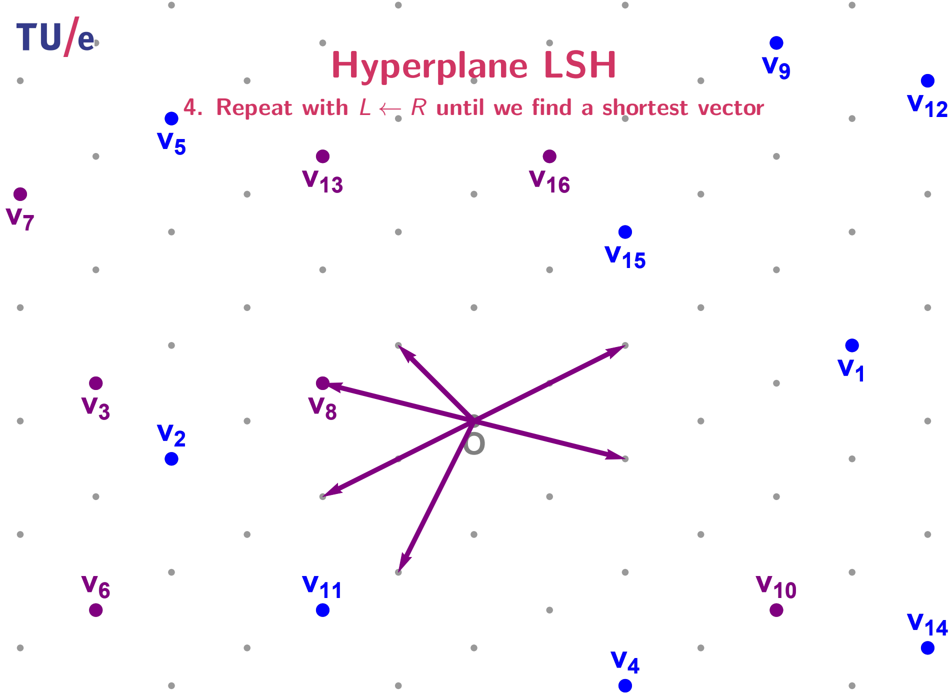

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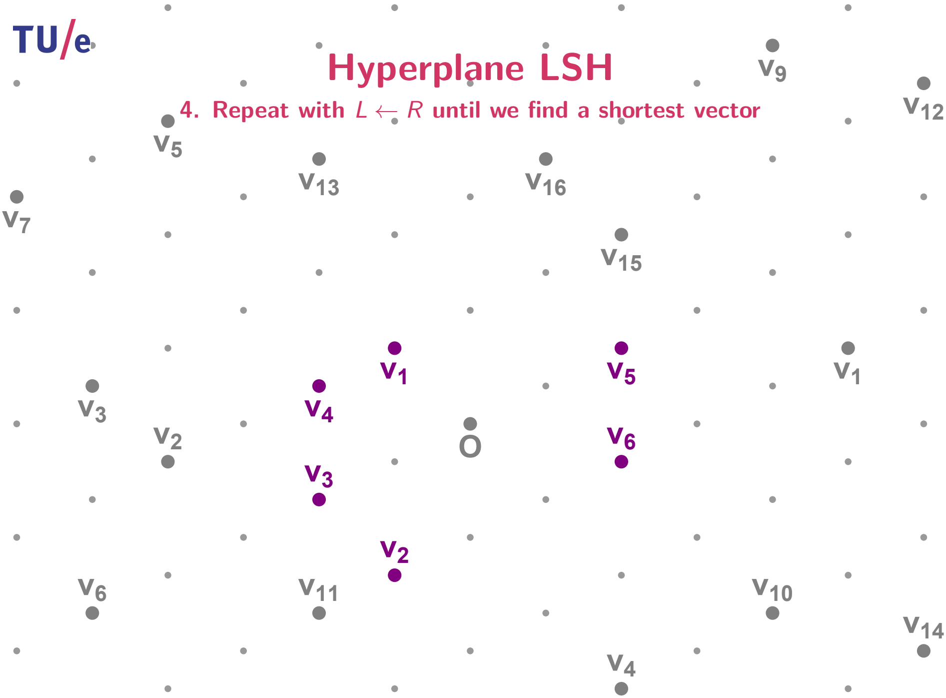

Hyperplane LSH

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  - $k = O(n)$: Number of hyperplanes, leading to 2^k regions
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  - Number of vectors: $2^{0.208n+o(n)}$
  - Number of hash tables: $2^{0.129n+o(n)}$
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Theorem

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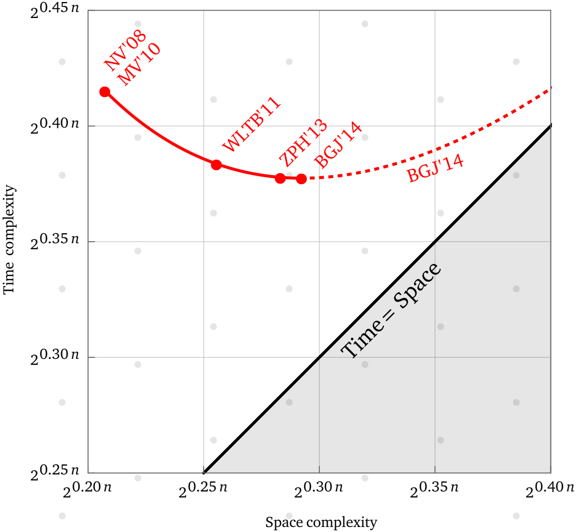
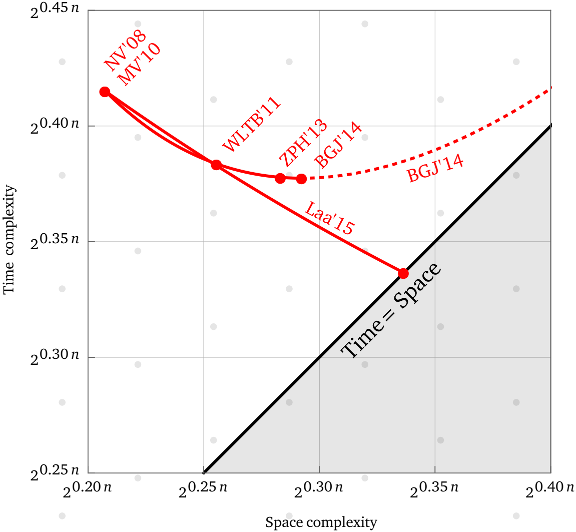
Space/time trade-off



