Technische Universiteit Eindhoven University of Technology

# Efficient Probabilistic Group Testing Based on Traitor Tracing

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# Problem: Blood Testing

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Antonino | | | | | | | | | | | | |
| Boris | | | | | | | | | | | | |
| Caroline | | | | | | | | | | | | |
| David | | | | | | | | | | | | |
| Eve | | | | | | | | | | | | |
| Fred | | | | | | | | | | | | |
| Gábor | | | | | | | | | | | | |
| Henry | | | | | | | | | | | | |

# Problem: Blood Testing

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Antonino | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | | | |
| Boris | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | | | | |
| Caroline | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | | | | |
| David | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | | | | |
| Eve | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | | | | |
| Fred | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | | | | |
| Gábor | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | | | | |
| Henry | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | |

# Problem: Blood Testing

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Antonino | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | | | |
| Boris | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | | | | |
| Caroline | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | | | | |
| David | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | | | | |
| Eve | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | | | | |
| Fred | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | | | | |
| Gábor | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | | | | |
| Henry | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | |
| Results | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | | | | |

# Problem: Blood Testing

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Antonino | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | | | |
| Boris | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | | | | |
| Caroline | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | | | | |
| David | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | | | | |
| Eve | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | | | | |
| Fred | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | | | | |
| Gábor | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | | | | |
| Henry | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | |
| Results | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | | | | |

# Solution: Using Pools

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Antonino | 0 | 0 | 0 | | | | | | | | | |
| Boris | 0 | 0 | 1 | | | | | | | | | |
| Caroline | 0 | 1 | 0 | | | | | | | | | |
| David | 0 | 1 | 1 | | | | | | | | | |
| Eve | 1 | 0 | 0 | | | | | | | | | |
| Fred | 1 | 0 | 1 | | | | | | | | | |
| Gábor | 1 | 1 | 0 | | | | | | | | | |
| Henry | 1 | 1 | 1 | | | | | | | | | |
| Results | | | | | | | | | | | | |

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# Solution: Using Pools

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Antonino | 0 | 0 | 0 | | | | | | | | | |
| Boris | 0 | 0 | 1 | | | | | | | | | |
| Caroline | 0 | 1 | 0 | | | | | | | | | |
| David | 0 | 1 | 1 | | | | | | | | | |
| Eve | 1 | 0 | 0 | | | | | | | | | |
| Fred | 1 | 0 | 1 | | | | | | | | | |
| Gábor | 1 | 1 | 0 | | | | | | | | | |
| Henry | 1 | 1 | 1 | | | | | | | | | |
| Results | 0 | 1 | 0 | | | | | | | | | |

# Solution: Using Pools

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Antonino | 0 | 0 | 0 | | | | | | | | | |
| Boris | 0 | 0 | 1 | | | | | | | | | |
| Caroline | 0 | 1 | 0 | | | | | | | | | |
| David | 0 | 1 | 1 | | | | | | | | | |
| Eve | 1 | 0 | 0 | | | | | | | | | |
| Fred | 1 | 0 | 1 | | | | | | | | | |
| Gábor | 1 | 1 | 0 | | | | | | | | | |
| Henry | 1 | 1 | 1 | | | | | | | | | |
| Results | 0 | 1 | 0 | | | | | | | | | |

# Problem: Multiple ($K$) “Defectives”

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Antonino | 0 | 0 | 0 | | | | | | | | | |
| Boris | 0 | 0 | 1 | | | | | | | | | |
| Caroline | 0 | 1 | 0 | | | | | | | | | |
| David | 0 | 1 | 1 | | | | | | | | | |
| Eve | 1 | 0 | 0 | | | | | | | | | |
| Fred | 1 | 0 | 1 | | | | | | | | | |
| Gábor | 1 | 1 | 0 | | | | | | | | | |
| Henry | 1 | 1 | 1 | | | | | | | | | |
| Results | | | | | | | | | | | | |

# Problem: Multiple ($K$) "Defectives"

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Antonino | 0 | 0 | 0 | | | | | | | | | |
| Boris | 0 | 0 | 1 | | | | | | | | | |
| Caroline | 0 | 1 | 0 | | | | | | | | | |
| David | 0 | 1 | 1 | | | | | | | | | |
| Eve | 1 | 0 | 0 | | | | | | | | | |
| Fred | 1 | 0 | 1 | | | | | | | | | |
| Gábor | 1 | 1 | 0 | | | | | | | | | |
| Henry | 1 | 1 | 1 | | | | | | | | | |
| Results | | | | | | | | | | | | |

