Sieving for shortest lattice vectors using nearest neighbor techniques

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Tools for Asymmetric Cryptanalysis, Bochum, Germany
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Lattices

What is a lattice?

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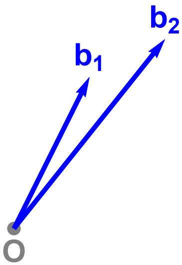

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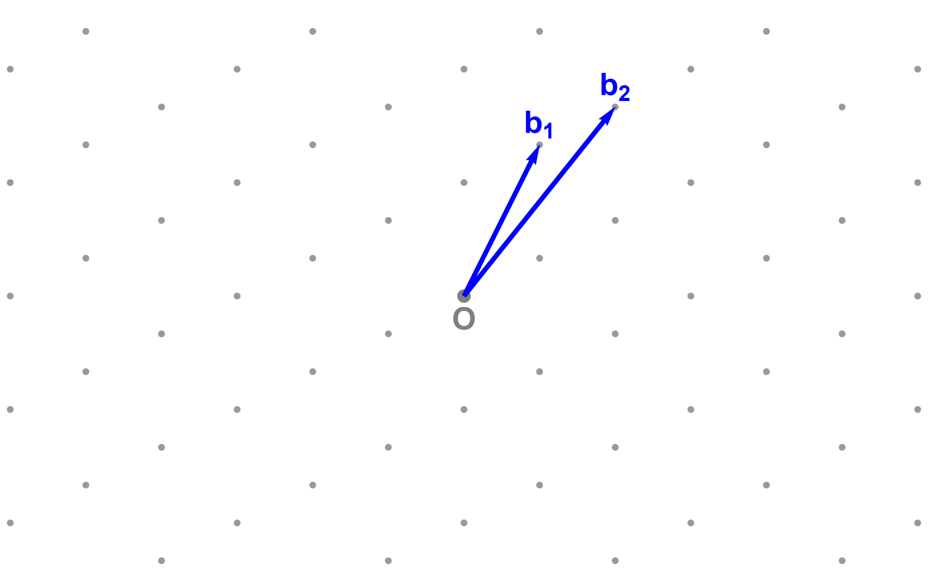

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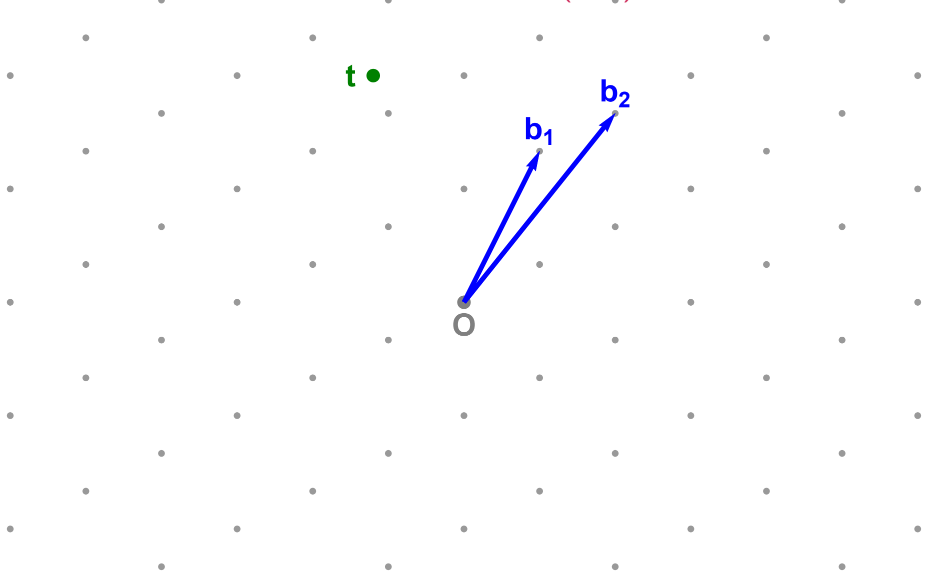
Closest Vector Problem (CVP)



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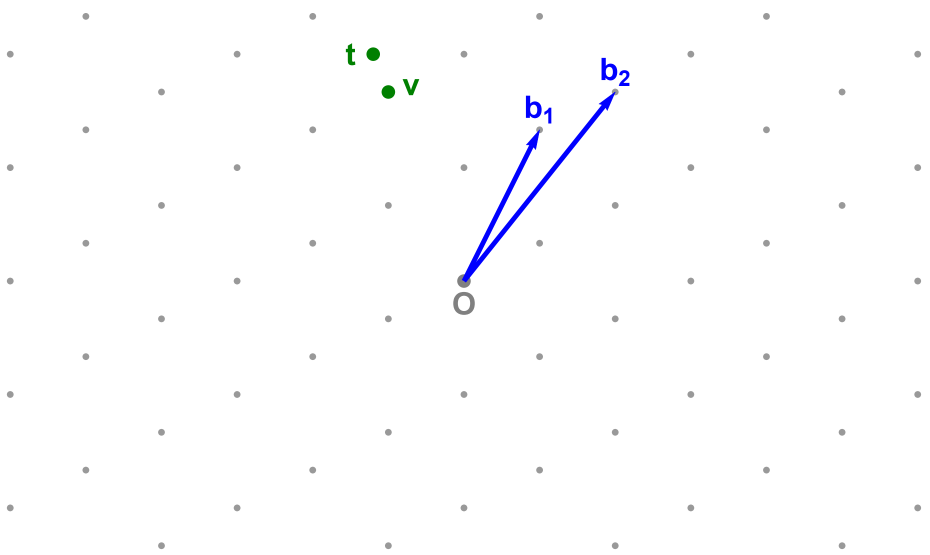

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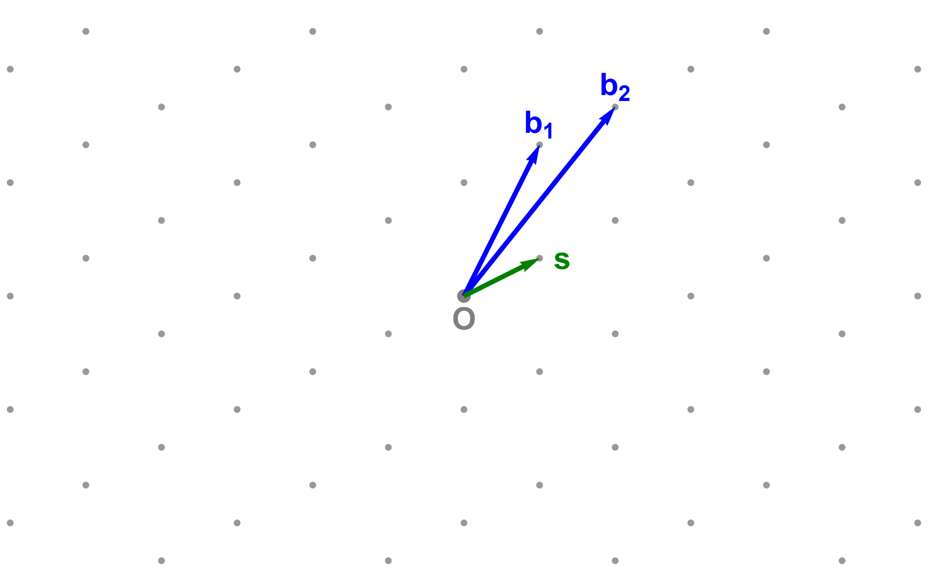

Lattices

Applications

- "Constructive cryptography": Lattice-based cryptosystems
  - Based on hard lattice problems (SVP, CVP, LWE, SIS)
  - NTRU cryptosystem [HPS98, . . . , HPSSWZ15]
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How hard are hard lattice problems such as SVP?

Lattices

Exact SVP algorithms

	Algorithm	$\log_2(\text{Time})$	$\log_2(\text{Space})$
Provable SVP	Enumeration [Poh81, Kan83, ..., MW15]	$\Omega(n \log n)$	$O(\log n)$
	AKS-sieve [AKS01, NV08, MV10, HPS11]	$3.398n$	$1.985n$
	ListSieve [MV10, MDB14]	$3.199n$	$1.327n$
	AKS-sieve-birthday [PS09, HPS11]	$2.648n$	$1.324n$
	ListSieve-birthday [PS09]	$2.465n$	$1.233n$
	Voronoi cell algorithm [AEVZ02, MV10b]	$2.000n$	$1.000n$
	Discrete Gaussians [ADRS15, ADS15, Ste16]	$1.000n$	$1.000n$
Heuristic SVP	Nguyen-Vidick sieve [NV08]	$0.415n$	$0.208n$
	GaussSieve [MV10, ..., IKMT14, BNvdP14]	$0.415n$?	$0.208n$
	Two-level sieve [WLTB11]	$0.384n$	$0.256n$
	Three-level sieve [ZPH13]	$0.3778n$	$0.283n$
	Overlattice sieve [BGJ14]	$0.3774n$	$0.293n$
	Hyperplane LSH [Laa15, MLB15]	$0.337n$	$0.208n$
	May and Ozerov's NNS method [BGJ15]	$0.311n$	$0.208n$
	Spherical LSH [LdW15]	$0.298n$	$0.208n$
	Cross-polytope LSH [BL15]	$0.298n$	$0.208n$
	Spherical filtering [BDGL16]	$0.293n$	$0.208n$

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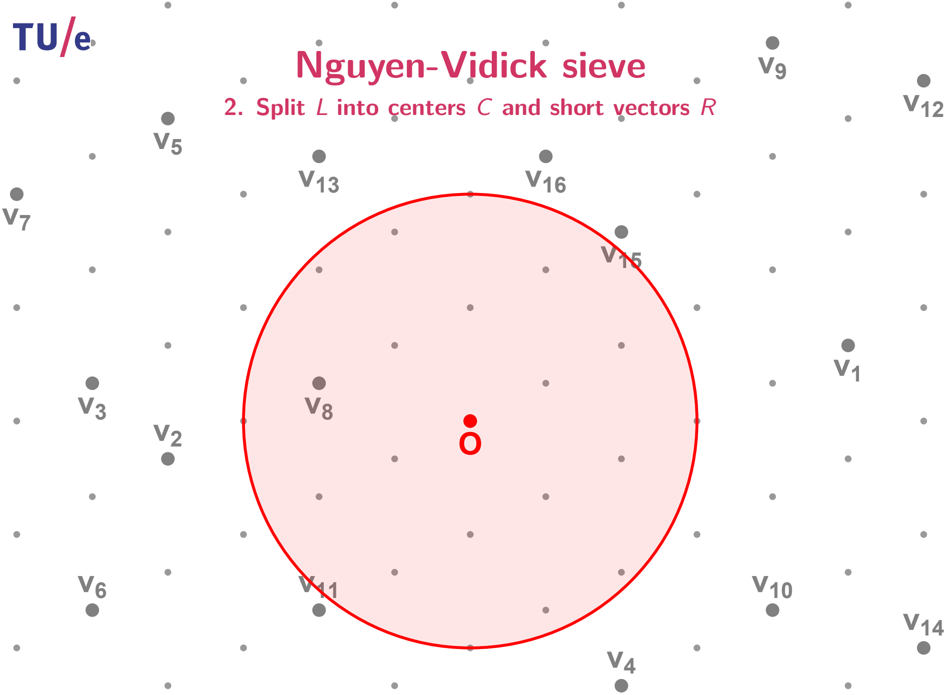

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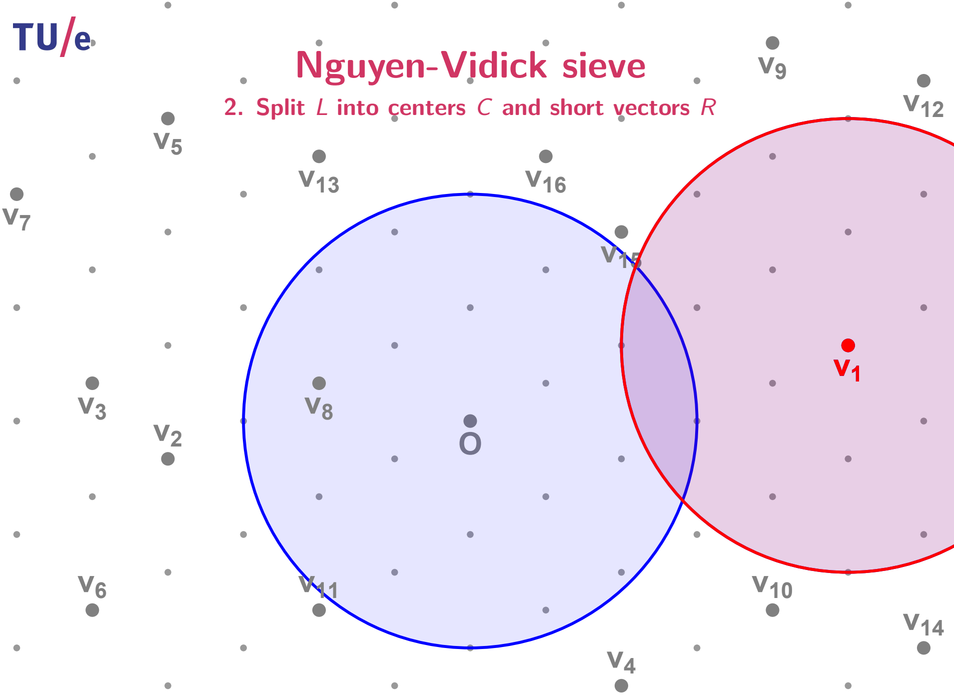

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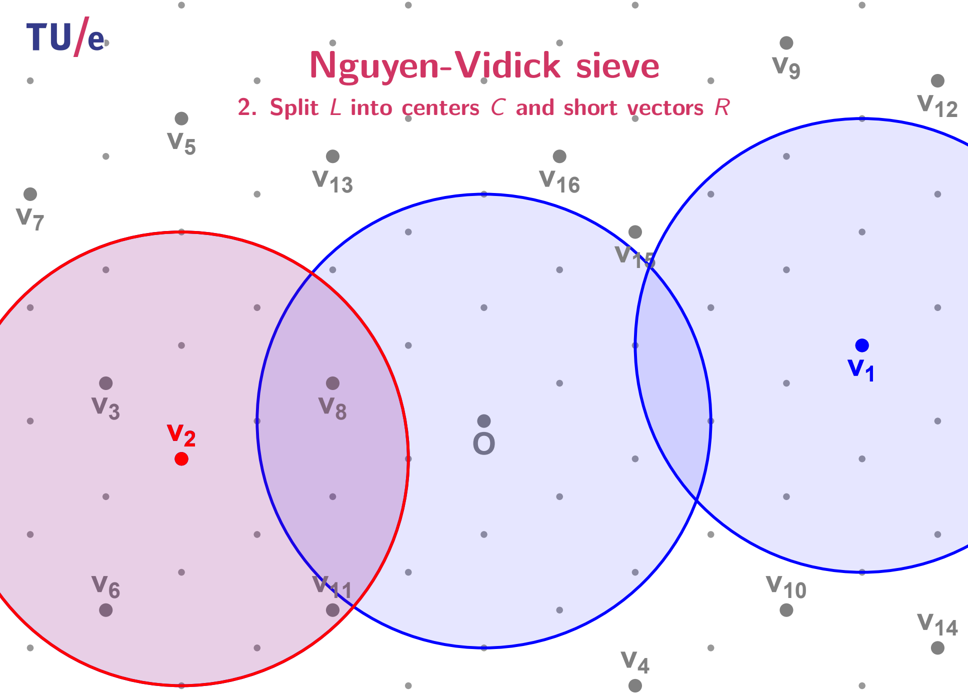

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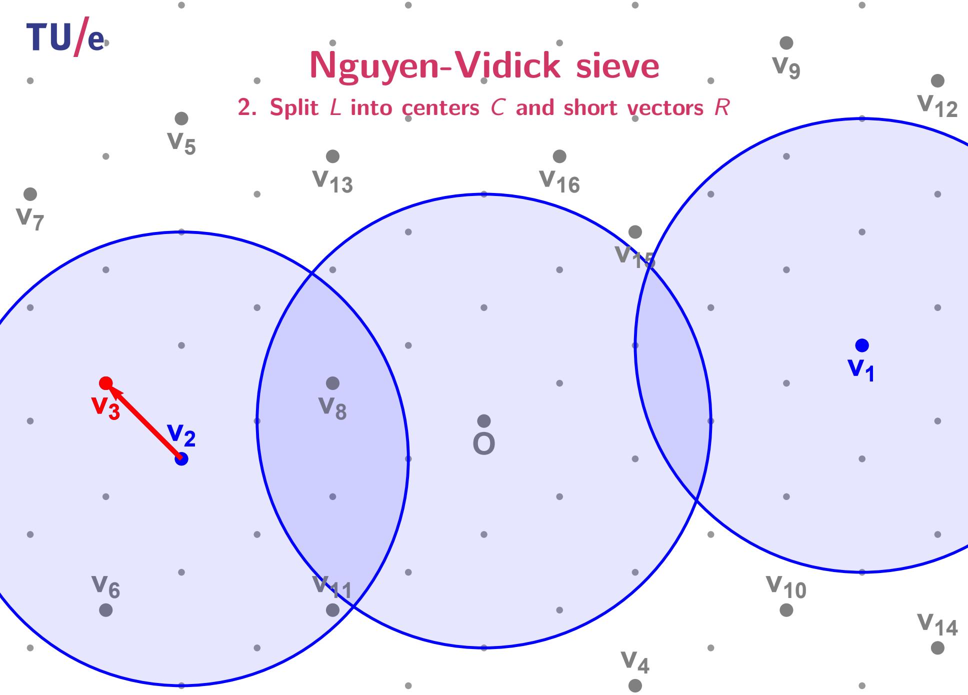

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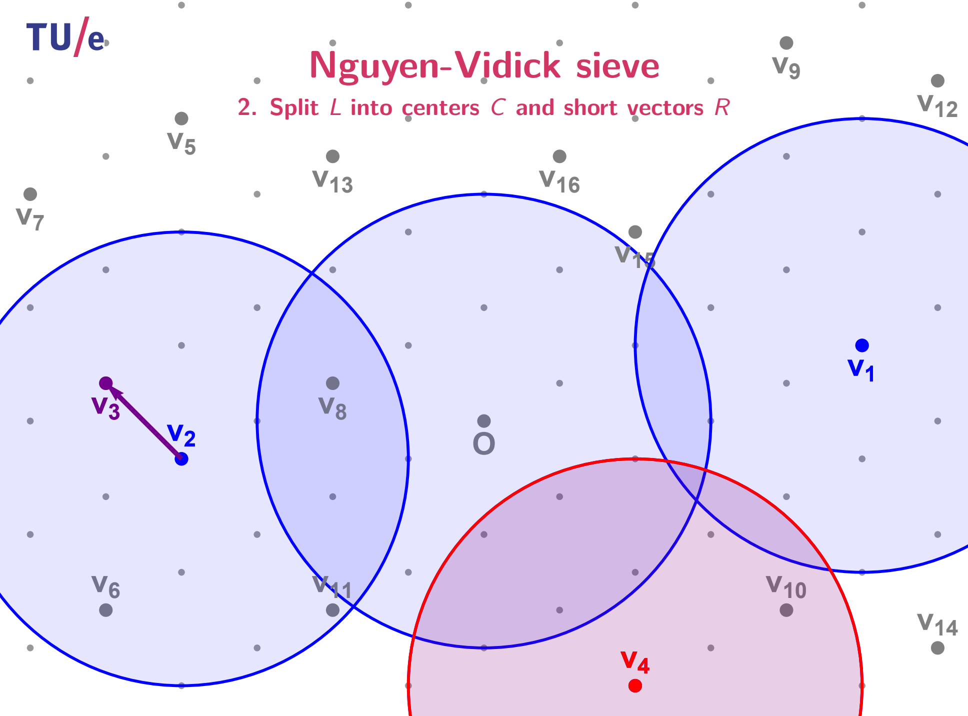

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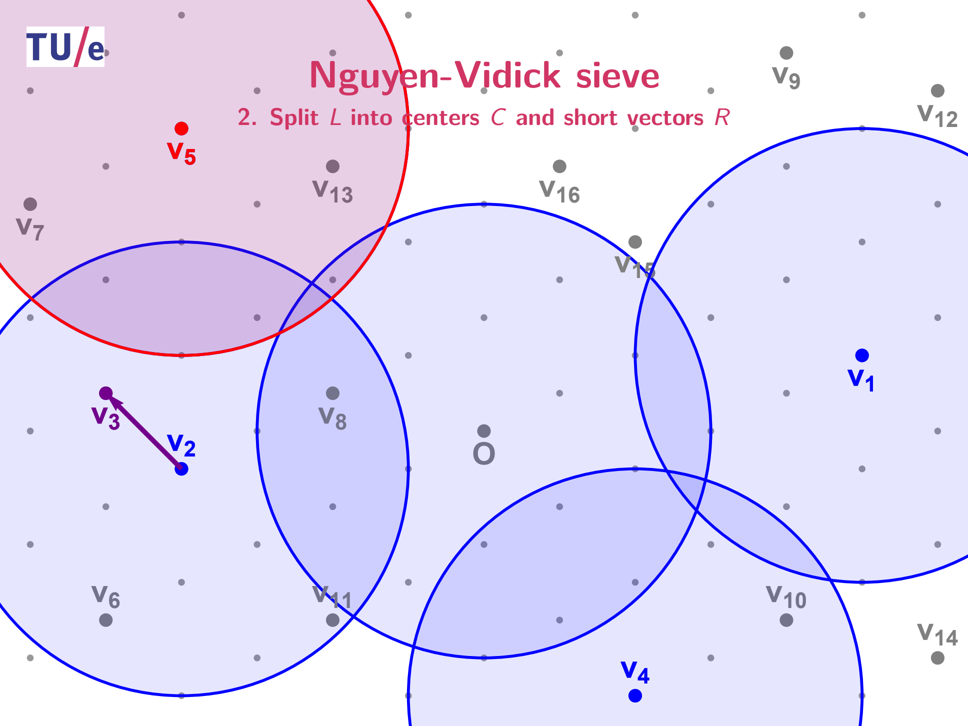

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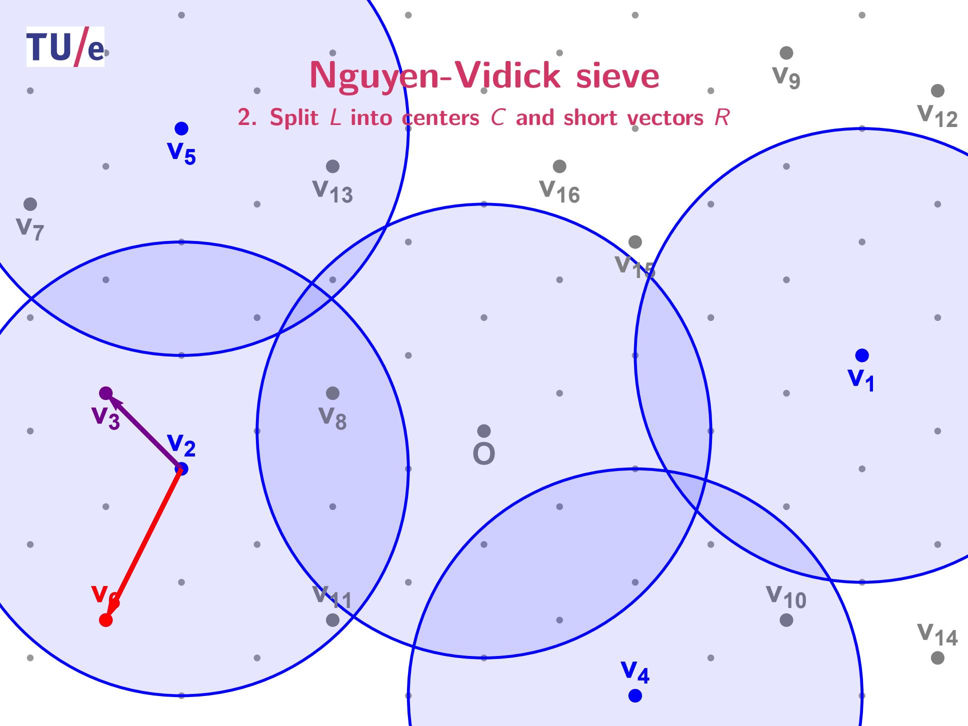

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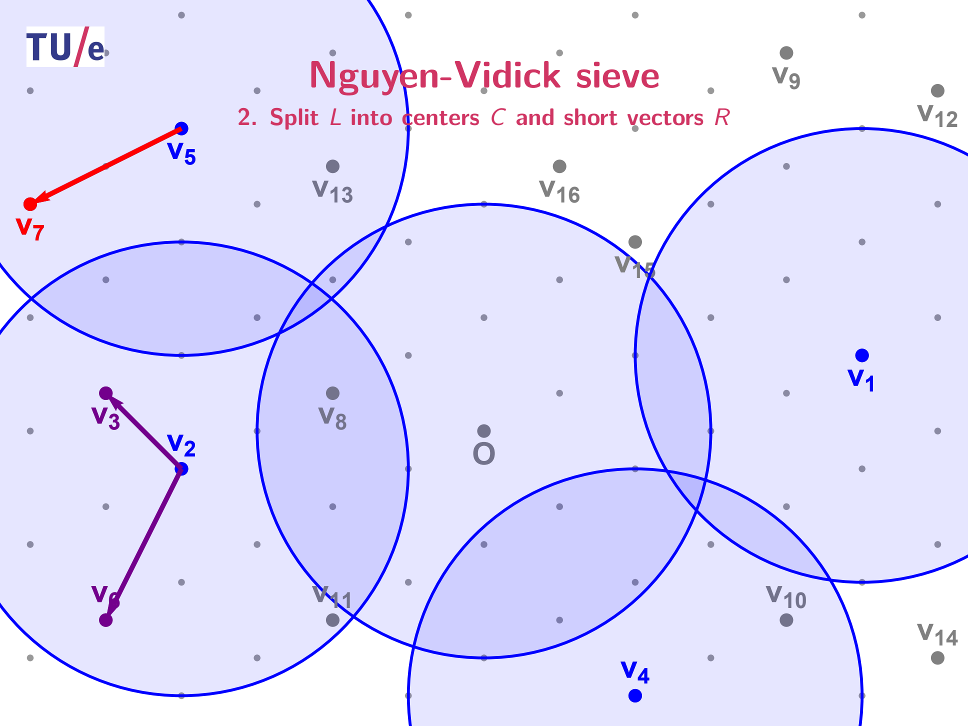

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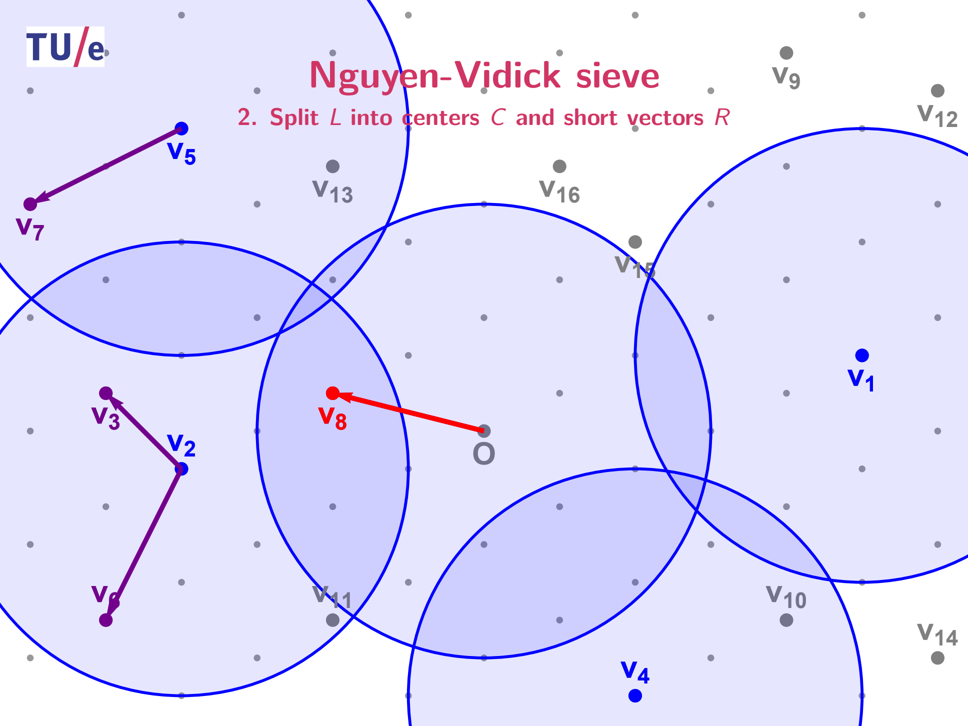

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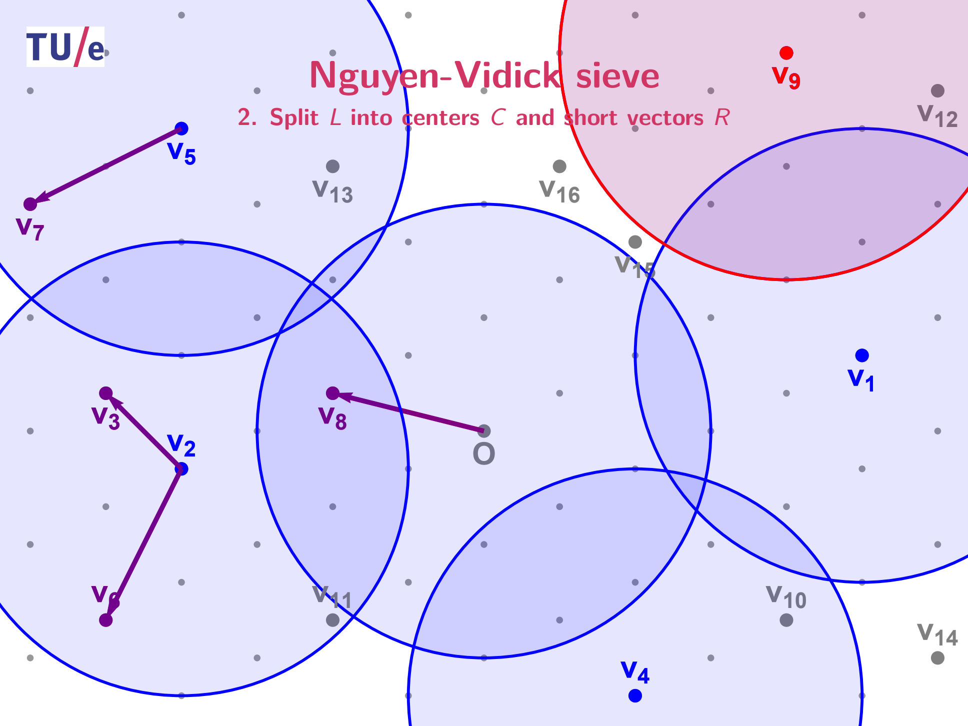

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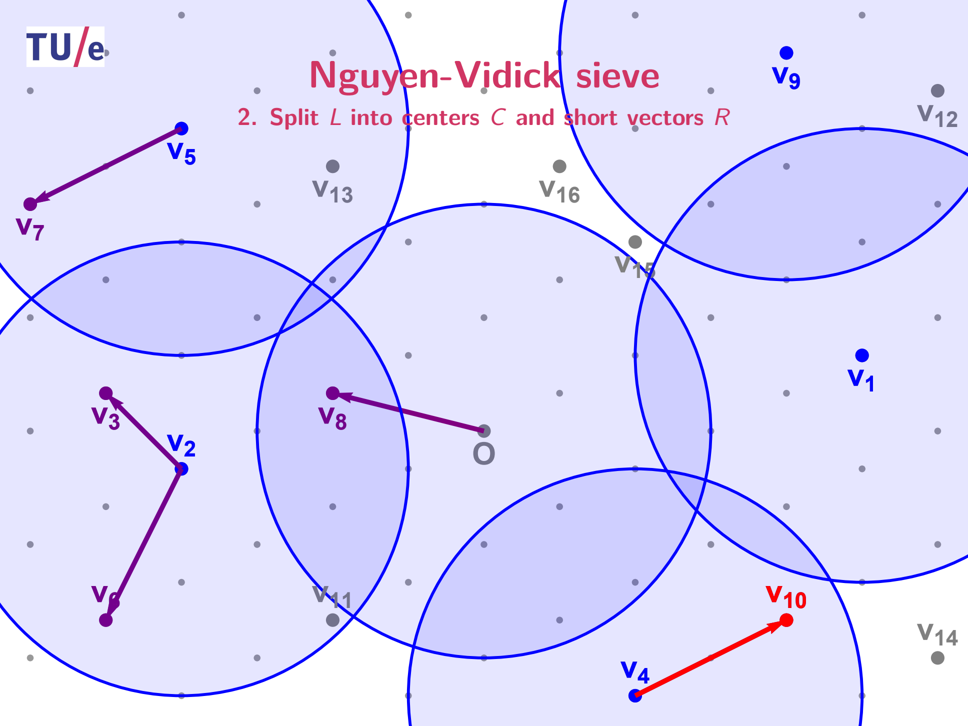

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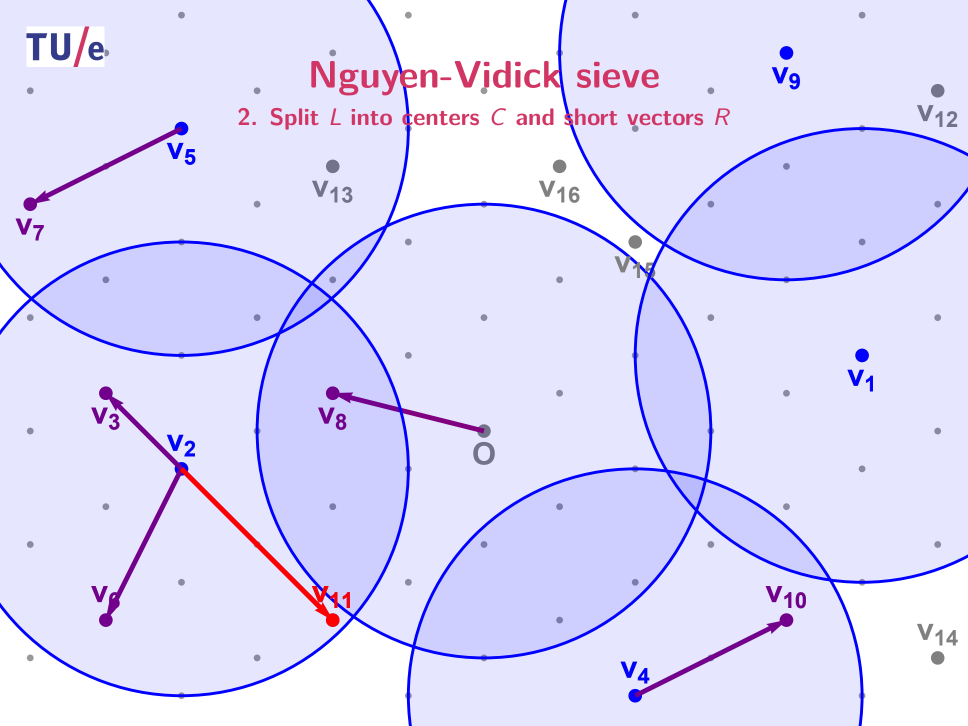