Hyperplane LSH

Space/time trade-off



Hyperplane LSH

1. Sample a list L of random lattice vectors

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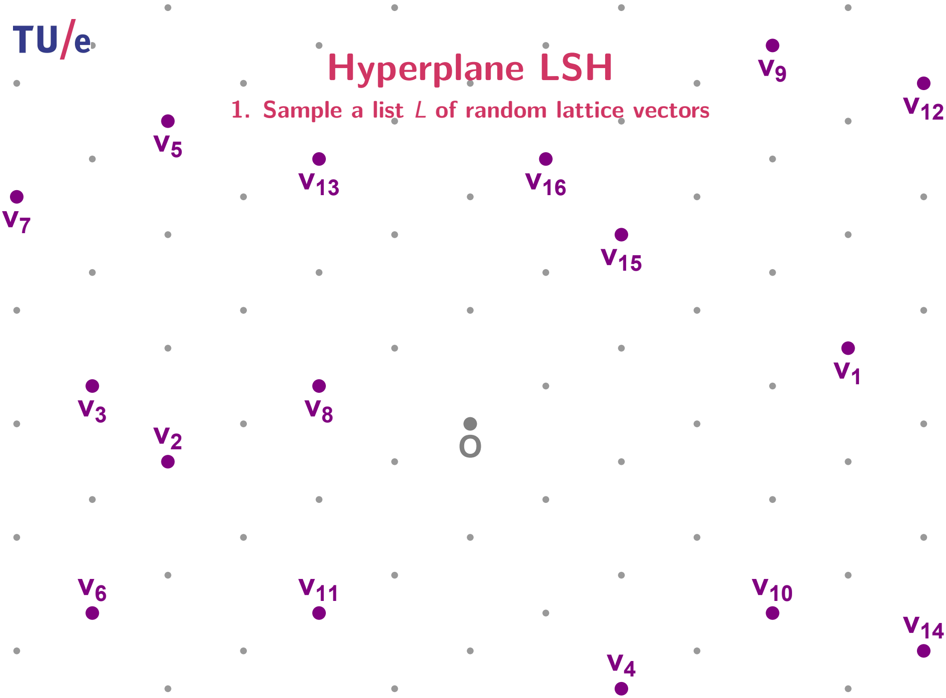
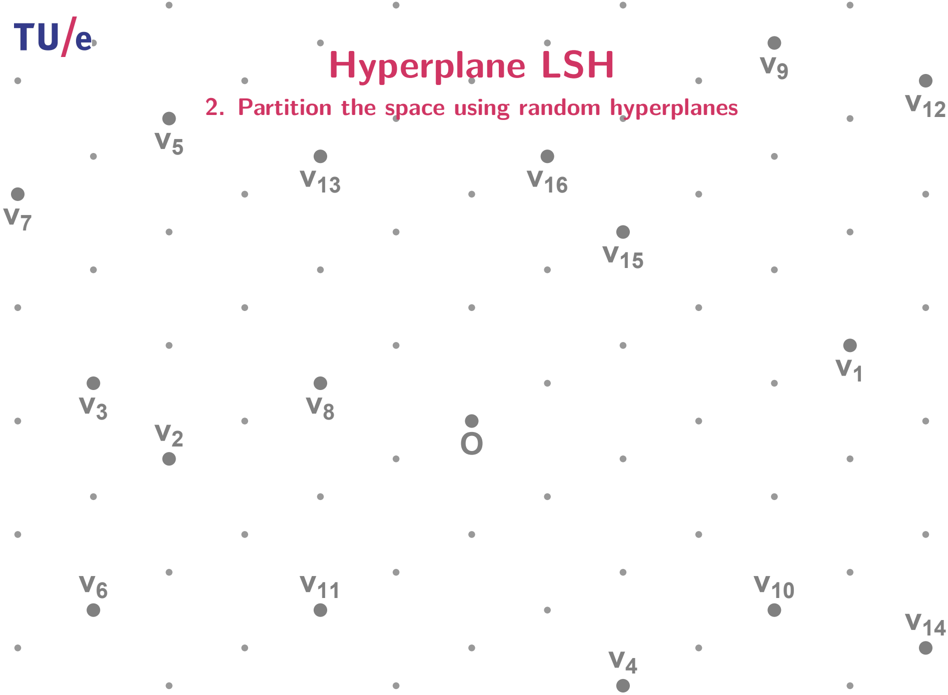