# Problem: Multiple ($K$) “Defectives”

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Antonino | 0 | 0 | 0 | | | | | | | | | |
| Boris | 0 | 0 | 1 | | | | | | | | | |
| Caroline | 0 | 1 | 0 | | | | | | | | | |
| David | 0 | 1 | 1 | | | | | | | | | |
| Eve | 1 | 0 | 0 | | | | | | | | | |
| Fred | 1 | 0 | 1 | | | | | | | | | |
| Gábor | 1 | 1 | 0 | | | | | | | | | |
| Henry | 1 | 1 | 1 | | | | | | | | | |
| Results | 1 | 1 | 1 | | | | | | | | | |

# Solution: Group Testing

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Antonino | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Boris | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Caroline | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| David | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Eve | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Fred | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Gábor | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Henry | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Results | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |

# Solution: Group Testing

| |  | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Antonino | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Boris | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Caroline | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| David | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Eve | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Fred | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Gábor | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Henry | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Results | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |

1. An algorithm to construct group testing matrices

# Solution: Group Testing

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Antonino | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Boris | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Caroline | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| David | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Eve | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Fred | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Gábor | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Henry | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| Results | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |

1. An algorithm to construct group testing matrices
2. An algorithm to link test results to infected people

# Solution: Group Testing

| |  |
|---|---|
| Antonino | |
| Boris | |
| Caroline | $X \in \{0,1\}^{N \times T}$ |
| David | |
| Eve | |
| Fred | |
| Gábor | |
| Henry | |
| Results | $y \in \{0,1\}^T$ |

1. An algorithm to construct group testing matrices
2. An algorithm to link test results to infected people

# Solution: Group Testing

| |  |
|---|---|
| Antonino<br>Boris<br>Caroline<br>David<br>Eve<br>Fred<br>Gábor<br>Henry | $X \in \{0, 1\}^{N \times T}$<br><br>deterministic: $T = \Omega(K^2 \log N)$ |
| Results | $y \in \{0, 1\}^{T}$ |

1. An algorithm to construct group testing matrices
2. An algorithm to link test results to infected people

# Solution: Group Testing

Antonino
Boris
Caroline
David
Eve
Fred
Gábor
Henry

$$X \in \{0,1\}^{N \times T}$$

deterministic: $T = \Omega(K^2 \log N)$

probabilistic: $T = \Theta(K \log N)$

Results $y \in \{0,1\}^T$

1. An algorithm to construct group testing matrices
2. An algorithm to link test results to infected people

# Solution: Group Testing

| | |
|---|---|
| Antonino<br>Boris<br>Caroline<br>David<br>Eve<br>Fred<br>Gábor<br>Henry | $X \in \{0,1\}^{N \times T}$<br><br>deterministic: $T = \Omega(K^2 \log N)$<br>probabilistic: $T = \Theta(K \log N)$ |
| Results | $y \in \{0,1\}^T$ |

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# Solution: Group Testing

1. An algorithm to construct group testing matrices
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# Solution: Group Testing

1. An algorithm to construct group testing matrices

2. An algorithm to link test results to infected people

# A Group Testing Framework

1. An algorithm to construct group testing matrices

2. An algorithm to link test results to infected people

# A Group Testing Framework

1. An algorithm to construct group testing matrices
    1a. For each test $i$, person $j$, choose $X_{j,i} = 1$ with prob. $p$.
    - Intuitively: $p \approx \frac{1}{K}$.
    - Precise value of $p$ depends on $N$, $K$, $\varepsilon$.
2. An algorithm to link test results to infected people

# A Group Testing Framework

1. An algorithm to construct group testing matrices
   1a. For each test $i$, person $j$, choose $X_{j,i} = 1$ with prob. $p$.
      - Intuitively: $p \approx \frac{1}{K}$.
      - Precise value of $p$ depends on $N$, $K$, $\varepsilon$.
2. An algorithm to link test results to infected people
   2a. For each test $i$, person $j$, compute $S_{j,i} = g(X_{j,i}, y_i)$.
      - Positive scores ($S_{j,i} > 0$) for matches ($X_{j,i} = y_i$).
      - Negative scores ($S_{j,i} < 0$) for differences ($X_{j,i} \neq y_i$).
      - Large scores ($|S_{j,i}| \gg 0$) for rare events.