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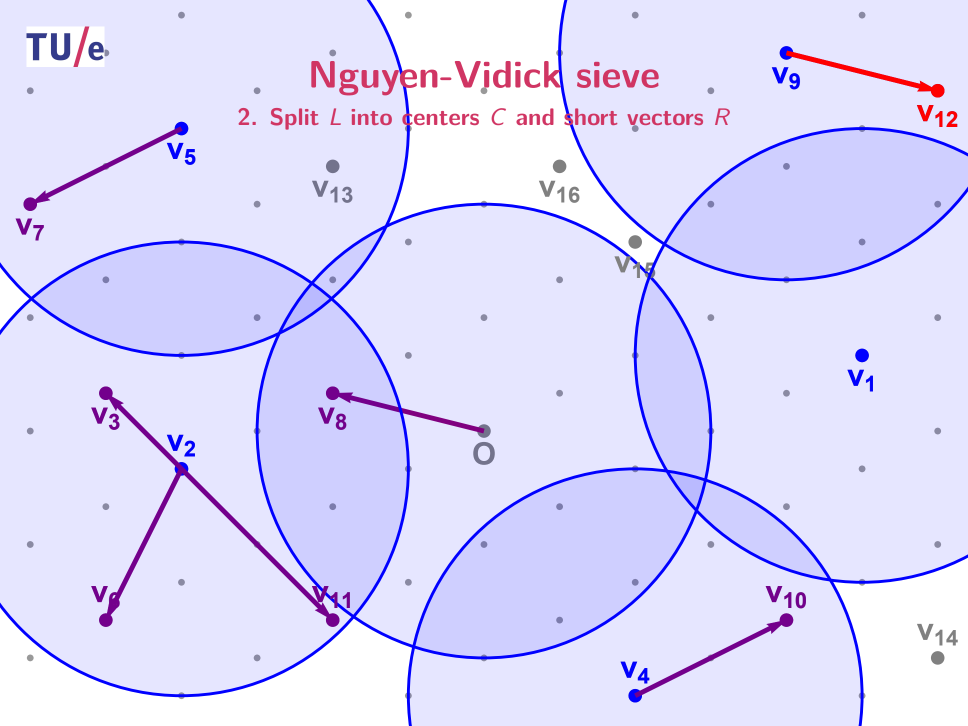

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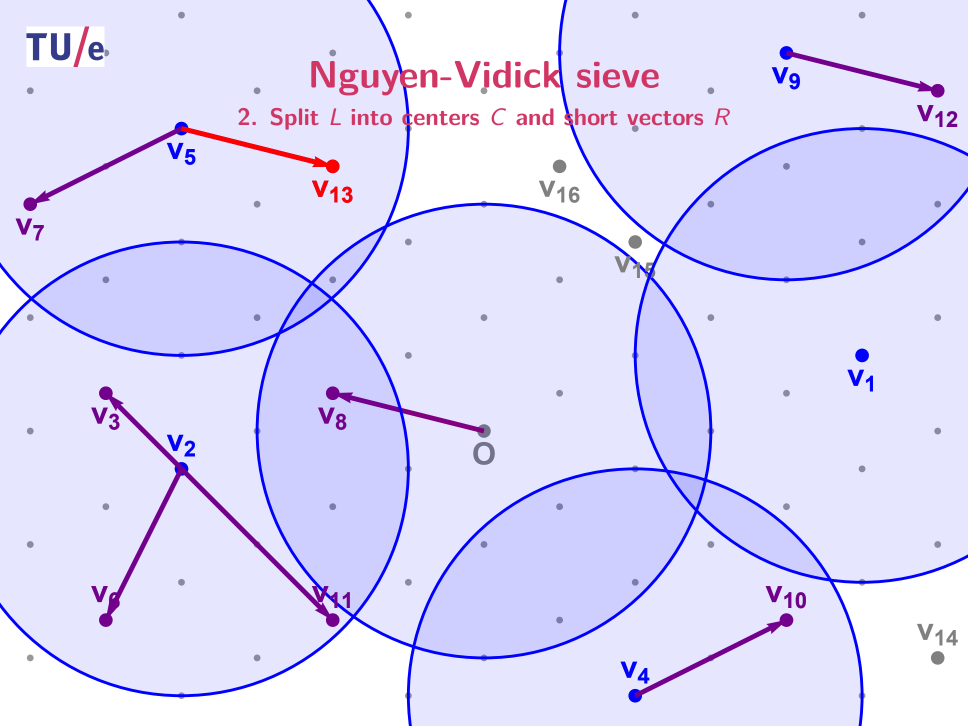

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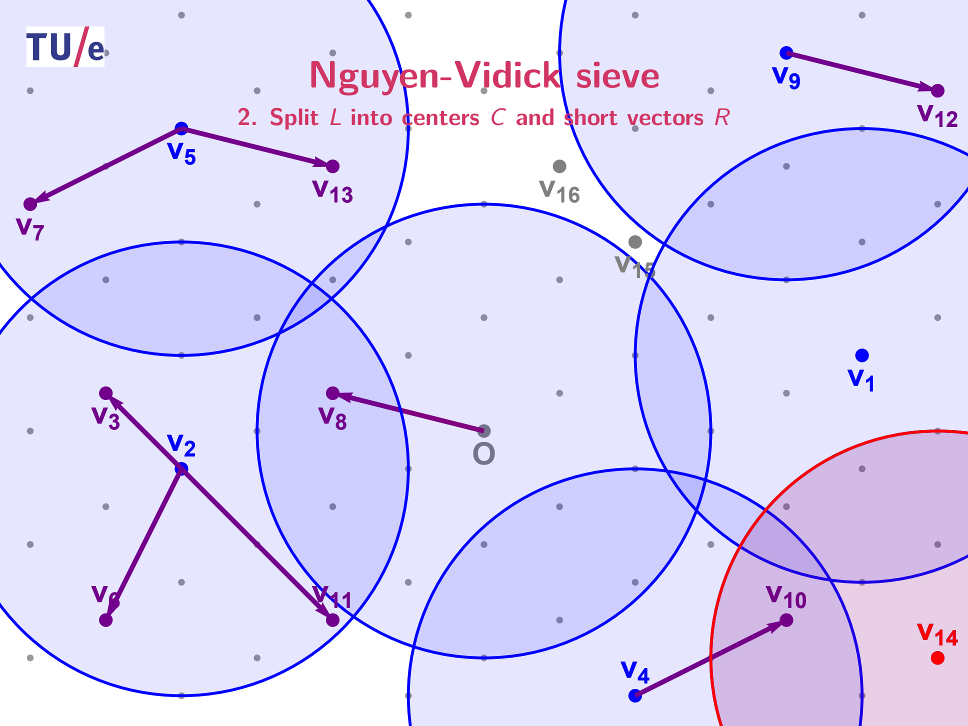

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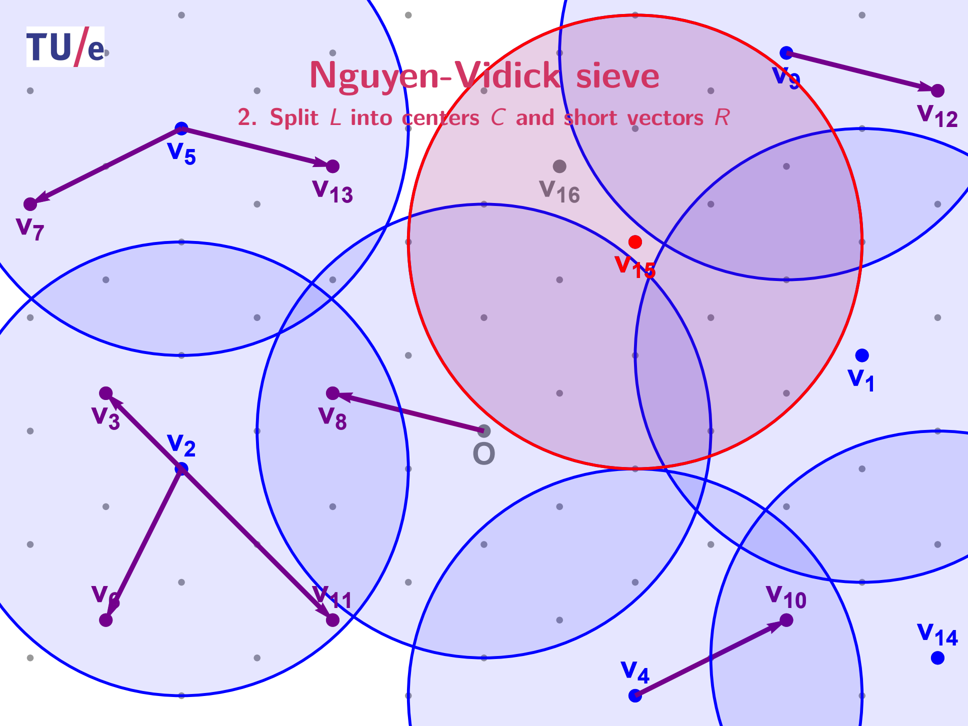

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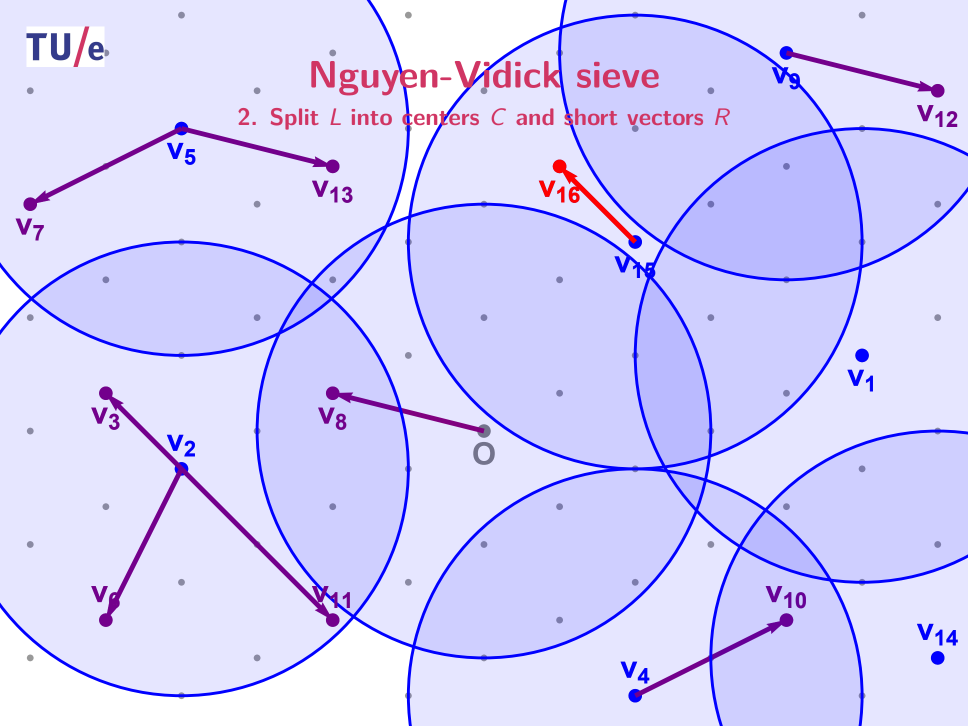

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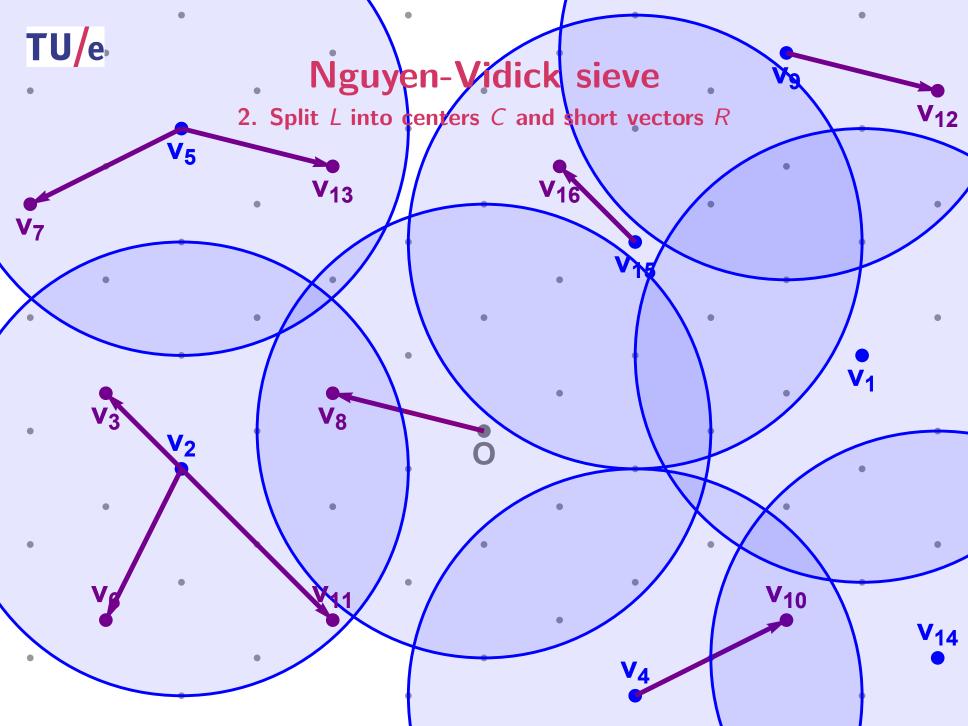

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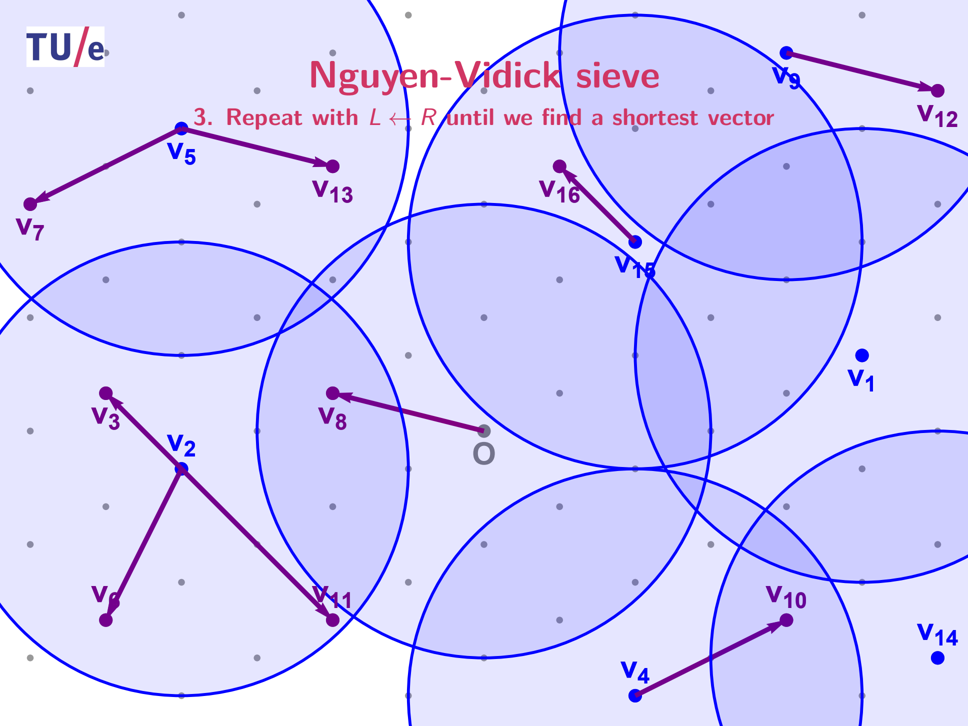

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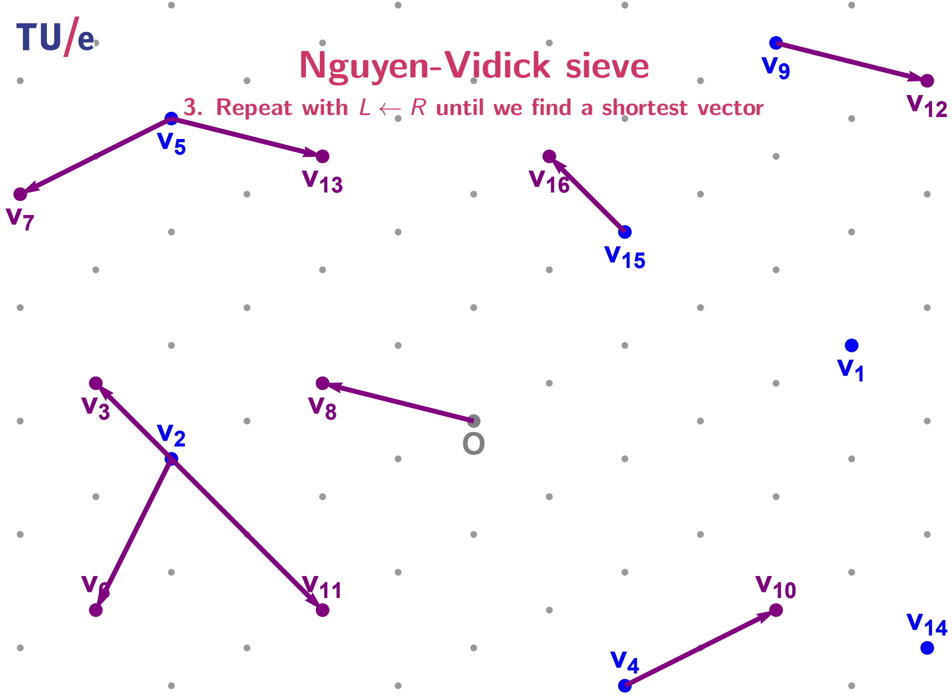

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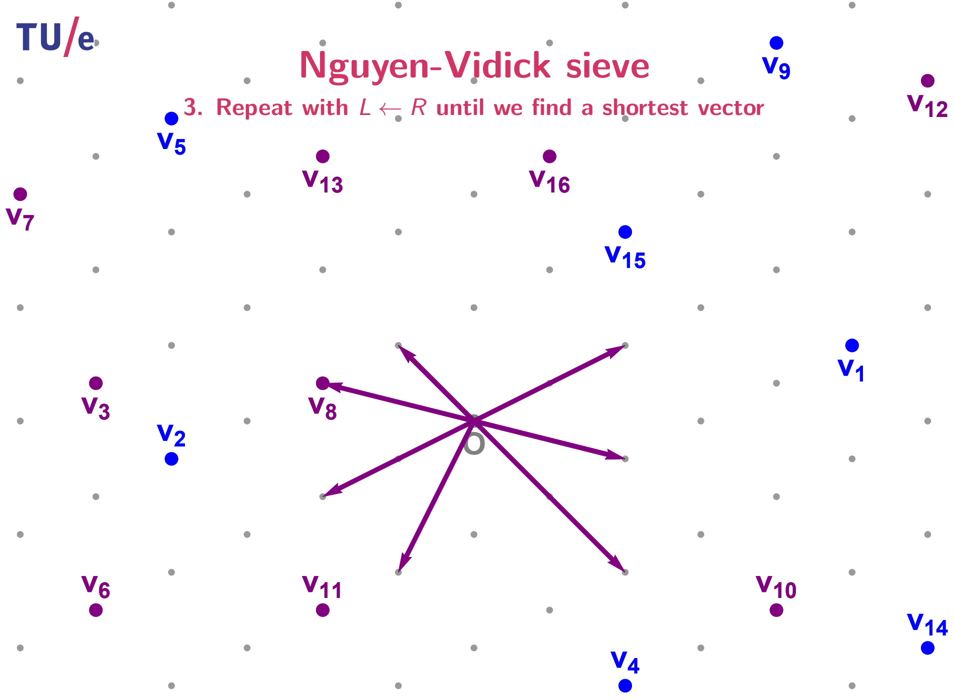

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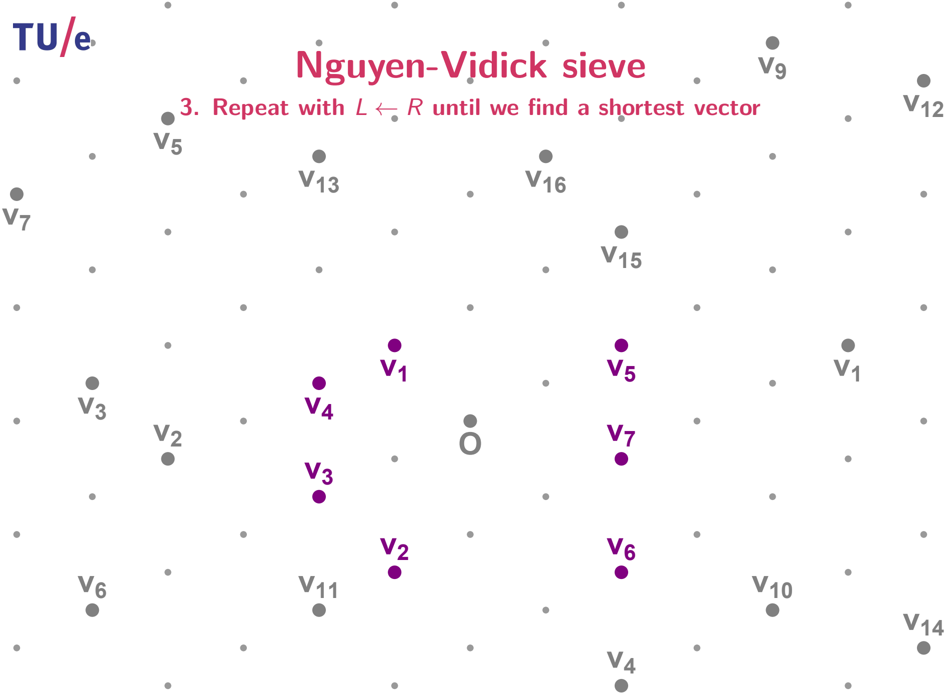

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- Space complexity: $\sqrt{4/3}^n \approx 2^{0.21n+o(n)}$ vectors
  - Need $\sqrt{4/3}^n$ vectors to cover all corners of $\mathbb{R}^n$

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  - Comparing a target vector to all centers: $2^{0.21n+o(n)}$
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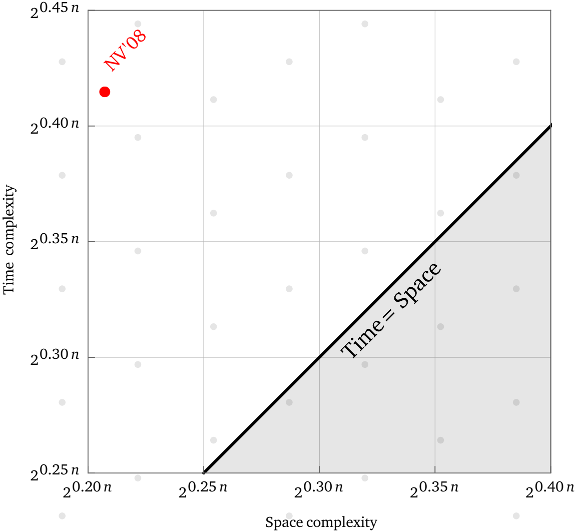
Heuristic (Nguyen and Vidick, J. Math. Crypt. '08)

The NV-sieve runs in time $2^{0.42n+o(n)}$ and space $2^{0.21n+o(n)}$.

Nguyen-Vidick sieve

Space/time trade-off



GaussSieve

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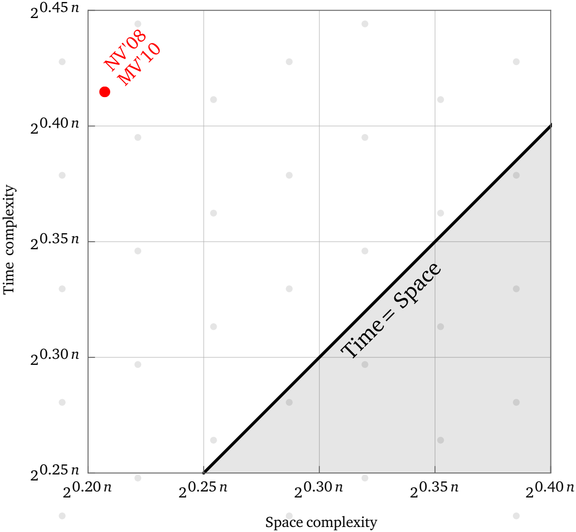

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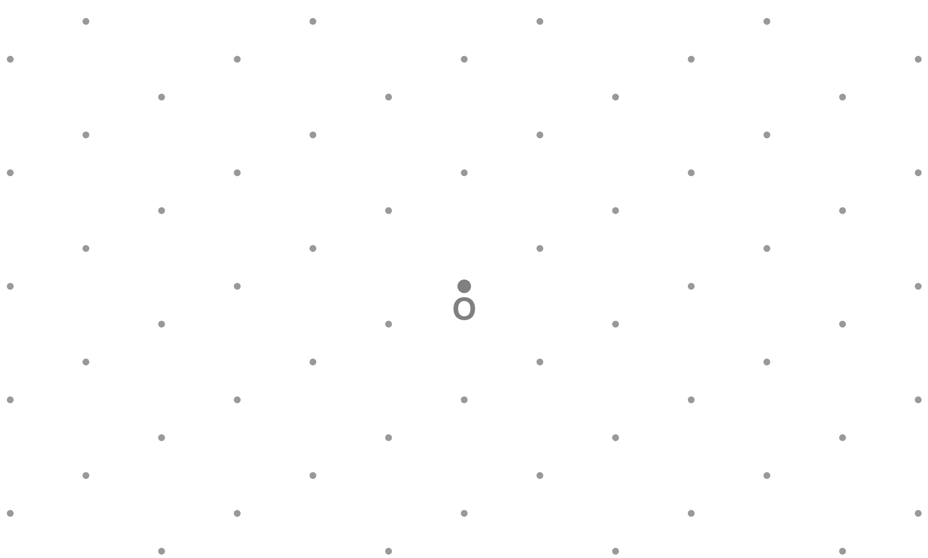

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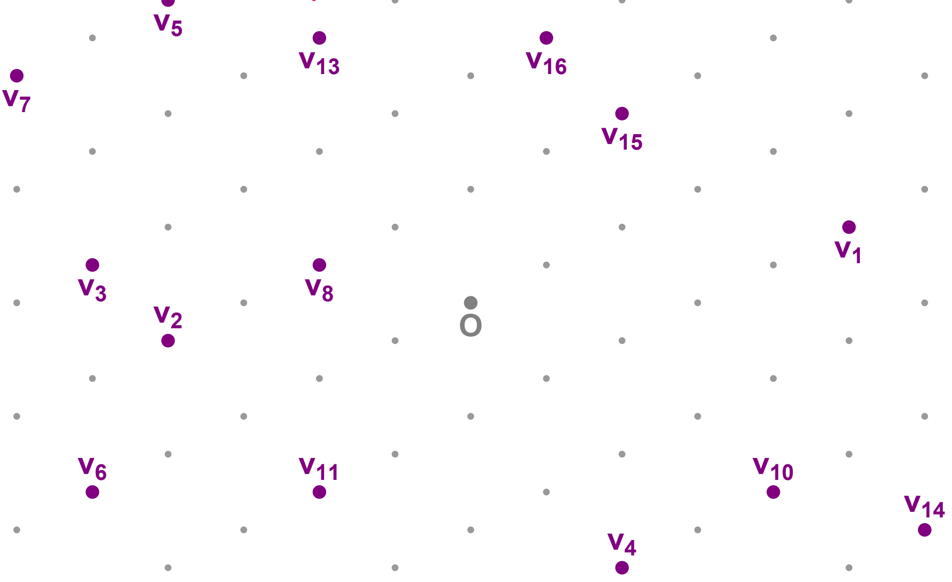

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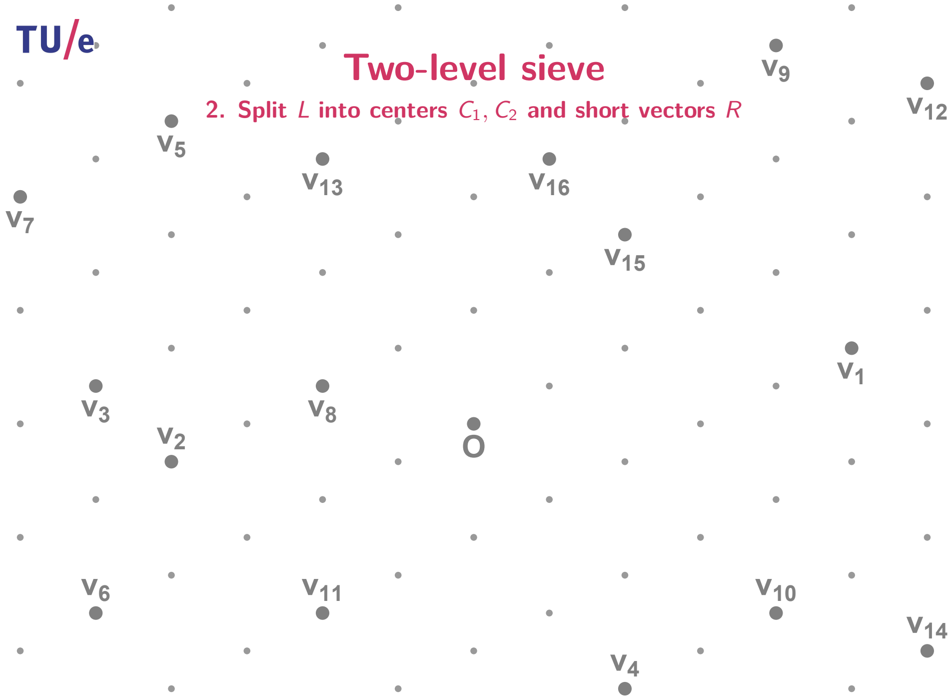

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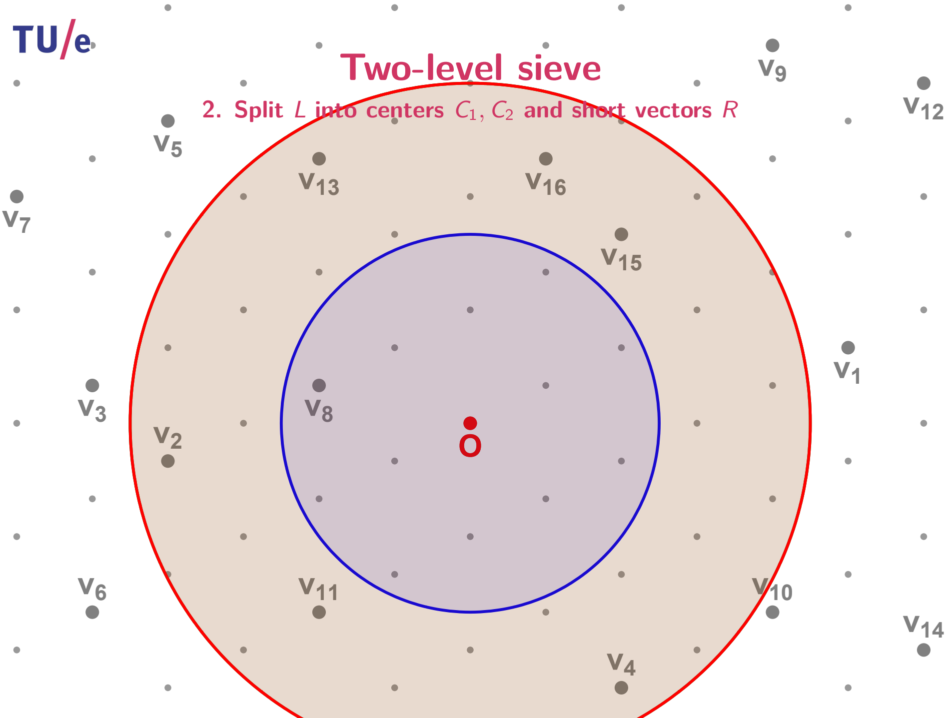

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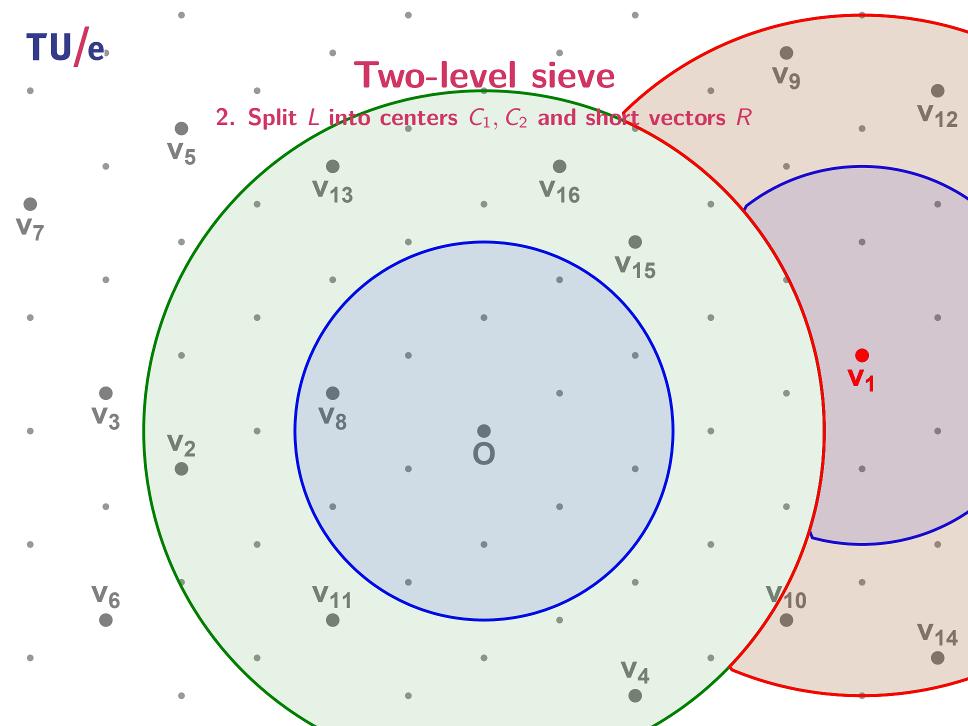