Hyperplane LSH

2. Partition the space using random hyperplanes



Hyperplane LSH

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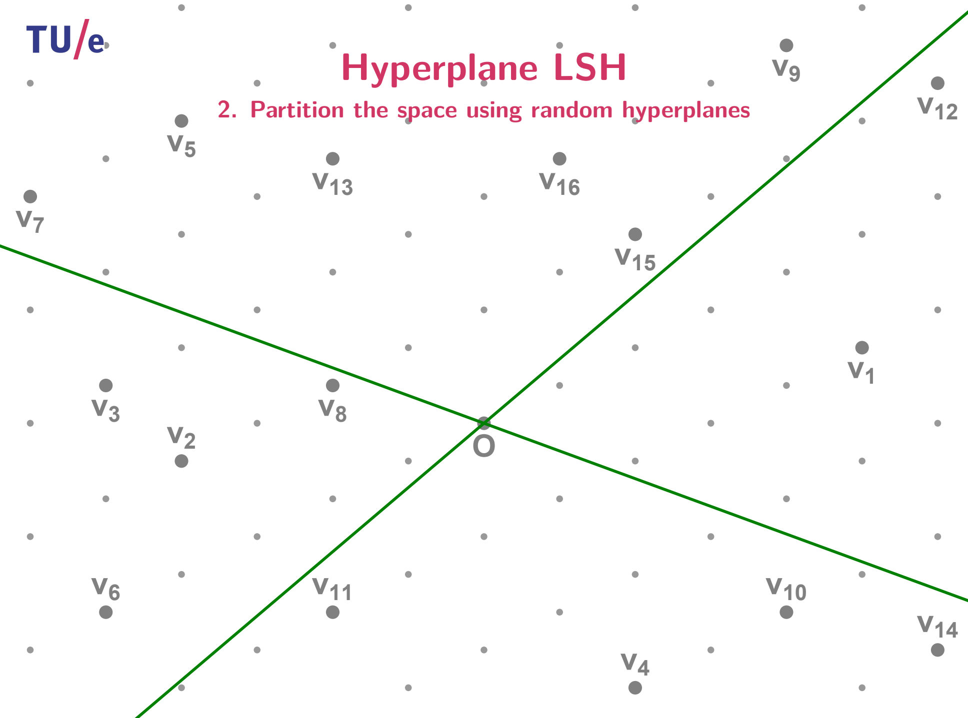
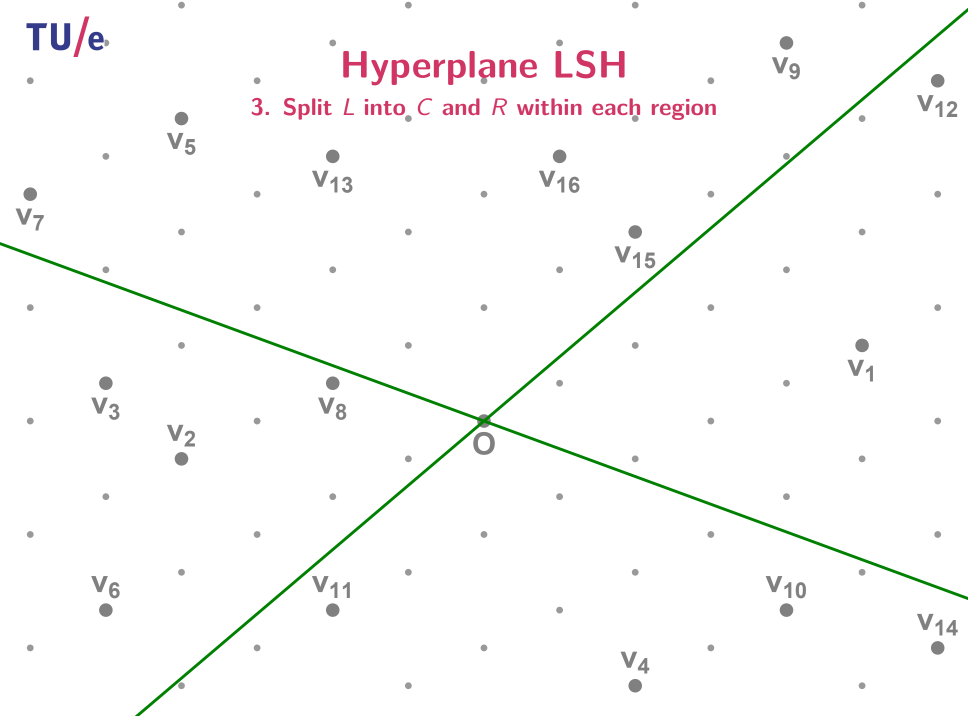


TU/e

Hyperplane LSH

3. Split L into C and R within each region



TU/e

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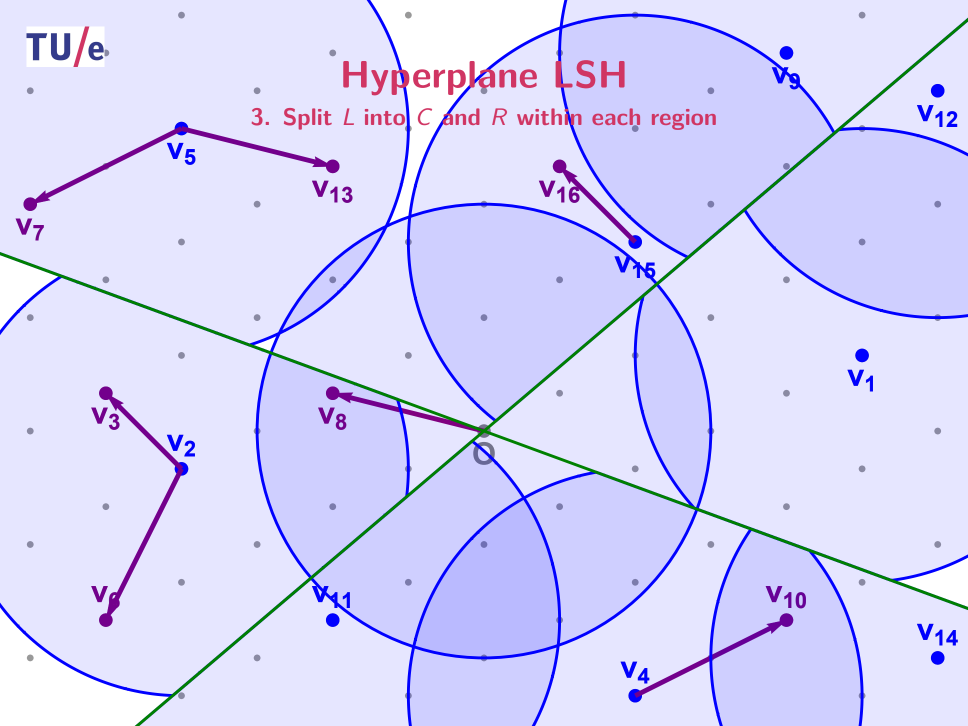
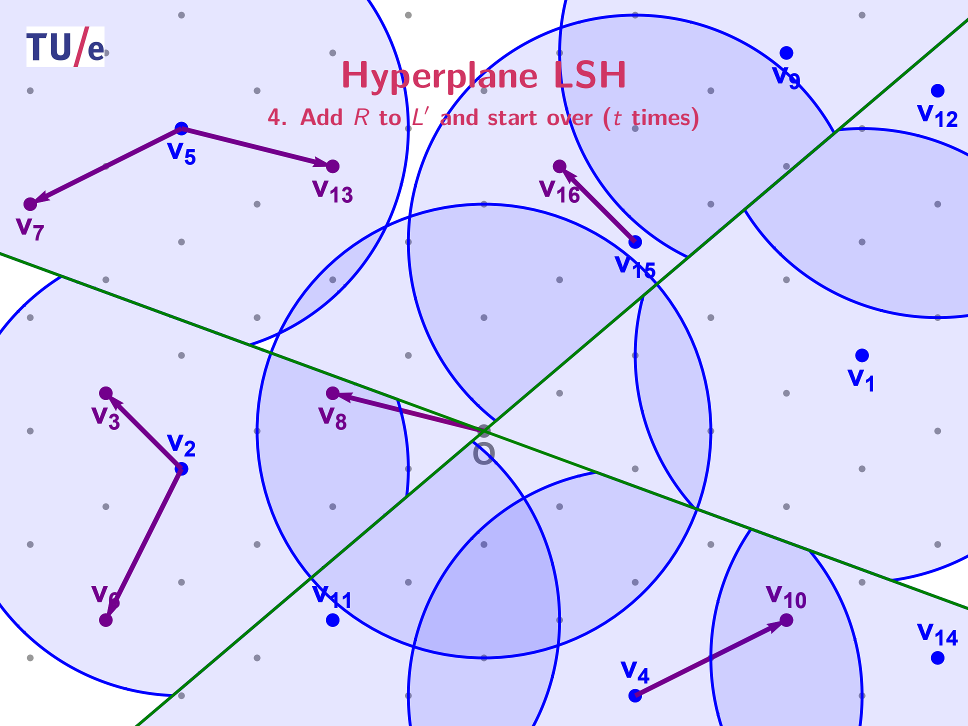