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     - Negative scores ($S_{j,i} < 0$) for differences ($X_{j,i} \neq y_i$).
     - Large scores ($|S_{j,i}| \gg 0$) for rare events.
   - 2b. Mark person $j$ infected iff $\sum_i S_{j,i} > Z$ (threshold).
     - Large $Z$: Fewer false positives, more false negatives.
     - Small $Z$: More false positives, fewer false negatives.

# A Group Testing Framework

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      - Large $Z$: Fewer false positives, more false negatives.
      - Small $Z$: More false positives, fewer false negatives.

Exact choices of $p$, $g$, and $Z$ depend on the model/parameters.

# Traditional Group Testing

1. An algorithm to construct group testing matrices
   1a. For each test $i$, person $j$, choose $X_{j,i} = 1$ with prob. $p$.

2. An algorithm to link test results to infected people
   2a. For each test $i$, person $j$, compute $S_{j,i} = g(X_{j,i}, y_i)$.

   2b. Mark person $j$ infected iff $\sum_i S_{j,i} > Z$ (threshold).

# Traditional Group Testing

1. An algorithm to construct group testing matrices
   - 1a. For each test $i$, person $j$, choose $X_{j,i} = 1$ with prob. $p$.

2. An algorithm to link test results to infected people
   - 2a. For each test $i$, person $j$, compute $S_{j,i} = g(X_{j,i}, y_i)$.

$$g(X_{j,i}, y_i) = \begin{cases} +p/(1-p), & \text{if } X_{j,i} = 0, y_i = 0, \\ -p(1-p)^{K-1}/(1-(1-p)^K), & \text{if } X_{j,i} = 0, y_i = 1, \\ -1, & \text{if } X_{j,i} = 1, y_i = 0, \\ +(1-p)^K/(1-(1-p)^K), & \text{if } X_{j,i} = 1, y_i = 1. \end{cases}$$

   - 2b. Mark person $j$ infected iff $\sum_i S_{j,i} > Z$ (threshold).

## Traditional Group Testing

1. An algorithm to construct group testing matrices
   1a. For each test $i$, person $j$, choose $X_{j,i} = 1$ with prob. $p$.

$$p = \mathrm{argmin}_q T(N, K, \varepsilon, q).$$

2. An algorithm to link test results to infected people
   2a. For each test $i$, person $j$, compute $S_{j,i} = g(X_{j,i}, y_i)$.

$$g(X_{j,i}, y_i) = \begin{cases} +p/(1-p), & \text{if } X_{j,i} = 0, y_i = 0, \\ -p(1-p)^{K-1}/(1-(1-p)^K), & \text{if } X_{j,i} = 0, y_i = 1, \\ -1, & \text{if } X_{j,i} = 1, y_i = 0, \\ +(1-p)^K/(1-(1-p)^K), & \text{if } X_{j,i} = 1, y_i = 1. \end{cases}$$

   2b. Mark person $j$ infected iff $\sum_i S_{j,i} > Z$ (threshold).

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   2b. Mark person $j$ infected iff $\sum_i S_{j,i} > Z$ (threshold).

$$Z = Z(N, K, \varepsilon, p),$$
$$T = T(N, K, \varepsilon, p).$$

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# Example: $N = 8$, $K = 3$, $\varepsilon = 10^{-2}$

1. An algorithm to construct group testing matrices
   1a. For each test $i$, person $j$, choose $X_{j,i} = 1$ with prob. $p$.

$$p = \operatorname{argmin}_q T(N, K, \varepsilon, q).$$

2. An algorithm to link test results to infected people
   2a. For each test $i$, person $j$, compute $S_{j,i} = g(X_{j,i}, y_i)$.

$$g(X_{j,i}, y_i) = \begin{cases} +p/(1-p), & \text{if } X_{j,i} = 0, y_i = 0, \\ -p(1-p)^{K-1}/(1-(1-p)^K), & \text{if } X_{j,i} = 0, y_i = 1, \\ -1, & \text{if } X_{j,i} = 1, y_i = 0, \\ +(1-p)^K/(1-(1-p)^K), & \text{if } X_{j,i} = 1, y_i = 1. \end{cases}$$

   2b. Mark person $j$ infected iff $\sum_i S_{j,i} > Z$ (threshold).