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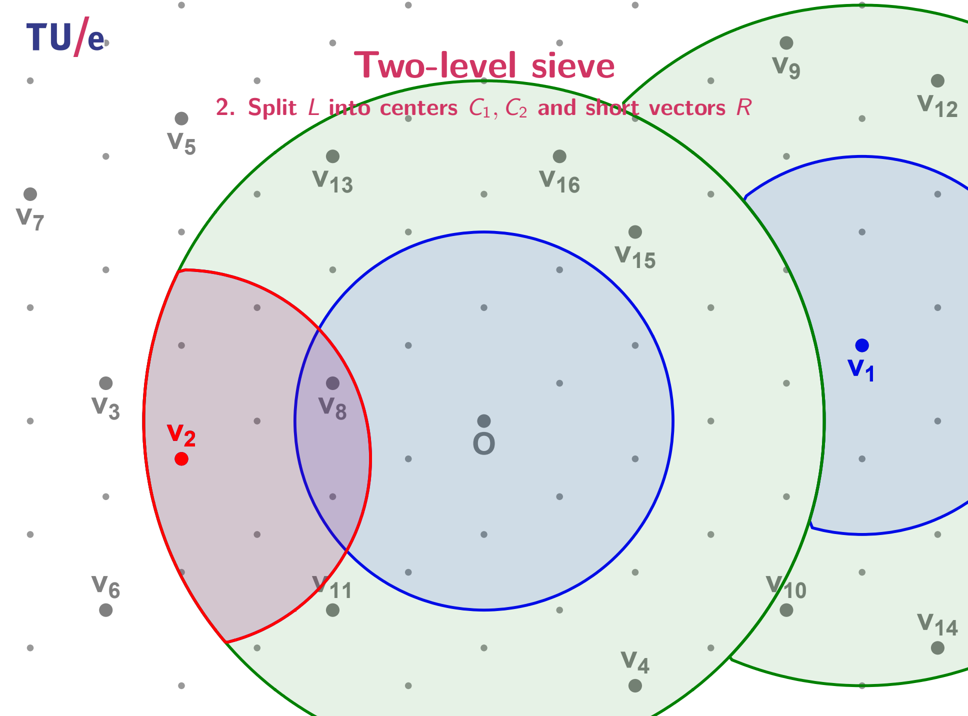

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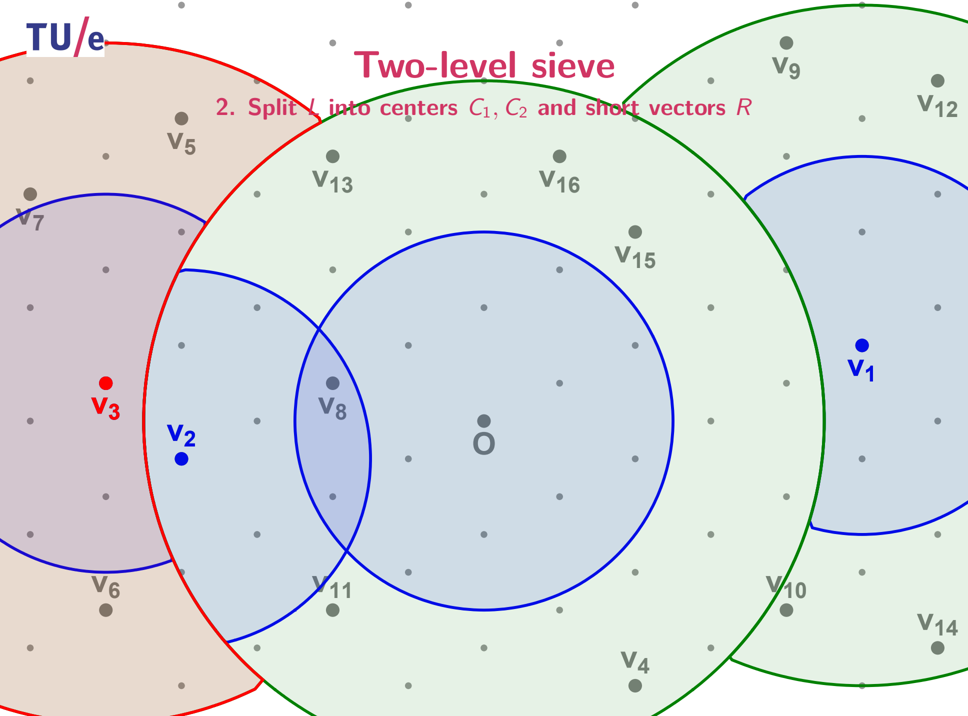

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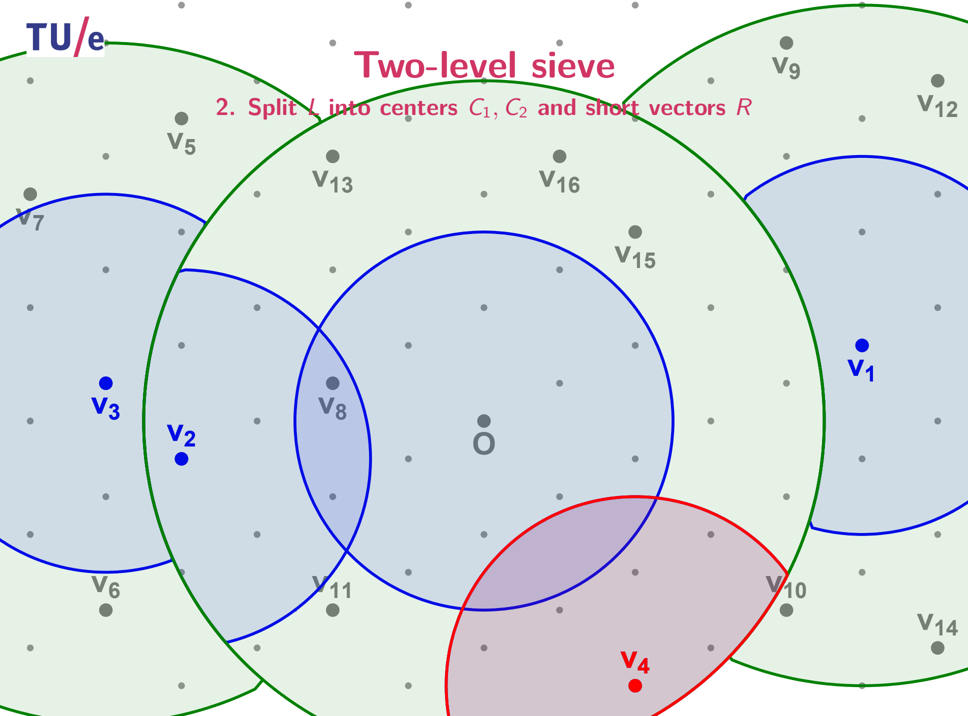

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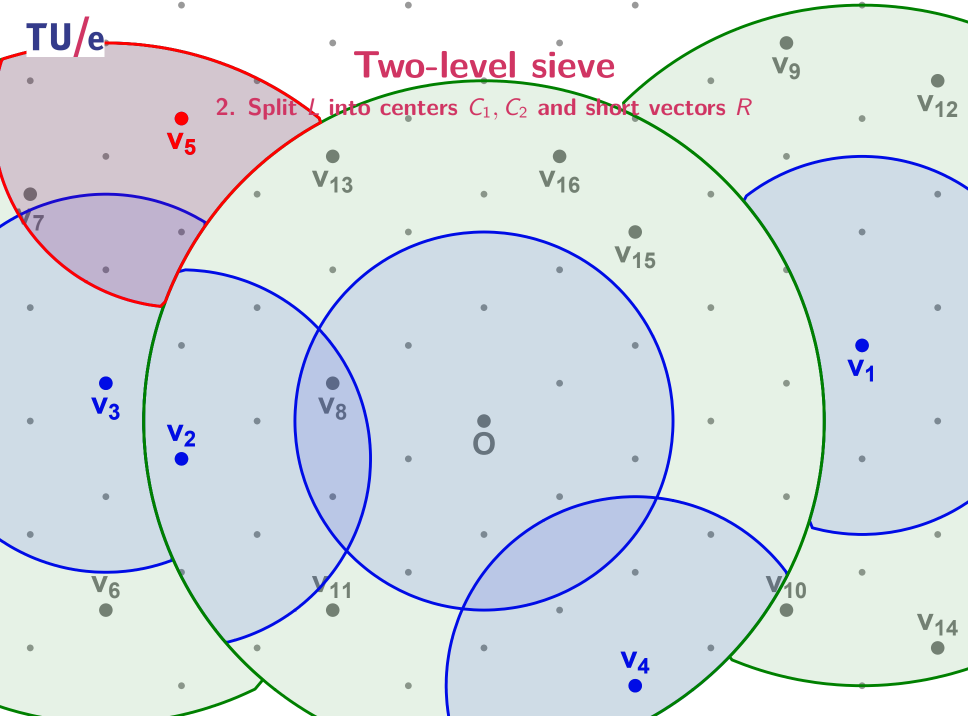

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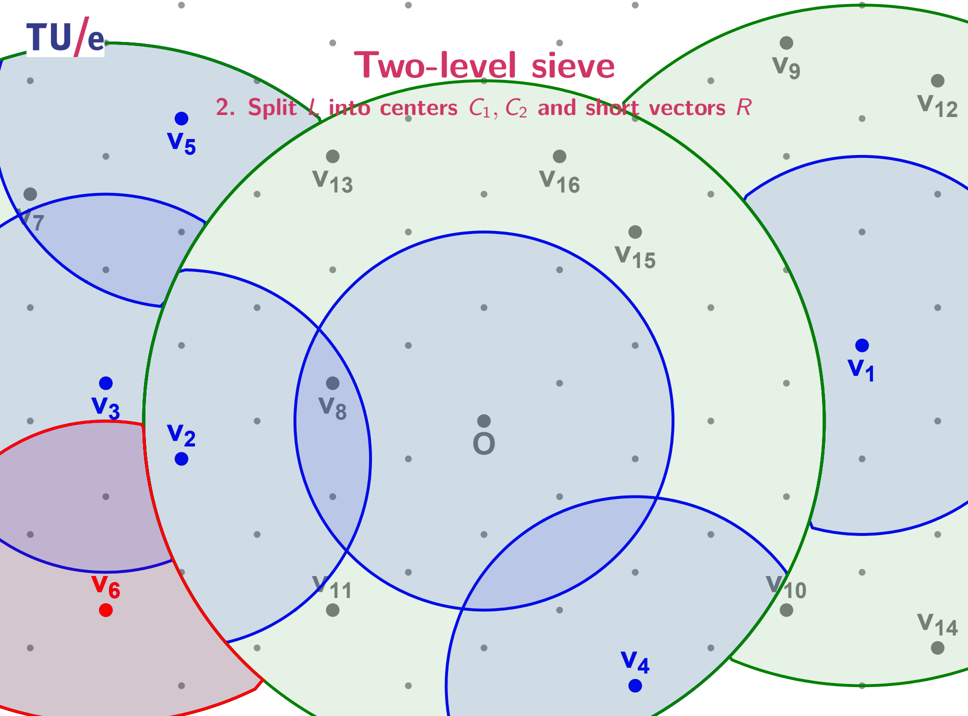

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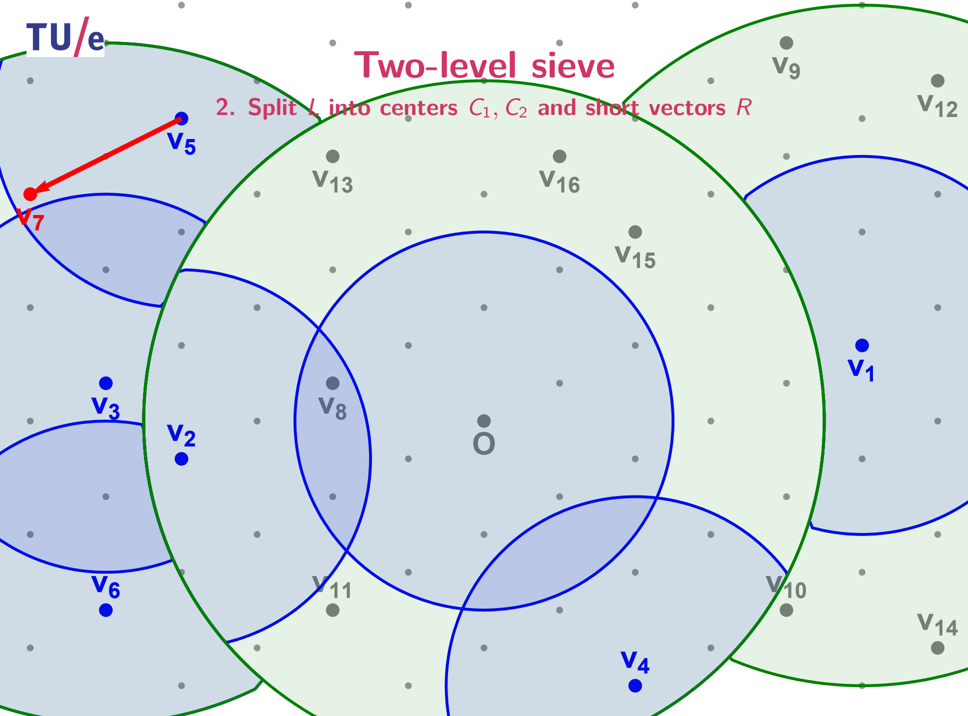

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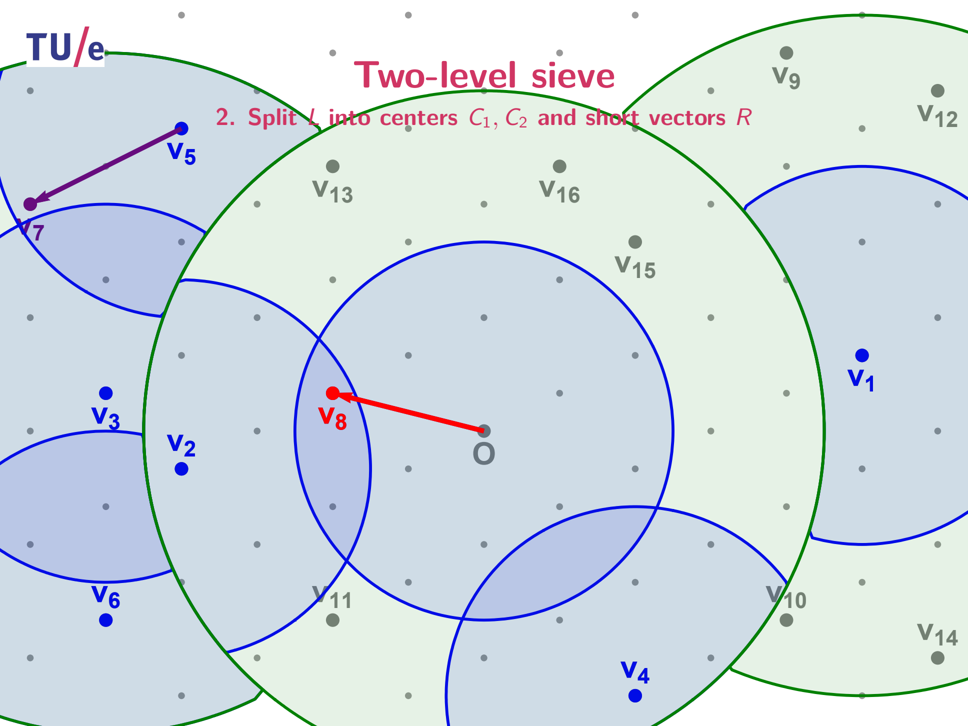

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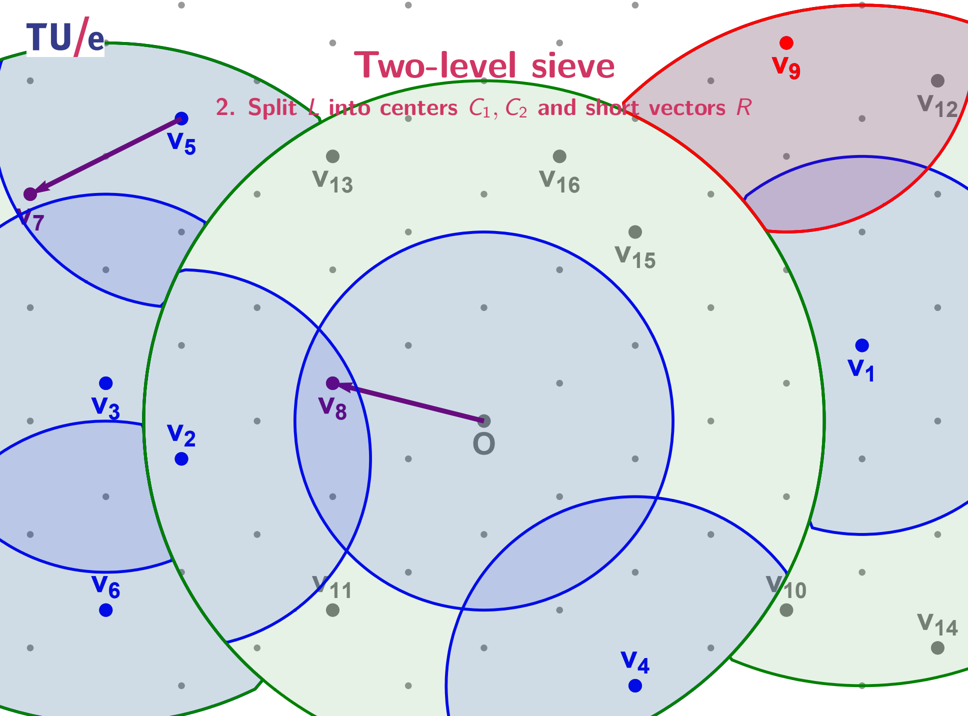

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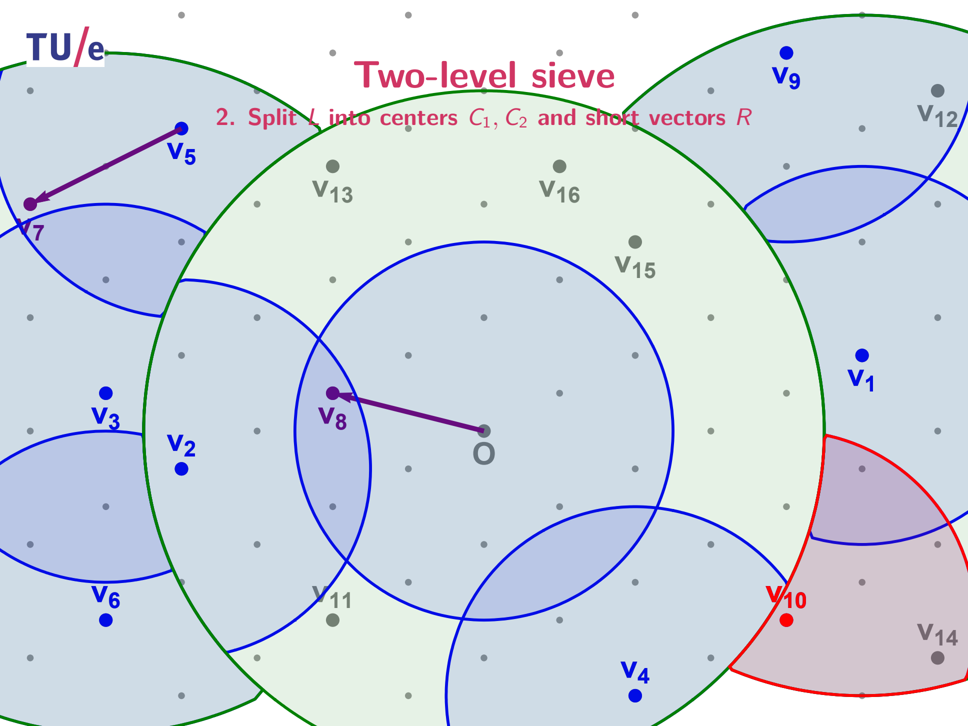

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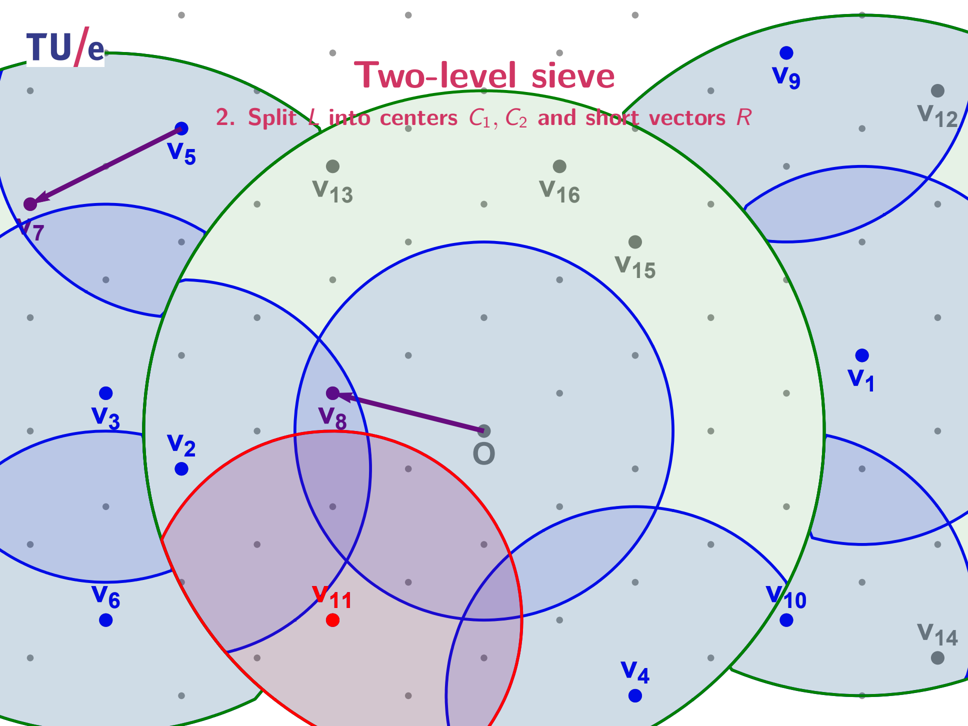

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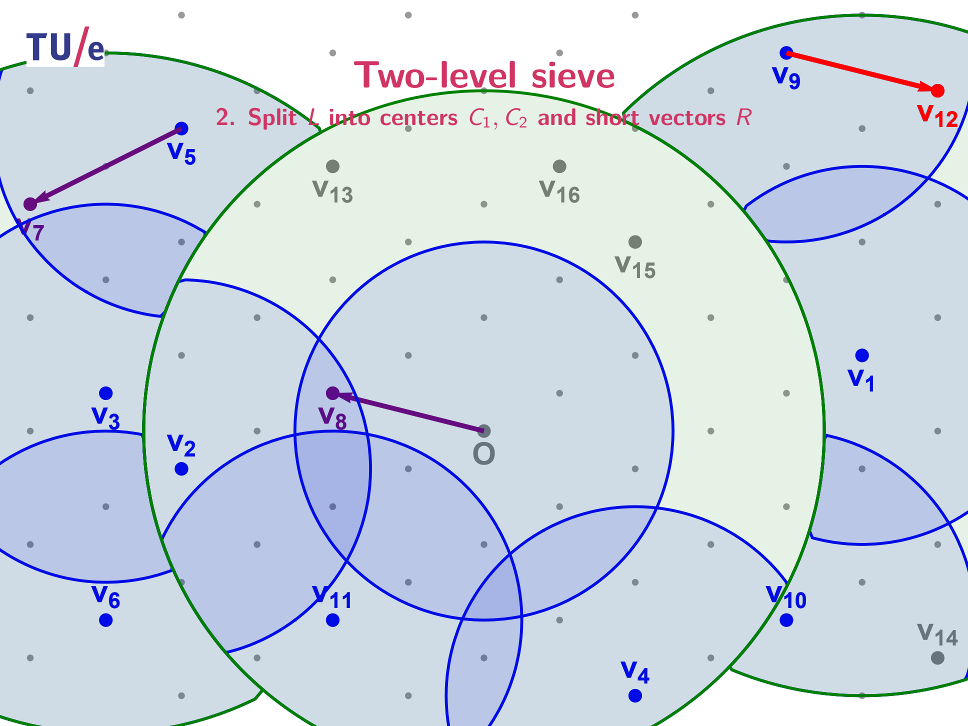

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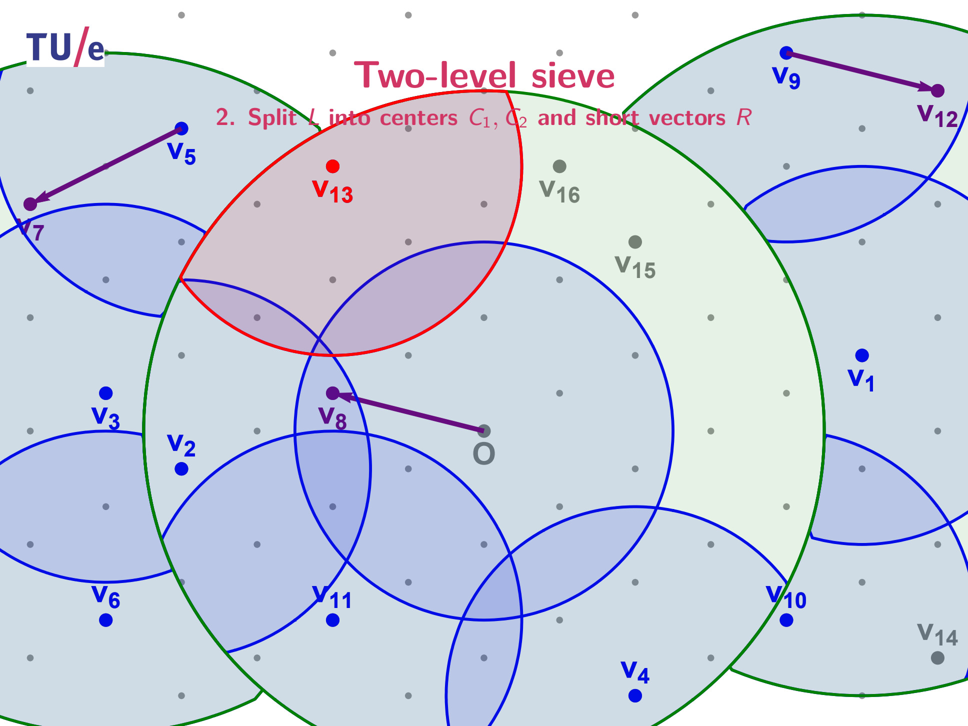

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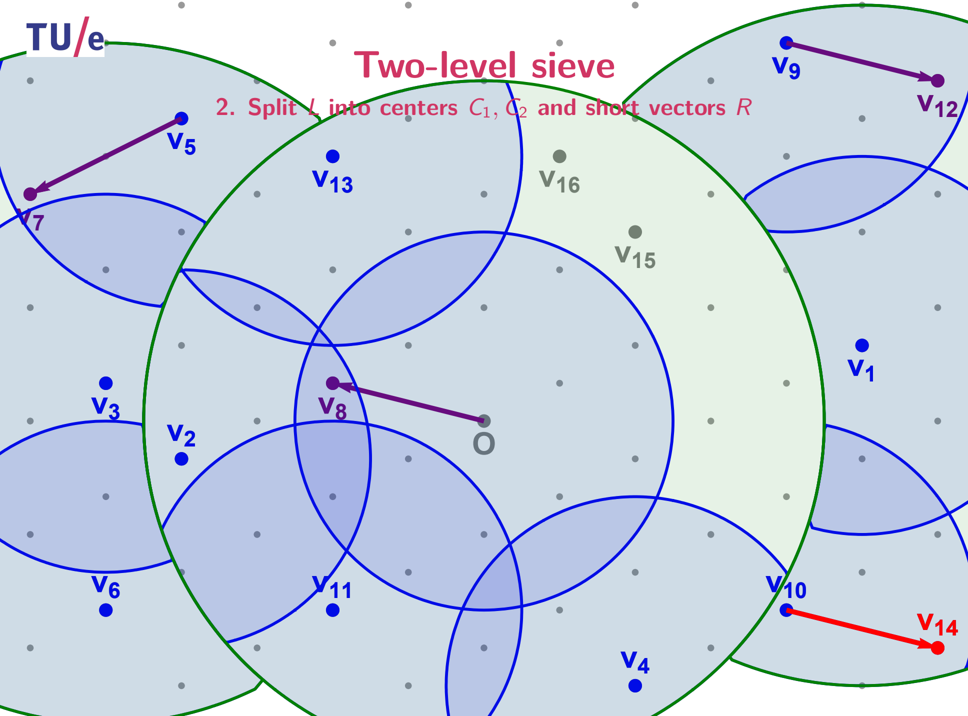

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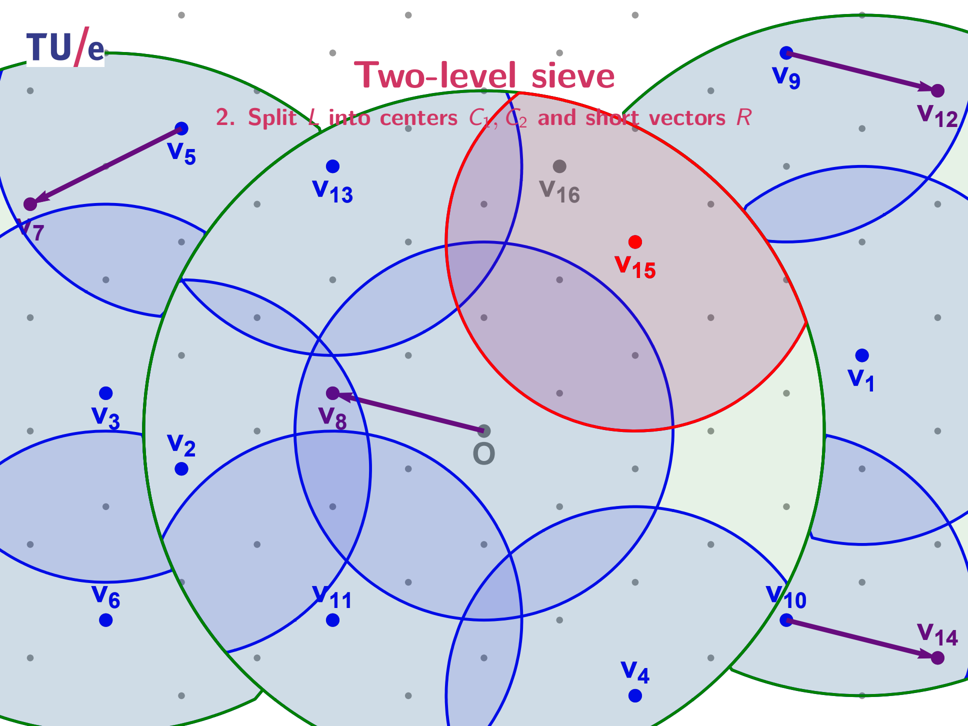

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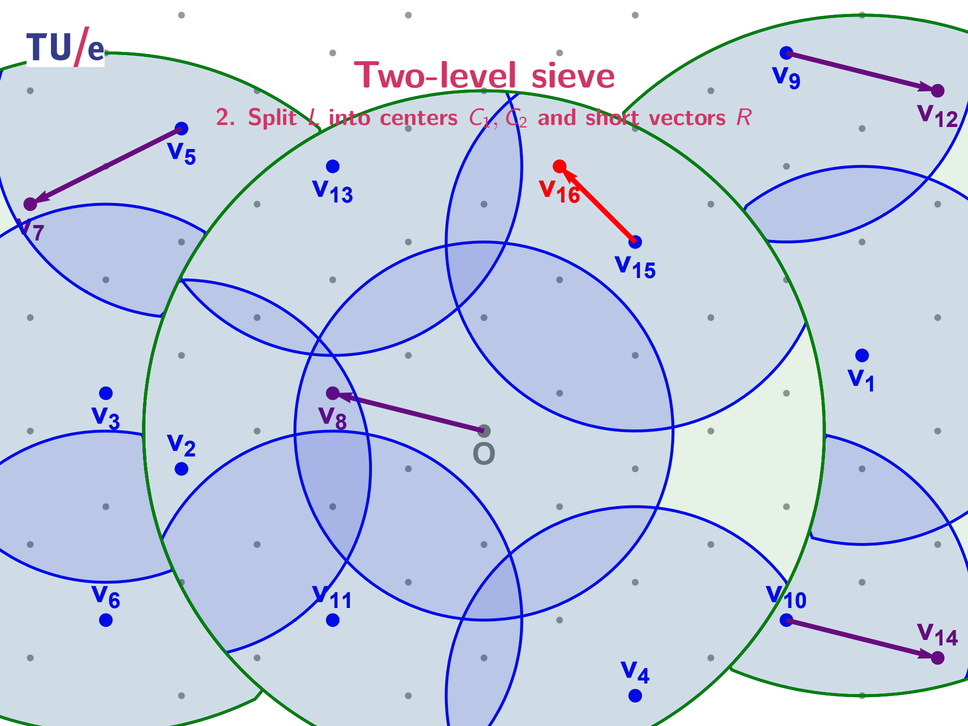

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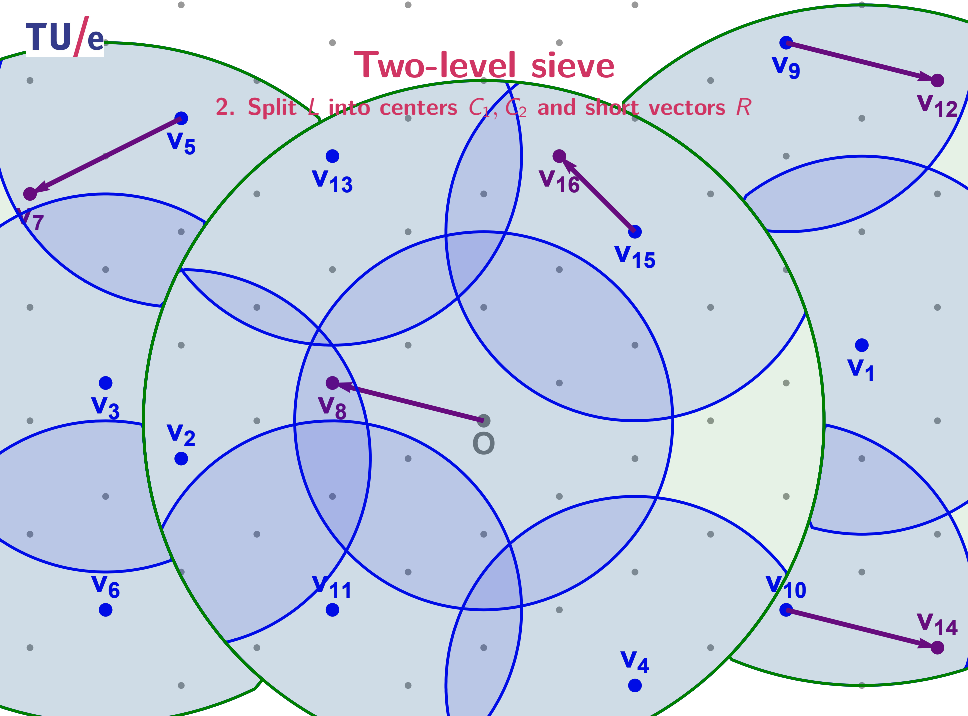