TU/e

Hyperplane LSH

4. Add R to L' and start over (t times)



Hyperplane LSH

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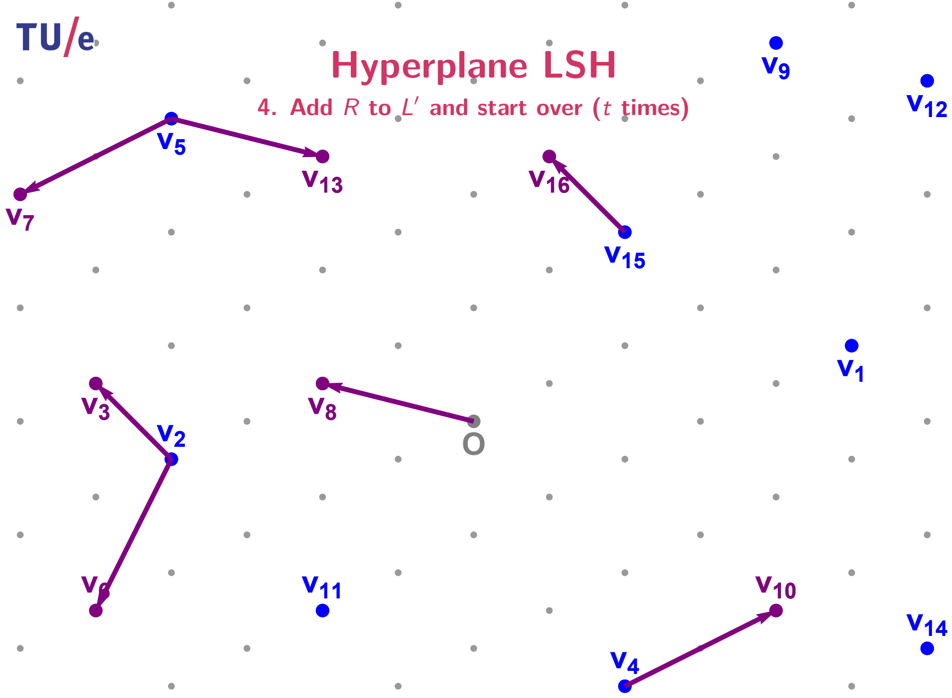

TU/e

Hyperplane LSH

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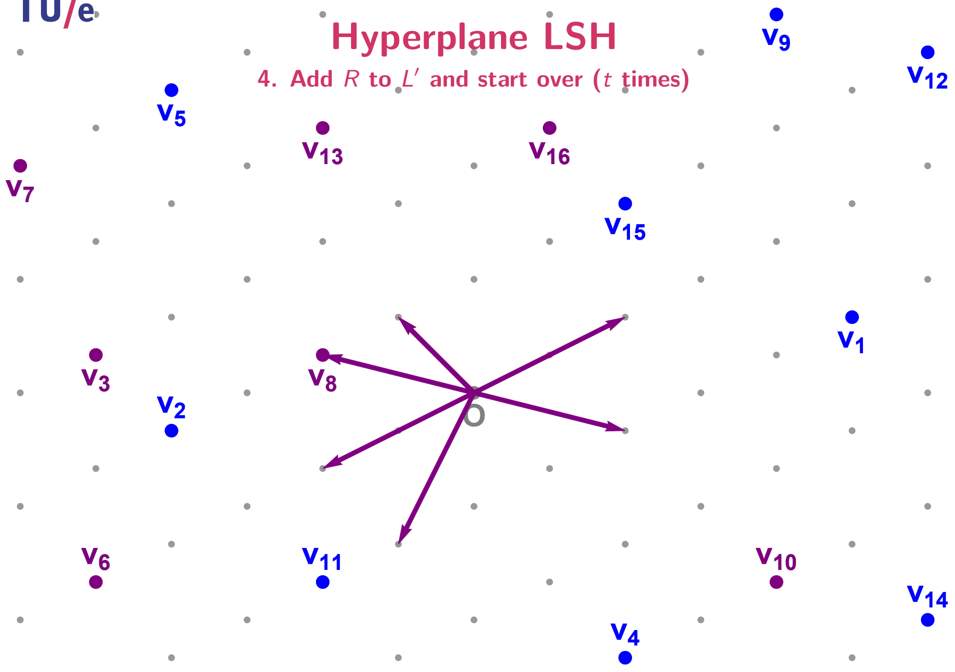
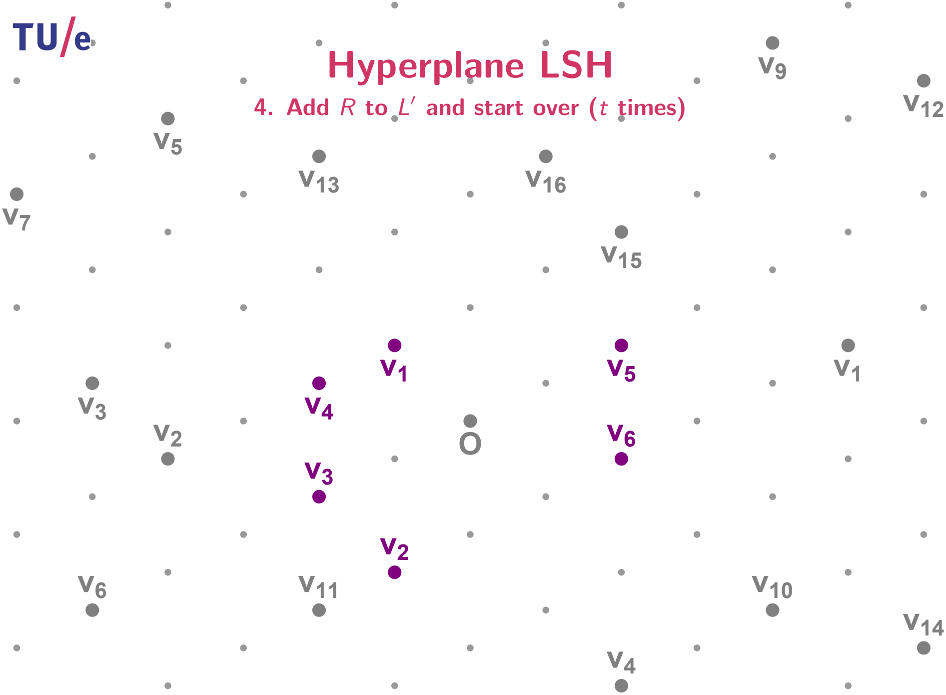