$$Z = Z(N, K, \varepsilon, p),$$
$$T = T(N, K, \varepsilon, p).$$

# Example: $N = 8$, $K = 3$, $\varepsilon = 10^{-2}$

1. An algorithm to construct group testing matrices
   1a. For each test $i$, person $j$, choose $X_{j,i} = 1$ with prob. $p$.

$$p = 0.25\ldots.$$

2. An algorithm to link test results to infected people
   2a. For each test $i$, person $j$, compute $S_{j,i} = g(X_{j,i}, y_i)$.

$$g(X_{j,i}, y_i) = \begin{cases} +p/(1-p), & \text{if } X_{j,i} = 0, y_i = 0, \\ -p(1-p)^{K-1}/(1-(1-p)^K), & \text{if } X_{j,i} = 0, y_i = 1, \\ -1, & \text{if } X_{j,i} = 1, y_i = 0, \\ +(1-p)^K/(1-(1-p)^K), & \text{if } X_{j,i} = 1, y_i = 1. \end{cases}$$

   2b. Mark person $j$ infected iff $\sum_i S_{j,i} > Z$ (threshold).

$$Z = Z(N, K, \varepsilon, p),$$
$$T = T(N, K, \varepsilon, p).$$

# Example: $N = 8$, $K = 3$, $\varepsilon = 10^{-2}$

1. An algorithm to construct group testing matrices
   1a. For each test $i$, person $j$, choose $X_{j,i} = 1$ with prob. $p$.

$$p = 0.25\dots.$$

2. An algorithm to link test results to infected people
   2a. For each test $i$, person $j$, compute $S_{j,i} = g(X_{j,i}, y_i)$.

$$g(X_{j,i}, y_i) = \begin{cases} +0.34\dots, & \text{if } X_{j,i} = 0, y_i = 0, \\ -0.24\dots, & \text{if } X_{j,i} = 0, y_i = 1, \\ -1, & \text{if } X_{j,i} = 1, y_i = 0, \\ +0.69\dots, & \text{if } X_{j,i} = 1, y_i = 1. \end{cases}$$

   2b. Mark person $j$ infected iff $\sum_i S_{j,i} > Z$ (threshold).

$$Z = Z(N, K, \varepsilon, p),$$
$$T = T(N, K, \varepsilon, p).$$

TU/e

# Example: $N = 8$, $K = 3$, $\varepsilon = 10^{-2}$

1. An algorithm to construct group testing matrices
   1a. For each test $i$, person $j$, choose $X_{j,i} = 1$ with prob. $p$.

$$p = 0.25\ldots.$$

2. An algorithm to link test results to infected people
   2a. For each test $i$, person $j$, compute $S_{j,i} = g(X_{j,i}, y_i)$.

$$g(X_{j,i}, y_i) = \begin{cases} +0.34\ldots, & \text{if } X_{j,i} = 0, y_i = 0, \\ -0.24\ldots, & \text{if } X_{j,i} = 0, y_i = 1, \\ -1, & \text{if } X_{j,i} = 1, y_i = 0, \\ +0.69\ldots, & \text{if } X_{j,i} = 1, y_i = 1. \end{cases}$$

   2b. Mark person $j$ infected iff $\sum_i S_{j,i} > Z$ (threshold).

$$Z = 22.62\ldots,$$

$$T = 160.$$

# Example: Group testing matrix

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | $\dots$ | |
| Antonino | $X_{1,1}$ | $X_{1,2}$ | $X_{1,3}$ | $X_{1,4}$ | $X_{1,5}$ | $\dots$ | $X_{1,160}$ |
| Boris | $X_{2,1}$ | $X_{2,2}$ | $X_{2,3}$ | $X_{2,4}$ | $X_{2,5}$ | $\dots$ | $X_{2,160}$ |
| Caroline | $X_{3,1}$ | $X_{3,2}$ | $X_{3,3}$ | $X_{3,4}$ | $X_{3,5}$ | $\dots$ | $X_{3,160}$ |
| David | $X_{4,1}$ | $X_{4,2}$ | $X_{4,3}$ | $X_{4,4}$ | $X_{4,5}$ | $\dots$ | $X_{4,160}$ |
| Eve | $X_{5,1}$ | $X_{5,2}$ | $X_{5,3}$ | $X_{5,4}$ | $X_{5,5}$ | $\dots$ | $X_{5,160}$ |
| Fred | $X_{6,1}$ | $X_{6,2}$ | $X_{6,3}$ | $X_{6,4}$ | $X_{6,5}$ | $\dots$ | $X_{6,160}$ |
| Gábor | $X_{7,1}$ | $X_{7,2}$ | $X_{7,3}$ | $X_{7,4}$ | $X_{7,5}$ | $\dots$ | $X_{7,160}$ |
| Henry | $X_{8,1}$ | $X_{8,2}$ | $X_{8,3}$ | $X_{8,4}$ | $X_{8,5}$ | $\dots$ | $X_{8,160}$ |
| Copy | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $\dots$ | $y_{160}$ |