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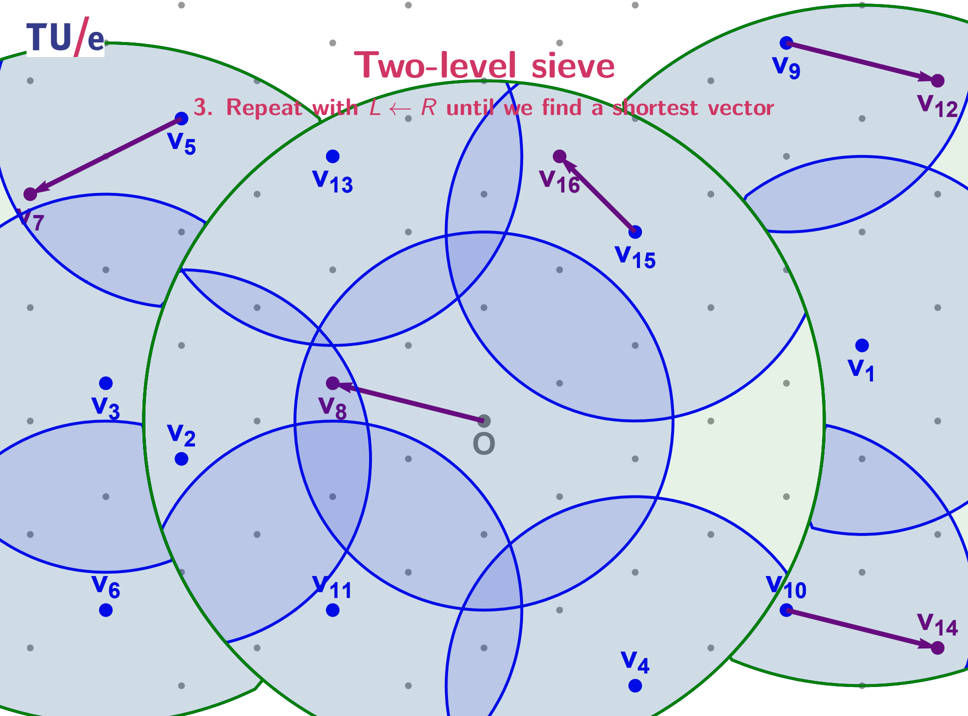

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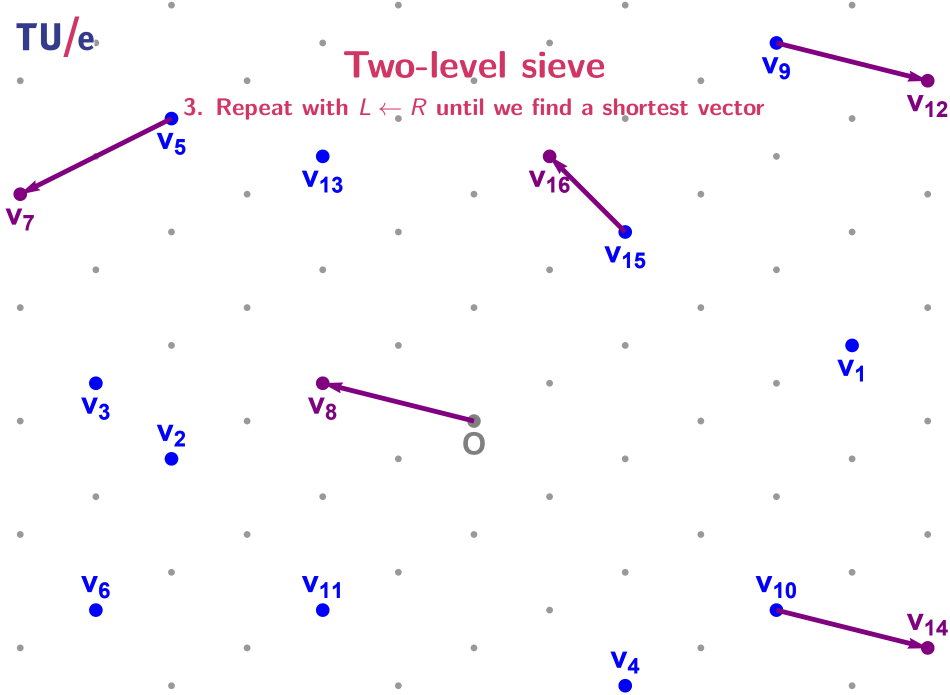

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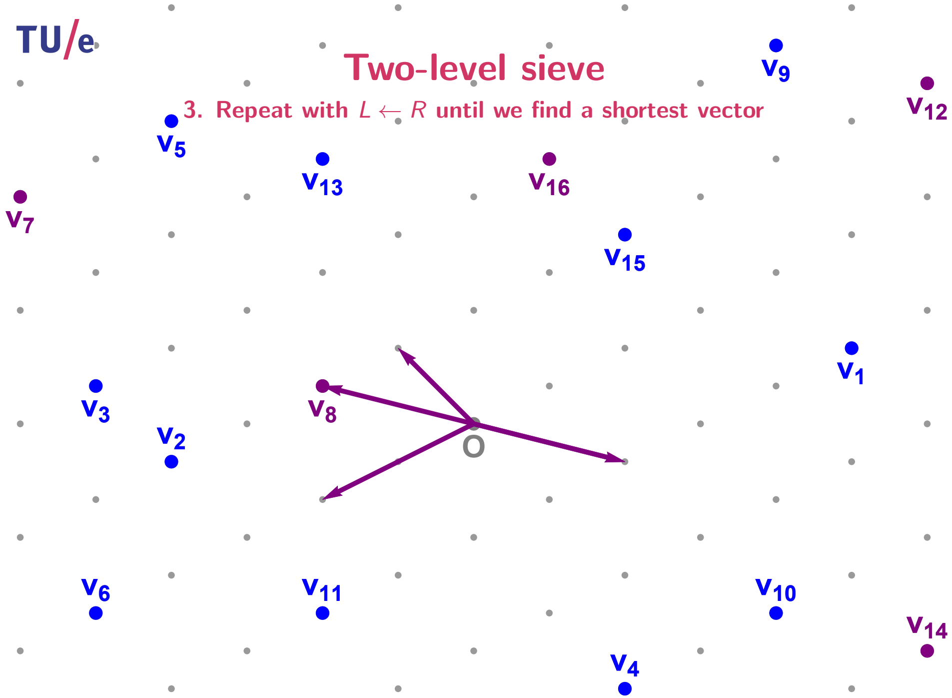

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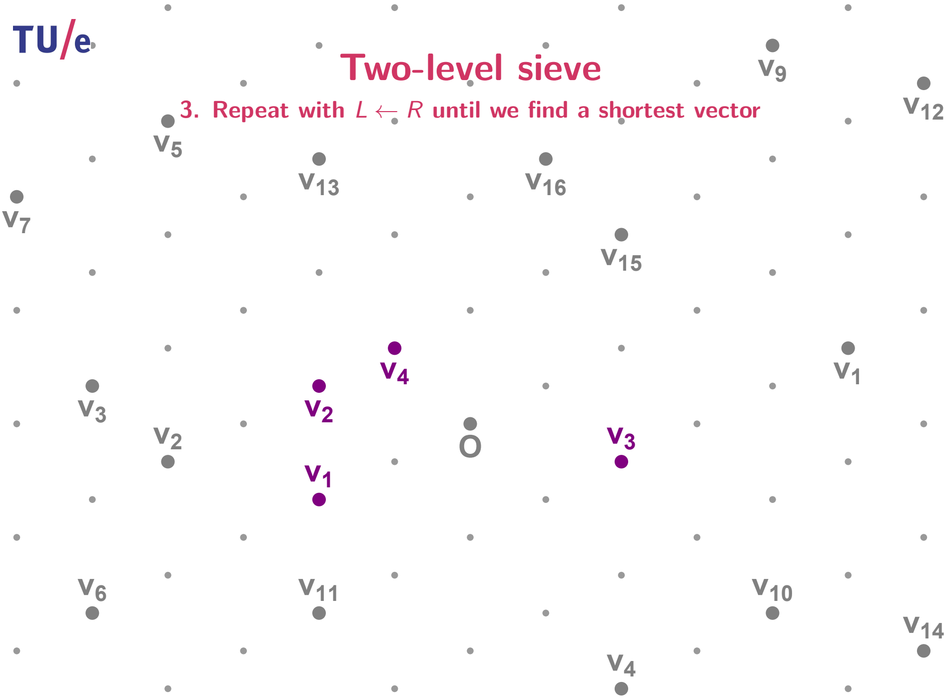
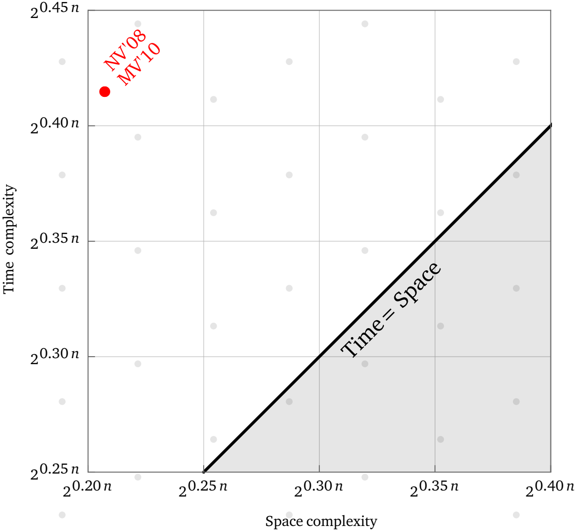


Two-level sieve

Space/time trade-off



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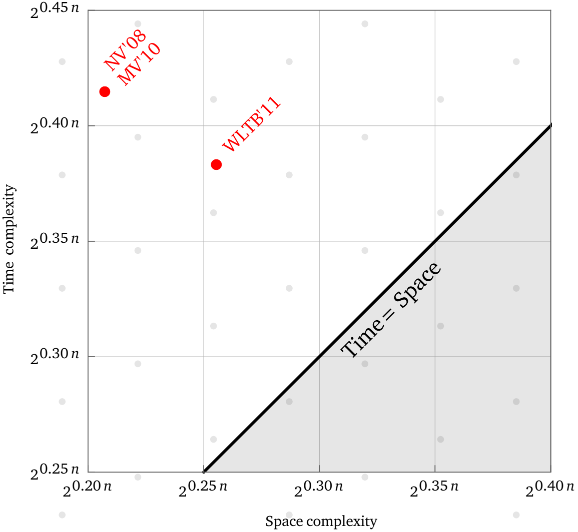

Three-level sieve

Overview

Heuristic (Nguyen and Vidick, J. Math. Crypt. '08)

The one-level sieve runs in time $2^{0.4150n}$ and space $2^{0.2075n}$.

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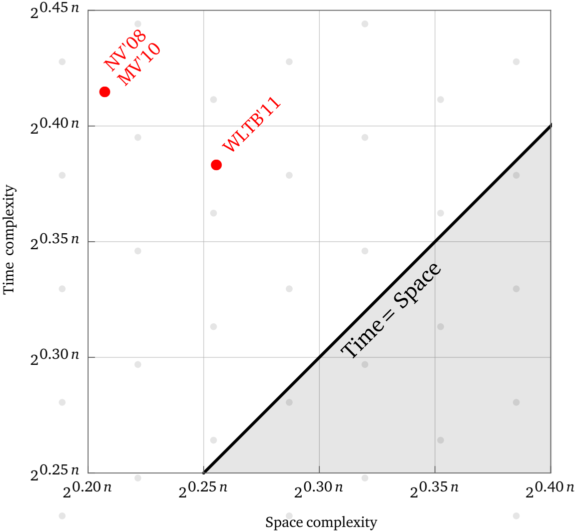
Conjecture

The four-level sieve runs in time $2^{0.3774n}$ and space $2^{0.2925n}$, and higher-level sieves are not faster than this.

Three-level sieve

Space/time trade-off



Three-level sieve

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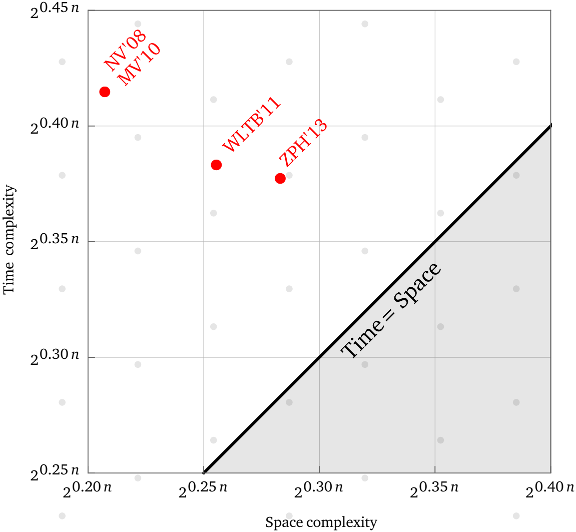
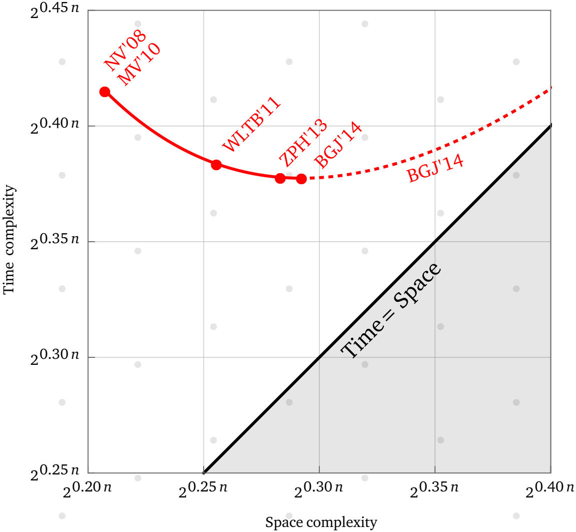


TU/e

Decomposition approach

Space/time trade-off



Hyperplane LSH

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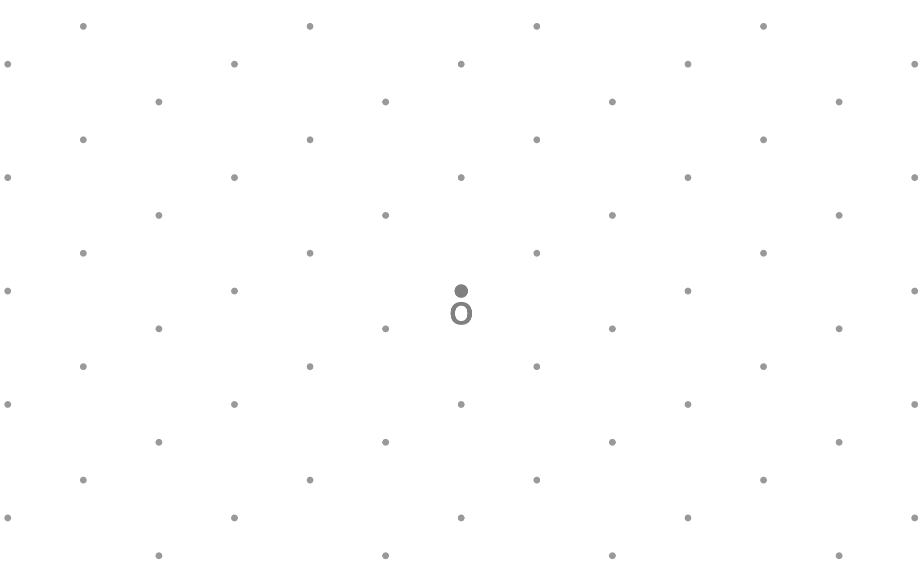

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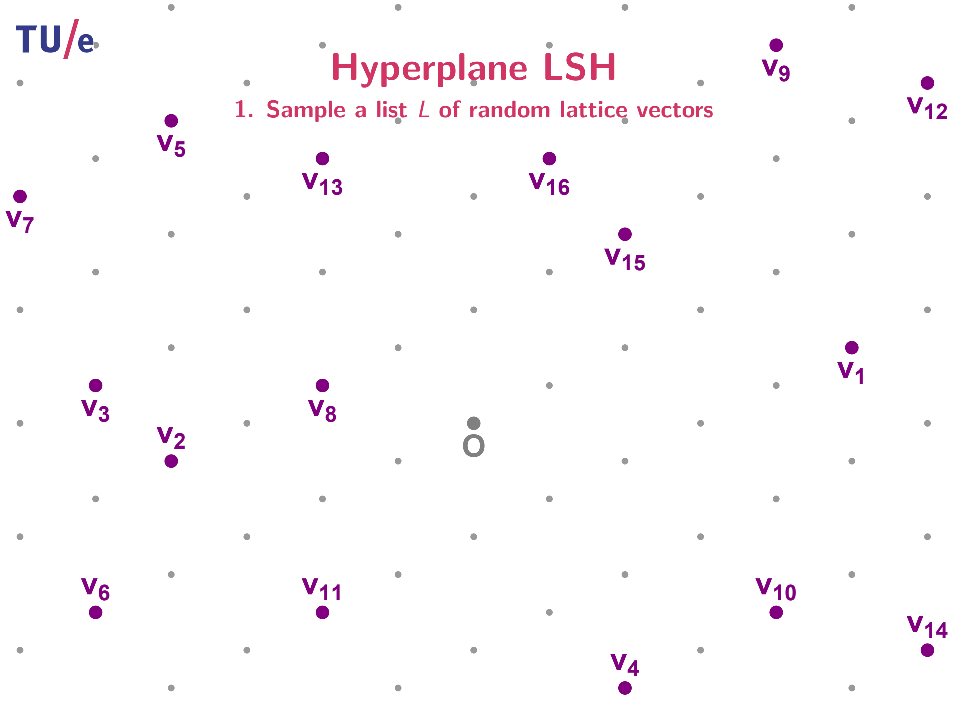
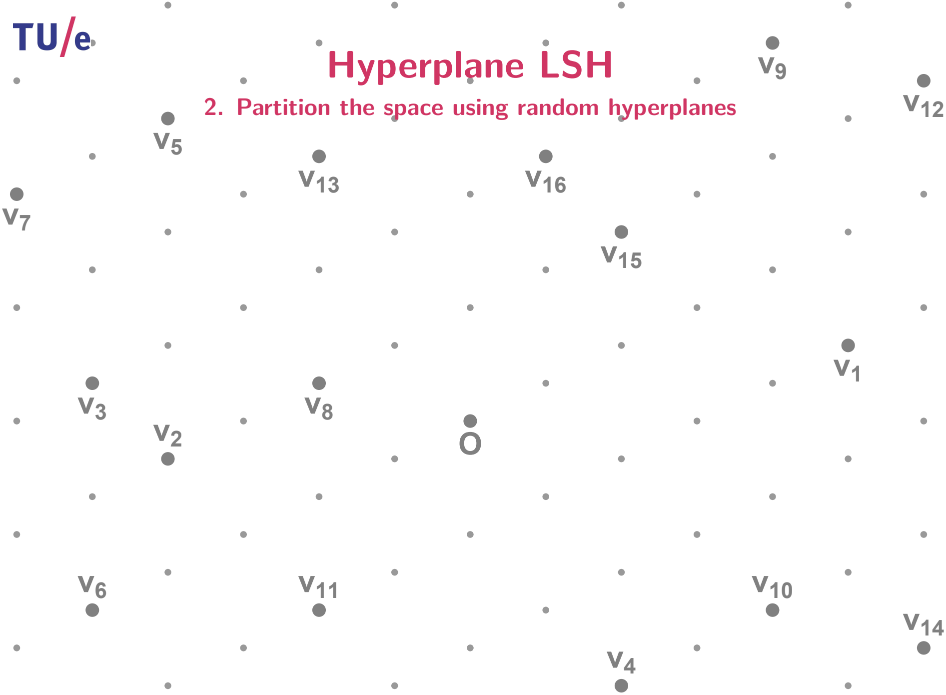


Hyperplane LSH

2. Partition the space using random hyperplanes



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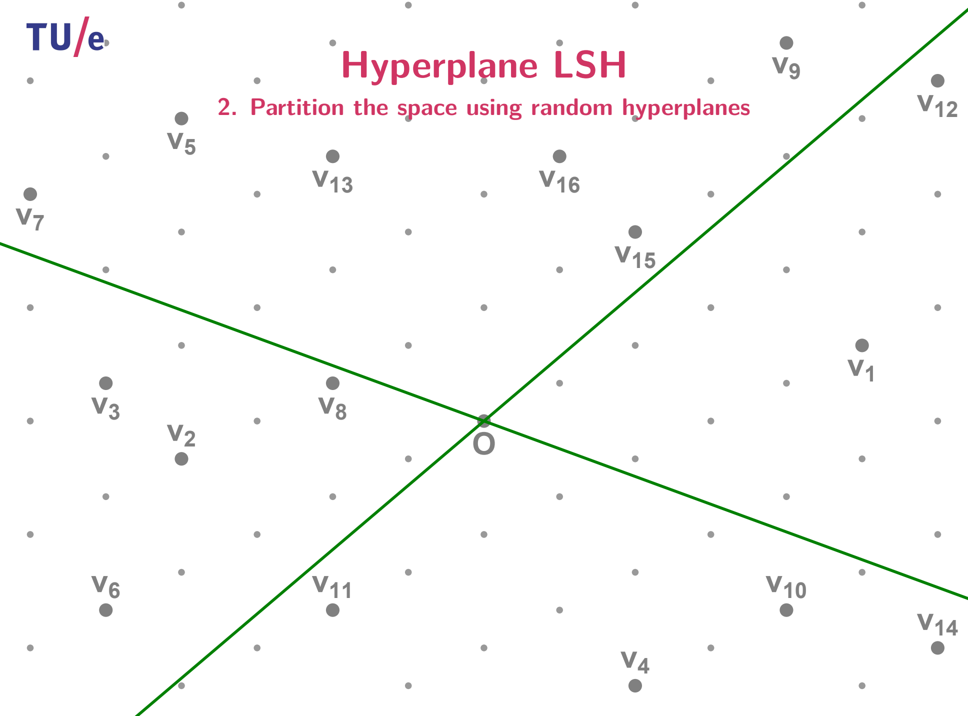

Hyperplane LSH

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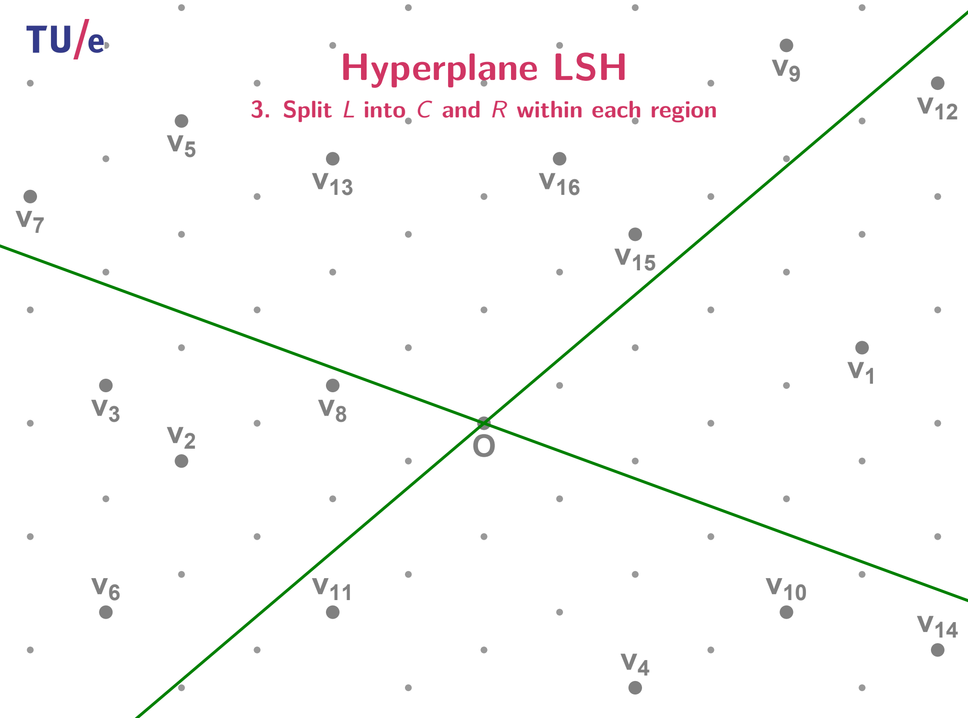

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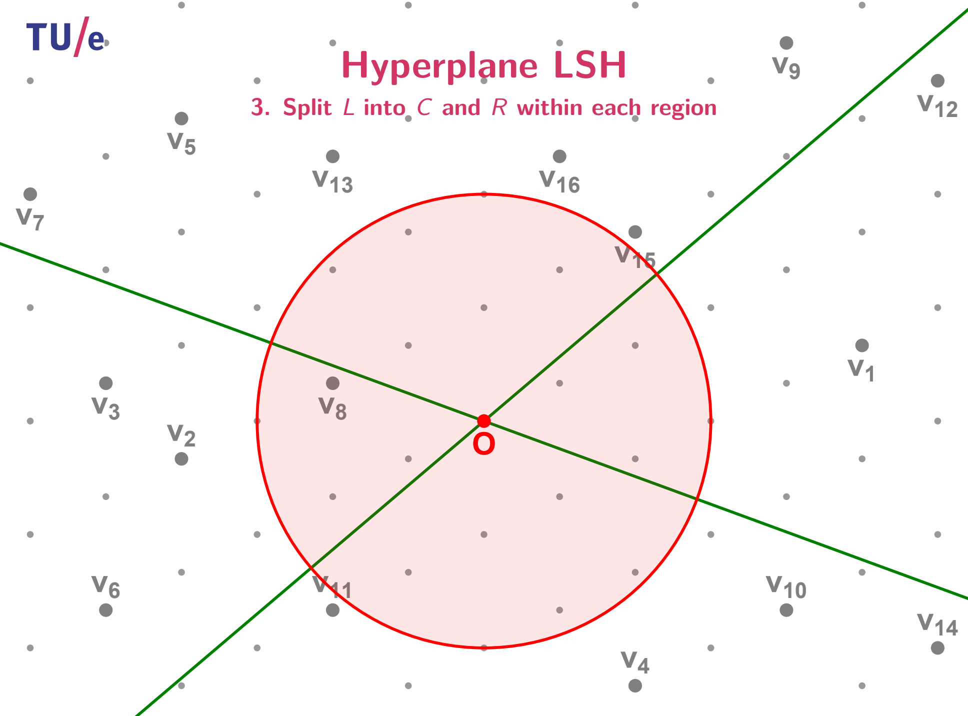

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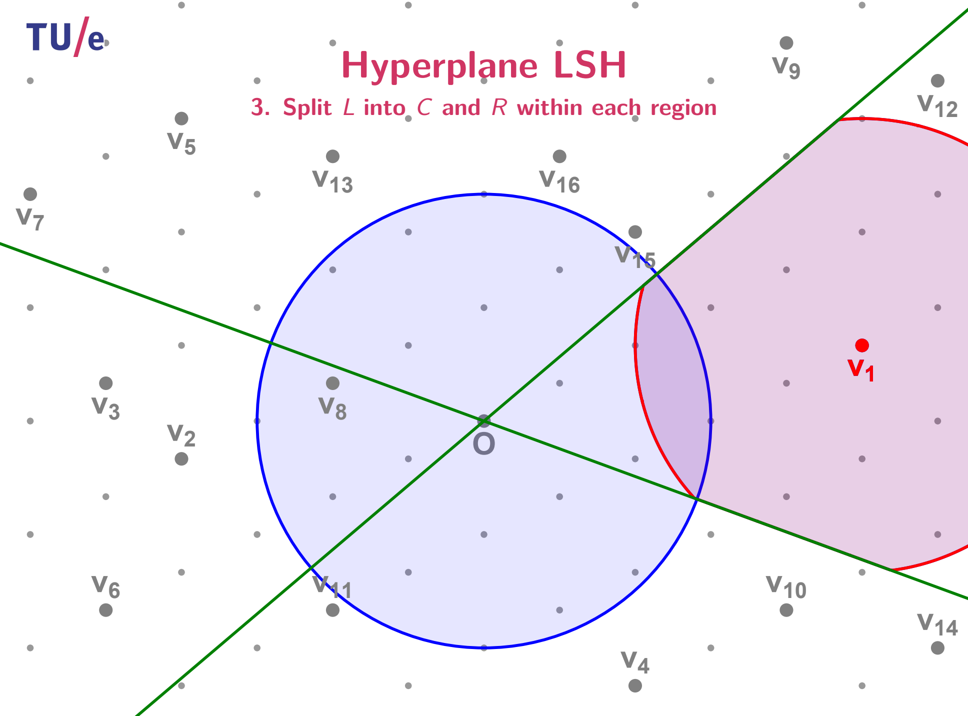

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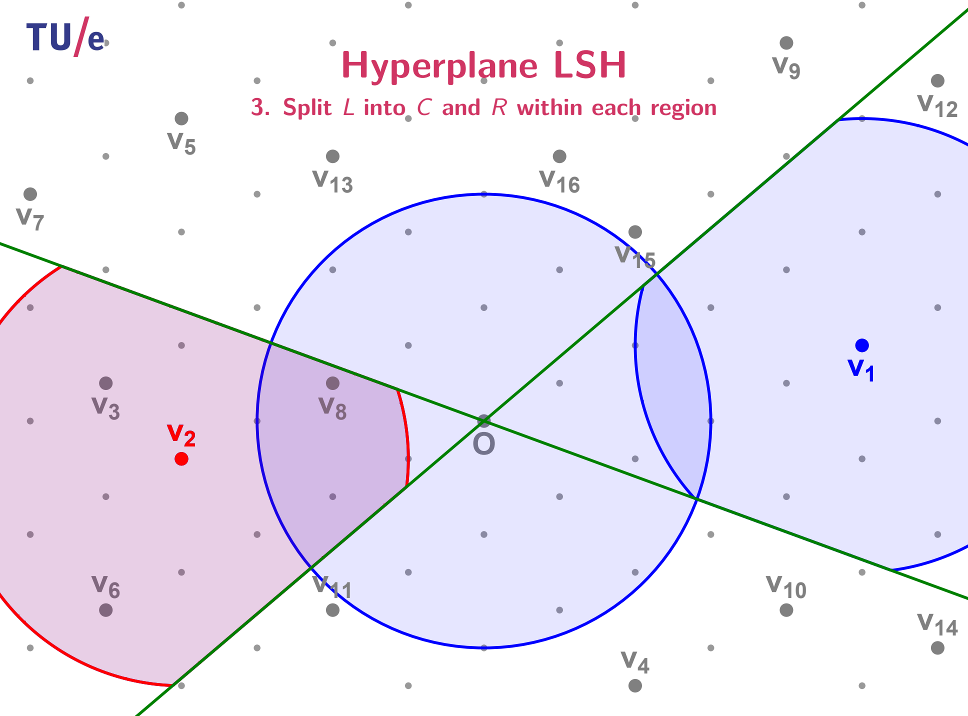

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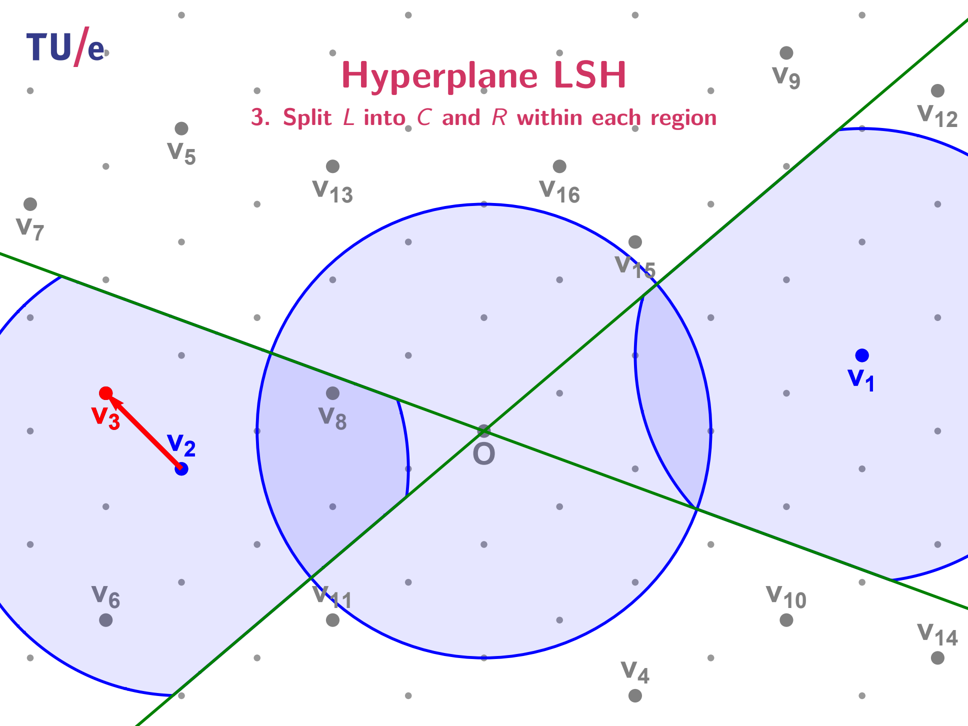

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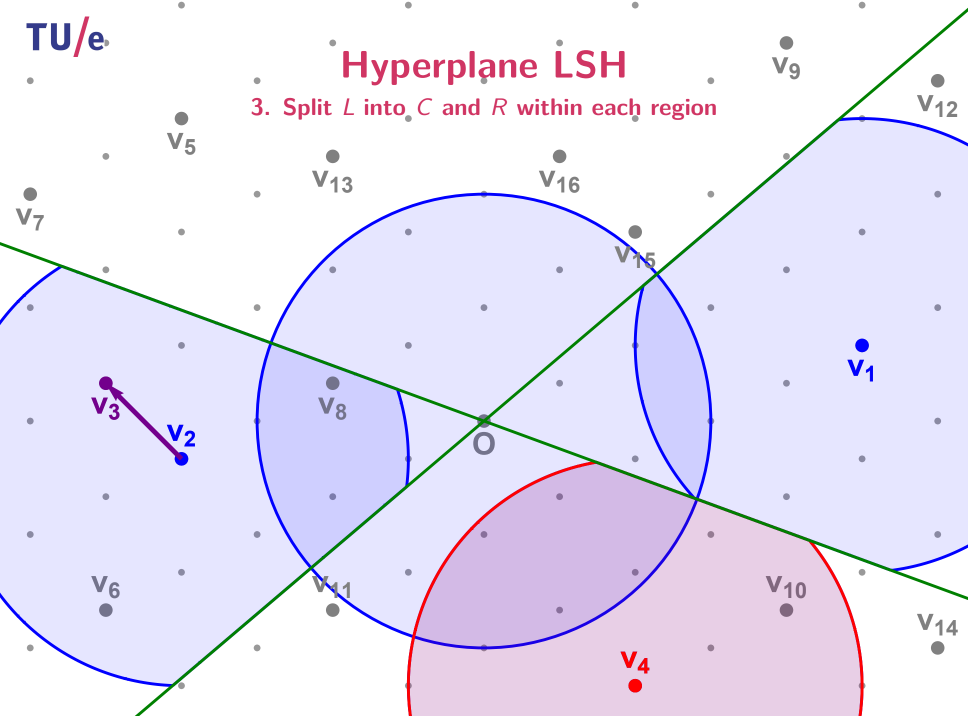