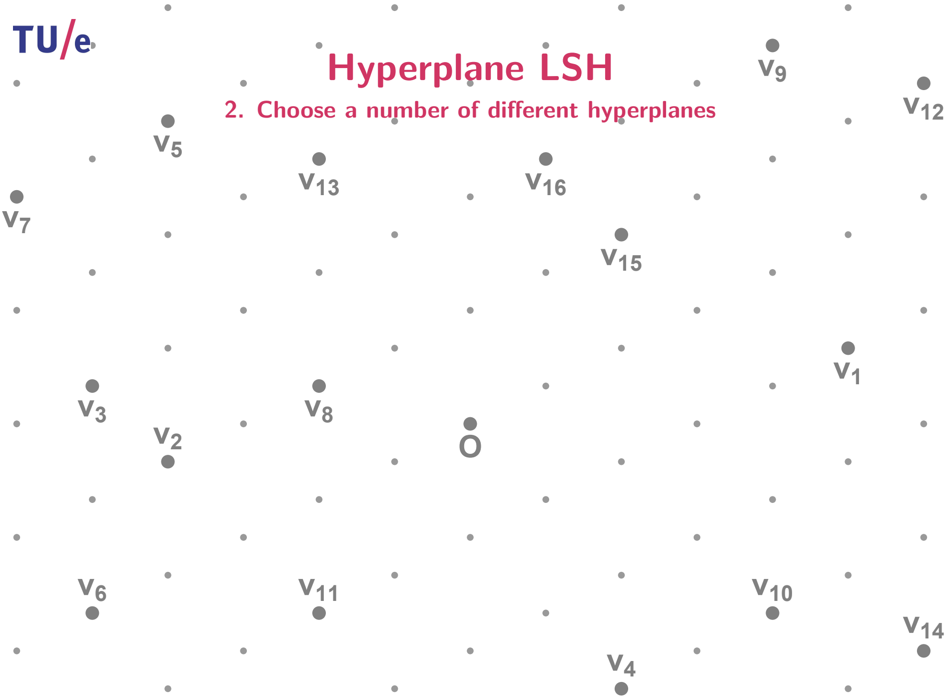
Hyperplane LSH

4. Add R to L' and start over (t times)



Hyperplane LSH

2. Choose a number of different hyperplanes



Hyperplane LSH

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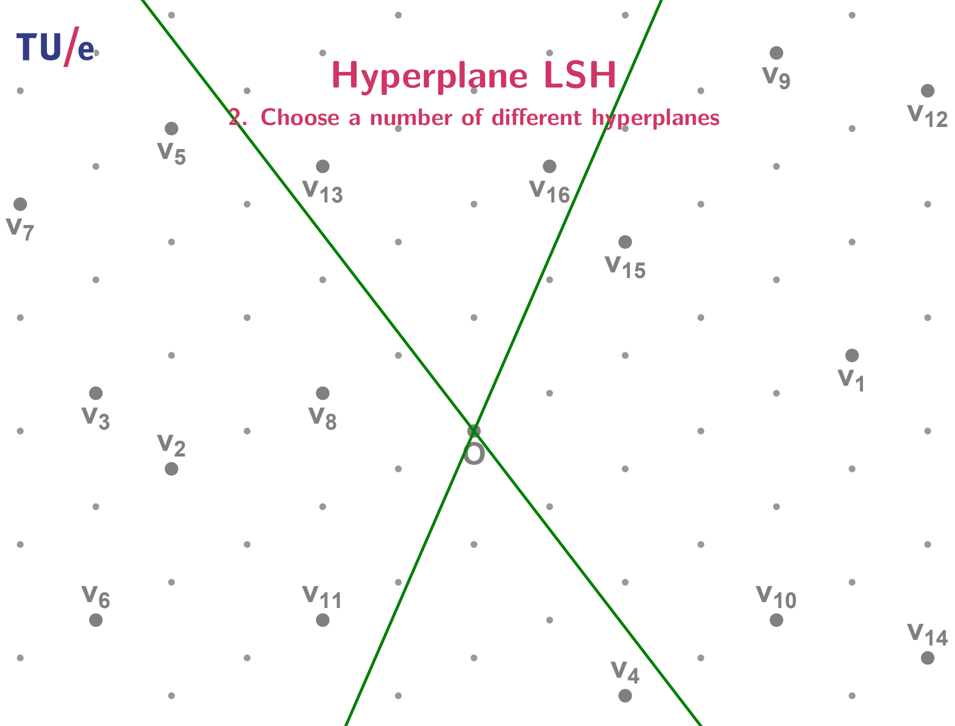
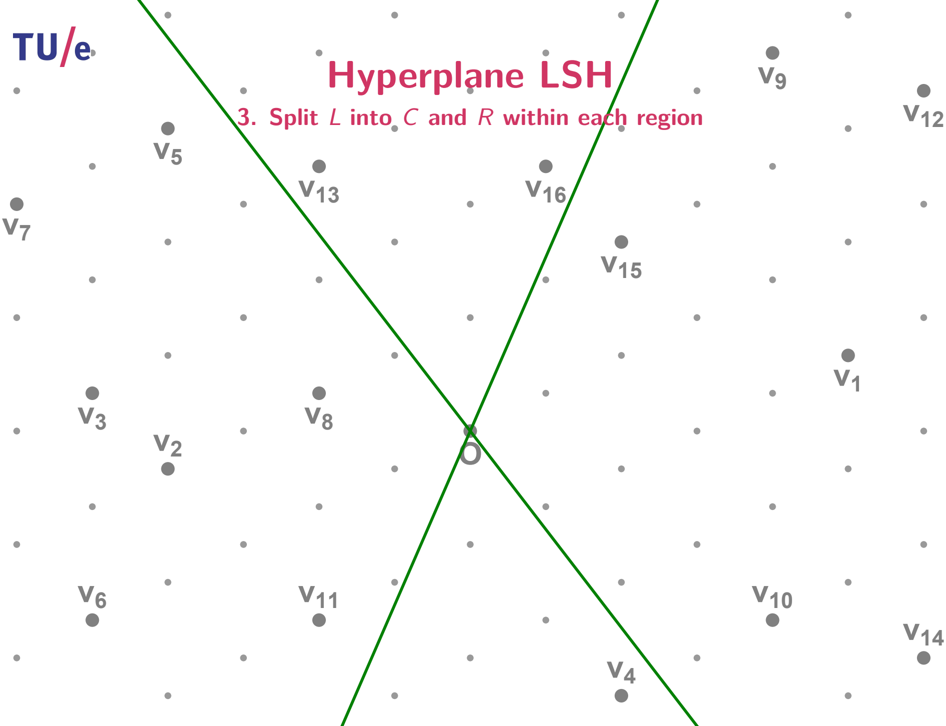