# Example: Group testing matrix

1a. For each test $i$, person $j$, set $X_{j,i} = 1$ with prob. $p$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | … | |
| Antonino | $X_{1,1}$ | $X_{1,2}$ | $X_{1,3}$ | $X_{1,4}$ | $X_{1,5}$ | … | $X_{1,160}$ |
| Boris | $X_{2,1}$ | $X_{2,2}$ | $X_{2,3}$ | $X_{2,4}$ | $X_{2,5}$ | … | $X_{2,160}$ |
| Caroline | $X_{3,1}$ | $X_{3,2}$ | $X_{3,3}$ | $X_{3,4}$ | $X_{3,5}$ | … | $X_{3,160}$ |
| David | $X_{4,1}$ | $X_{4,2}$ | $X_{4,3}$ | $X_{4,4}$ | $X_{4,5}$ | … | $X_{4,160}$ |
| Eve | $X_{5,1}$ | $X_{5,2}$ | $X_{5,3}$ | $X_{5,4}$ | $X_{5,5}$ | … | $X_{5,160}$ |
| Fred | $X_{6,1}$ | $X_{6,2}$ | $X_{6,3}$ | $X_{6,4}$ | $X_{6,5}$ | … | $X_{6,160}$ |
| Gábor | $X_{7,1}$ | $X_{7,2}$ | $X_{7,3}$ | $X_{7,4}$ | $X_{7,5}$ | … | $X_{7,160}$ |
| Henry | $X_{8,1}$ | $X_{8,2}$ | $X_{8,3}$ | $X_{8,4}$ | $X_{8,5}$ | … | $X_{8,160}$ |
| Copy | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | … | $y_{160}$ |

# Example: Group testing matrix

1a. For each test $i$, person $j$, set $X_{j,i} = 1$ with prob. $p$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | ... | |
| Antonino | 0 | 0 | 0 | 0 | 0 | ... | 0 |
| Boris | 1 | 0 | 1 | 1 | 1 | ... | 1 |
| Caroline | 0 | 0 | 0 | 1 | 0 | ... | 0 |
| David | 0 | 0 | 1 | 1 | 1 | ... | 0 |
| Eve | 0 | 0 | 0 | 0 | 0 | ... | 0 |
| Fred | 1 | 0 | 1 | 0 | 0 | ... | 0 |
| Gábor | 0 | 0 | 1 | 0 | 0 | ... | 0 |
| Henry | 0 | 0 | 0 | 0 | 1 | ... | 0 |
| Copy | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | ... | $y_{160}$ |

# Example: Running the tests

Infected samples determine the test results.

| | | | | | | ... | |
|---|---|---|---|---|---|---|---|
| Antonino | . | . | . | . | . | ... | . |
| Boris | . | . | . | . | . | ... | . |
| Caroline | 0 | 0 | 0 | 1 | 0 | ... | 0 |
| David | . | . | . | . | . | ... | . |
| Eve | 0 | 0 | 0 | 0 | 0 | ... | 0 |
| Fred | . | . | . | . | . | ... | . |
| Gábor | . | . | . | . | . | ... | . |
| Henry | 0 | 0 | 0 | 0 | 1 | ... | 0 |
| Results | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | ... | $y_{160}$ |

$$\text{Infected} = \{\text{Caroline}, \text{Eve}, \text{Henry}\}$$

# Example: Running the tests

Infected samples determine the test results.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | ... | |
| Antonino | . | . | . | . | . | ... | . |
| Boris | . | . | . | . | . | ... | . |
| Caroline | 0 | 0 | 0 | 1 | 0 | ... | 0 |
| David | . | . | . | . | . | ... | . |
| Eve | 0 | 0 | 0 | 0 | 0 | ... | 0 |
| Fred | . | . | . | . | . | ... | . |
| Gábor | . | . | . | . | . | ... | . |
| Henry | 0 | 0 | 0 | 0 | 1 | ... | 0 |
| Results | 0 | 0 | 0 | 1 | 1 | ... | 0 |