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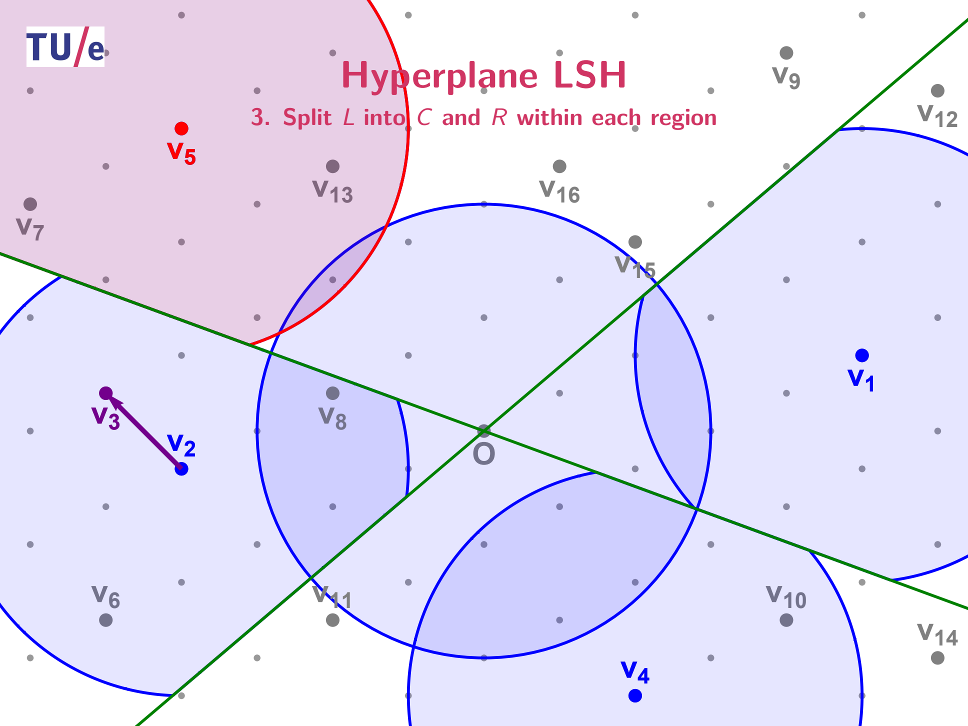

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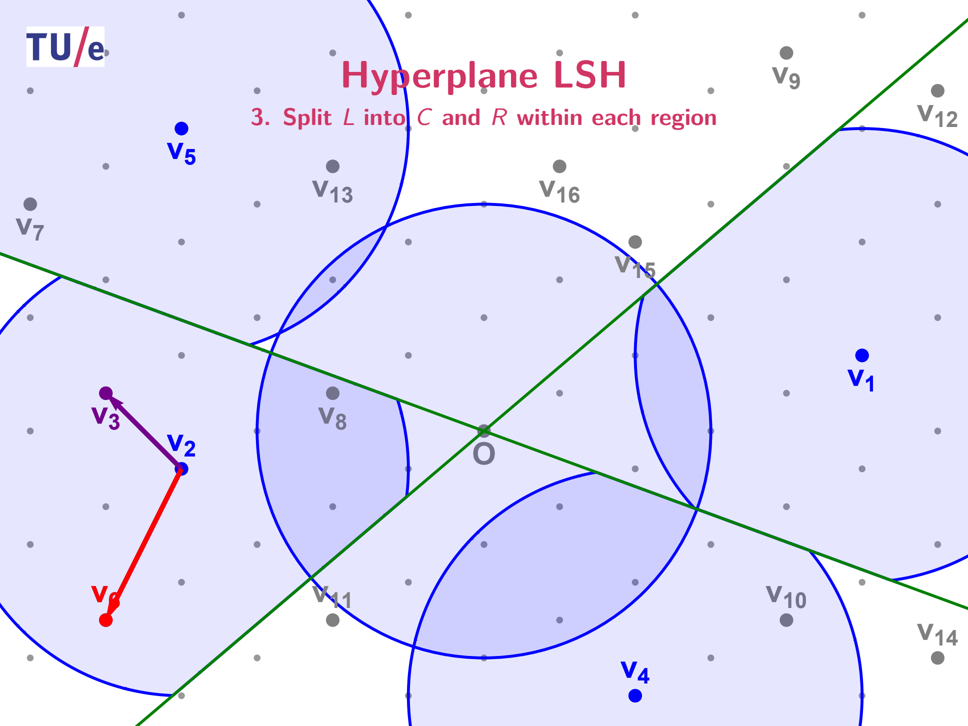

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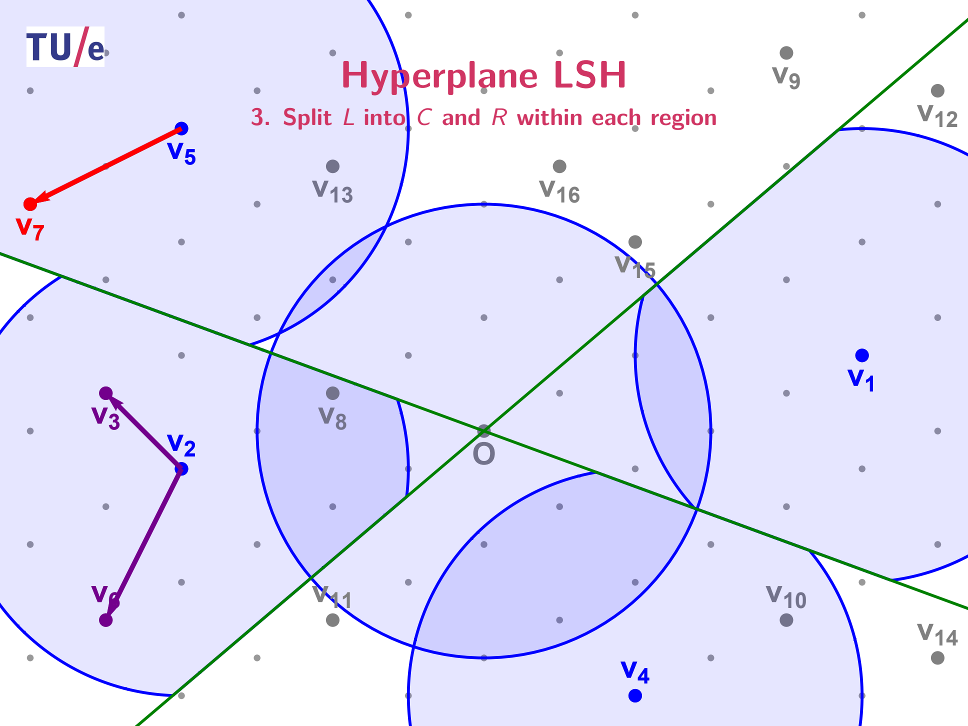

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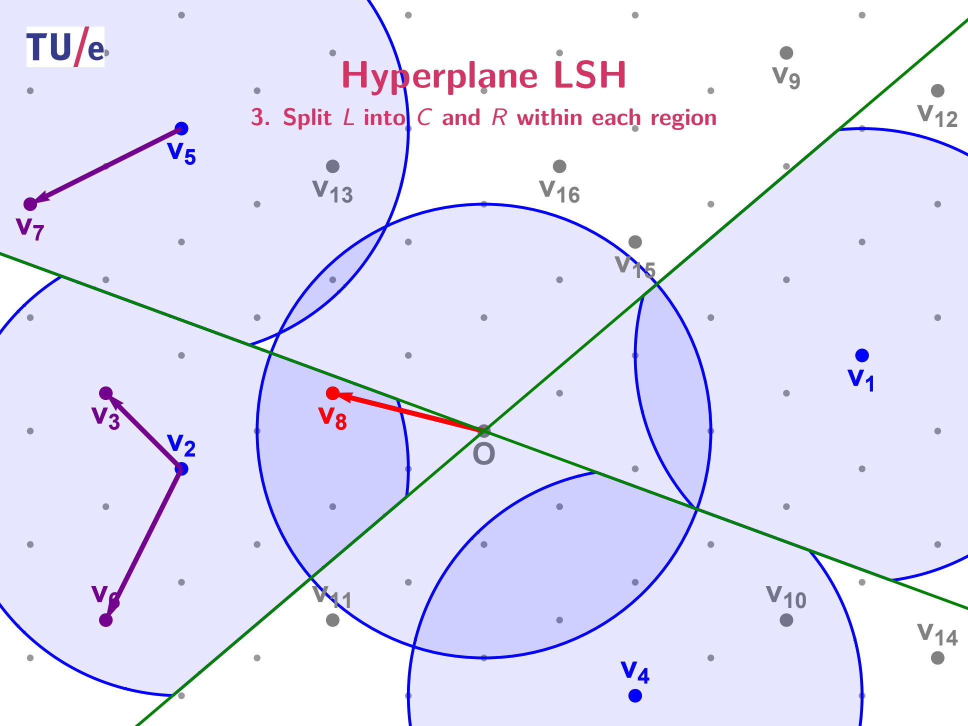

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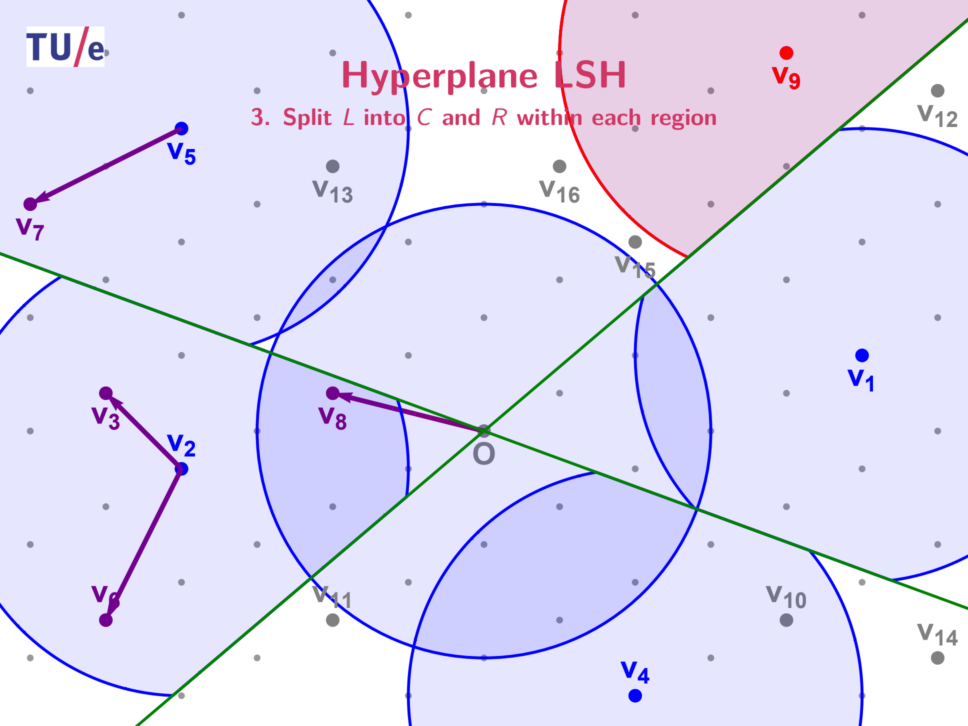

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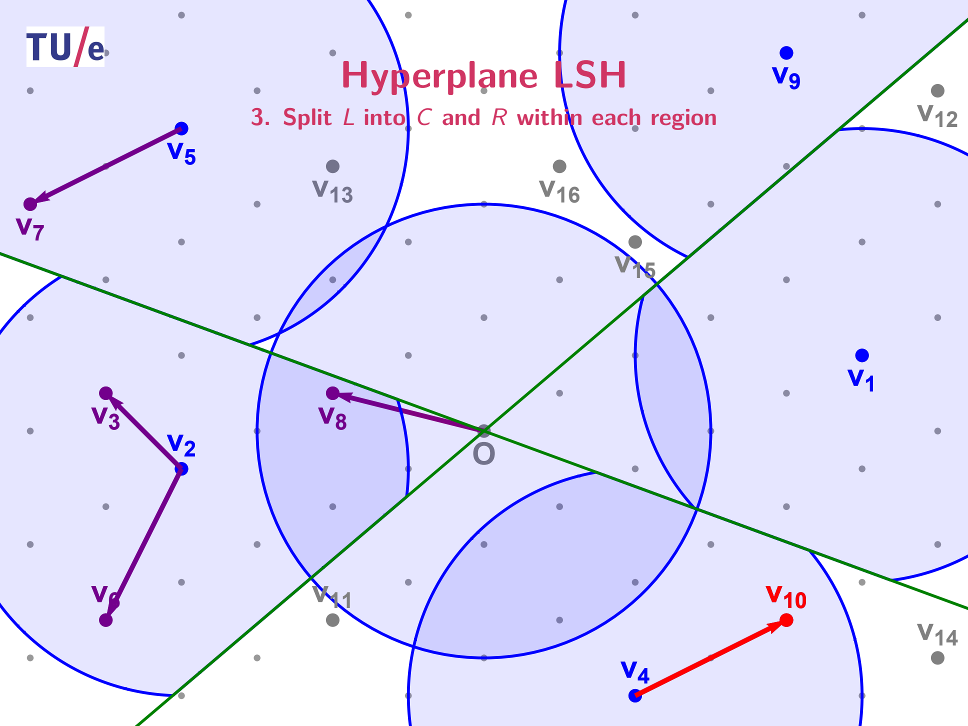

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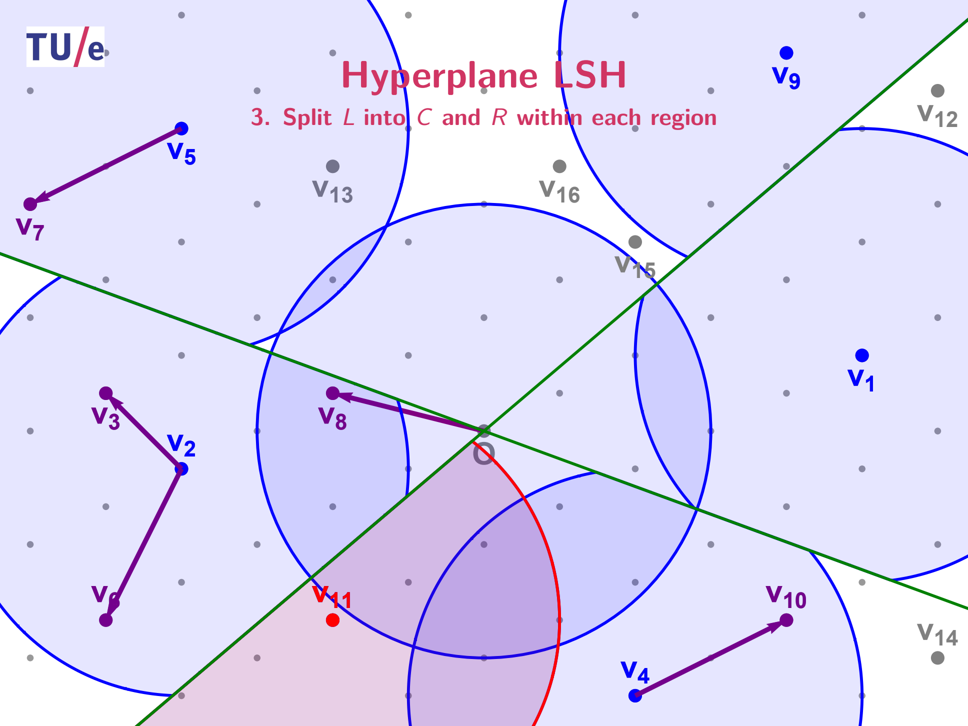

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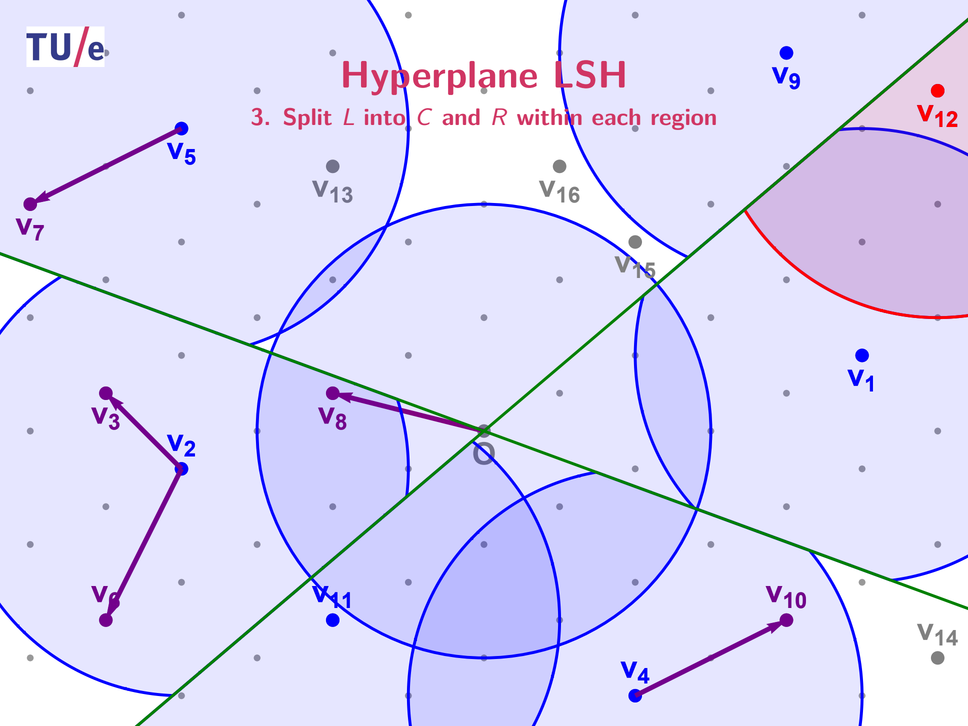

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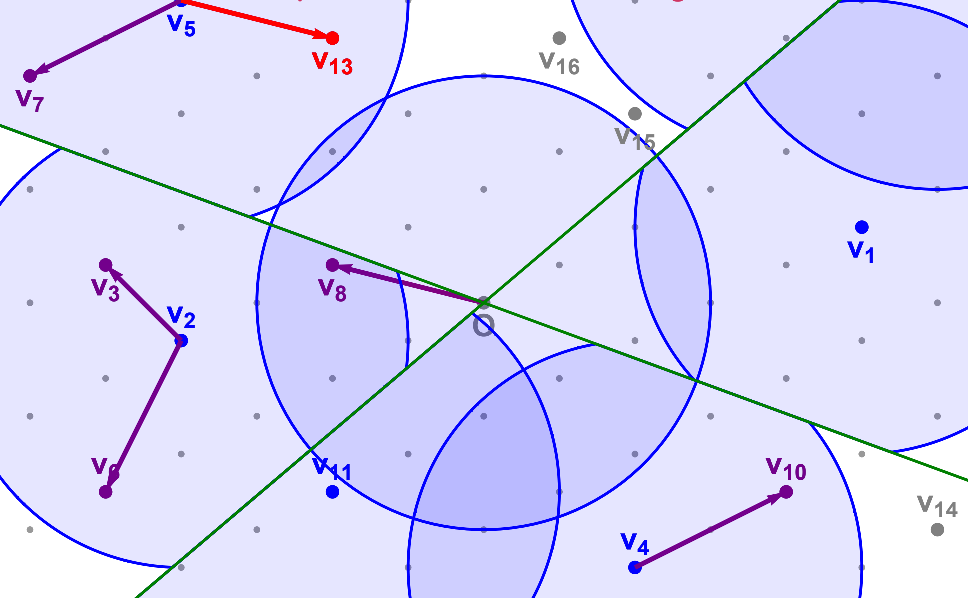

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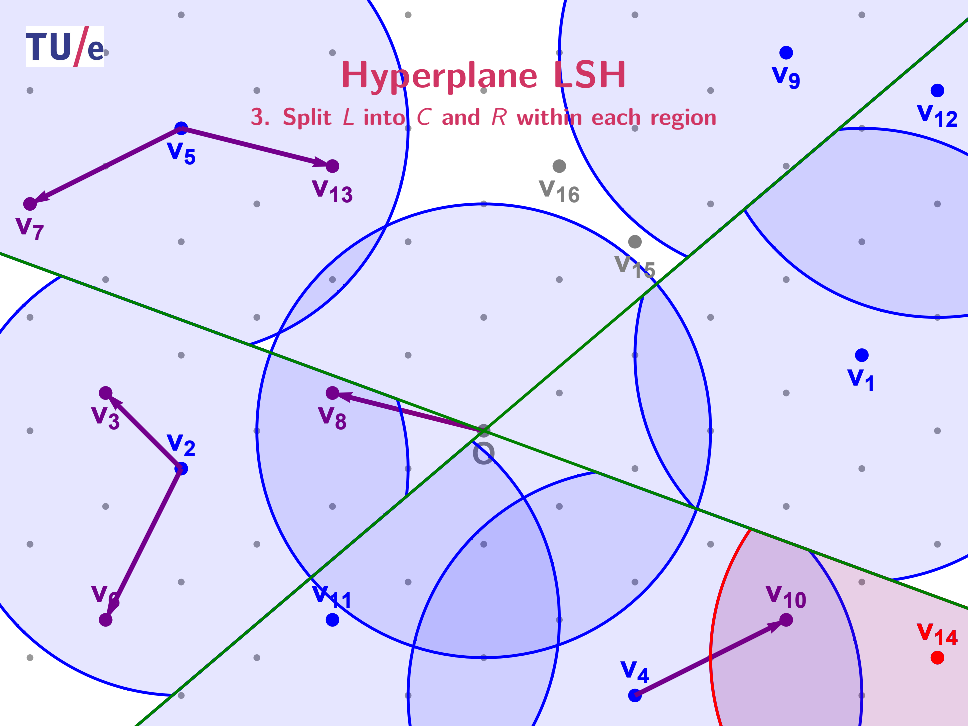

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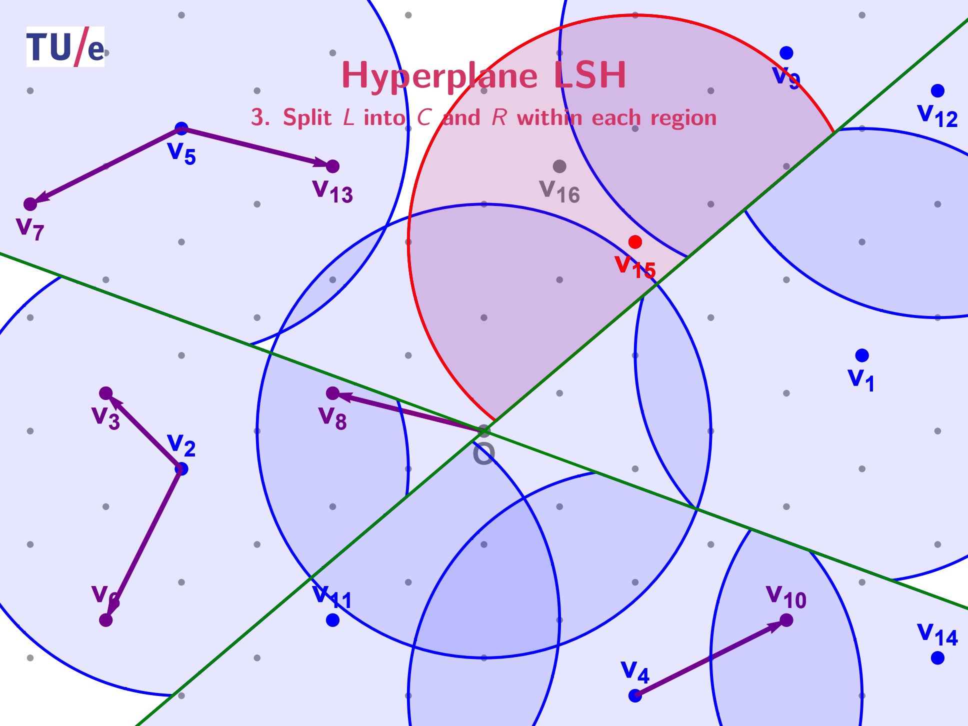

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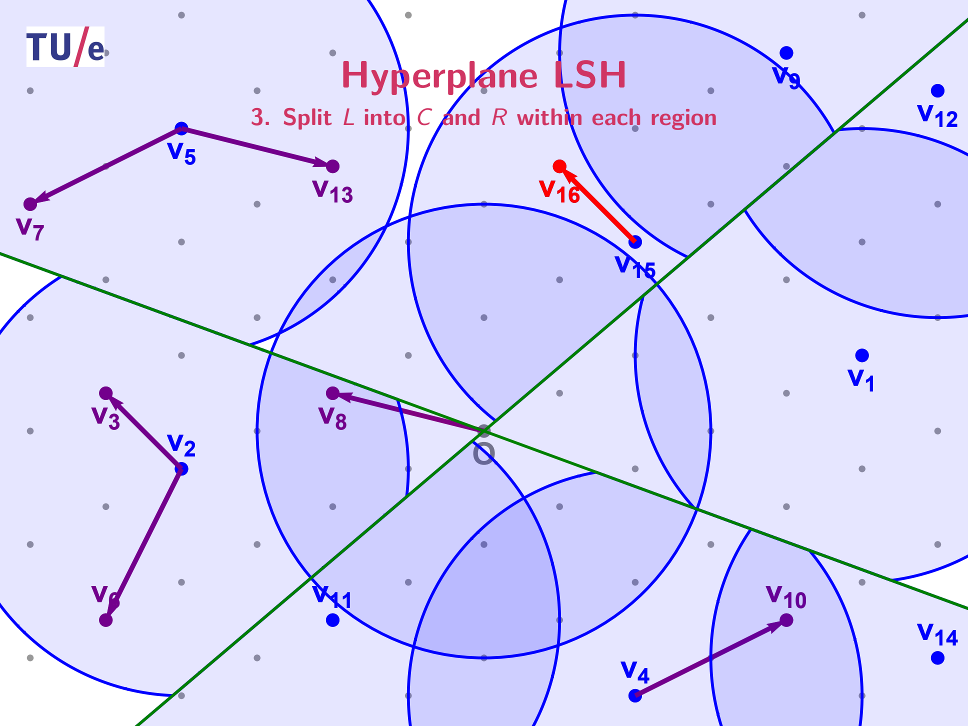

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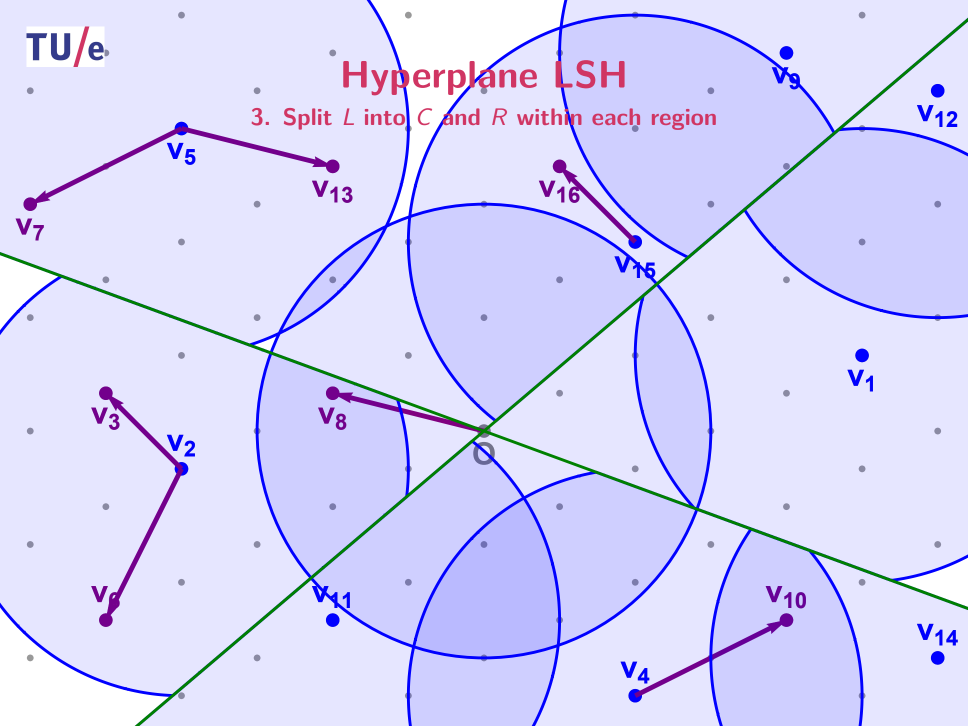

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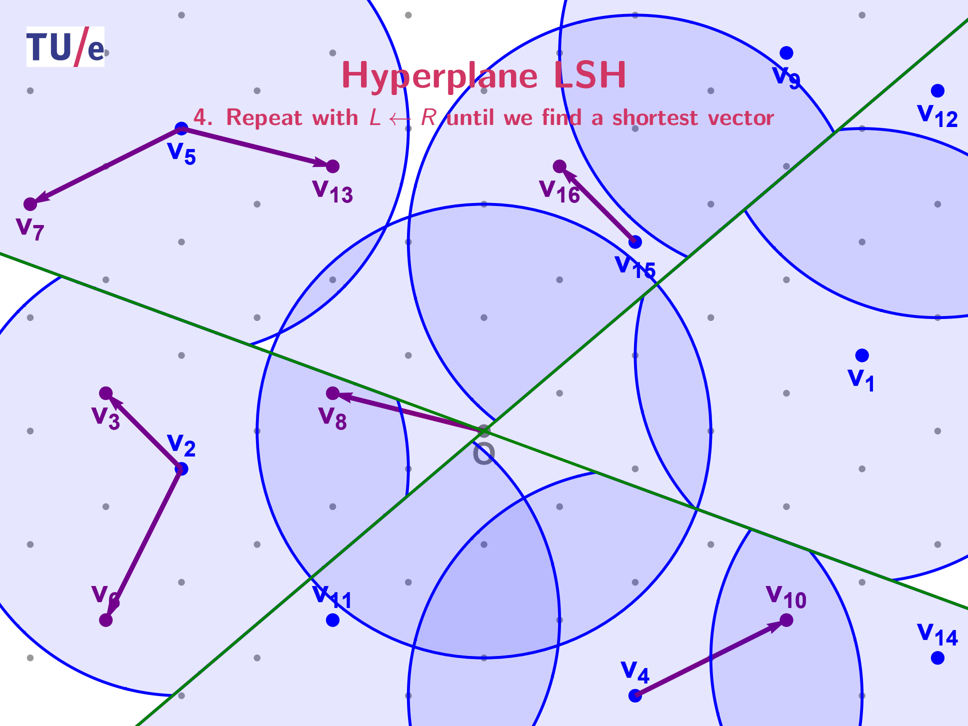

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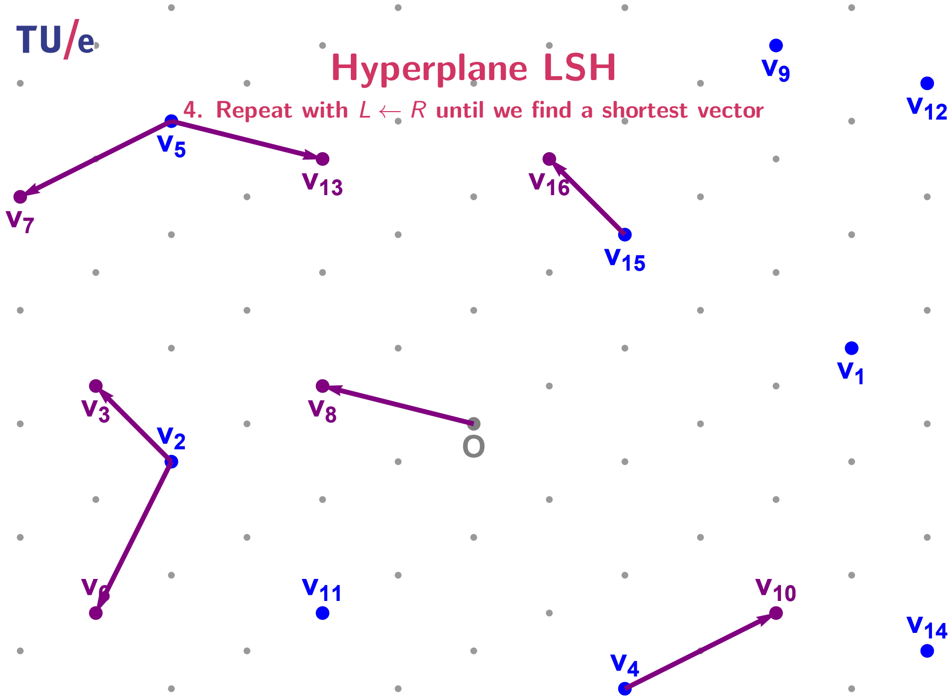

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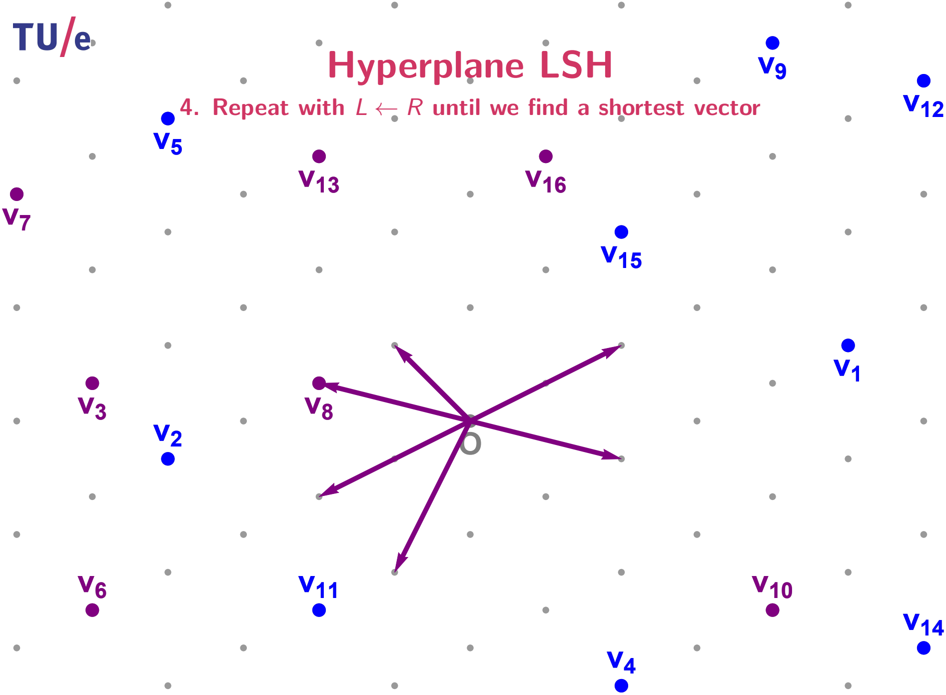

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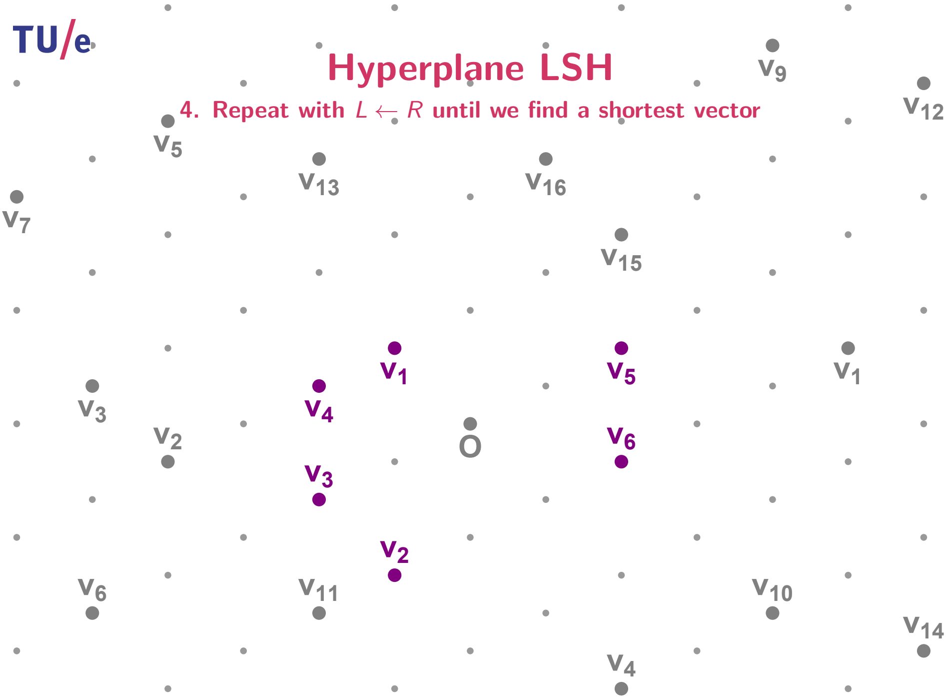

Hyperplane LSH

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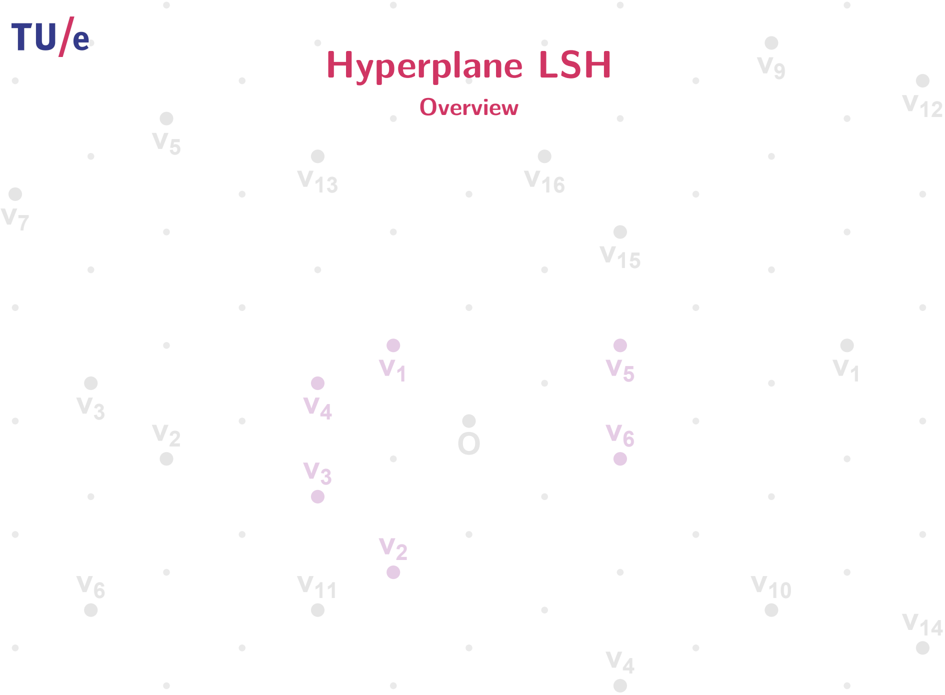

Hyperplane LSH

Overview

- Two parameters to tune
  - $k = O(n)$: Number of hyperplanes, leading to 2^k regions
  - $t = 2^{O(n)}$: Number of different, independent “hash tables”

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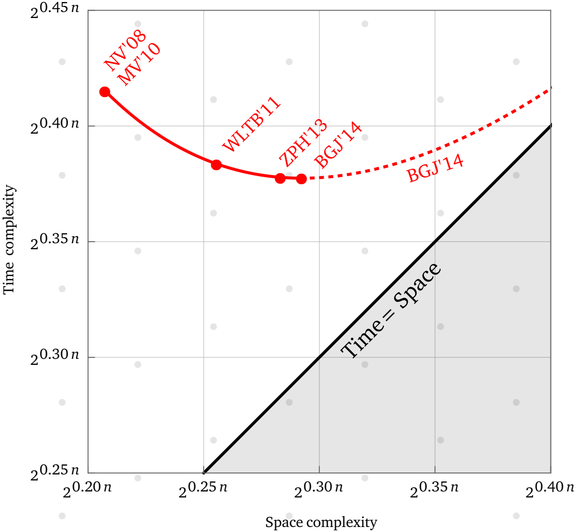
Theorem

Sieving with hyperplane LSH heuristically solves SVP in time and space $2^{0.337n+o(n)}$.