Hyperplane LSH

3. Split L into C and R within each region



Hyperplane LSH

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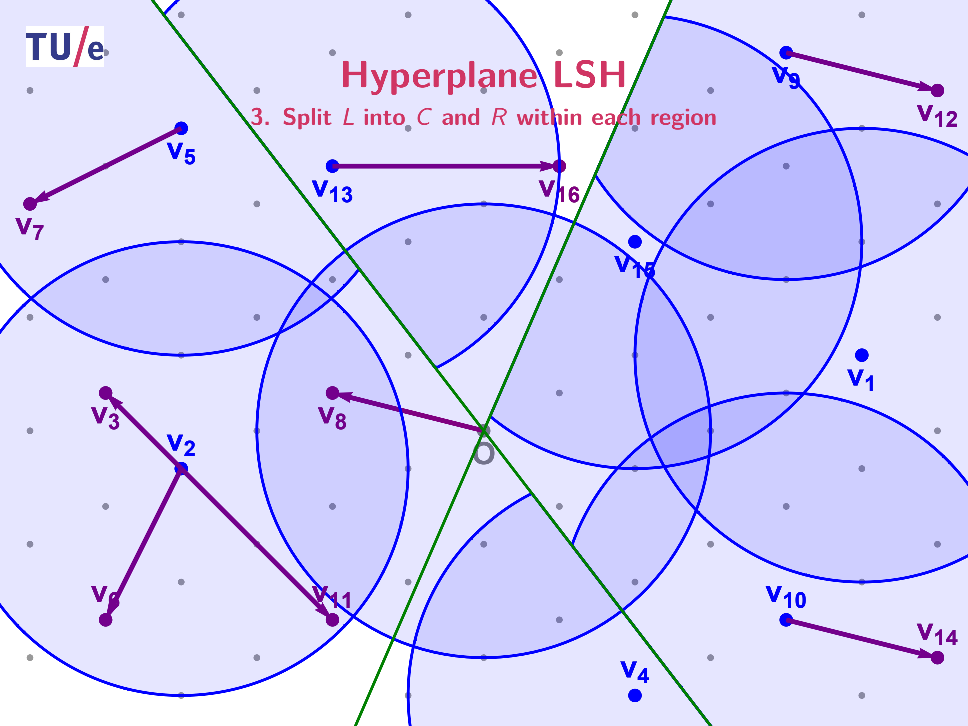
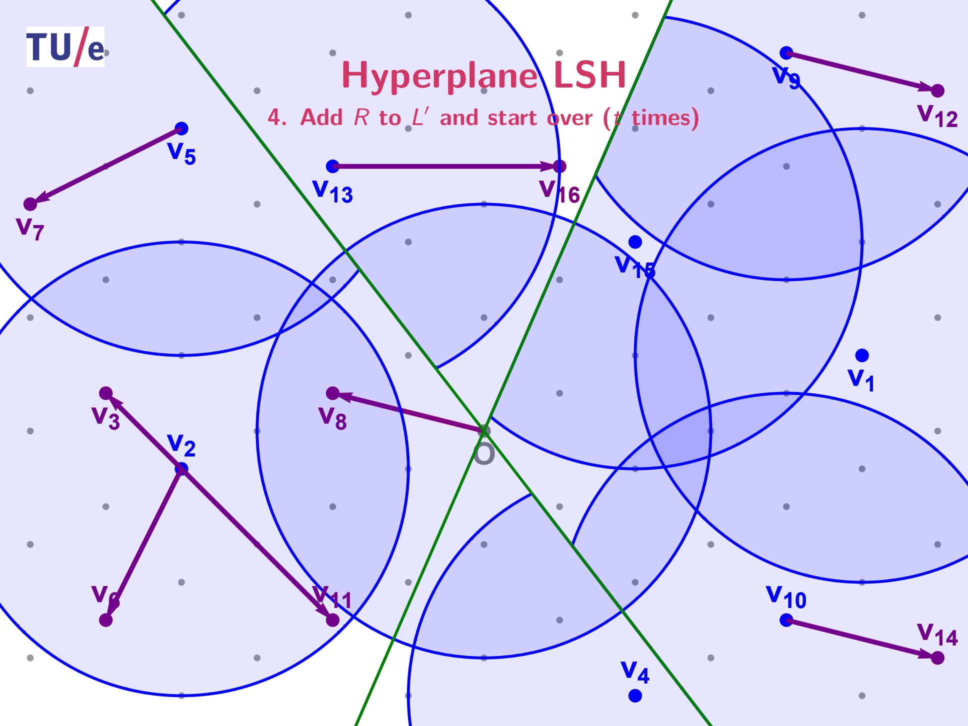