$$\text{Infected} = \{\text{Caroline}, \text{Eve}, \text{Henry}\}$$

# Example: Scores

We perform the tests and the results come back.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | ... | |
| Antonino | 0 | 0 | 0 | 0 | 0 | ... | 0 |
| Boris | 1 | 0 | 1 | 1 | 1 | ... | 1 |
| Caroline | 0 | 0 | 0 | 1 | 0 | ... | 0 |
| David | 0 | 0 | 1 | 1 | 1 | ... | 0 |
| Eve | 0 | 0 | 0 | 0 | 0 | ... | 0 |
| Fred | 1 | 0 | 1 | 0 | 0 | ... | 0 |
| Gábor | 0 | 0 | 1 | 0 | 0 | ... | 0 |
| Henry | 0 | 0 | 0 | 0 | 1 | ... | 0 |
| Results | 0 | 0 | 0 | 1 | 1 | ... | 0 |

$$\text{Infected} = \{\text{Caroline}, \text{Eve}, \text{Henry}\}$$

# Example: Scores

2a. For each test $i$, person $j$, compute $S_{j,i} = g(X_{j,i}, y_i)$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | ... | |
| Antonino | 0 | 0 | 0 | 0 | 0 | ... | 0 |
| Boris | 1 | 0 | 1 | 1 | 1 | ... | 1 |
| Caroline | 0 | 0 | 0 | 1 | 0 | ... | 0 |
| David | 0 | 0 | 1 | 1 | 1 | ... | 0 |
| Eve | 0 | 0 | 0 | 0 | 0 | ... | 0 |
| Fred | 1 | 0 | 1 | 0 | 0 | ... | 0 |
| Gábor | 0 | 0 | 1 | 0 | 0 | ... | 0 |
| Henry | 0 | 0 | 0 | 0 | 1 | ... | 0 |
| Results | 0 | 0 | 0 | 1 | 1 | ... | 0 |

$$\text{Infected} = \{\text{Caroline}, \text{Eve}, \text{Henry}\}$$

TU/e

# Example: Scores

2a. For each test $i$, person $j$, compute $S_{j,i} = g(X_{j,i}, y_i)$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Antonino | +0.3 | +0.3 | +0.3 | −0.2 | −0.2 | ... | +0.3 |
| Boris | −1.0 | +0.3 | −1.0 | +0.7 | +0.7 | ... | −1.0 |
| Caroline | +0.3 | +0.3 | +0.3 | +0.7 | −0.2 | ... | +0.3 |
| David | +0.3 | +0.3 | −1.0 | +0.7 | +0.7 | ... | +0.3 |
| Eve | +0.3 | +0.3 | +0.3 | −0.2 | −0.2 | ... | +0.3 |
| Fred | −1.0 | +0.3 | −1.0 | −0.2 | −0.2 | ... | +0.3 |
| Gábor | +0.3 | +0.3 | −1.0 | −0.2 | −0.2 | ... | +0.3 |
| Henry | +0.3 | +0.3 | +0.3 | −0.2 | +0.7 | ... | +0.3 |
| Results | 0 | 0 | 0 | 1 | 1 | ... | 0 |

$$\text{Infected} = \{\text{Caroline}, \text{Eve}, \text{Henry}\}$$

# Example: Scores

2b. Mark person $j$ infected iff $\sum_i S_{j,i} > Z$ (threshold).

| |  |  |  |  |  | ... |  | $\sum_i S_{j,i}$ |
|---|---|---|---|---|---|---|---|---|
| Antonino | +0.3 | +0.3 | +0.3 | −0.2 | −0.2 | ... | +0.3 | 0 |
| Boris | −1.0 | +0.3 | −1.0 | +0.7 | +0.7 | ... | −1.0 | 0 |
| Caroline | +0.3 | +0.3 | +0.3 | +0.7 | −0.2 | ... | +0.3 | 0 |
| David | +0.3 | +0.3 | −1.0 | +0.7 | +0.7 | ... | +0.3 | 0 |
| Eve | +0.3 | +0.3 | +0.3 | −0.2 | −0.2 | ... | +0.3 | 0 |
| Fred | −1.0 | +0.3 | −1.0 | −0.2 | −0.2 | ... | +0.3 | 0 |
| Gábor | +0.3 | +0.3 | −1.0 | −0.2 | −0.2 | ... | +0.3 | 0 |
| Henry | +0.3 | +0.3 | +0.3 | −0.2 | +0.7 | ... | +0.3 | 0 |
| Results | 0 | 0 | 0 | 1 | 1 | ... | 0 | |