Hyperplane LSH

Space/time trade-off



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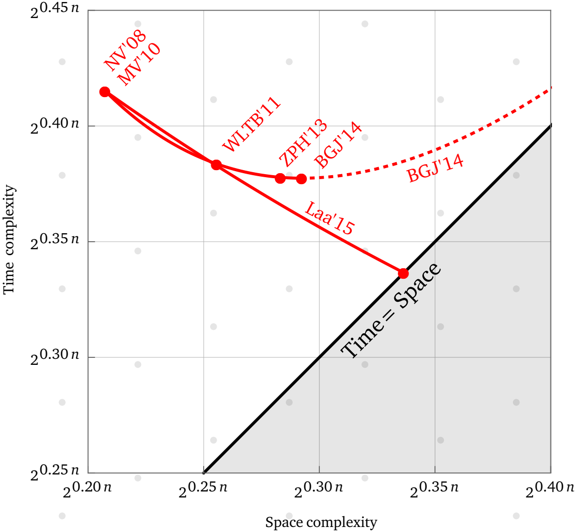

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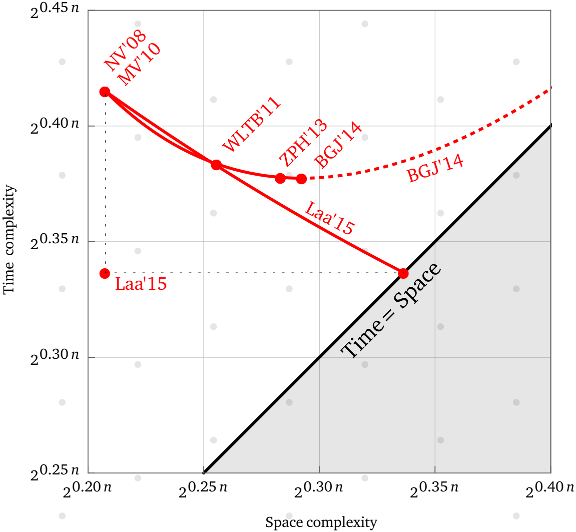

Spherical LSH

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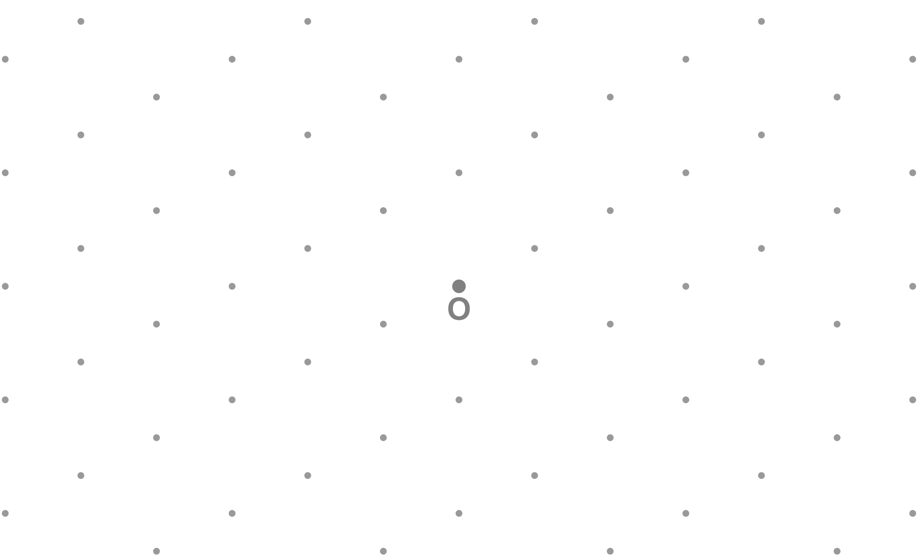

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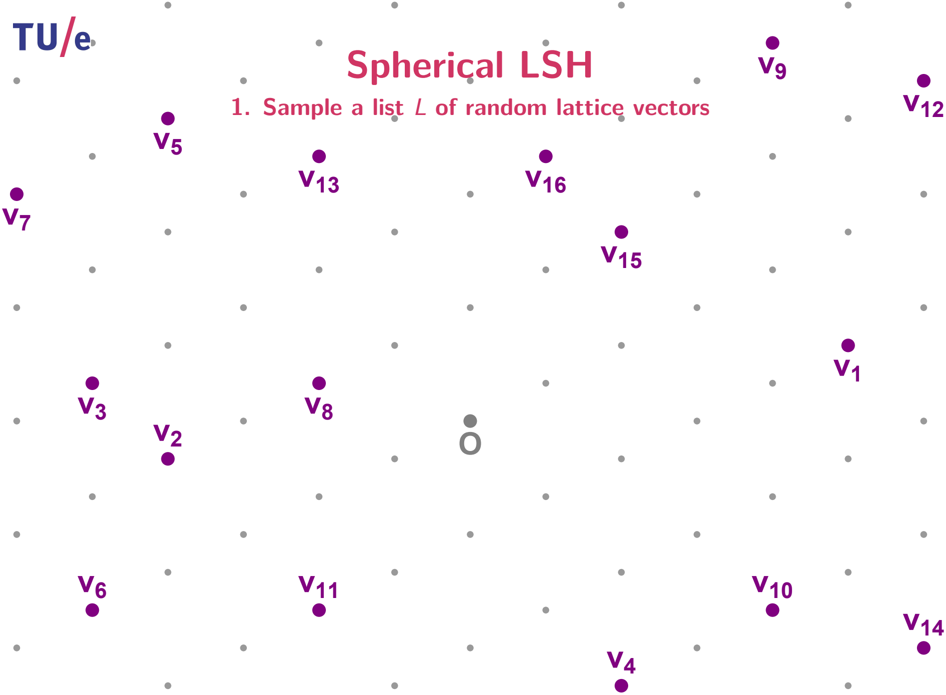

Spherical LSH

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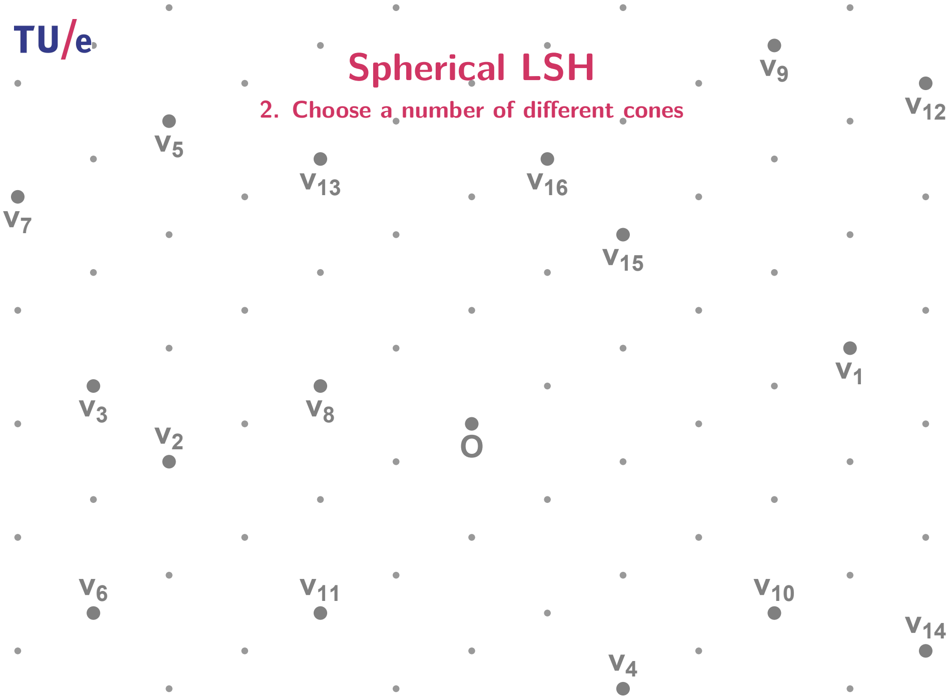

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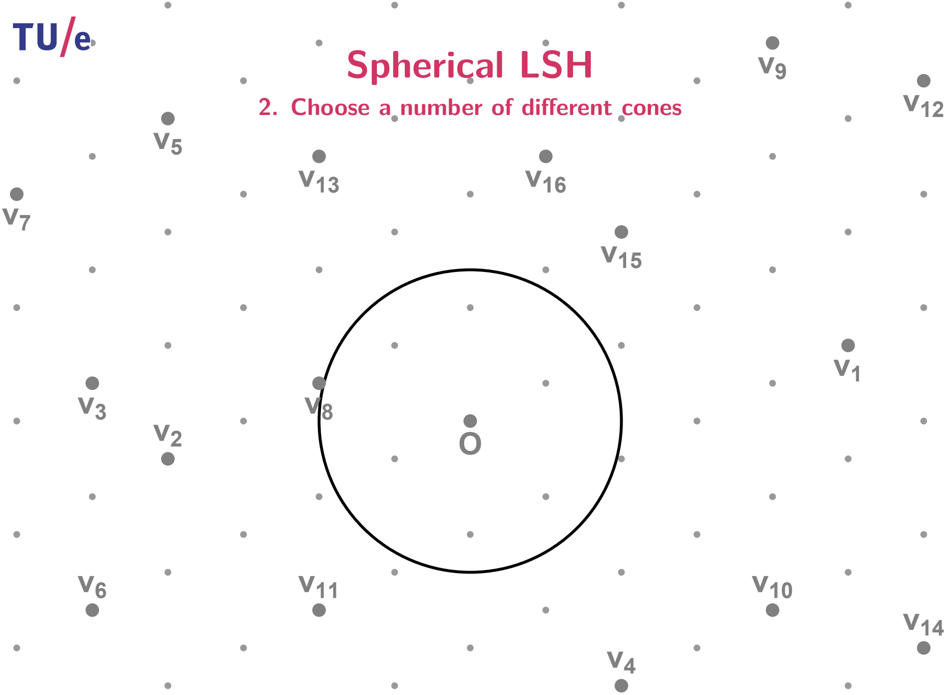

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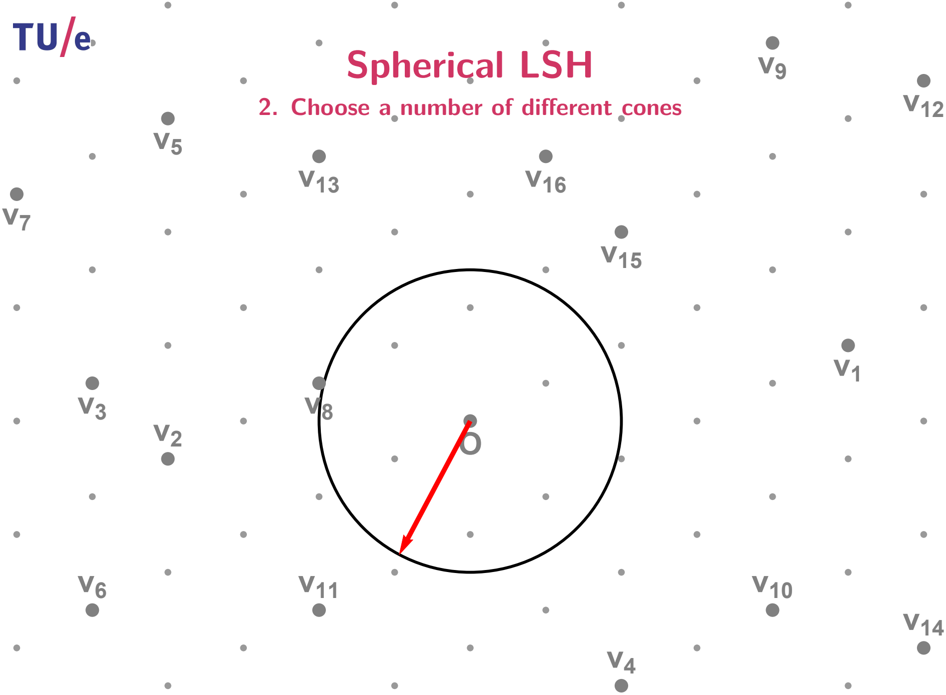

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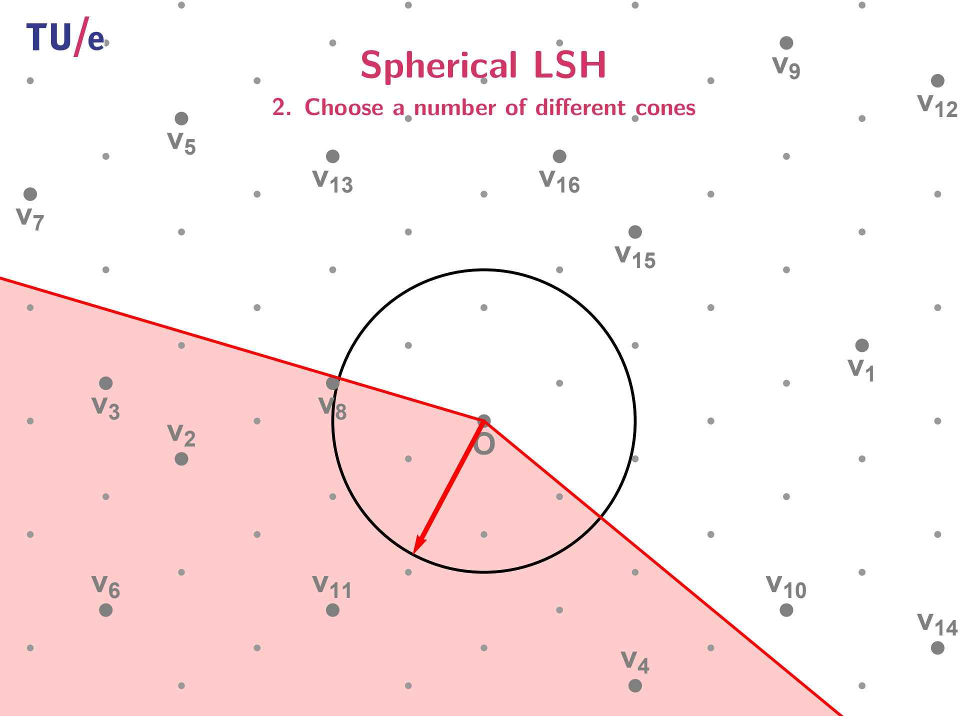

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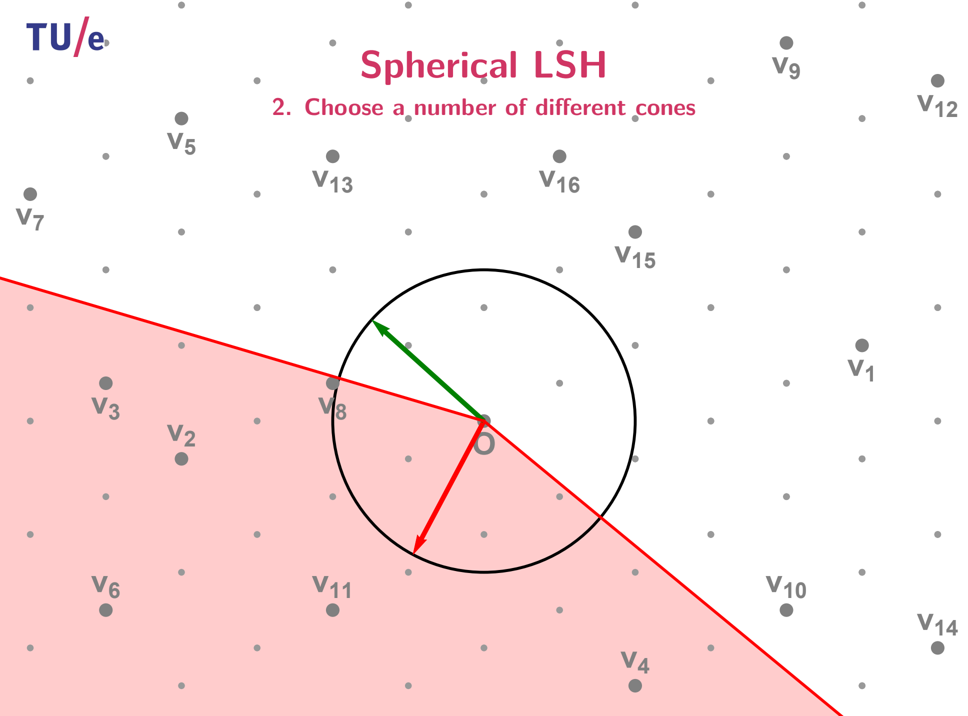

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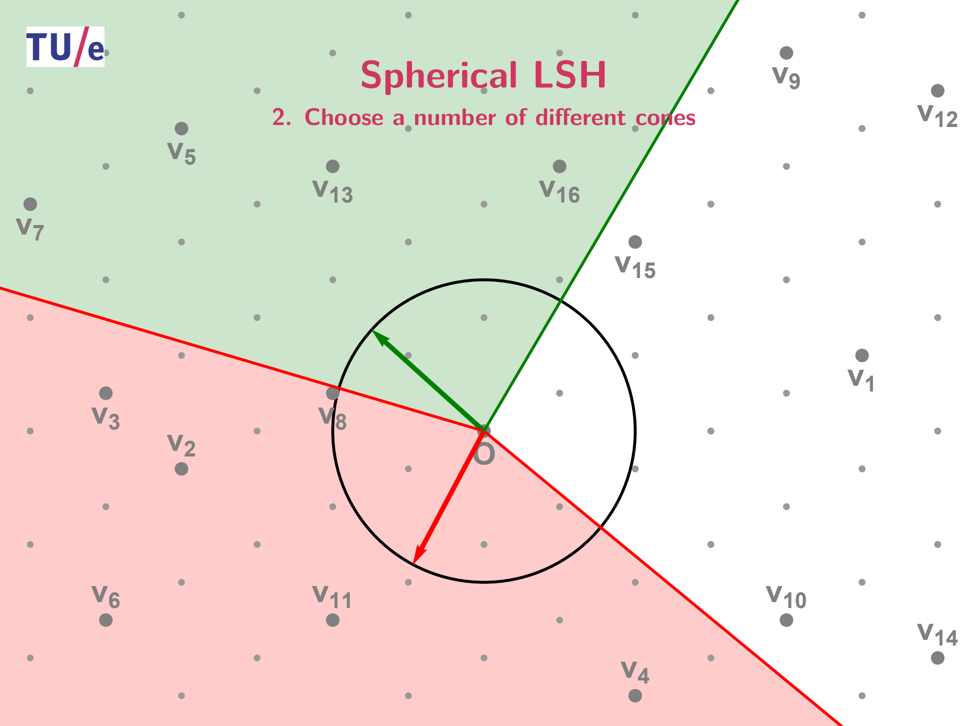

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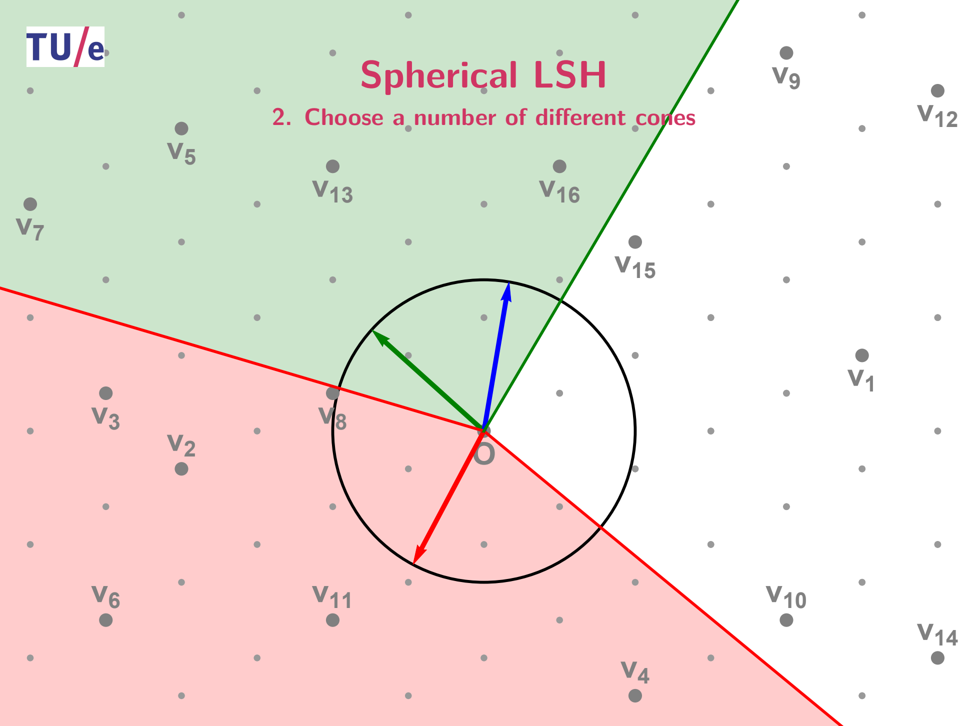

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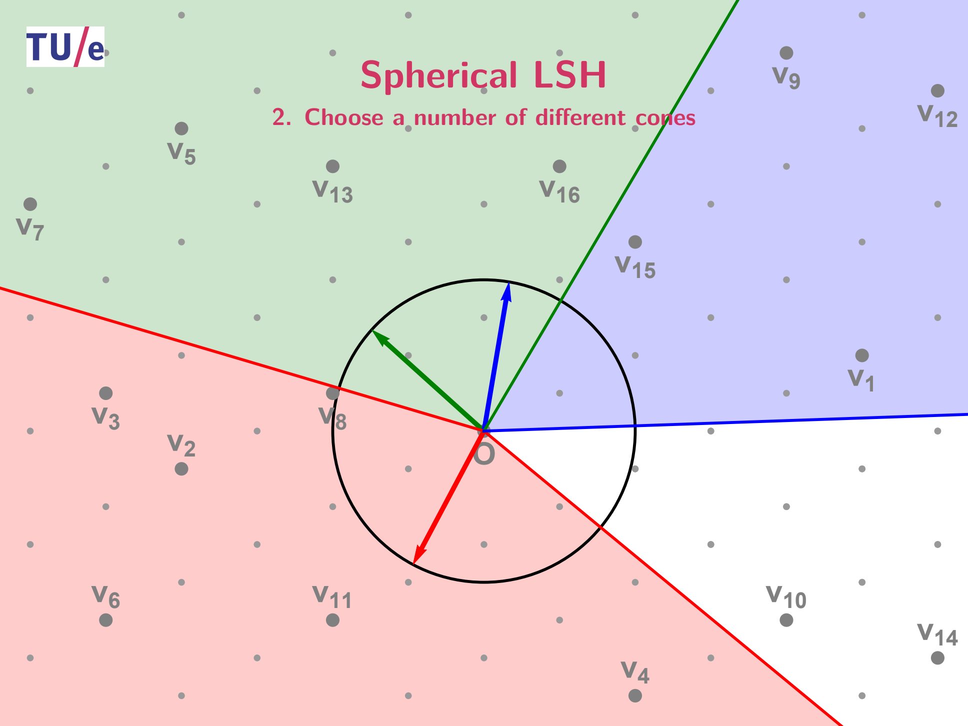

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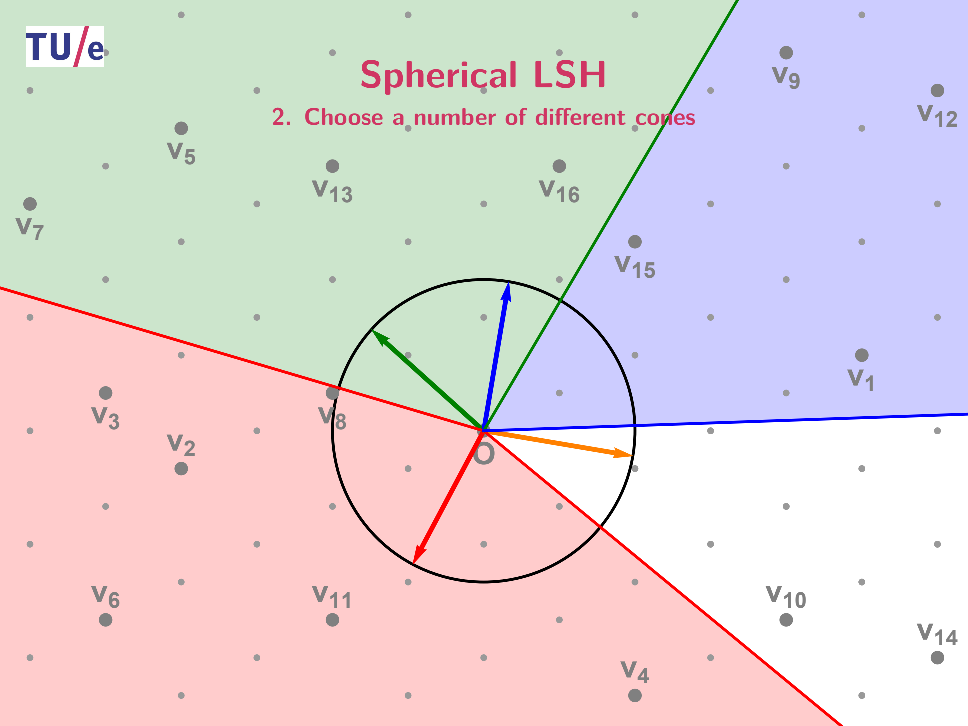

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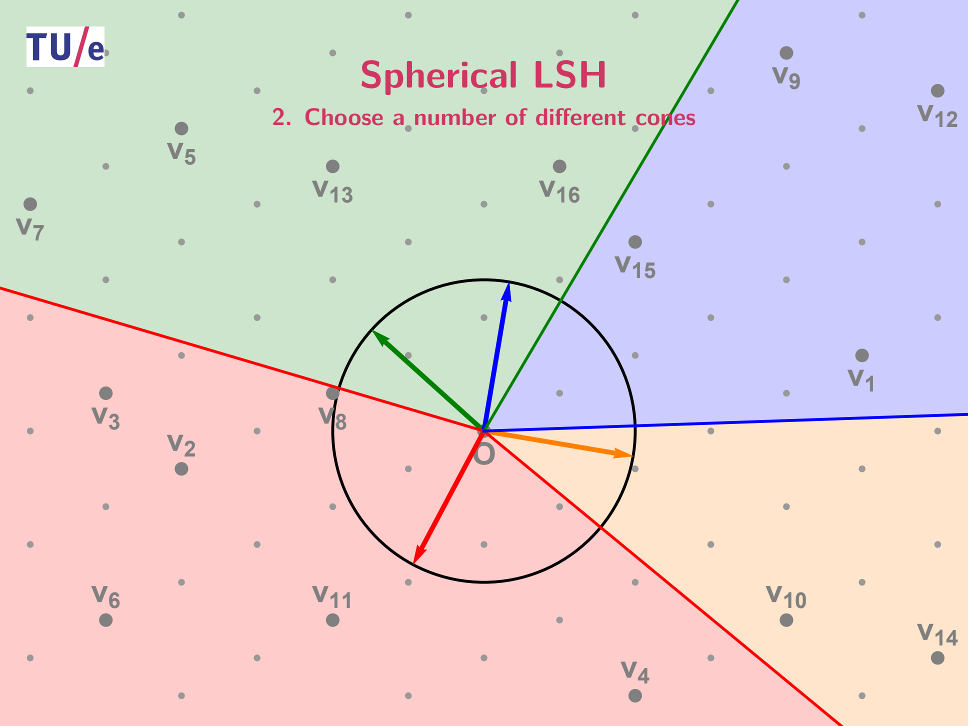

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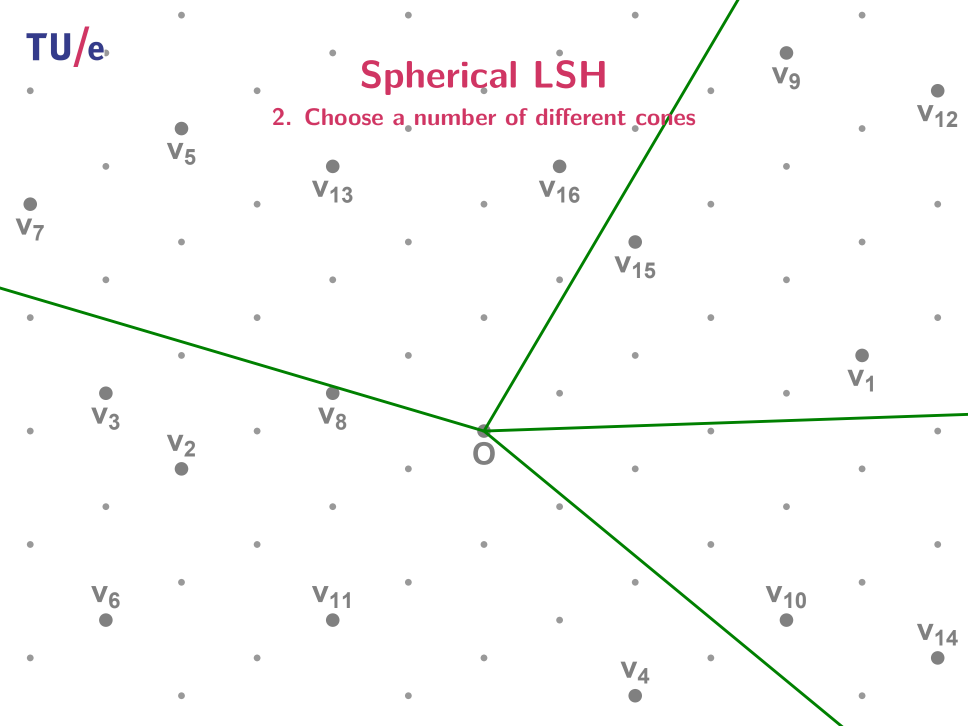