Hyperplane LSH

4. Add R to L' and start over (t times)



Hyperplane LSH

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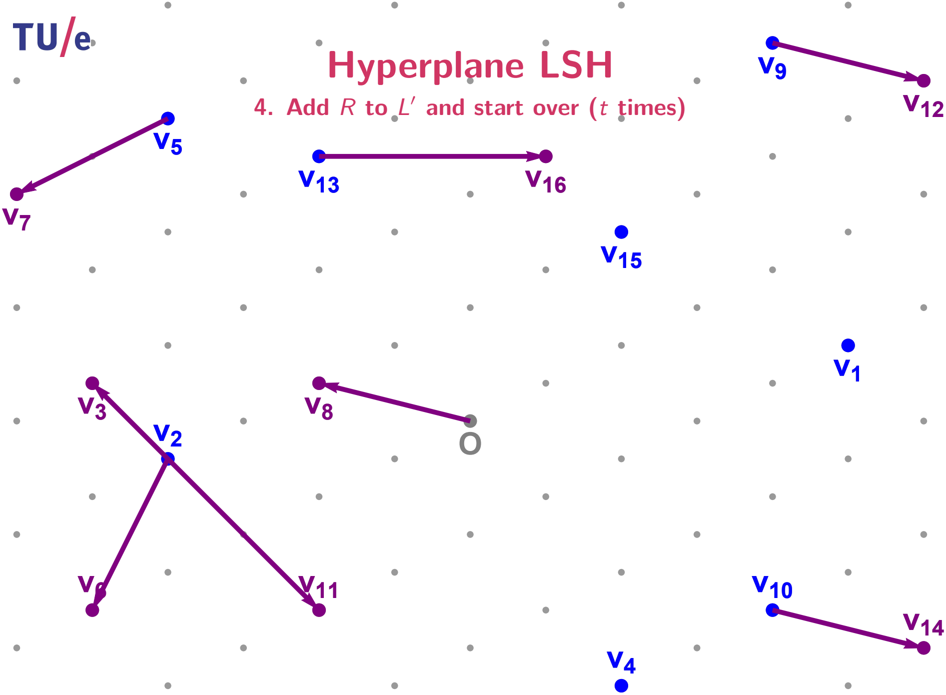

TU/e

Hyperplane LSH

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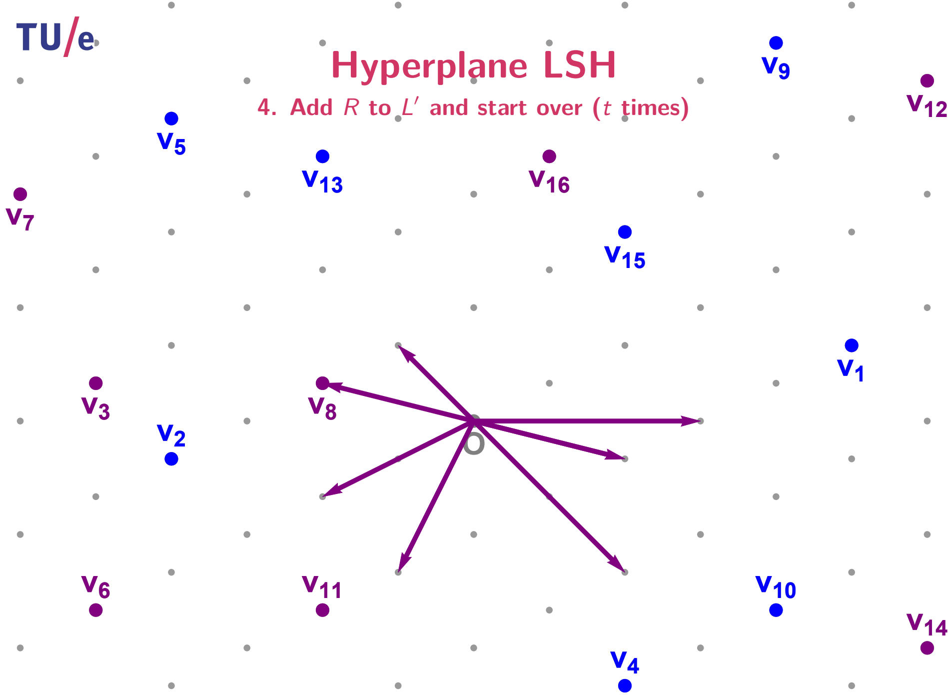