Infected = {Caroline, Eve, Henry}

# Example: Scores

2b. Mark person $j$ infected iff $\sum_i S_{j,i} > Z$ (threshold).

| | | | | | | ... | | $\sum_i S_{j,i}$ |
|---|---|---|---|---|---|---|---|---|
| Antonino | +0.3 | +0.3 | +0.3 | −0.2 | −0.2 | ... | +0.3 | −5 |
| Boris | −1.0 | +0.3 | −1.0 | +0.7 | +0.7 | ... | −1.0 | −12 |
| Caroline | +0.3 | +0.3 | +0.3 | +0.7 | −0.2 | ... | +0.3 | +41 |
| David | +0.3 | +0.3 | −1.0 | +0.7 | +0.7 | ... | +0.3 | −3 |
| Eve | +0.3 | +0.3 | +0.3 | −0.2 | −0.2 | ... | +0.3 | +38 |
| Fred | −1.0 | +0.3 | −1.0 | −0.2 | −0.2 | ... | +0.3 | +10 |
| Gábor | +0.3 | +0.3 | −1.0 | −0.2 | −0.2 | ... | +0.3 | −1 |
| Henry | +0.3 | +0.3 | +0.3 | −0.2 | +0.7 | ... | +0.3 | +40 |
| Results | 0 | 0 | 0 | 1 | 1 | ... | 0 | |

Infected = {Caroline, Eve, Henry}

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# Example: Scores

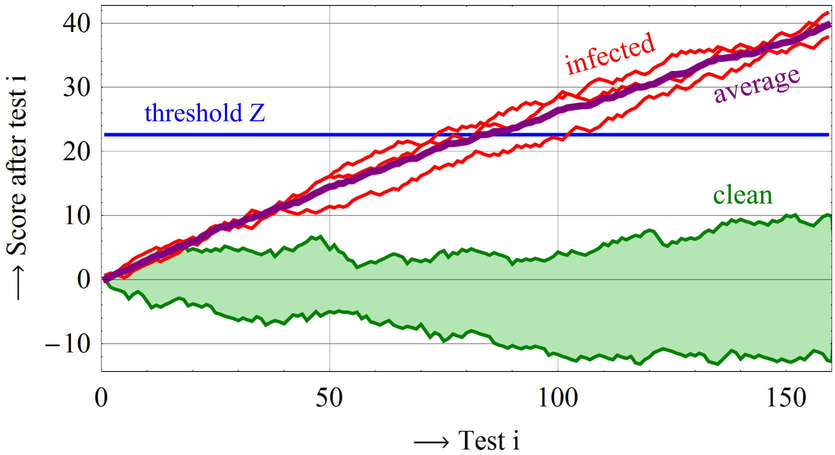
2b. Mark person $j$ infected iff $\sum_i S_{j,i} > Z$ (threshold).

| | | | | | | ... | | $\sum_i S_{j,i}$ |
|---|---|---|---|---|---|---|---|---|
| Antonino | $+0.3$ | $+0.3$ | $+0.3$ | $-0.2$ | $-0.2$ | ... | $+0.3$ | $-5$ |
| Boris | $-1.0$ | $+0.3$ | $-1.0$ | $+0.7$ | $+0.7$ | ... | $-1.0$ | $-12$ |
| Caroline | $+0.3$ | $+0.3$ | $+0.3$ | $+0.7$ | $-0.2$ | ... | $+0.3$ | $+41$ |
| David | $+0.3$ | $+0.3$ | $-1.0$ | $+0.7$ | $+0.7$ | ... | $+0.3$ | $-3$ |
| Eve | $+0.3$ | $+0.3$ | $+0.3$ | $-0.2$ | $-0.2$ | ... | $+0.3$ | $+38$ |
| Fred | $-1.0$ | $+0.3$ | $-1.0$ | $-0.2$ | $-0.2$ | ... | $+0.3$ | $+10$ |
| Gábor | $+0.3$ | $+0.3$ | $-1.0$ | $-0.2$ | $-0.2$ | ... | $+0.3$ | $-1$ |
| Henry | $+0.3$ | $+0.3$ | $+0.3$ | $-0.2$ | $+0.7$ | ... | $+0.3$ | $+40$ |
| Results | 0 | 0 | 0 | 1 | 1 | ... | 0 | |