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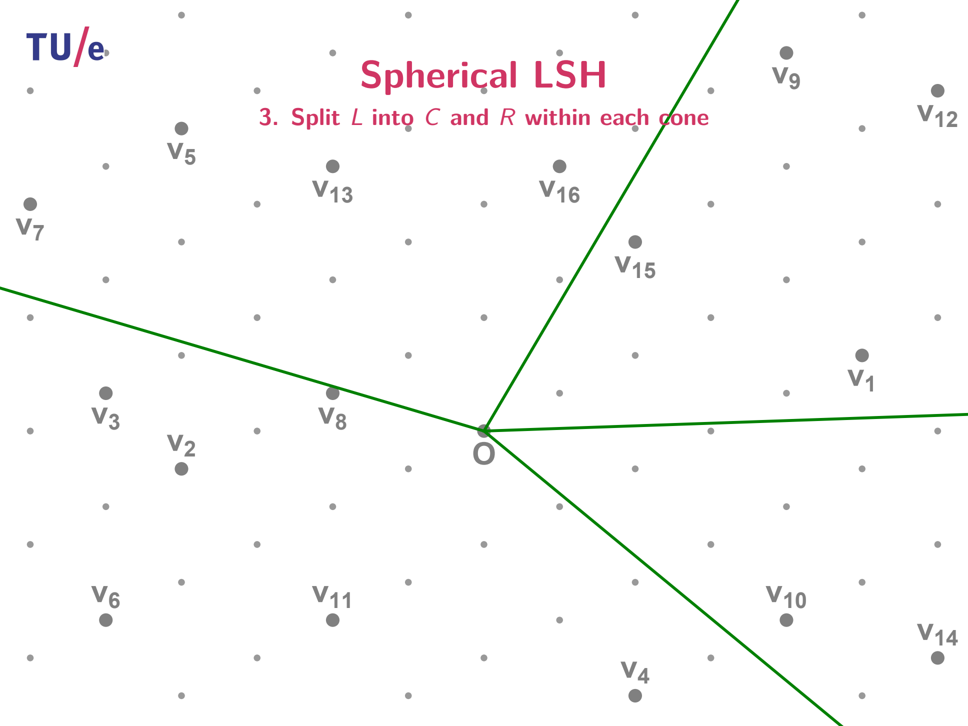

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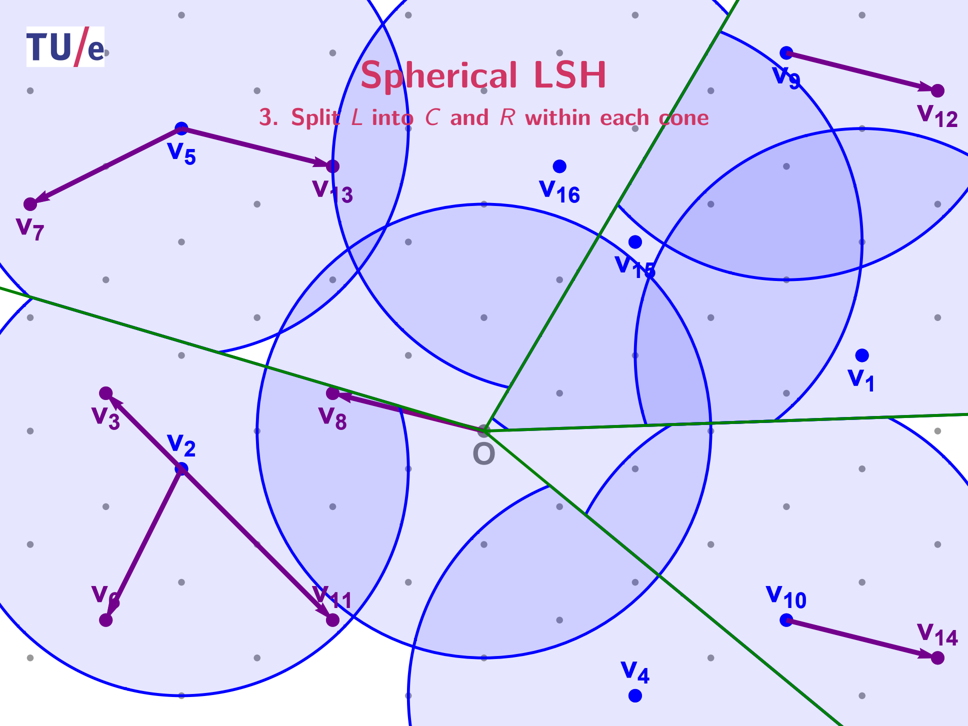

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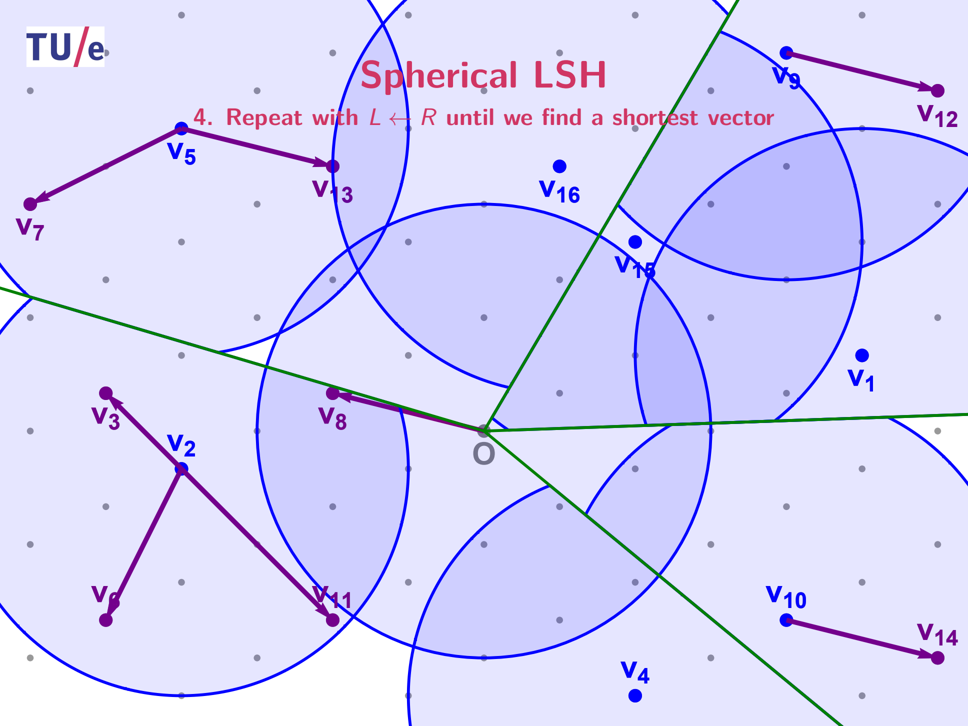

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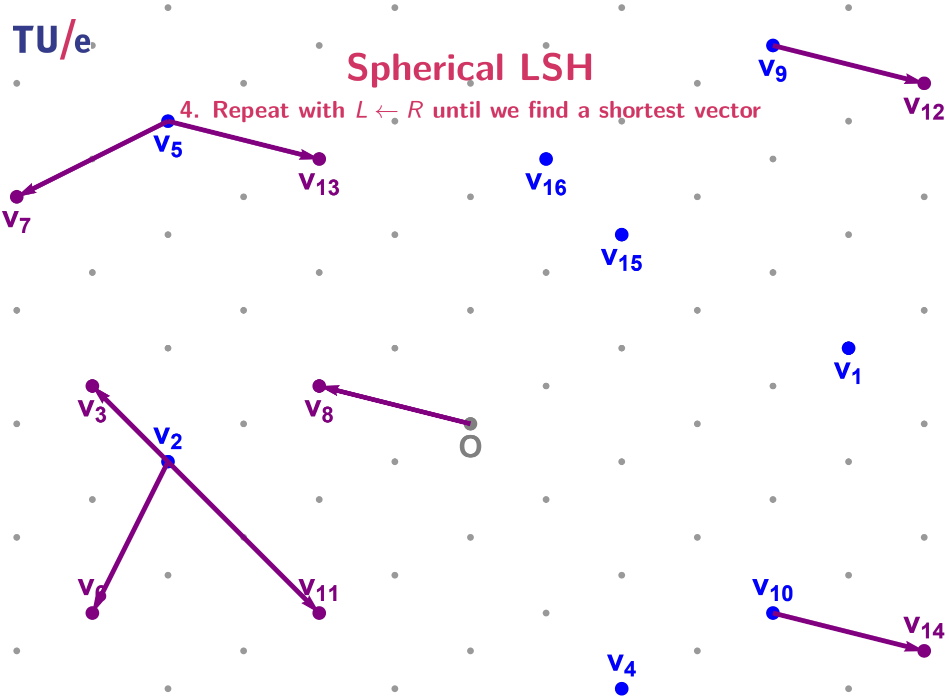

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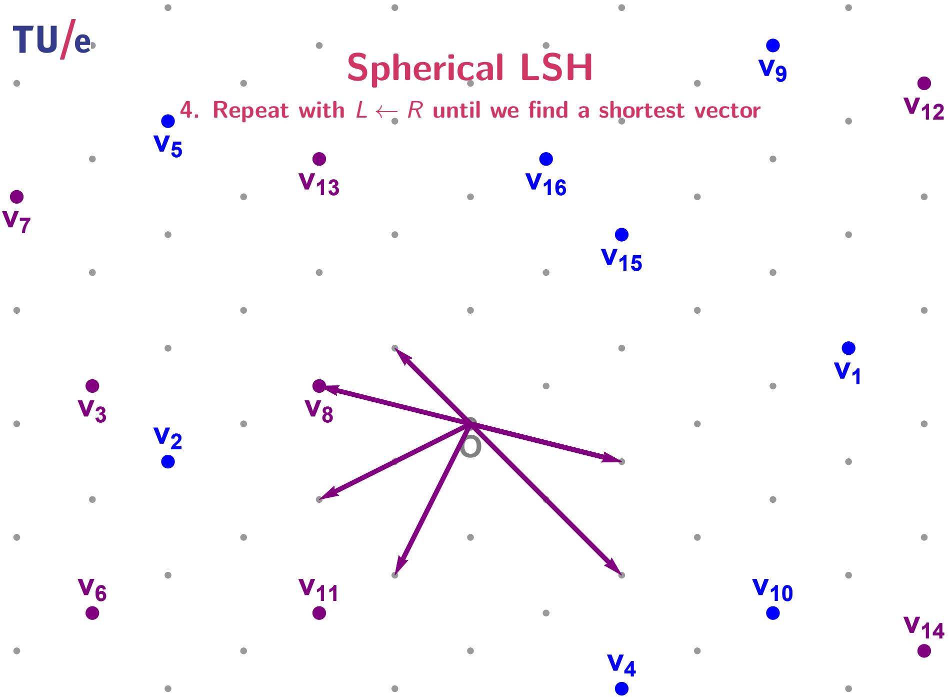

Spherical LSH

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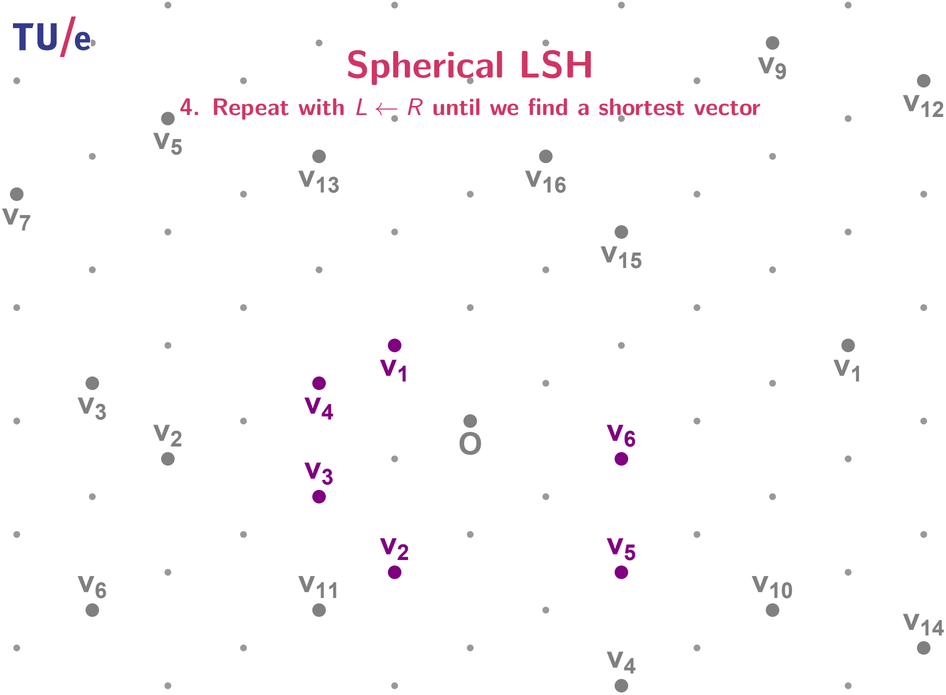
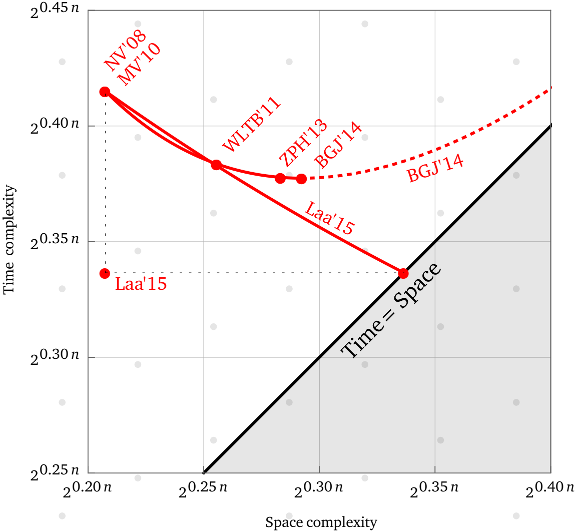


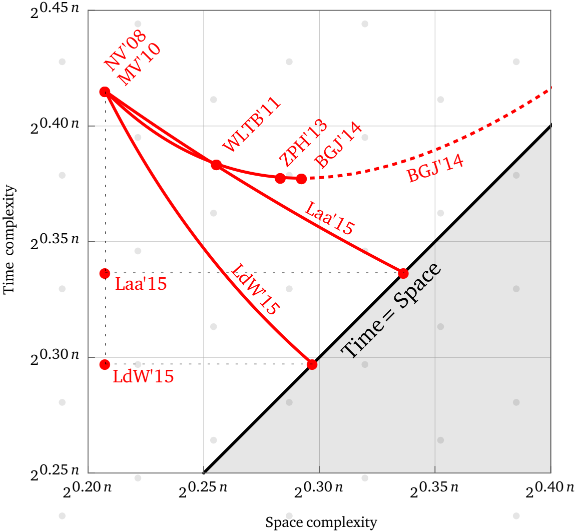
Spherical LSH

Space/time trade-off



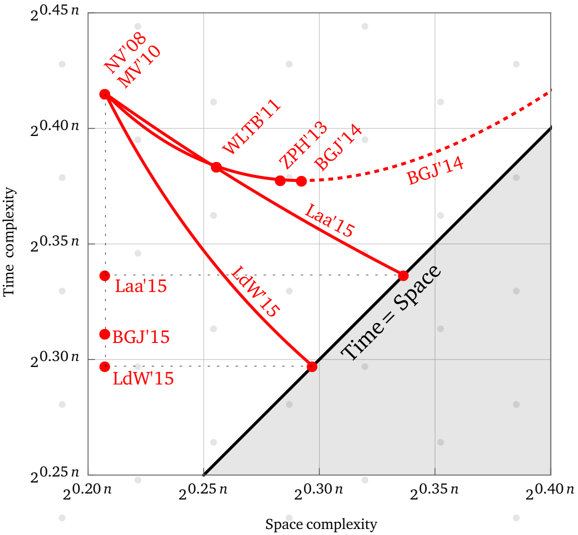
Spherical LSH

Space/time trade-off



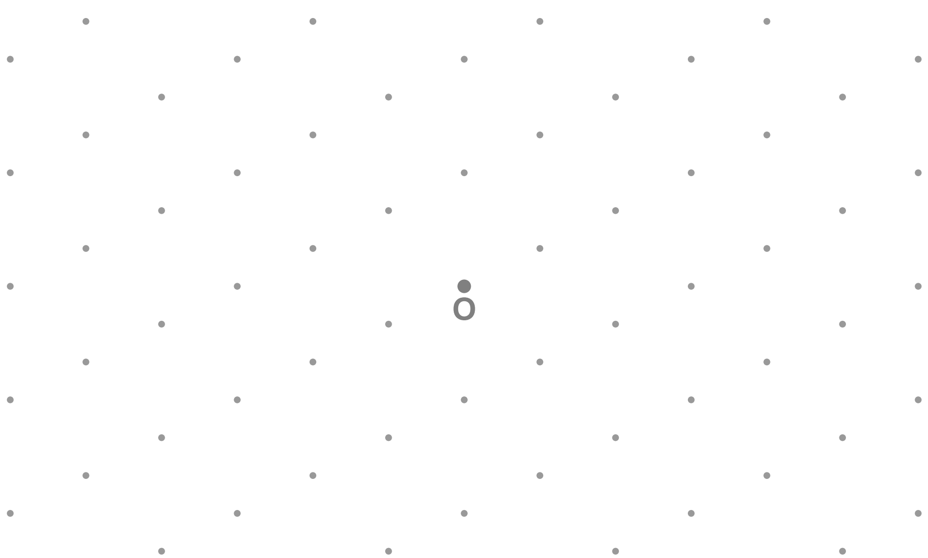
May and Ozerov's NNS method

Space/time trade-off



Cross-Polytope LSH

1. Sample a list L of random lattice vectors



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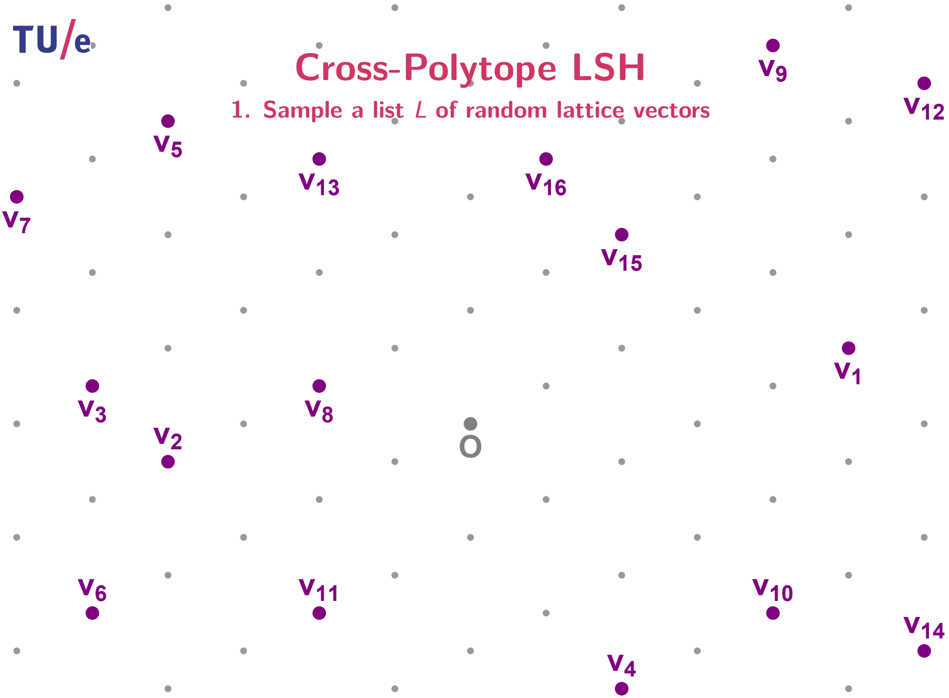
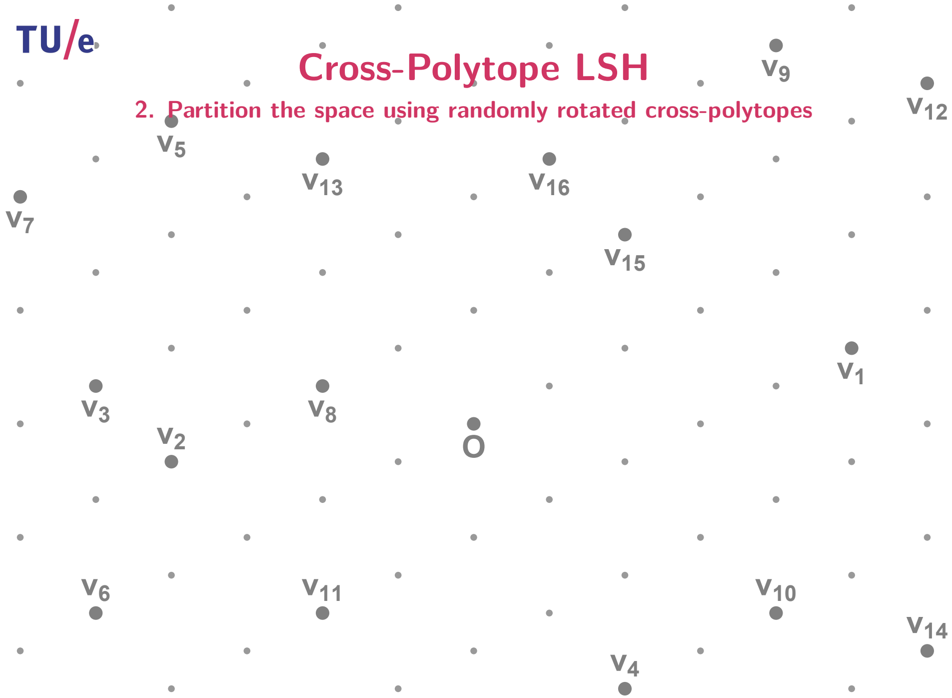


Cross-Polytope LSH

2. Partition the space using randomly rotated cross-polytopes



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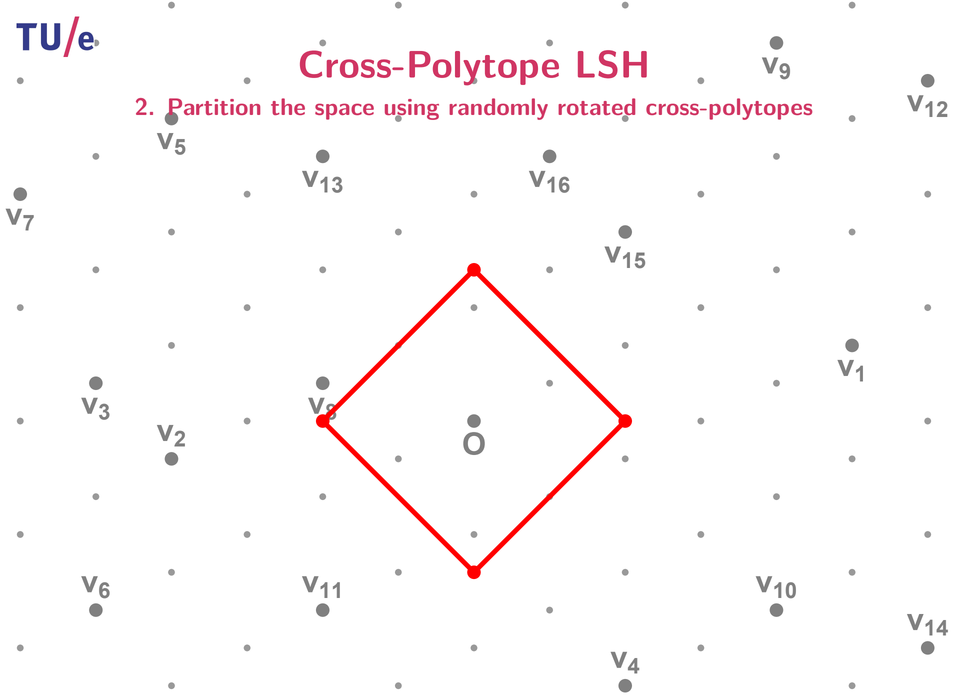

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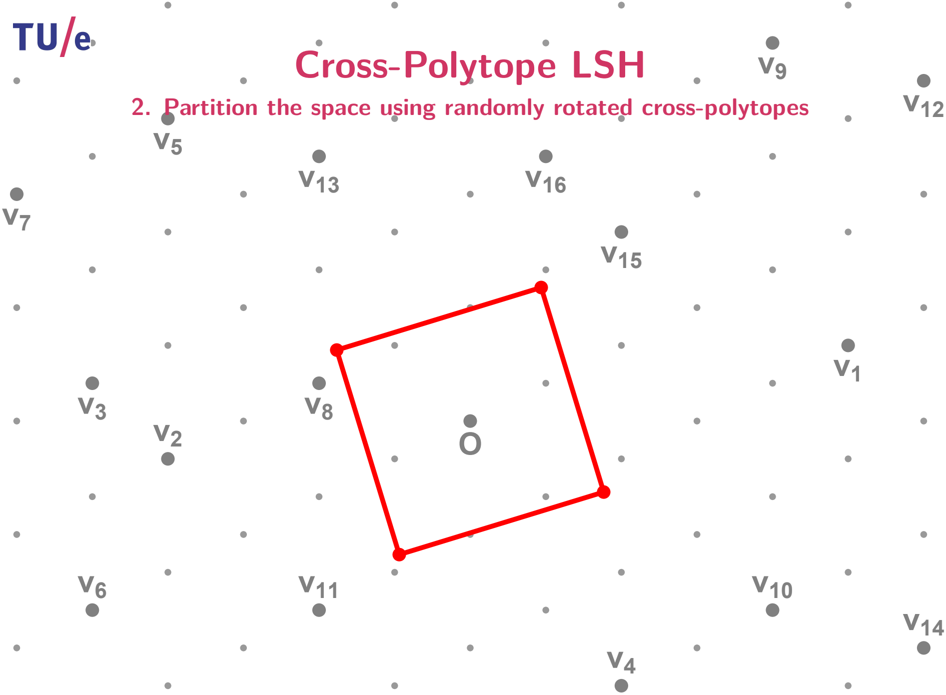

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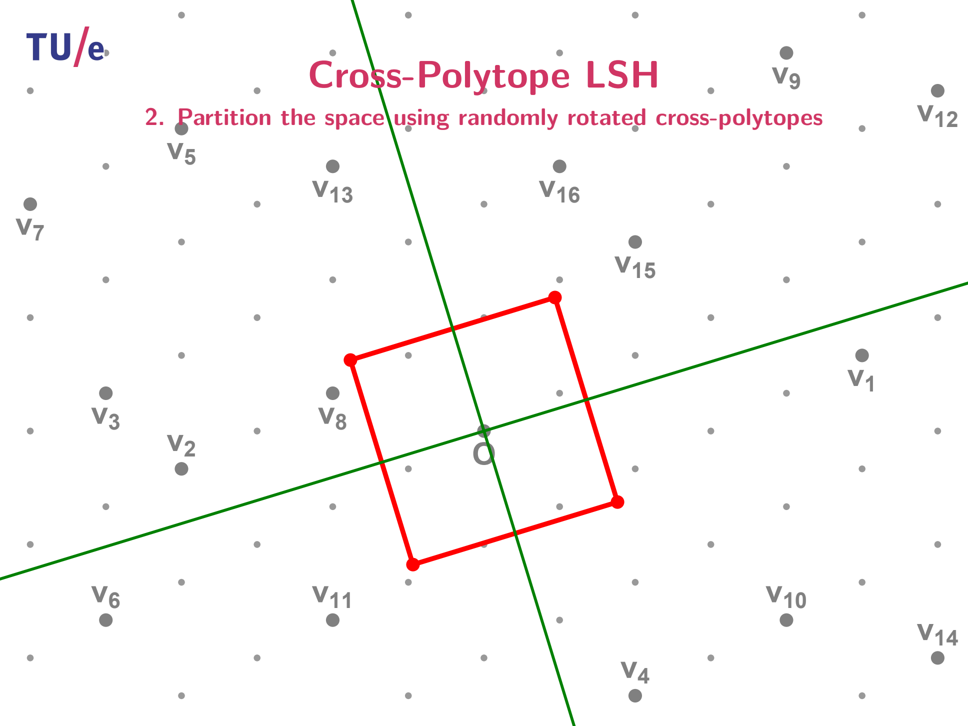

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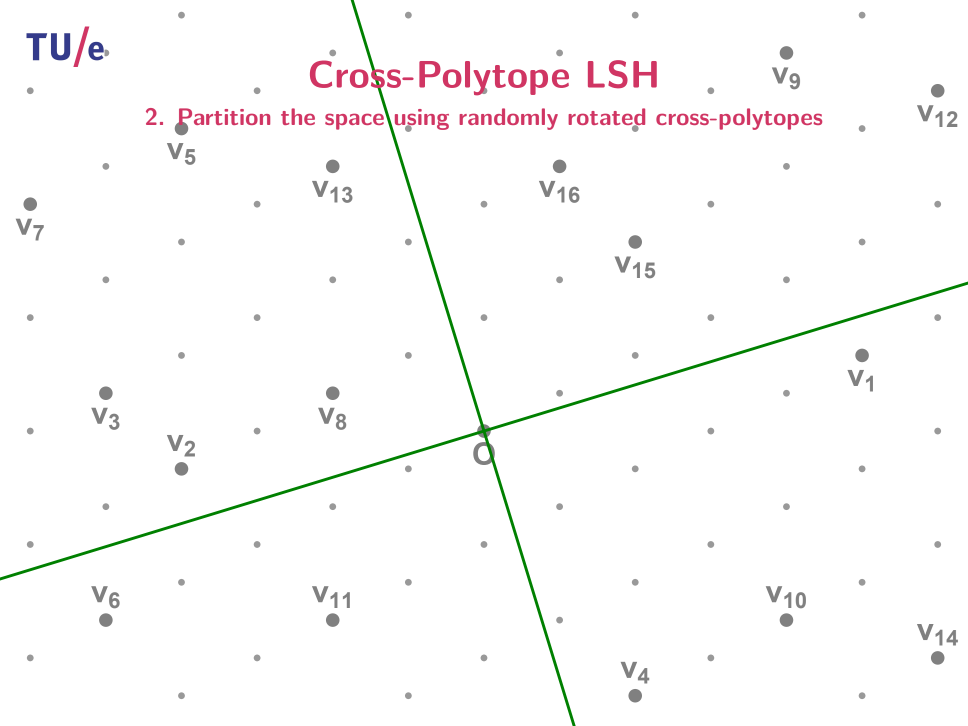
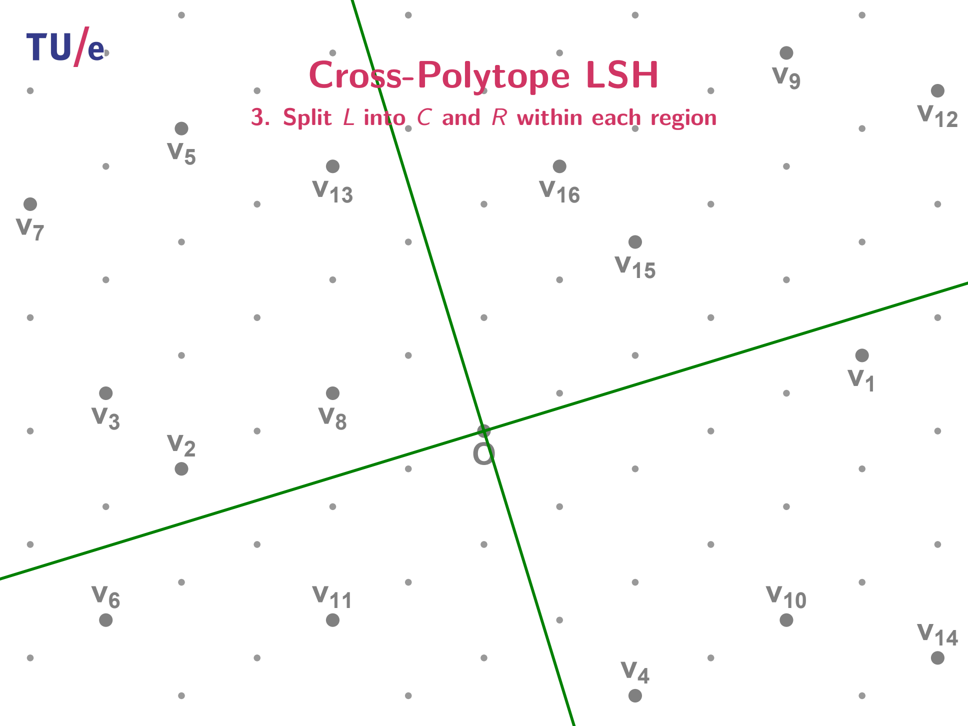


Cross-Polytope LSH

3. Split L into C and R within each region



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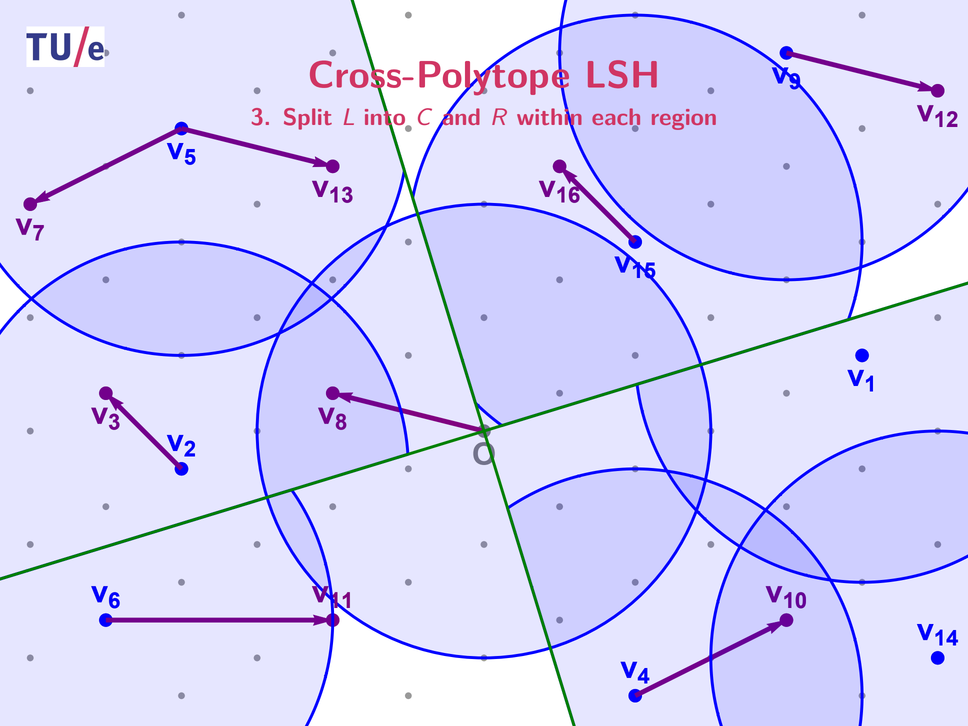

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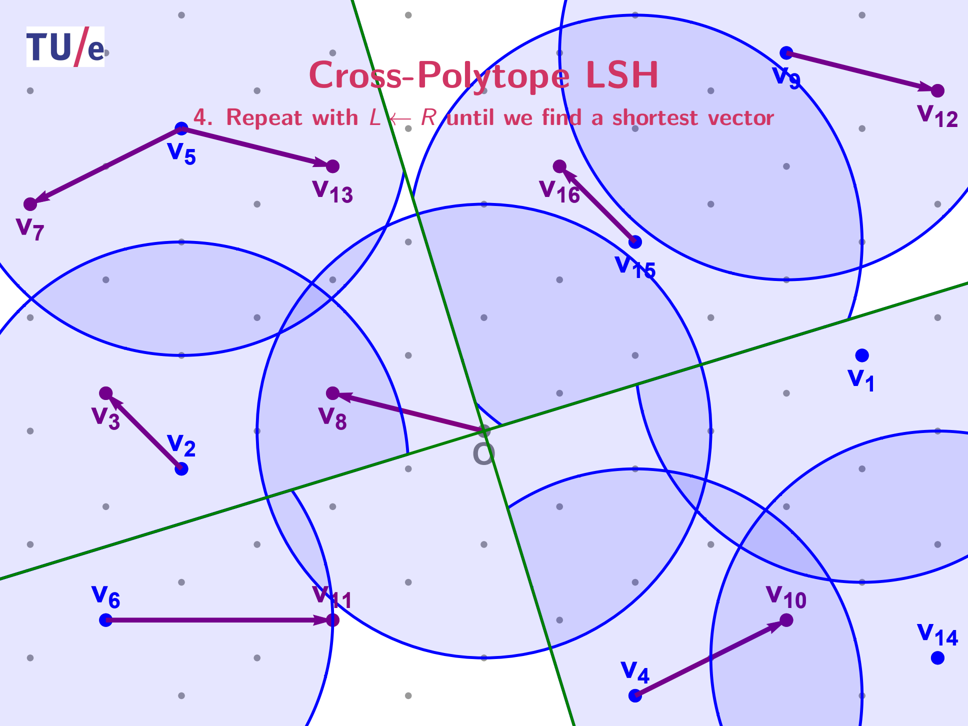