Hyperplane LSH

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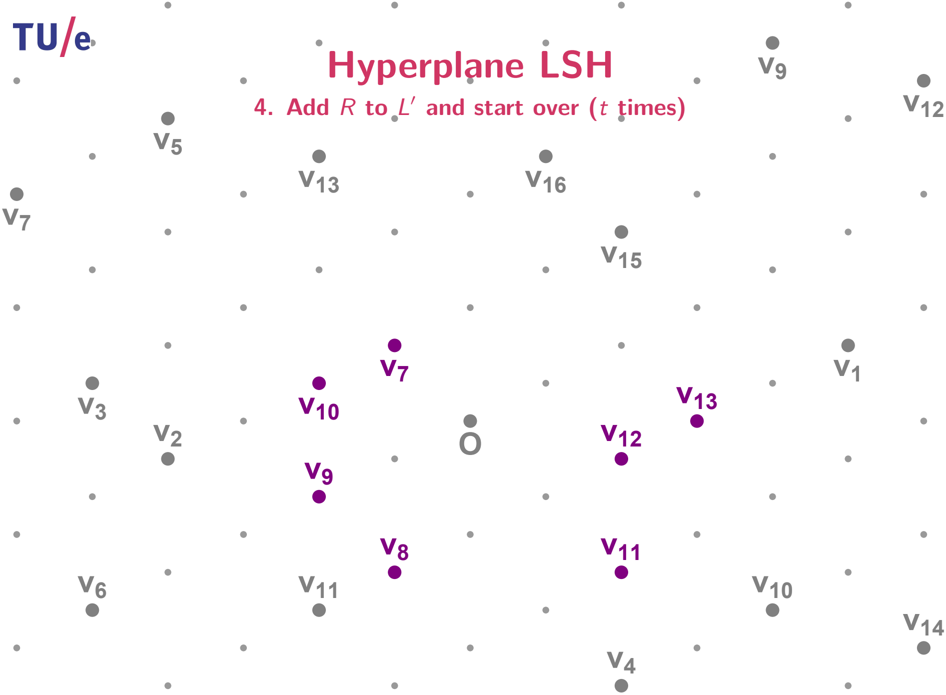
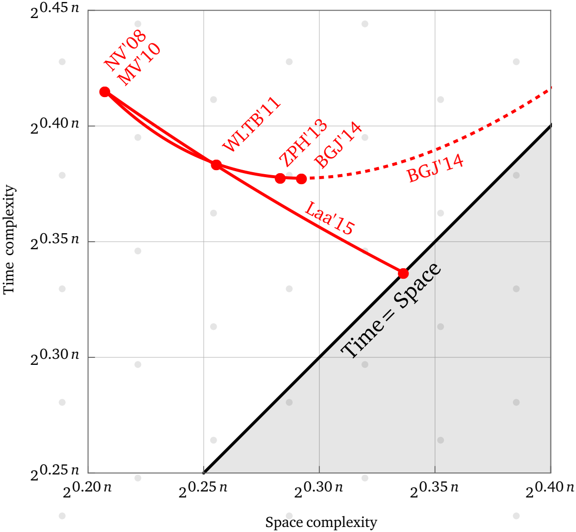


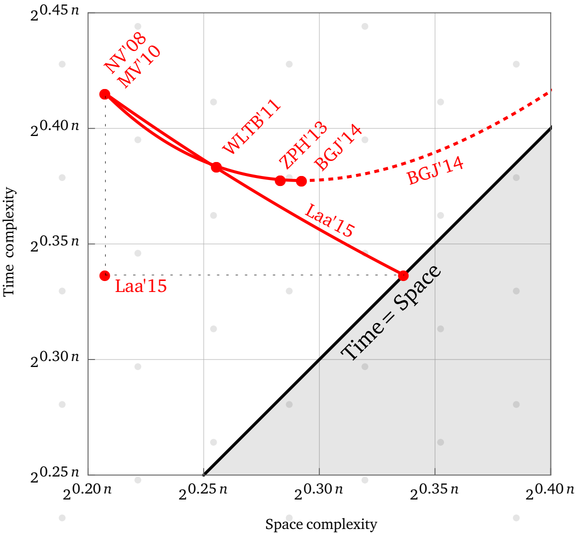
Hyperplane LSH

Space/time trade-off



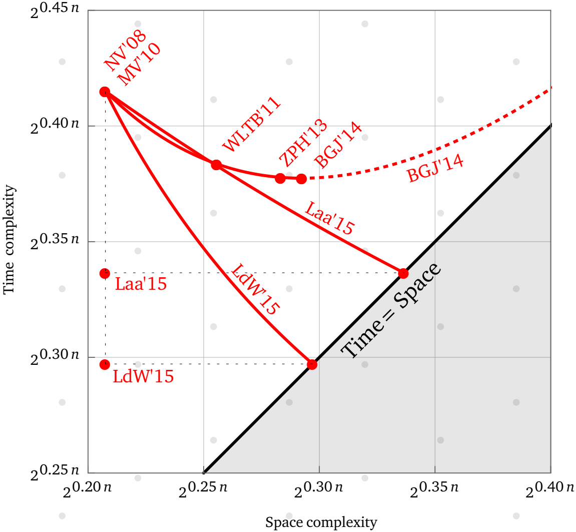
Hyperplane LSH

Space/time trade-off



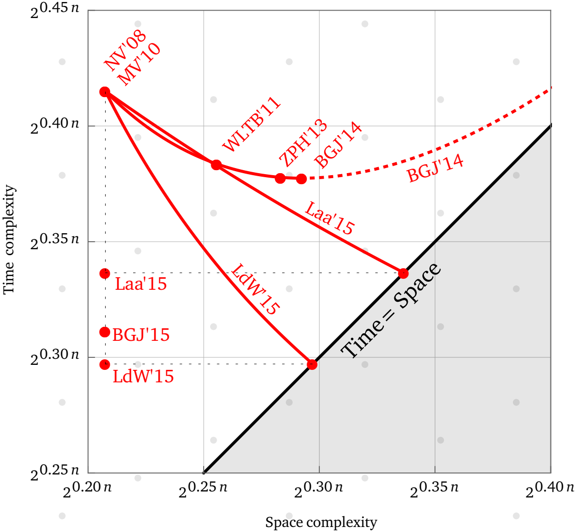
Spherical LSH

Space/time trade-off



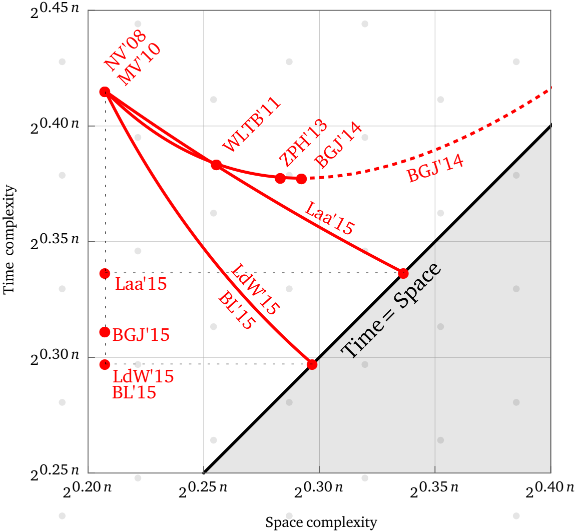
May and Ozerov's NNS method

Space/time trade-off



Cross-polytope LSH

Space/time trade-off



Questions?

[vdP'12]