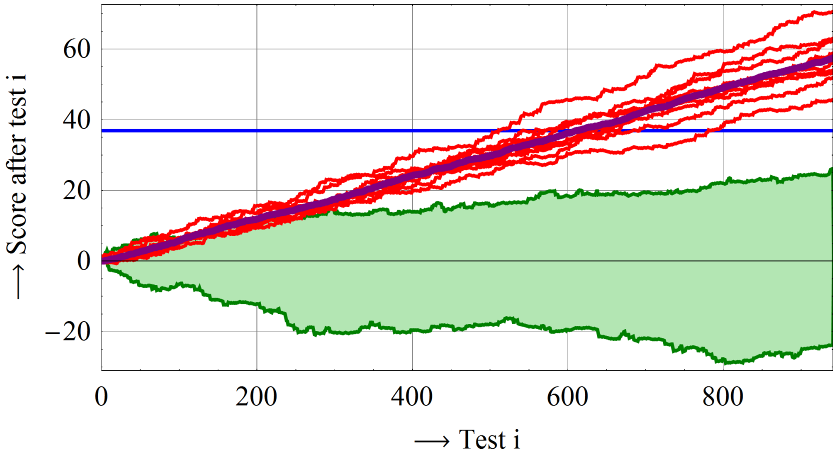
Infected = {Caroline, Eve, Henry}

Marked = {Caroline, Eve, Henry}

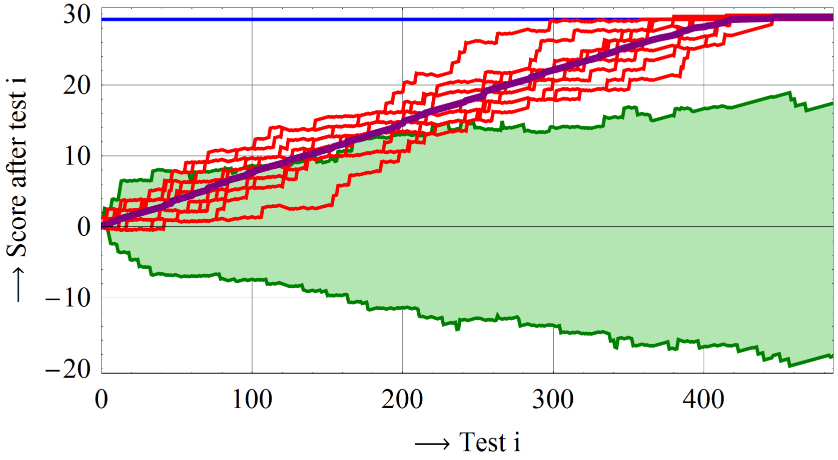
# Example: Scores



# Larger Example: Non-Adaptive



# Larger Example: Adaptive



# Framework: Other Models

Traditional group testing

- Positive test result iff at least one tested is infected

Noisy group testing

- Dilution: Clean sample testing positive
- Additive: Infected sample testing negative
- Combined: Any wrong test result
- ...

Threshold group testing

- Majority: Positive iff more than $\ell$ infected
- Bernoulli: Few infected tested, random result
- Linear: More infected, more positive results
- ...

# Framework: Other Models

Traditional group testing

- Tests required: $T \sim 2K \ln N$

Noisy group testing

- Dilution: $T \sim 2K \ln N/(1-r)$
- Additive: $T \sim 2K \ln N/(1-\sqrt{2r})$
- Combined: $T \sim 2K \ln N/(1-\sqrt{2r})$
- ...

Threshold group testing

- Majority: $T \sim \pi K \ln N$
- Bernoulli: $T \sim 4K \ln N$
- Linear: $T \sim 2K^2 \ln N$
- ...

# Conclusion

Framework for probabilistic group testing

- Score-based construction
- Speed-ups in the adaptive setting (see paper)
- Versatile construction (see paper)

Results when applied to common models:

- Traditional model: $T \sim 2K \ln N$
- Dilution noise: $T \sim 2K \ln N/(1-r)$
- Additive noise: $T \sim 2K \ln N/(1-\sqrt{2r})$
- Threshold models: $T = \Theta(K \ln N)$

# Questions?