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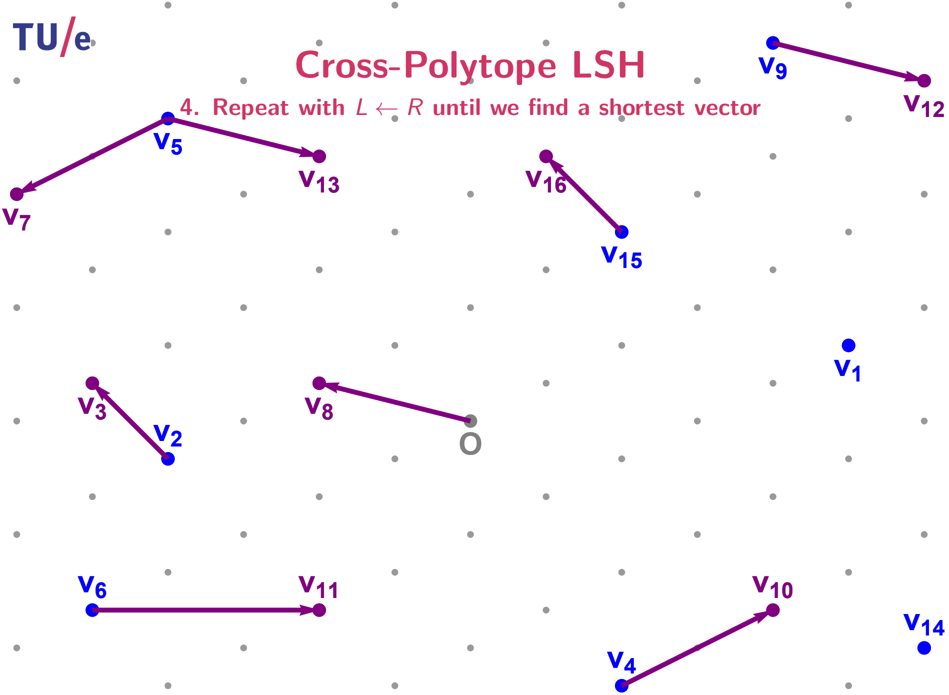

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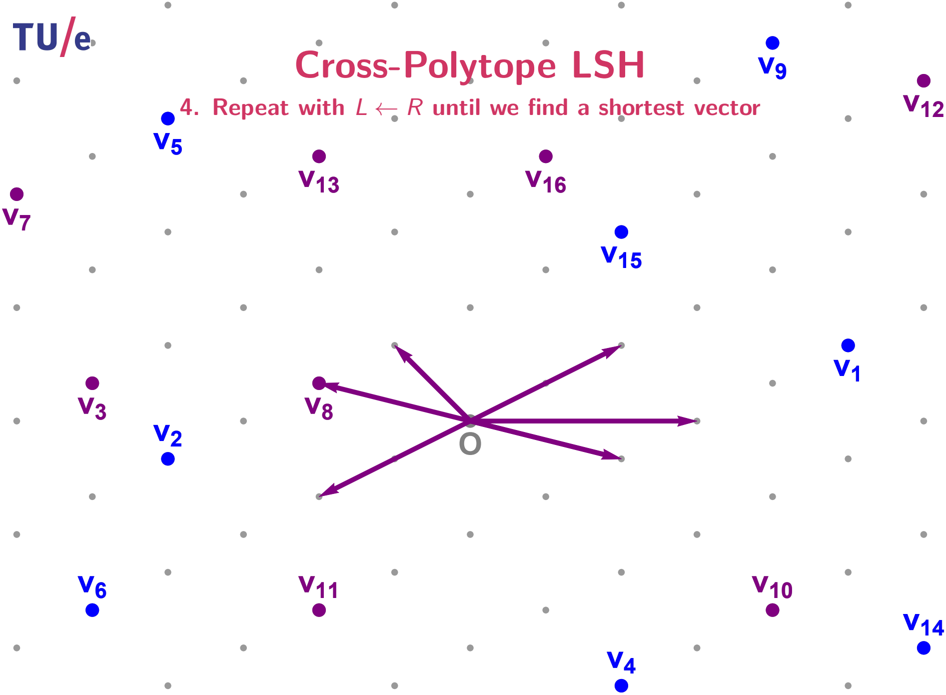

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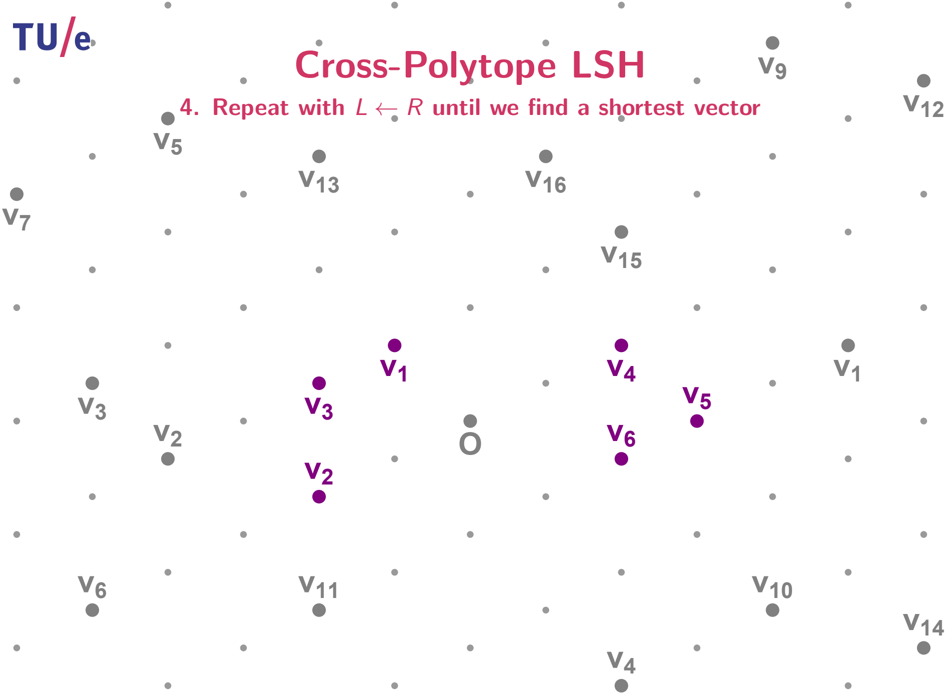
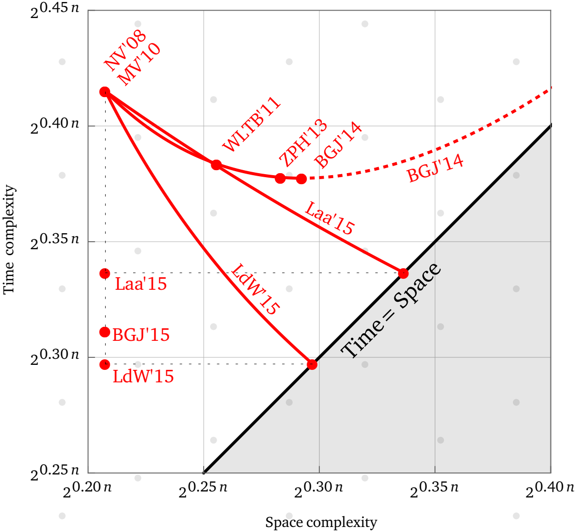


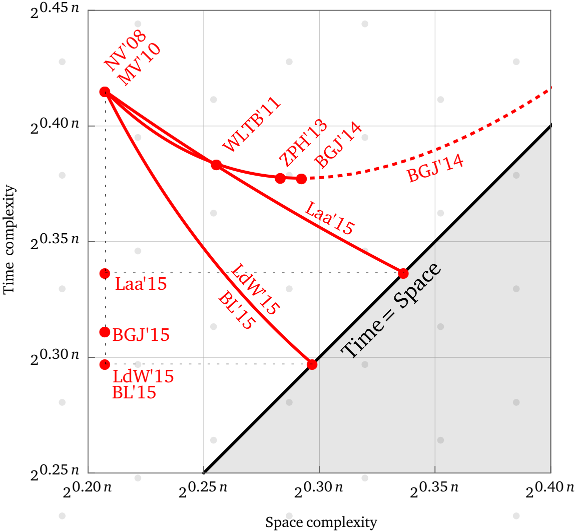
Cross-Polytope LSH

Space/time trade-off



Cross-Polytope LSH

Space/time trade-off



Spherical filtering

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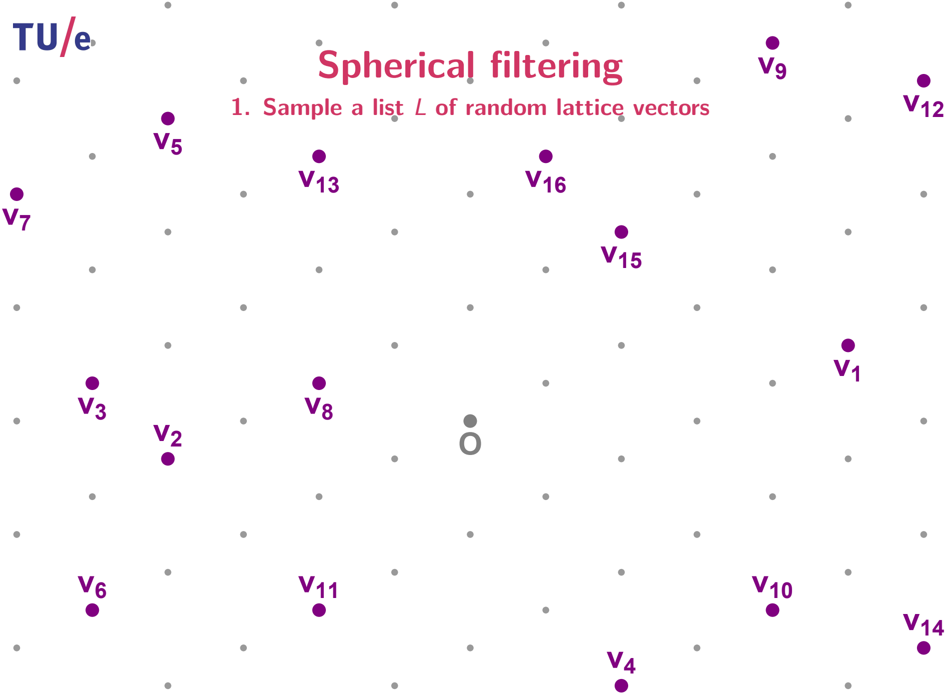
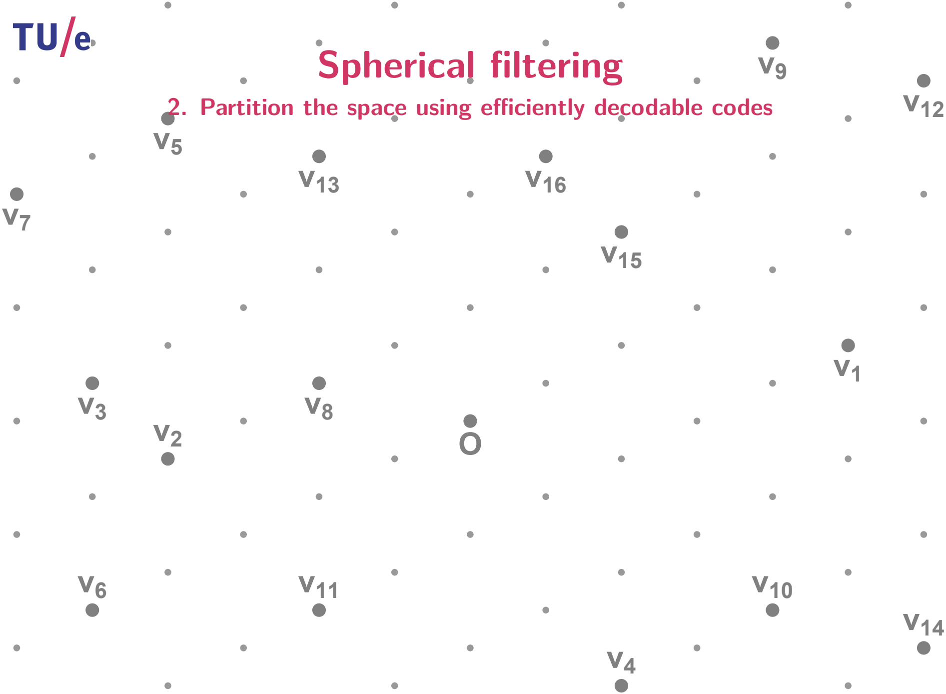


Spherical filtering

2. Partition the space using efficiently decodable codes



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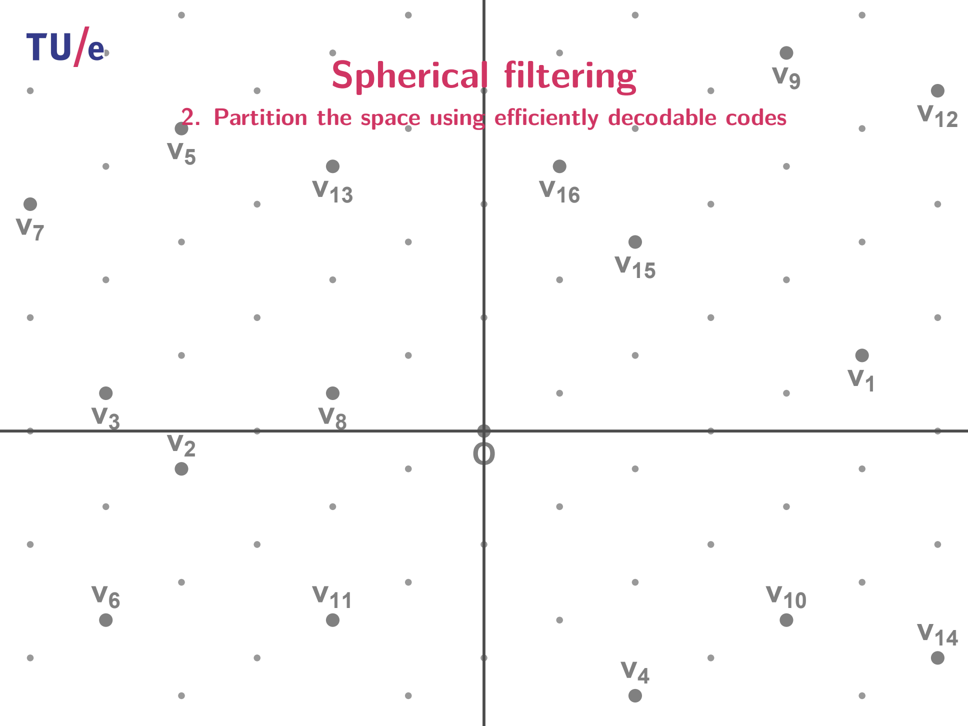

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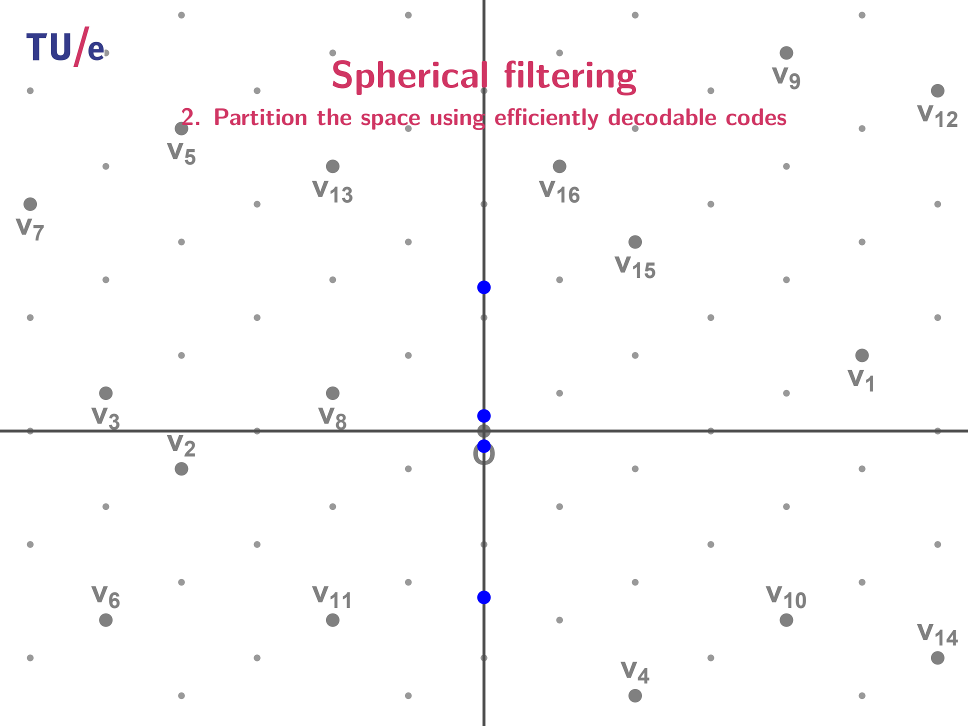

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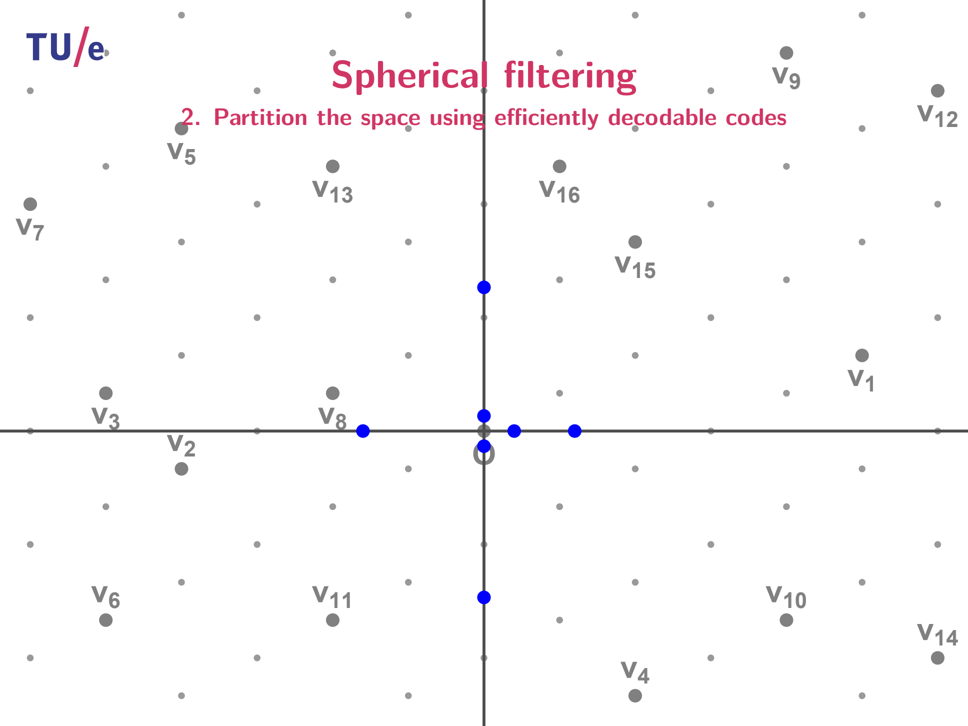

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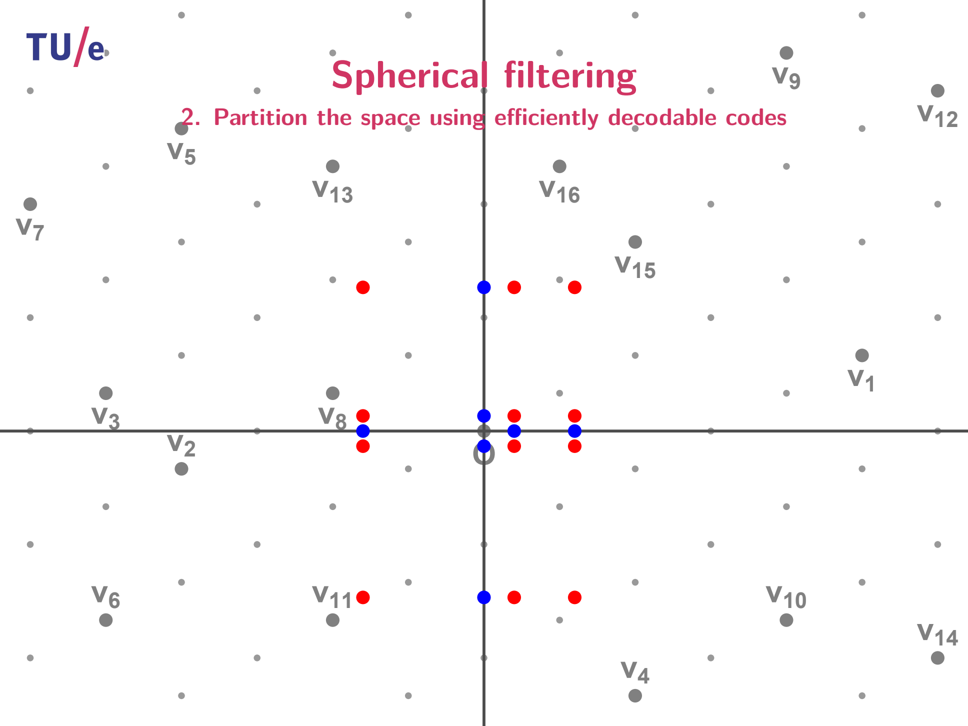

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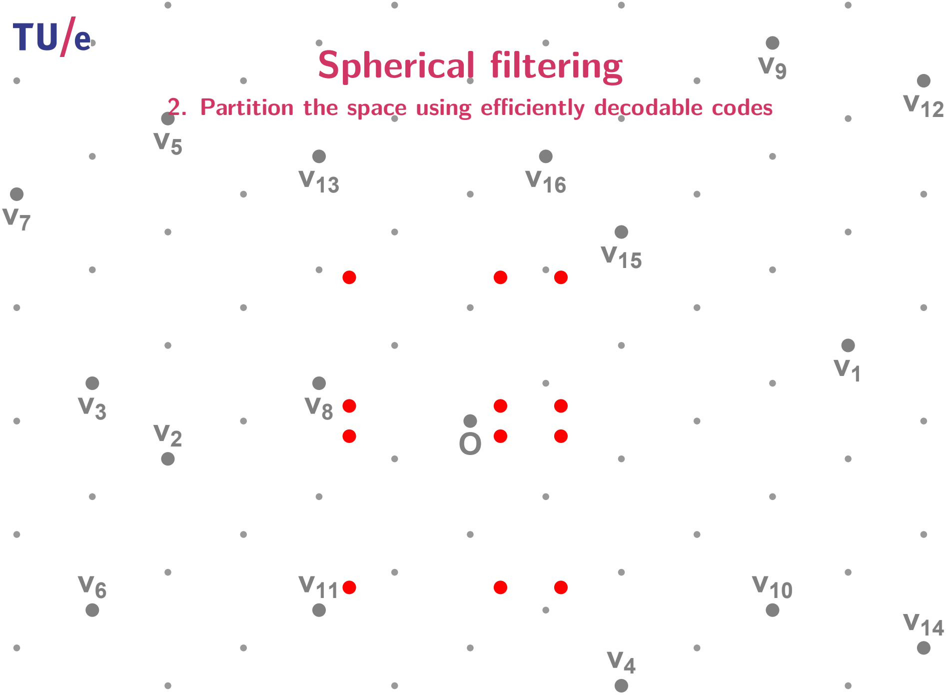

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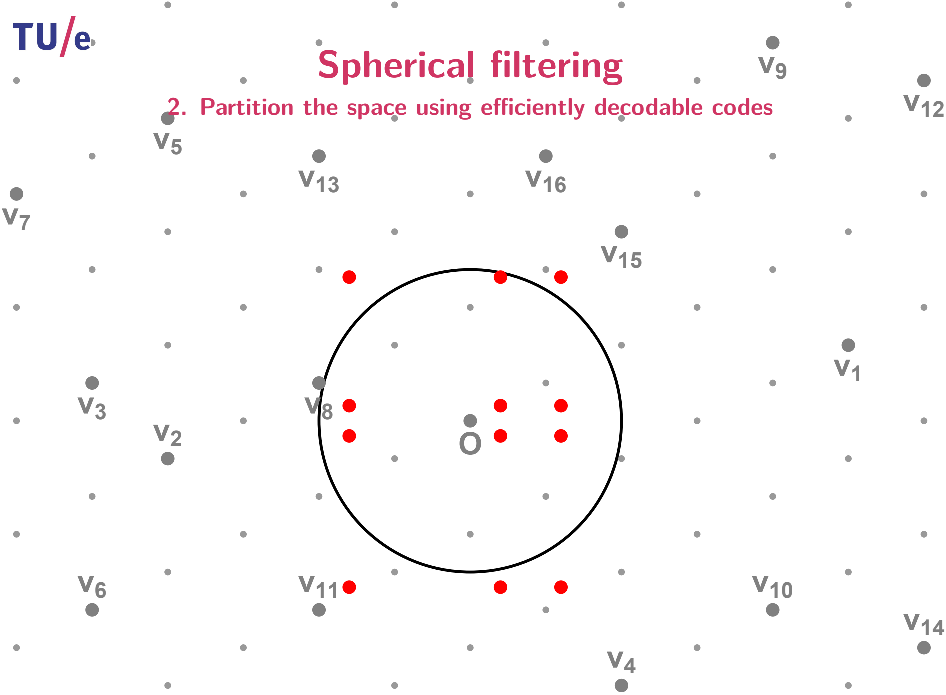

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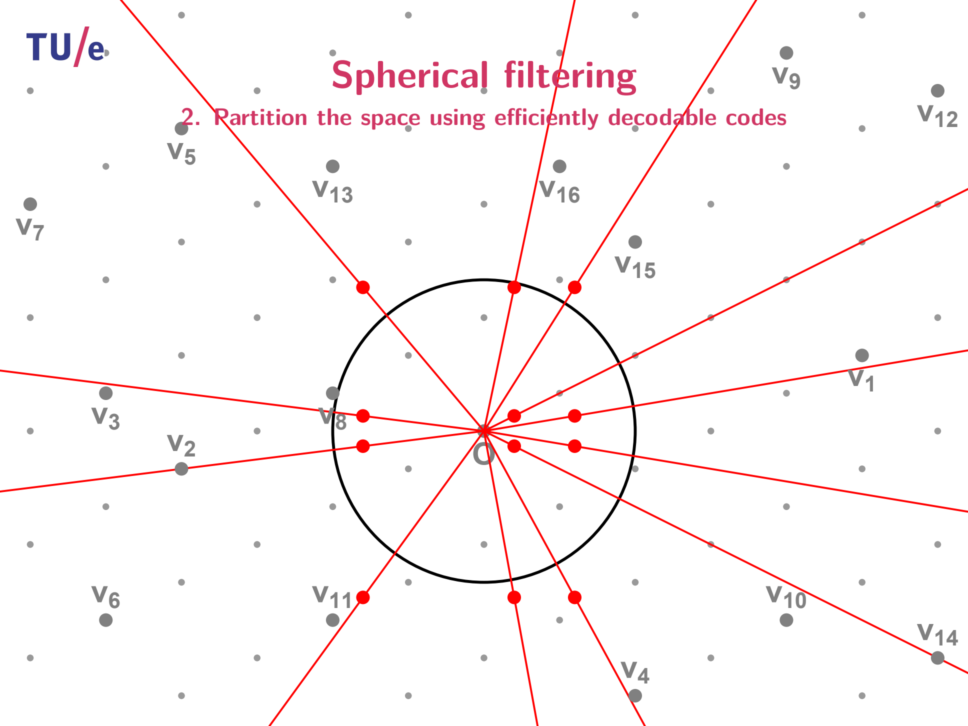

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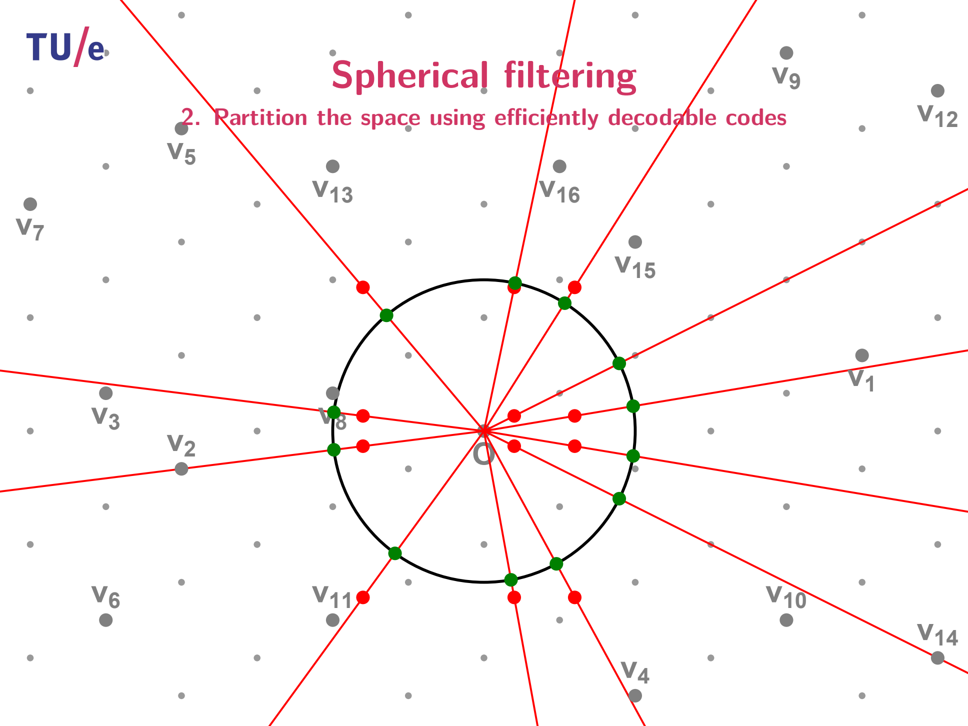

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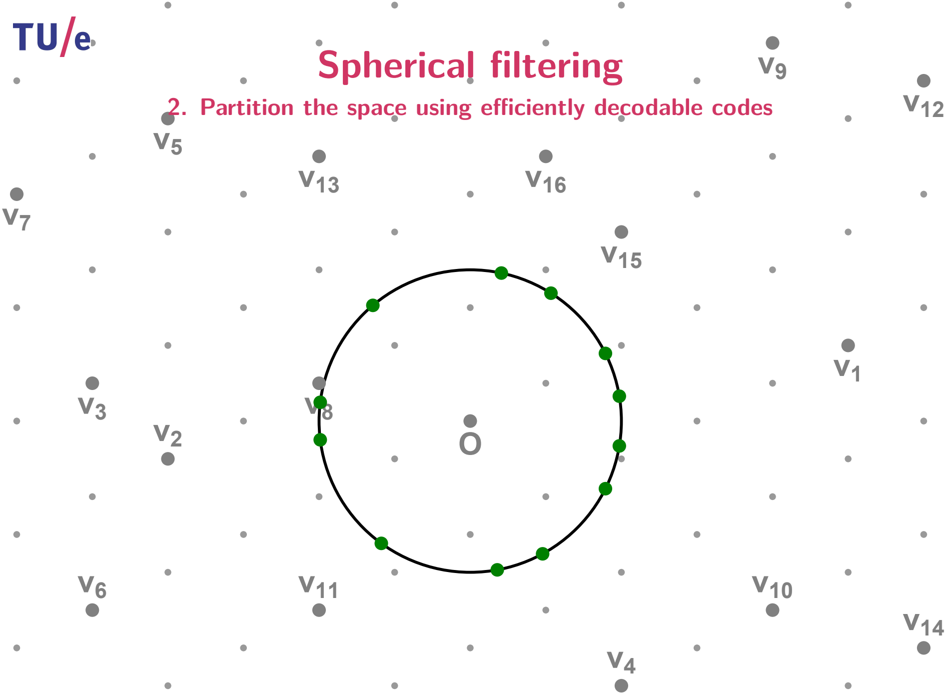
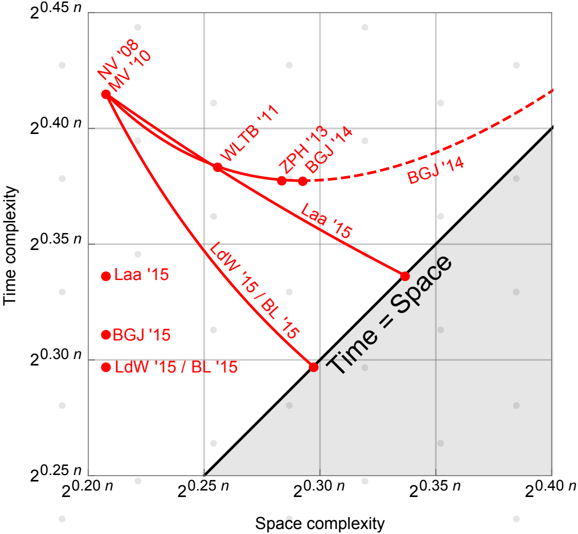


TU/e

Spherical filtering

Space/time trade-off



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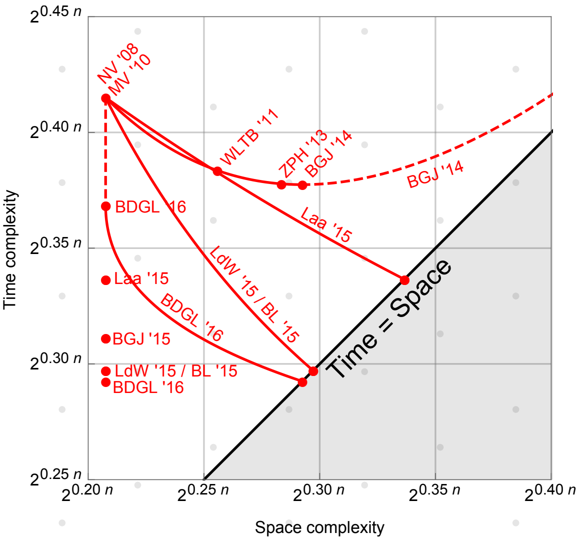
Space/time trade-off



Questions

[vdP'12]